User Manual for **BBCPOP**: A Sparse Doubly Nonnegative Relaxation of Polynomial Optimization Problems with Binary, Box and Complementarity Constraints

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March 2018

**Abstract**

BBCPOP proposed in [4] is a MATLAB implementation of a hierarchy of sparse doubly nonnegative (DNN) relaxations of a class of polynomial optimization (minimization) problems (POPs) with binary, box and complementarity constraints. Given a POP in the class and a relaxation order (or a hierarchy level), BBCPOP constructs a simple conic optimization problem (COP), which serves as a DNN relaxation of the POP, and then solves the COP by applying the bisection and projection (BP) method [6, 5]. The software package **BBCPOP**, this manual, and a test set of POPs are available at https://sites.google.com/site/bbcpop1/.

**Key words.** Polynomial optimization problems, Doubly nonnegative relaxation, Bisection and projection method, Large-scale problems, MATLAB software package.

**AMS Classification.** 90C20, 90C22, 90C25, 90C26.

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1 Introduction

BBCPOP proposed in [4] is a MATLAB software package to compute tight lower bounds for global optimal values of a class of polynomial optimization (minimization) problems (POPs) with binary, box and complementarity constraints. As shown in Figure 1, BBCPOP.m constructs a hierarchy of sparse DNN relaxations of a POP in the class and solves them by applying the bisection and projection method [6, 1, 5] and the accelerated proximal gradient (APG) method [2]. These two methods were described as BP Algorithm and APGR Algorithm (an enhanced version of the APG method) in [4], respectively.

Let \( f_0 \) be a real valued polynomial defined on the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), \( I_{\text{bin}} \) and \( I_{\text{box}} \) a partition of \( N \equiv \{1, 2, \ldots, n\} \), i.e., \( I_{\text{bin}} \cup I_{\text{box}} = N \) and \( I_{\text{bin}} \cap I_{\text{box}} = \emptyset \), and \( C \) a family of subsets of \( N \). Each POP in the class is described as

\[
\zeta^* = \min_{x} \left\{ f_0(x) \mid x_i \in \{0, 1\} \ (i \in I_{\text{bin}}) \ (\text{box constraint}), \ x_j \in [0, 1] \ (j \in I_{\text{box}}) \ (\text{binary constraint}), \ \prod_{j \in C} x_j = 0 \ (C \in C) \ (\text{complementarity constraint}) \right\}.
\]

As an illustrative example, we consider the following POP with five variables \( x_1, x_2, x_3, x_4 \) and \( x_5 \):

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \equiv 0.5x_1 - 1.8x_3 - 2.2x_5 + 3x_3^2 + x_1x_2x_4 + 1.3x_2x_4x_5 \\
\text{subject to} & \quad x_2x_3 = 0, \ x_3x_4 = 0, \ x_1, x_2 \in \{0, 1\}, \ x_3, x_4, x_5 \in [0, 1]
\end{align*}
\]

In this case, \( I_{\text{bin}} = \{1, 2\} \), \( I_{\text{box}} = \{3, 4, 5\} \) and \( C = \{\{2, 3\}, \{3, 4\}\} \). We can easily compute the optimal solution \( x^* = (0, 0, 0, 3, 0, 1) \) and the optimal value \( \zeta^* = -2.47 \).

The input of BBCPOP.m consists of the data for POP (1), the relaxation order \( \omega \) which determines the hierarchy level of DNN relaxation to be constructed, and parameters which control the execution of BBCPOP.m. With the input data, BBCPOP.m constructs a simple conic optimization problem (COP):

\[
y_0^* = \max_{y_0, Y_1, Y_2} \{ y_0 \mid Q_0 - y_0H_0 = Y_1 + Y_2, \ Y_1 \in K_1^*, \ Y_2 \in K_2^* \},
\]
which serves as a DNN relaxation of POP (1) (hence \( y_0^* \leq \zeta^* \)). Here \( H_0 \) denotes a constant vector in a linear space \( V \) (= the Cartesian product of symmetric matrix spaces) endowed with an inner product, \( K_1, K_2 \subset V \) closed convex cones, and \( y_0 \in \mathbb{R}, Y_1 \in V, Y_2 \in V \) variables. The output of BBCPOP.m is an approximate optimal solution \((y_0, Y_1, Y_2)\) of (3) such that \( y_0 \leq y_0^* \); hence \( y_0 \leq \zeta^* \) is guaranteed.

The degree of POP (1) is defined as \( \max \{ \deg f_0, |C| \ (C \in C) \} \), where \( |C| \) denotes the number of elements in \( C \ (C \in C) \). The relaxation order \( \omega \) needs to be a positive integer not less than the half of the degree of POP (1). Hence the minimum relaxation order that can be taken is

\[ \omega_{\text{min}} = \text{the smallest positive integer not less than the half of the degree of POP (1)} \]

As we take a larger \( \omega \), we can expect a tighter lower bound \( y_0 \) (= the approximate optimal value of (3)) for \( \zeta^* \), but COP (3) to be solved by the BP Algorithm becomes larger, so that longer execution time is required. For many application problems in practice, taking \( \omega_{\text{min}} \) for the relaxation order \( \omega \) is sufficient to obtain a tight lower bound for \( \zeta^* \).

In the above example (2), we see that \( \deg f_0 = 3, |\{2, 3\}| = 2 \) and \( |\{3, 4\}| = 2 \). Hence \( \omega_{\text{min}} = 2 \leq \omega \).

In Section 2, we present how to describe POP (1) for the input of the main function BBCPOP.m. In Section 3, we illustrate the execution of BBCPOP.m and present the details on the output of BBCPOP.m. Section 4 lists the parameters which control the execution of BBCPOP.m, and Section 5 some main functions contained in the BBCPOP software package.

## 2 Representing polynomial optimization problems

POP (1) is described by

\[ \text{objPoly, l01 and lcomp,} \]

which are input arguments of BBCPOP.m. Let \( \text{term} f_0 \) denote the number of terms of \( f_0 \). We employ a simplified SparsePOP format to describe the objective function of POP (1):

\[ \text{objPoly.supports} = \text{a set of supports of } f_0(x), \quad \text{term} f_0 \times n \text{ matrix.} \]

\[ \text{objPoly.coef} = \text{coefficients, column vector of dimension term} f_0. \]

For the original SparsePOP format, see [7]. The functions simplifyPoly.m, addPoly.m and multiplyPoly.m in the directory polyTools/ can be used when describing an objective function \( f_0(x) \) in the simplified SparsePOP format.

The partition \( I_{\text{bin}} \cup I_{\text{box}} \) of \( N = \{1, 2, \ldots, n\} \) is described by the \( n \)-dimensional row vector \( \text{l01} \) such that

\[ \text{l01}_j = \begin{cases} \text{true} & \text{if } j \in I_{\text{bin}}, \\ \text{false} & \text{otherwise, i.e., } j \in I_{\text{box}}. \end{cases} \]

The family \( C \) of subsets of \( N \), which represents the complementarity condition in POP (1), is represented by \( \text{lcomp} \). Suppose that \( C \) consists of \( m \) nonempty subsets \( C_1, \ldots, C_m \) of \( N \). Then we set \( \text{lcomp} \) to be the \( m \times n \) matrix such that

\[ \text{lcomp}_{ij} = \begin{cases} \text{true} & \text{if } j \in C_i, \\ \text{false} & \text{otherwise, i.e., } j \notin C_i \end{cases} \]
\( i = 1, \ldots, m, \ j = 1, \ldots, n \). If \( C = \emptyset \), set \( \text{lcomp} = [ ] \).

The three input arguments \( \text{objPoly}, \text{I01} \) and \( \text{lcomp} \) for POP (2) are described as follows:

```matlab
function \([\text{objPoly}, \text{I01}, \text{lcomp}] = \text{example1};
    \text{objPoly}.supports = [...
    \[1 \ 0 \ 0 \ 0; \ 0 \ 1 \ 0 \ 0; \ 0 \ 0 \ 0 \ 1; \ 0 \ 0 \ 2 \ 0; \ 1 \ 1 \ 0 \ 1; \ 0 \ 1 \ 0 \ 1 \ 1];
    \text{objPoly}.coef = \[0.5; \ -1.8; \ -2.2; \ 3; \ 1; \ 1.3];
    \text{lcomp} = \text{logical}([0 \ 1 \ 1 \ 0 \ 0; \ 0 \ 0 \ 1 \ 1 \ 0]);
    \text{I01} = \text{logical}([1 \ 1 \ 0 \ 0 \ 0]);
end
```

3 Executing BBCPOP

A lower bound for the optimal value of POP (2) can be computed by BBCPOP.m as follows:

```matlab
>>[\text{objPoly}, \text{I01}, \text{lcomp}] = \text{example1};
>>\text{relaxOrder} = 2; \text{params} = []; \% the default parameters are used
>>[\text{sol}, \text{info}] = \text{BBCPOP(\text{objPoly}, \text{I01}, \text{lcomp}, \text{relaxOrder}, \text{params});}
```

The following is shown on the screen as the BBCPOP.m terminates at 17 iterations.

Original
iter=17:y0=-2.469903e+00, LBv=-2.470066e+00, [LB, UB]=[-2.470066e+00, -2.469903e+00]

Scaled

<table>
<thead>
<tr>
<th>LBv</th>
<th>y0</th>
<th>UB</th>
<th>UB-LBv</th>
<th>relnormX</th>
<th>iter</th>
<th>b_yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1.000000e+21</td>
<td>+5.800000e+00</td>
<td>+5.800000e+00</td>
<td>+9.80e+00</td>
<td>101</td>
<td>90</td>
</tr>
<tr>
<td>1</td>
<td>-4.536533e+01</td>
<td>+5.800000e+00</td>
<td>+5.800000e+00</td>
<td>+9.80e+00</td>
<td>4.90e+00</td>
<td>431</td>
</tr>
<tr>
<td>2</td>
<td>-1.867610e+01</td>
<td>+9.000000e-01</td>
<td>+9.000000e-01</td>
<td>+4.90e+00</td>
<td>4.31e-01</td>
<td>56</td>
</tr>
<tr>
<td>3</td>
<td>-6.974229e+00</td>
<td>-1.550000e+00</td>
<td>-1.550000e+00</td>
<td>+2.45e+00</td>
<td>9.58e-02</td>
<td>218</td>
</tr>
<tr>
<td>4</td>
<td>-2.775000e+00</td>
<td>-2.775000e+00</td>
<td>-2.775000e+00</td>
<td>+1.22e+00</td>
<td>0.00e+00</td>
<td>17</td>
</tr>
<tr>
<td>5</td>
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<td>-2.162500e+00</td>
<td>-2.162500e+00</td>
<td>+6.12e-02</td>
<td>3.06e-02</td>
<td>201</td>
</tr>
<tr>
<td>6</td>
<td>-2.476112e+00</td>
<td>-2.468750e+00</td>
<td>-2.468750e+00</td>
<td>+3.14e-01</td>
<td>1.21e-04</td>
<td>350</td>
</tr>
<tr>
<td>7</td>
<td>-2.476112e+00</td>
<td>-2.319306e+00</td>
<td>-2.319306e+00</td>
<td>+1.57e-01</td>
<td>1.48e-02</td>
<td>274</td>
</tr>
<tr>
<td>8</td>
<td>-2.476112e+00</td>
<td>-2.397709e+00</td>
<td>-2.397709e+00</td>
<td>+7.84e-02</td>
<td>7.06e-03</td>
<td>180</td>
</tr>
<tr>
<td>9</td>
<td>-2.476112e+00</td>
<td>-2.436911e+00</td>
<td>-2.436911e+00</td>
<td>+3.92e-02</td>
<td>3.22e-03</td>
<td>301</td>
</tr>
<tr>
<td>10</td>
<td>-2.476112e+00</td>
<td>-2.456512e+00</td>
<td>-2.456512e+00</td>
<td>+1.96e-02</td>
<td>1.31e-03</td>
<td>294</td>
</tr>
<tr>
<td>11</td>
<td>-2.476112e+00</td>
<td>-2.466312e+00</td>
<td>-2.466312e+00</td>
<td>+9.80e-03</td>
<td>3.58e-04</td>
<td>301</td>
</tr>
<tr>
<td>12</td>
<td>-2.471212e+00</td>
<td>-2.466312e+00</td>
<td>-2.466312e+00</td>
<td>+4.90e-03</td>
<td>0.00e+00</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>-2.471212e+00</td>
<td>-2.468762e+00</td>
<td>-2.468762e+00</td>
<td>+2.45e-03</td>
<td>1.20e-04</td>
<td>301</td>
</tr>
<tr>
<td>14</td>
<td>-2.470066e+00</td>
<td>-2.469987e+00</td>
<td>-2.469987e+00</td>
<td>+1.30e-03</td>
<td>1.29e-06</td>
<td>339</td>
</tr>
</tbody>
</table>
The value $-2.470039e+00$ is the approximate optimal value of (2) obtained by BP Algorithm, which is a valid lower bound of the optimal value $\zeta^* = -2.47$ of POP (2).

The output sol is a structure containing the solution information for (3):

- $y0init$: 2.1272
- $LBv$: -2.4700
- $LB$: -2.4700
- $UB$: -2.4700
- $Y1$: [1226 double]
- $Y2$: [1226 double]

Here $y0init$ corresponds to the initial value of $y^u_0$, and $LBv$, $LB$, $UB$, $Y1$, $Y2$ to the terminal values of $y^v_0$, $y^e$, $y^u_0$, $\hat{Y}_1$ and $\hat{Y}_2$, respectively, in BP Algorithm in [4]. The output info is a structure for some of the execution information of BP.m (BP Algorithm) and APGR.m (APGR Algorithm):

- iterBP: 17
- timeBP: 1.5884
- termcodeBP: 2
- iterAPGR: 3545

Here iterBP denotes the number of iteration of BP.m, timeBP its execution time, termcodeBP its termination code and iterAPGR the total number of iterations of APGR.m. BP.m stops when

(i) the difference of the upper bound $y^u_0$ and the lower bound $y^e_0$ becomes smaller than the prescribed parameter params.delta,

(ii) $(y^u_0 - y^e_0)/\max\{1.0, |y^e_0|, |y^u_0|\}$ is smaller than the prescribed parameter params.delta2

(iii) the reduction of the length of the interval $[y^u_0 - y^e_0]$ gets smaller than $\max\{\text{params.delta}, \text{params.delta2}\}$.

(iv) the iteration of BP.m exceeds the prescribed parameter params.maxiterBP, or

(v) the execution time exceeds the prescribed parameter params.maxtimeBP.

These terminations are associated with the termination code 1,2,3,0,-1, respectively.

4 Parameters

In addition to objPoly, l01, lcomp and relaxOrder for describing a POP, the MATLAB function BBCPOP.m has params as an input argument. It is a structure consisting of many parameters that control the performance of the function. Table [1] shows the list of parameters used in BBCPOP.m. The default values of parameters are given in the MATLAB function defaultParamBP.m. They can be modified if necessary.
Table 1: The fields of \textit{params}, default values and possible values

<table>
<thead>
<tr>
<th>field of params</th>
<th>default</th>
<th>usage, possible values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters for DNN relaxation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>sparseSW</td>
<td>1</td>
<td>Set 0 for dense DNN relaxation. Set 1 for sparse DNN relaxation.</td>
</tr>
<tr>
<td>Parameters for BP and APGR Algorithms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>maxtimeBP</td>
<td>20000</td>
<td>the maximum execution time</td>
</tr>
<tr>
<td>maxiterBP</td>
<td>40</td>
<td>the maximum iteration for BP Algorithm</td>
</tr>
<tr>
<td>maxiterAPGR</td>
<td>20000</td>
<td>the maximum iteration for APGR Algorithm</td>
</tr>
<tr>
<td>delta1</td>
<td>1e-4</td>
<td>the relative tolerance for BP Algorithm Set 0 if $\delta$ is used.</td>
</tr>
<tr>
<td>delta</td>
<td>0</td>
<td>the absolute tolerance for BP Algorithm Set $\delta \in (0,1)$ if $\zeta^*$ is integer. Set 0 otherwise.</td>
</tr>
<tr>
<td>printyes</td>
<td>2</td>
<td>print level, 0,1,2 or 3</td>
</tr>
<tr>
<td>UbdObjVal</td>
<td>$f_0(0)$</td>
<td>the upper bound for the optimal value $\zeta^*$ of POP (1)</td>
</tr>
<tr>
<td>UbdIX</td>
<td>$= \rho$ given in Assumption (A1) of [4]. To specify this parameter, see Sections 2.3 and 4.1 of [4].</td>
<td></td>
</tr>
</tbody>
</table>

5 Some functions

We list some functions contained in the BBCPOP software package. Important subfunctions of BBCPOP.m are listed in Section 5.1, and functions for some test instances are listed in Section 5.2.

5.1 Main subfunctions of BBCPOP.m

relaxation/BBCPOPtoDNN.m constructs COP (3), which serves as a DNN relaxation of [1].
solver/solveDNN.m solves COP (3) by applying BP Algorithm and APGR Algorithm.
solver/BP/BP.m is an implementation of BP Algorithm in [4].
solver/BP/APGR.m is an implementation of APGR Algorithm in [4].

5.2 Functions for POP instances

All the functions below output \textit{objPoly}, \textit{l01}, \textit{lcomp}, \textit{relaxOrder} $= \omega_{\text{min}}$ and \textit{params}, which can be used as input of BBCPOP.m.

instances/POPrandom/genPOPdense.m generates a dense POP instance with binary, box and complementarity constraints.
\[ \text{instances/POPrandom/genPOParrow.m} \text{ generates a sparse POP instance (whose objective function } f_0(x) \text{ has an arrow type sparsity pattern Hessian matrix) with binary, box and complementarity constraints.} \]

\[ \text{instances/POPrandom/genPOPchordal.m} \text{ generates a sparse POP instance (whose objective function } f_0(x) \text{ has a sparse Hessian matrix characterized by a chordal graph) with binary, box and complementarity constraints.} \]

\[ \text{instances/QAP/qapreadBP.m} \text{ generates a QOP with box and complementarity constraints which is induced from a Lagrangian relaxation of a QAP instance from QAPLIB [3].} \]

\[ \text{instances/BIQ/biqreadBP.m} \text{ generates a QOP with box and complementarity constraints which is induced from a Lagrangian relaxation of a binary/box constrained QOP instance from BIQMAC [8].} \]

\begin{enumerate}
\item \[ \text{References} \]
\end{enumerate}


