Variational Analysis and Optimization of Sweeping Processes with Controlled Moving Sets

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Abstract. This paper briefly overviews some recent and very fresh results on a rather new class of dynamic optimization problems governed by the so-called sweeping (Moreau) processes with controlled moving sets. Uncontrolled sweeping processes have been known in dynamical systems and applications starting from 1970s while control problems for them have drawn attention of mathematicians, applied scientists, and practitioners quite recently. We discuss here such problems and major results achieved in their theory and applications.

Key words. Variational analysis, dynamic optimization, sweeping processes, optimal control, discrete approximations, generalized differentiation

AMS subject classifications. 49J52, 49J53, 49K24, 49M25, 90C30

1 Introduction and Initial Discussions

The basic sweeping process ("processus du rafle") was introduced by Jean-Jacques Moreau in the 1970s to describe some quasistatic mechanical problems; see [34] and the book [27] for more details. Besides the original motivations, models of this type have found significant applications to elastoplasticity [19], hysteresis [24, 23], electric circuits [1], traffic equilibria [26, 38], and various other areas of applied sciences and operations research. For its own sake, sweeping process theory has become an important area of nonlinear and variational analysis with numerous mathematical achievements and challenging open questions; see, e.g., [15, 25] and the references therein.

The basic sweeping process is described by the dissipative differential inclusion

$$\dot{x}(t) \in -N\big(x(t); C(t)\big) \quad \text{a.e.} \quad t \in [0, T], \tag{1}$$

where $N(x;\Omega)$ stands for the normal cone to a convex set $\Omega \subset \mathbb{R}^n$ at x defined by

$$N(x;\Omega) := \begin{cases} \big\{v \in \mathbb{R}^n \big|\ \langle v, u - x\rangle \le 0 \ \text{ for all } \ u \in \Omega\big\} & \text{if } \ x \in \Omega, \\ \emptyset & \text{otherwise,} \end{cases} \tag{2}$$

and where the convex variable set $C(t)$ continuously evolves in time. It has been realized that the Cauchy problem $x(0) = x_0 \in C(0)$ for (1) admits a *unique* solution (see, e.g., [15]), and hence there is no sense to consider optimization problems for the sweeping differential inclusion (1). This is totally different from the well-developed optimal control theory for Lipschitzian differential inclusions of the type

$$\dot{x}(t) \in F\big(x(t)\big) \quad \text{a.e.} \quad t \in [0, T], \tag{3}$$

which arises from the classical one for controlled differential equations

$$\dot{x}(t) = f\big(x(t), u(t)\big), \quad u(t) \in U \quad \text{a.e.} \quad t \in [0, T] \tag{4}$$

with $F(x) := f(x, U) = \{y \in \mathbb{R}^n |\ y = f(x, u) \ \text{ for some } \ u \in U\}$ in (3); see, e.g., the books [29, 39] with the references therein as well as more recent publications devoted to optimal control of (3).

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To the best of our knowledge, there are three approaches in the literature to introduce control actions in the sweeping process frameworks and then to conduct optimization with respect to these controls and the corresponding sweeping trajectories. The *first approach* considers controls in additive *perturbations* on the right-hand side of (1) without changing the moving set $C(t)$. The results obtained in this direction mostly concern existence theorems and relaxation procedures while not optimality conditions; see [18] and the recent papers [3, 11] with the references therein. The *second approach* developed in [6] and then partly extended in [2] introduces controls in an ordinary differential equation associated with the sweeping process over a *given* set $C(t) \subset \mathbb{R}^n$. The obtained results provide necessary optimality conditions for the continuous-time problem in [6] and for the approximating finite-difference systems in [2].

The *third approach* to optimal control of the sweeping process (1), as well as its modifications and extensions, employs a control parametrization *directly in the sweeping set* $C(t)$ making it dependent on control actions. It has been initiated in [12] for the case of a controlled hyperplane $C(t)$ and then has been developed in a number of subsequent publications. The author has been strongly involved in this line of research, which applies the *method of discrete approximations* accomplished in [28, 30] for optimal control of Lipschitzian differential inclusions. Developing this method in order to cover highly non-Lipschitzian ones associated with the sweeping dynamics has been a challenging goal of our approach, which makes a bridge between finite-dimensional and infinite-dimensional optimization as well as between static and dynamic expects of optimization and control. It is largely based on employing powerful tools of first-order and second-order variational analysis and generalized differentiation.

The current paper is mostly devoted to discussions, implementations, and applications of the major results obtained in this direction. It is organized as follows. In Section 2 we present some preliminaries from variational analysis and generalized differentiation needed for the further material. Section 3 concerns optimization of sweeping processes with moving controlled sets given by polyhedra. We also discuss there optimal control models containing control actions in both moving sets and external perturbations.

Section 4 deals with optimal control problems for nonconvex sweeping processes described via uniformly prox-regular moving sets. The results presented there are motivated by applications to optimization of the crowd motion model, which is well recognized in traffic equilibria and operations research. In Section 5 we discuss new results on necessary optimality conditions in the extended Euler-Lagrange and Hamiltonian forms for controlled sweeping processes with moving sets given by inverse images.

Section 6 is devoted to applications to selected practical models arising in crowd motions, elastoplasticity, and hysteresis. In the concluding Section 7 we discuss some challenging open problems.

Throughout the paper we use standard notation of variational analysis and control theory; see, e.g., [29, 36, 39]. Recall that $\mathbb{N} := \{1, 2, \ldots\}$, that A^* stands for the transposed/adjoint matrix to A, and that $\mathbb{B}$ denotes the closed unit ball of the space in question.

2 Preliminaries from Generalized Differentiation

Employing the geometric approach to generalized differentiation [29, 31], we start with our basic concept of the normal cone to a locally closed set that induces the corresponding generalized differentiability notions for nonsmooth functions and (single-valued and set-valued) mappings. Note that our discrete approximation technique requires considering normals to nonconvex sets even in the case of sweeping processes generated by convex moving sets as in (1). The (basic, limiting, Mordukhovich) *normal cone* to an arbitrary locally closed set $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \mathbb{R}^n$ is defined by

$$N(\bar{x}; \Omega) := \begin{cases} \big\{v \in \mathbb{R}^n \big|\ \exists x_k \to \bar{x},\ \alpha_k \geq 0,\ w_k \in \Pi(x_k; \Omega),\ \alpha_k(x_k - w_k) \to v & \text{if } \bar{x} \in \Omega, \\ \emptyset & \text{otherwise,} \end{cases} \tag{5}$$

where $\Pi(x; \Omega)$ stands for the Euclidean projector of x onto Ω. When Ω is convex, the normal cone (5) reduces to the one (2) in the sense of convex analysis, but in general the multifunction $x \rightrightarrows N(x; \Omega)$ is nonconvex-valued while satisfying a *full calculus* together with the associated subdifferential of extended-real-valued functions and coderivative of set-valued mappings considered below. Such a calculus is due to *variational/extremal principles* of variational analysis; see [29, 31, 36] for more details.

Given a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ and a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$ from its graph

$$\operatorname{gph} F := \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^q \big|\ y \in F(x)\big)\big\},$$

the *coderivative* $D^*F(\bar{x},\bar{y})\colon \mathbb{R}^q \rightrightarrows \mathbb{R}^n$ of F at $(\bar{x},\bar{y})$ is defined by

$$D^*F(\bar{x},\bar{y})(u) := \big\{v \in \mathbb{R}^n \big|\ (v,-u) \in N\big((\bar{x},\bar{y}); \operatorname{gph} F\big)\big\}, \quad u \in \mathbb{R}^q, \tag{6}$$

where $\bar{y}$ is omitted in the notation if $F\colon \mathbb{R}^n \to \mathbb{R}^q$ is single-valued. If furthermore F is $\mathcal{C}^1$-smooth around $\bar{x}$ (or merely strictly differentiable at this point), we have $D^*F(\bar{x})(v) = \{\nabla F(\bar{x})^* v\}$ via the adjoint Jacobian matrix. In general, the coderivative (6) is a positively homogeneous multifunction satisfying comprehensive calculus rules and providing complete characterizations of major well-posedness properties in variational analysis related to Lipschitzian stability, metric regularity, and linear openness; see [29, 36].

For an extended-real-valued function $\phi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (\infty,\infty]$ finite at $\bar{x}$, i.e., with $\bar{x} \in \operatorname{dim}\phi$, the (first-order) *subdifferential* of ϕ at $\bar{x}$ is defined geometrically by

$$\partial\phi(\bar{x}) := \{v \in \mathbb{R}^n \big|\ (v,-1) \in N\big((\bar{x},\phi(\bar{x})); \operatorname{epi}\phi\big)\} \tag{7}$$

via the normal cone (5) to the epigraphical set $\operatorname{epi}\phi := \{(x,\alpha) \in \mathbb{R}^{n+1} \big|\ \alpha \geq \phi(x)\}$. If $\phi(x) := \delta_\Omega(x)$, the indicator function of a set Ω that equals to 0 for $x \in \Omega$ and to ∞ otherwise, we get $\partial\phi(\bar{x}) = N(\bar{x};\Omega)$. Given further $\bar{v} \in \partial\phi(\bar{x})$, the *second-order subdifferential* (or generalized Hessian) $\partial^2\phi(\bar{x},\bar{v})\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of ϕ at $\bar{x}$ relative to $\bar{v}$ is defined as the coderivative of the first-order subdifferential by

$$\partial^2\phi(\bar{x},\bar{v})(u) := (D^*\partial\phi)(\bar{x},\bar{v})(u), \quad u \in \mathbb{R}^n, \tag{8}$$

where $\bar{v} = \nabla\phi(\bar{x})$ is omitted when ϕ is differentiable at $\bar{x}$. If ϕ is $\mathcal{C}^2$-smooth around $\bar{x}$, then (8) reduces to the classical (symmetric) Hessian matrix

$$\partial^2\phi(\bar{x})(u) = \big\{\nabla^2\phi(\bar{x})u\big\} \text{ for all } u \in \mathbb{R}^n.$$

For applications in this paper we also need partial versions of the above subdifferential constructions for functions of two variables $\phi\colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$. Consider the *partial first-order subdifferential* mapping $(x,w) \mapsto \partial_x\phi(x,w)$ for $\varphi(x,w)$ with respect to x by

$$\partial_x\phi(x,w) := \big\{ \text{ set of subgradients } v \in \mathbb{R}^n \text{ of } \phi_w := \phi(\cdot,w) \text{ at } x\big\} = \partial\phi_w(x)$$

and then, picking $(\bar{x},\bar{w}) \in \operatorname{dom}\phi$ and $\bar{v} \in \partial_x\phi(\bar{x},\bar{w})$, define the *partial second-order subdifferential* of ϕ with respect to x at $(\bar{x},\bar{w})$ relative to $\bar{v}$ by

$$\partial^2_x\phi(\bar{x},\bar{w},\bar{v})(u) := \big(D^*\partial_x\phi\big)(\bar{x},\bar{w},\bar{v})(u) \text{ for all } u \in \mathbb{R}^n. \tag{9}$$

If ϕ is $\mathcal{C}^2$-smooth around $(\bar{x},\bar{w})$, we have the representation

$$\partial^2\phi(\bar{x},\bar{w})(u) = \big\{\big(\nabla^2_{xx}\phi(\bar{x},\bar{w})^*u, \nabla^2_{xw}\phi(\bar{x},\bar{w})^*u\big)\big\} \text{ for all } u \in \mathbb{R}^n.$$

Consider further the *parametric constraint system*

$$S(w) := \big\{x \in \mathbb{R}^n \big|\ \psi(x,w) \in \Theta\big\}, \quad w \in \mathbb{R}^m, \tag{10}$$

generated by a vector function $\psi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s$ and a set $\Theta \subset \mathbb{R}^s$. We associate with (10) the *normal cone mapping* $\mathcal{N}\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ by

$$\mathcal{N}(x,w) := N\big(x; S(w)\big) \text{ for } x \in S(w). \tag{11}$$

It is easy to see that the mapping $\mathcal{N}$ in (11) admits the composite representation

$$\mathcal{N}(x,w) = \partial_x\phi(x,w) \text{ with } \phi(x,w) := \big(\delta_\Theta \circ \psi\big)(x,w) \tag{12}$$

via ψ and the indicator function δ_Θ of the set Θ. It follows directly from (12) due to the second-order subdifferential construction (9) that

$$\partial_x^2\phi(\bar{x},\bar{w},\bar{v})(u)=D^*\mathcal{N}(\bar{x},\bar{w},\bar{v})(u)\ \text{ for any }\ \bar{v}\in\mathcal{N}(\bar{x},\bar{w})\ \text{ and }\ u\in\mathbb{R}^n.$$

The following second-order chain rule can be derived from [33, Theorem 3.1] applied to the composition in (12). It plays an important role in the subsequent applications to controlled sweeping processes.

Theorem 2.1 (second-order subdifferential chain rule). *Let ψ be $\mathcal{C}^2$-smooth around $(\bar{x},\bar{w})$ with the partial Jacobian matrix $\nabla_x\psi(\bar{x},\bar{w})$ of full rank. Then for each $\bar{v}\in\mathcal{N}(\bar{x},\bar{w})$ there is a unique vector $\bar{p}\in N_\Theta(\psi(\bar{x},\bar{w})):=N(\psi(\bar{x},\bar{w});\Theta)$ satisfying*

$$\nabla_x\psi(\bar{x},\bar{w})^*\bar{p}=\bar{v}$$

and such that the coderivative of the normal cone mapping is computed for all $u\in\mathbb{R}^n$ by

$$D^*\mathcal{N}(\bar{x},\bar{w},\bar{v})(u)=\begin{bmatrix}\nabla_{xx}^2\langle\bar{p},\psi\rangle(\bar{x},\bar{w})\\ \nabla_{xw}^2\langle\bar{p},\psi\rangle(\bar{x},\bar{w})\end{bmatrix}u+\nabla\psi(\bar{x},\bar{w})^*D^*N_\Theta\big(\psi(\bar{x},\bar{w}),\bar{p}\big)\big(\nabla_x\psi(\bar{x},\bar{w})u\big).$$

Thus Theorem 2.1 reduces the calculation of $D^*\mathcal{N}$ to that of D^*N_Θ, which has been computed via the given data for broad classes of sets Θ; see, e.g., [20, 31, 32, 33] for more details and references.

3 Sweeping Processes with Moving Controlled Polyhedra

In this section we discuss two different classes of optimal control problems of a *polyhedral* type. The first class contains control actions changing both normal directions and positions of polyhedra, while the other one employs actions controlling boundaries of polyhedra as well as additive external perturbations of the sweeping dynamics. Both models admit (and are largely motivated by) valuable applications.

3.1 Optimization over Shapes of Polyhedra

This part is based on [14], where the following optimal control problem (P) is considered. Given an extended-real-valued terminal cost function $\varphi\colon\mathbb{R}^n\to\overline{\mathbb{R}}$ and a running cost $\ell\colon[0,T]\times\mathbb{R}^{2(n+nm+m)}\to\overline{\mathbb{R}}$, minimize the Bolza-type functional

$$J[x,u,b]\colon=\varphi\big(x(T)\big)+\int_0^T\ell\big(t,x(t),u(t),b(t),\dot{x}(t),\dot{u}(t),\dot{b}(t)\big)\,dt \tag{13}$$

over the controlled sweeping dynamics described by

$$\dot{x}(t)\in-N\big(x(t);C(t)\big)\ \text{ for a.e. }\ t\in[0,T],\quad x(0):=x_0\in C(0) \tag{14}$$

with the inequality and equality constraint defined by

$$C(t):=\big\{x\in\mathbb{R}^n\big|\ \langle u_i(t),x\rangle\le b_i(t),\ i=1,\ldots,m\big\} \tag{15}$$

$$\text{with }\ \|u_i(t)\|=1\ \text{ for all }\ t\in[0,T],\ i=1,\ldots,m, \tag{16}$$

where the controls actions $u(\cdot)=\big(u_1(\cdot),\ldots,u_m(\cdot)\big)$ and $b(\cdot)=\big(b_1(\cdot),\ldots,b_m(\cdot)\big)$ are absolutely continuous on $[0,T]$, the final time T is fixed, and the absolutely continuous trajectories $x(\cdot)$ of the differential inclusion are understood in the standard sense of Carathéodory. It follows from the normal cone definition (2) that the sweeping inclusion (14) automatically yields the (implicit) *state constraints*

$$x(t)\in C\big(u(t),b(t)\big)\ \text{ for all }\ t\in[0,T]. \tag{17}$$

A particular case of problem (P) in (13)–(16) was partially investigated in [12] for the sweeping process generated by a moving affine hyperplane whose normal direction and boundary were acting as control functions. The study in [12] was confined to considering cost functionals independent of time, control, and control velocities with imposing a rather restrictive assumption on the uniform Lipschitzian continuity of feasible controls needed there for the truncation to bounded differential inclusions.

Paper [14] also addresses the following *parametric perturbation* (P^τ) of the original problem (P) with the control constraints in (16) replaced by

$$\|u_i(t)\| = 1 \;\text{ on }\; [\tau, T-\tau] \;\text{ and }\; \frac{1}{2} \le \|u_i(t)\| \le \frac{3}{2} \;\text{ on }\; [0,\tau) \cup (T-\tau, T],\; i = 1, \ldots, m,$$

where the time endpoint perturbation parameter $\tau > 0$ is arbitrarily small, and so (P^τ) is not much different from (P). The purpose of the equality constraint relaxation on the small intervals adjacent to the time endpoints is to avoid *degeneracy* of necessary optimality conditions, which otherwise may hold for all the feasible solutions under some choice of nontrivial dual elements. Such a degeneracy phenomenon for necessary optimality conditions of the Pontryagin Maximum Principle type has been discovered and well investigated in control theory with *inequality* state constraints; in particular, for Lipschitzian and compact-valued differential inclusions as in [5, 39].

Both problems (P) and (P^τ) are highly nonstandard in optimal control theory while dealing with controlled differential equations and Lipschitzian differential inclusions. To proceed with deriving necessary optimality conditions, we develop in [13, 14] a new version of the *method of discrete approximations*, which significantly modifies the Lipschitzian ones in [28, 30], establishes the $W^{1,2}$*-strong convergence* of discrete optimal solutions piecewise linearly extended to $[0, T]$, and reduces discrete-time approximation problems to those in nonsmooth mathematical programming with many geometric constraints. Then we apply powerful techniques of variational analysis, based on the *extremal principle* and *generalized differential calculus*, to derive necessary optimality condition in the obtained mathematical programming and discrete-time optimization problems, and finally derive necessary optimality conditions for local minimizers in the original sweeping optimal control problems by passing to the limit from those for discrete approximations. To efficiently implement this scheme, we employ *second-order computations* of (8) and (9) for mappings associated with the sweeping process that are given in terms of its original data. Second-order calculus rules as in Theorem 2.1 and its modifications play a crucial role in our implementations.

This approach allows us to establish necessary optimality conditions in the *extended Euler-Lagrange form* for the so-called *intermediate local minimizers* of the sweeping control problems formulated above. We refer the reader to [14] for the all the details on this development. The results of the aforementioned type will be presented in Subsection 5.2 for a more general version of problem (P) from (13)–(17).

3.2 Polyhedral Sweeping Process with Controlled Perturbations

The main goal of [7, 8] is to study a parametric family of sweeping optimal control problems with controls acting in *both* polyhedral moving sets and additive perturbations. Consider the problem:

$$\text{minimize } \; J[x,u,a] := \varphi\big(x(T)\big) + \int_0^T \ell\big(t, x(t), u(t), a(t), \dot{x}(t), \dot{u}(t), \dot{a}(t)\big)\,dt \tag{18}$$

over control pairs $(u(\cdot), a(\cdot))$ with $u(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$, $a(\cdot) \in W^{1,2}([0,T];\mathbb{R}^d)$ and the corresponding trajectories $x(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$ of the differential inclusion

$$-\dot{x}(t) \in N\big(x(t); C(t)\big) + f\big(x(t), a(t)\big) \;\text{ for a.e. }\; t \in [0,T], \quad x(0) := x_0 \in C(0), \tag{19}$$

where $x_0 \in \mathbb{R}^n$ and $T > 0$ are fixed, and where the moving convex set $C(t)$ is given by

$$C(t) := C + u(t) \;\text{ with }\; C := \big\{x \in \mathbb{R}^n \big|\; \langle x_i^*, x\rangle \le 0 \;\text{ for all }\; i = 1, \ldots, m\big\} \tag{20}$$

with the fixed generating vectors x_i^* of the convex polyhedron C in (20). Besides the dynamic constraints (19), we consider the pointwise constraints on u-controls defined by

$$\begin{cases} \|u(t)\| = r \;\text{ for all }\; t \in [\tau, T-\tau], \\ r - \tau \le \|u(t)\| \le r + \tau \;\text{ for all }\; t \in [0,\tau) \cup (T-\tau, T] \end{cases} \tag{21}$$

with the parameter $\tau \in [0, \overline{\tau}]$ depending on $\overline{\tau} := \min\{r, T\}$ and fixed $r > 0$. Note that by definition (2) the differential inclusion in (19) yields the pointwise constraints of another type

$$\big\langle x_i^*, x(t) - u(t)\big\rangle \le 0 \text{ for all } t \in [0, T] \text{ and } i = 1, \ldots, m. \tag{22}$$

The parametric family of problems (18)–(22) is different from the problems in Subsection 3.1, but it can also be investigated by the appropriate version of the method of discrete approximations developed in [7, 8]. In this way we obtain in [7] the $W^{1,2}$-strong convergence of well-posed discrete approximations with deriving necessary optimality conditions for discrete-time problems by using tools of variational analysis and generalized differentiation. The passage to the limit accomplished in [8] gives us necessary optimality conditions in the extended Euler-Lagrange form for the continuous-time sweeping process with applications to the controlled corridor version of the crowd motion model; see Subsection 6.1.

4 Perturbed Prox-Regular Sweeping Processes

The polyhedral description of the controlled moving set in (20) allows us to apply the obtained optimality conditions only to the corridor version of the crowd motion model. Among our main motivations for further developments in [9] was to perform optimal control for the more realistic planar version of the crowd motion model. In this case, the corresponding controlled moving sets can be adequately described in the *nonconvex* (and hence nonpolyhedral) form as

$$C(t) := C + u(t) = \bigcap_{i=1}^{m} C_i + u(t) \text{ with } C_i := \big\{x \in \mathbb{R}^n \big|\, g_i(x) \ge 0\big\} \text{ for all } i = 1, \ldots, m \tag{23}$$

defined by some convex and $\mathcal{C}^2$-smooth functions $g_i \colon \mathbb{R}^n \to \mathbb{R}$. Such a description implies that the set $C(t)$ in (23) is *uniformly prox-regular*, which is the concept well understood in variational analysis and geometric measure theory; see, e.g., [15] and the references therein. The optimal control problem considered in [9] is formulated as follows: minimize the cost (18) over the control actions $u(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ and $a(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^d)$ generating the corresponding trajectories $x(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ of the sweeping inclusion (19) with the controlled moving set (23) under the pointwise constraints on the controls

$$r_1 \le \|u(t)\| \le r_2 \text{ for all } t \in [0, T] \text{ with } 0 < r_1 \le r_2,$$

where the normal cone to the nonconvex set (23) in (19) is understood in the sense of (5). Similarly to the previous models, we have the implicit mixed (control-state) constraints

$$g_i\big(x(t) - u(t)\big) \ge 0 \text{ for all } t \in [0, T] \text{ and } i = 1, \ldots, m.$$

Developing a suitable version of the method of discrete approximations married to machinery of variational analysis and generalized differentiation leads us to verifiable necessary optimality conditions obtained in [9] in the extended Euler-Lagrange form; see Theorem 5.2 for the results of this type.

5 Extended Euler-Lagrange and Hamiltonian Formalisms

The results of this section are mostly based on the brand new paper [22], which is devoted to a general class of controlled sweeping processes with moving sets given as inverse images of closed subsets of finite-dimensional spaces under nonlinear differentiable mappings dependent on both state and control variables. Among other areas, our investigation of such problems is motivated by applications to rate-independent systems arising in hysteresis and related areas.

5.1 Problem Formulation and Standing Assumptions

We address here the sweeping control systems modeled as

$$\dot{x}(t) \in f\big(t, x(t)\big) - N\big(g(x(t)); C(t, u(t))\big) \text{ a.e. } t \in [0, T], \quad x(0) = x_0 \in C\big(0, u(0)\big), \tag{24}$$

where the controlled moving set is given by

$$C(t,u) := \big\{x \in \mathbb{R}^n \big|\ \psi(t,x,u) \in \Theta\big\}, \quad (t,u) \in [0,T] \times \mathbb{R}^m, \tag{25}$$

with $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $g\colon \mathbb{R}^n \to \mathbb{R}^n$, $\psi\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s$, and $\Theta \subset \mathbb{R}^s$. The problem (P) under consideration consists of minimizing the cost functional

$$\text{minimize } J[x,u] := \varphi\big(x(T)\big) + \int_0^T \ell\big(t, x(t), u(t), \dot{x}(t), \dot{u}(t)\big)dt \tag{26}$$

over absolutely continuous control actions $u(\cdot)$ and the corresponding absolutely continuous trajectories $x(\cdot)$ of the sweeping differential inclusion (24) generated by the controlled moving set (25). It follows from (24) due to (5) that the optimal control problem (P) intrinsically contains the pointwise constraints on both state and control functions given by

$$\psi\big(t, g(x(t)), u(t)\big) \in \Theta \ \text{ for all } \ t \in [0,T].$$

Our *standing assumptions* are as follows:

(H1) There exits $L_f > 0$ such that $\|f(t,x) - f(t,y)\| \le L_f\|x-y\|$ for all $x, y \in \mathbb{R}^n,\ t \in [0,T]$ and the mapping $t \mapsto f(t,x)$ is a.e. continuous on $[0,T]$ for each $x \in \mathbb{R}^n$.

(H2) There exits $L_g > 0$ such that $\|g(x) - g(y)\| \le L_g\|x-y\|$ for all $x, y \in \mathbb{R}^n$.

(H3) For each $(t,u) \in [0,T] \times \mathbb{R}^m$, the mapping $\psi_{t,u}(x) := \psi(t,x,u)$ is $\mathcal{C}^2$-smooth around the reference points with the surjective derivative $\nabla\psi_{t,u}(x)$ satisfying

$$\|\nabla\psi_{t,u}(x) - \nabla\psi_{t,v}(x)\| \le L_\psi\|u-v\|$$

with the uniform Lipschitz constant L_ψ. Furthermore, the mapping $t \mapsto \psi(t,x)$ is a.e. continuous on $[0,T]$ for each $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

(H4) There are a number $\tau > 0$ and a mapping $\vartheta\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ locally Lipschitz continuous and uniformly bounded on bounded sets such that for all $t \in [0,T]$, $\bar{v} \in N(\psi_{(t,\bar{u})}(\bar{x});\Theta)$, and $x \in \psi^{-1}_{(t,u)}(\Theta)$ with $u := \bar{u} + \vartheta(x-\bar{x}, x, \bar{x}, \bar{u})$ there exists $v \in N(\psi_{(t,u)}(x);\Theta)$ satisfying $\|v-\bar{v}\| \le \tau\|x-\bar{x}\|$.

(H5) The cost functions $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty,\infty)$ and $\ell(t,\cdot)\colon \mathbb{R}^{2(n+m)} \to \overline{\mathbb{R}}$ in (26) are bounded from below and lower semicontinuous around a given feasible solution to (P) for a.e. $t \in [0,T]$, while the integrand ℓ is a.e. continuous in t and is uniformly majorized by a summable function on $[0,T]$.

(H6) The set Θ in (25) is locally closed around the reference points.

Assumption (H4) is technical and seems to be the most restrictive. As shown in [22], it holds automatically in the polyhedral setting of [14] as well as in rather standard nonconvex settings.

5.2 Extended Euler-Lagrange Conditions

The method of discrete approximations allows us to derive necessary optimality conditions of the extended Euler-Lagrange type for *two types* of local minimizers in problem (P) formulated in Subsection 5.1. The first type treats the trajectory and control components of the optimal pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the same way and reduces in fact to the *intermediate* $W^{1,2}$*-minimizers* introduced in [28] in the general framework of differential inclusions and then studied in [7]–[9], [12]–[14], [30], [39], and other publications. The second type seems to be *new in control theory*; it treats control and trajectory components differently and applies to problems (P) whose running costs do not depend on control velocities.

Definition 5.1 (local minimizers for controlled sweeping processes). *Let the pair* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be feasible to problem* (P) *under the standing assumptions made.*

(i) *We say that* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be a* LOCAL $W^{1,2} \times W^{1,2}$-MINIMIZER *for* (P) *if* $\bar{x}(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$, $\bar{u}(\cdot) \in W^{1,2}([0,T];\mathbb{R}^m)$, *and*

$$J[\bar{x},\bar{u}] \le J[x,u] \ \textit{ for all } \ x(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n) \ \textit{ and } \ u(\cdot) \in W^{1,2}([0,T];\mathbb{R}^m) \tag{27}$$

sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ *in the norm topology of the corresponding spaces in* (27).

(ii) *Let the running cost* $\ell(\cdot)$ *in* (13) *do not depend on* $\dot{u}$*. We say that the pair* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be a* LOCAL $W^{1,2} \times \mathcal{C}$-MINIMIZER *for* (P) *if* $\bar{x}(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$*,* $\bar{u}(\cdot) \in \mathcal{C}([0,T];\mathbb{R}^m)$*, and*

$$J[\bar{x}, \bar{u}] \le J[x, u] \;\; \text{for all} \;\; x(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n) \;\; \text{and} \;\; u(\cdot) \in \mathcal{C}([0,T];\mathbb{R}^m) \tag{28}$$

sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ *in the norm topology of the corresponding spaces in* (28).

It has been well recognized in the calculus of variations and optimal control, starting with pioneering studies by Bogolyubov and Young, that limiting procedures of dealing with continuous-time dynamics involving time derivatives require a certain *relaxation stability*, which means that the value of cost functionals does not change under the convexification of the dynamics and running cost with respect to velocity variables; see, e.g., [17, 31, 39] for more details and references. In sweeping control theory, such issues have been investigated in [18, 37] for controlled sweeping processes somewhat different from (P).

To consider an appropriate relaxation of our problem (P), denote

$$F = F(t, x, u) := f(t, x) - N\big(g(x); C(t, u)\big) \tag{29}$$

and formulate the *relaxed optimal control problem* (R) as a counterpart of (P) with the replacement of the cost functional (13) by the convexified one

$$\text{minimize } \widehat{J}[x, u] := \varphi\big(x(T)\big) + \int_0^T \widehat{\ell}_F\big(t, x(t), u(t), \dot{x}(t), \dot{u}(t)\big)dt,$$

where $\widehat{\ell}(t, x, u, \cdot, \cdot)$ is defined as the largest l.s.c. convex function majorized by $\ell(t, x, u, \cdot, \cdot)$ on the convex closure of the set F in (29) with $\widehat{\ell} := \infty$ otherwise. Then we say that the pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a *relaxed local* $W^{1,2} \times W^{1,2}$*-minimizer* for (P) if in additions to the conditions of Definition 5.1(i) we have $J[\bar{x}, \bar{u}) = \widehat{J}[\bar{x}, \bar{u}]$. Similarly we define a *relaxed local* $W^{1,2} \times \mathcal{C}$*-minimizer* for (P) in the setting of Definition 5.1(ii). Note that, in contrast to the original problem (P), the convexified structure of the relaxed problem (R) provides an opportunity to the establish the *existence* of global optimal solutions in the prescribed classes of controls and trajectories. It is not a goal of this paper, but we refer the reader to [9, Theorem 4.1] and [37, Theorem 4.2] for some particular settings of controlled sweeping processes in the classes of $W^{1,2} \times W^{1,2}$ and $W^{1,2} \times \mathcal{C}$ feasible pairs $(\bar{x}(\cdot), \bar{u}(\cdot))$, respectively.

There is clearly no difference between the problems (P) and (R) if the normal cone in (29) is convex and the integrand ℓ in (13) is convex with respect to velocity variables. On the other hand, the measure continuity/nonatonomicity on $[0, T]$ and the differential inclusion structure of the sweeping process (24) create the environment where any local minimizer of the types under consideration is also a relaxed one. Without delving into details here, we just mention that the possibility to derive such a *local relaxation stability* from [37, Theorem 4.2] for *strong* local (in the $\mathcal{C}$-norm) minimizers of (P), provided that the controlled moving set $C(t, u)$ in (25) is convex and continuously depends on its variables.

The following major result is derived by passing to the limit from discrete approximations and employing second-order calculus rules as in Theorem 2.1. We refer the reader to the proof of [22, Theorem 4.3] and the previous results therein for precise arguments. For simplicity the theorem is formulated in the case of (P) where $g(x) := x$, $f := 0$ while ψ and ℓ do not depend on t.

Theorem 5.2 (extended Euler-Lagrange optimality conditions). *Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be a local minimizer for problem* (P) *of the types specified below. In addition to the standing assumptions, suppose that* $\psi = \psi(x, u)$ *is* $\mathcal{C}^2$*-smooth with respect to both variables while* φ *and* ℓ *are locally Lipschitzian around the corresponding components of the optimal solution. The following assertions hold:*

(i) *If* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is a relaxed local* $W^{1,2} \times W^{1,2}$*-minimizer, then there exist a multiplier* $\lambda \ge 0$*, an adjoint arc* $p(\cdot) = (p^x, p^u) \in W^{1,2}([0,T];\mathbb{R}^n \times \mathbb{R}^m)$*, a signed vector measure* $\gamma \in C^*([0,T];\mathbb{R}^s)$*, as well as pairs* $(w^x(\cdot), w^u(\cdot)) \in L^2([0,T];\mathbb{R}^n \times \mathbb{R}^m)$ *and* $(v^x(\cdot), v^u(\cdot)) \in L^\infty([0,T];\mathbb{R}^n \times \mathbb{R}^m)$ *with*

$$\big(w^x(t), w^u(t), v^x(t), v^u(t)\big) \in \text{co}\, \partial\ell\big(\bar{x}(t), \bar{u}(t), \dot{\bar{x}}(t), \dot{\bar{u}}(t)\big) \;\; \text{a.e.} \;\; t \in [0, T] \tag{30}$$

satisfying the collection of necessary optimality conditions:

• Primal-dual dynamic relationships*:*

$$\dot{p}(t)=\lambda w(t)+\begin{bmatrix}\nabla^2_{xx}\langle\eta(t),\psi\rangle\big(\bar{x}(t),\bar{u}(t)\big)\\ \nabla^2_{xw}\langle\eta(t),\psi\rangle\big(\bar{x}(t),\bar{u}(t)\big)\end{bmatrix}\big(-\lambda v^x(t)+q^x(t)\big)\ \ a.e.\ \ t\in[0,T], \tag{31}$$

$$q^u(t)=\lambda v^u(t)\ \ a.e.\ \ t\in[0,T], \tag{32}$$

where $\eta(\cdot)\in L^2([0,T];\mathbb{R}^s)$ *is a uniquely defined vector function determined by the representation*

$$\dot{\bar{x}}(t)=-\nabla_x\psi\big(\bar{x}(t),\bar{u}(t)\big)^*\eta(t)\ \ a.e.\ \ t\in[0,T] \tag{33}$$

with $\eta(t)\in N(\psi(\bar{x}(t),\bar{u}(t));\Theta)$*, and where* $q\colon[0,T]\to\mathbb{R}^n\times\mathbb{R}^m$ *is a function of bounded variation on* $[0,T]$ *with its left-continuous representative given, for all* $t\in[0,T]$ *except at most a countable subset, by*

$$q(t)=p(t)-\int_{[t,T]}\nabla\psi\big(\bar{x}(\tau),\bar{u}(\tau)\big)^*d\gamma(\tau). \tag{34}$$

• Measured coderivative condition*: Considering the* t*-dependent outer limit*

$$\mathop{\rm Lim\,sup}_{|B|\to 0}\frac{\gamma(B)}{|B|}(t):=\Big\{y\in\mathbb{R}^s\Big|\ \exists\ \ sequence\ \ B_k\subset[0,1]\ \ with\ \ t\in\mathbb{B}_k,\ |B_k|\to 0,\ \frac{\gamma(B_k)}{|B_k|}\to y\Big\} \tag{35}$$

over Borel subsets $B\subset[0,1]$ *with the Lebesgue measure* $|B|$*, for a.e.* $t\in[0,T]$ *we have*

$$D^*N_\Theta\big(\psi(\bar{x}(t),\bar{u}(t)),\eta(t)\big)\big(\nabla_x\psi(\bar{x}(t),\bar{u}(t))(q^x(t)-\lambda v^x(t))\big)\cap\mathop{\rm Lim\,sup}_{|B|\to 0}\frac{\gamma(B)}{|B|}(t)\neq\emptyset. \tag{36}$$

• Transversality condition *at the right endpoint:*

$$-\big(p^x(T),p^u(T)\big)\in\lambda\big(\partial\varphi(\bar{x}(T)),0\big)+\nabla\psi\big(\bar{x}(T),\bar{u}(T)\big)N_\Theta\big((\bar{x}(T),\bar{u}(T)\big). \tag{37}$$

• Measure nonatomicity condition*: Whenever* $t\in[0,T)$ *with* $\psi(\bar{x}(t),\bar{u}(t))\in\operatorname{int}\Theta$ *there is a neighborhood* V_t *of* t *in* $[0,T]$ *such that* $\gamma(V)=0$ *for any Borel subset* V *of* V_t*.*

• Nontriviality condition*:*

$$\lambda+\sup_{t\in[0,T]}\|p(t)\|+\|\gamma\|\neq 0\ \ with\ \ \|\gamma\|:=\sup_{\|x\|_{C([0,T]}=1}\int_{[0,T]}x(s)d\gamma. \tag{38}$$

(ii) *If* $(\bar{x}(\cdot),\bar{u}(\cdot))$ *is a relaxed local* $W^{1,2}\times\mathcal{C}$*-minimizer, then all the conditions* (31)–(38) *in* (i) *hold with the replacement of the quadruple* $(w^x(\cdot),w^u(\cdot),v^x(\cdot),v^u(\cdot))$ *in* (30) *by the triple* $(w^x(\cdot),w^u(\cdot),v^x(\cdot))\in L^2([0,T];\mathbb{R}^n)\times L^2([0,T];\mathbb{R}^m)\times L^\infty([0,T];\mathbb{R}^n)$ *satisfying the inclusion*

$$\big(w^x(t),w^u(t),v^x(t)\big)\in\operatorname{co}\partial\ell\big(\bar{x}(t),\bar{u}(t),\dot{\bar{x}}(t)\big)\ \ a.e.\ \ t\in[0,T].$$

(iii) *Suppose in addition that* $\eta(T)$ *is well defined and that* $\theta=0$ *is the only vector for which*

$$\theta\in D^*N_\Theta\big(\psi(\bar{x}(T),\bar{u}(T)),\eta(T)\big)(0),\quad \nabla\psi\big(\bar{x}(T),\bar{u}(T)\big)^*\theta\in\nabla\psi\big(\bar{x}(T),\bar{u}(T)\big)N_\Theta\big(\bar{x}(T),\bar{u}(T)\big).$$

Then the above necessary optimality conditions hold with the enhanced nontriviality

$$\lambda+\operatorname{mes}\big\{t\in[0,T]\big|\ q(t)\neq 0\big\}+\|q(0)\|+\|q(T)\|>0.$$

5.3 New Hamiltonian Formalism and Maximum Principle

Note that the necessary optimality conditions obtained in Theorem 5.2 as well as in our previous papers do not contain the formalism of the Pontryagin Maximum Principle (PMP) [35] (i.e., the maximization of the corresponding Hamiltonian function) established in classical optimal control of (4) and then extended to optimal control problems for Lipschitzian differential inclusions of type (3).

To the best of our knowledge, necessary optimality conditions involving the maximization of the corresponding Hamiltonian were first obtained for sweeping control systems in [6], where the authors considered a sweeping process with a strictly smooth, convex, and solid set $C(t) \equiv C$ in (1) while with control functions entering linearly an adjacent ordinary differential equation. Further results with the maximum condition for global (as in [6]) minimizers were derived in [4] for the sweeping control system

$$\dot{x}(t) \in f\big(x(t), u(t)\big) - N\big(x(t); C(t)\big) \ \text{ a.e. } \ t \in [0, T], \tag{39}$$

where measurable controls $u(t)$ enter the additive smooth term f while the uncontrolled moving set $C(t)$ is compact, uniformly prox-regular regular, and possesses a $\mathcal{C}^3$-smooth boundary for each $t \in [0, T]$ under some other assumptions. The very recent paper [16] also concerns a (generally nonautonomous) sweeping control system in form (39) and derives necessary optimality conditions of the PMP type for global minimizers provided that the convex, solid, and compact set $C(t) \equiv C$ therein is defined by $C := \{x \in \mathbb{R}^n \,|\, \psi(x) \le 0\}$ via a $\mathcal{C}^2$-smooth function ψ under other assumptions, which are partly differ from [4]. The *penalty-type* approximation methods developed in [4], [6], and [16] are different from each other, significantly based on the *smoothness* of uncontrolled moving sets while being totally distinct from the method of discrete approximations employed in our previous papers and in what follows.

The next result is the first in the literature on the validity of a PMP counterpart in optimal control of sweeping processes with control-dependent moving sets. It addresses the above problem (P) with

$$\Theta = h^{-1}(\mathbb{R}^l_-) := \big\{z \in \mathbb{R}^s \big|\, h(z) \in \mathbb{R}^l_-\big\} \tag{40}$$

in (25) defined via a smooth mapping $h\colon \mathbb{R}^s \to \mathbb{R}^l$ and is based on the *precise computation* of the second-order construction $D^*N_{\mathbb{R}^s_-}$ taken from [32]. Consider the index set

$$I(x, u) := \big\{i \in \{1, \ldots, s\} \big|\, \psi_i(x, u) = 0\big\}.$$

It follows from assumption (H3) that for each $v \in -N(x; C(u))$ there is a unique collection $\{\alpha_i\}_{i \in I(x,u)}$ with $\alpha_i \le 0$ and $v = \sum_{i \in I(x,u)} \alpha_i [\nabla_x \psi(x, u)]_i$. Given $\nu \in \mathbb{R}^s$, define the vector $[\nu, v] \in \mathbb{R}^n$ by

$$[\nu, v] := \sum_{i \in I(x,u)} \nu_i \alpha_i \big[\nabla_x \psi(x, u)\big]_i$$

and introduce the (new) *modified Hamiltonian* function

$$H_\nu(x, u, p) := \sup\big\{\big\langle [\nu, v], p\big\rangle \big|\, v \in -N\big(x; C(u)\big)\big\}, \quad (x, u, p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n. \tag{41}$$

Note that if $h(z) := Az - b$ in (40), the full Jacobian rank assumption in (H3) corresponds to the classical *linear independence constraint qualification* (LICQ). In this linear case of $h(\cdot)$, we can improve the LICQ condition in the aforementioned coderivative evaluation by the weaker *positive LICQ* (PLICQ) meaning that arbitrary linear combinations of gradients in LICQ are replaced by those with nonnegative coefficients. Now we are ready to formulate a PMP counterpart for (P) via the new Hamiltonian (41).

Theorem 5.3 (maximum principle in sweeping optimal control). *Consider the control problem (P) in the frameworks of Theorem 5.2 with the set Θ given by (40), where $h\colon \mathbb{R}^s \to \mathbb{R}^l$ is $\mathcal{C}^2$-smooth around the local optimal solution $\bar{z}(t) := (\bar{x}(t), \bar{u}(t))$ for all $t \in [0, T]$. Suppose that either $\nabla h(\bar{z}(t))$ is surjective, or $h(\cdot)$ is linear and the PLICQ assumption is fulfilled at $\bar{z}(t)$ on $[0, T]$. Then, in addition to the necessary optimality conditions of Theorem 5.2, the maximum condition*

$$\big\langle \big[\nu(t), \dot{\bar{x}}(t)\big], q^x(t) - \lambda v^x(t)\big\rangle = H_{\nu(t)}\big(\bar{x}(t), \bar{u}(t), q^x(t) - \lambda v^x(t)\big) = 0 \ \ \textit{a.e.} \ \ t \in [0, T].$$

holds with a measurable vector function $\nu\colon [0,T]\to\mathbb{R}^s$ *satisfying the inclusion*

$$\nu(t)\in D^*N_{\mathbb{R}^s_-}\big(h(\psi(\bar{x}(t),\bar{u}(t))),\mu(t)\big)\big(\nabla_x\psi(\bar{x}(t),\bar{u}(t))(q^x(t)-\lambda v^x(t))\big)\cap\mathop{\rm Lim\,sup}_{|B|\to 0}\frac{\gamma(B)}{|B|}(t)$$

for a.e. $t\in[0,T]$, *where* Lim sup *is defined in* (35), *and where* $\mu\colon[0,T]\to\mathbb{R}^l$ *is measurable with*

$$\mu(t)\in N_{\mathbb{R}^l_-}\big(h(\psi(\bar{x}(t),\bar{u}(t))\big)\quad and\quad \eta(t)=\nabla h\big(\psi(\bar{x}(t),\bar{u}(t)\big)^*\mu(t)\quad a.e.\quad t\in[0,T].$$

As shown in [22], a conventional form of the maximum principle with replacing the new Hamiltonian function (41) by the standard Hamiltonian

$$H(x,u,p):=\sup\big\{\langle p,v\rangle\big|\ v\in -N\big(x;C(u)\big)\big\},\quad (x,u,p)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n.$$

fails as a necessary optimality condition even for global minimizers of (P).

6 Selected Applications

This section briefly discusses some recent applications of the obtained necessary optimality conditions for controlled sweeping processes to dynamical models of practical interest. We start with a class of optimal control problems for the *crowd motion model in a corridor*, which can be completely solved by using the results for polyhedral sweeping processes with controlled perturbations outlined in Subsection 3.2.

6.1 Optimal Control of Crowd Motions: Corridor Model

The original motivation for the crowd motion model relates to the study of local interactions between participants in order to describe the dynamics of pedestrian traffic. By now this model has been successfully used to investigate more general classes of problems in operations research, socioeconomics, etc.

The microscopic form of the crowd motion model is based on the following *two postulates*. On one hand, each individual has a *spontaneous* velocity that he/she intends to implement in the absence of other participants. On the other hand, the *actual* velocity must be taken into account. The latter one is incorporated via a projection of the spontaneous velocity into the set of admissible/feasible velocities, i.e., those which do not violate certain nonoverlapping constraints. A mathematical description of the uncontrolled microscopic crowd motion model is given [26, 38] in the *sweeping process* form with the subsequent usage therein for numerical simulations and various applications.

In the *corridor version* of the crowd motion model the crucial *nonoverlapping condition* is written as

$$Q_0=\big\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n,\ x_{i+1}-x_i\ge 2R\big\},\tag{42}$$

where $n\ge 2$ indicates the number of participants identified with rigid disks of the same radius R in a corridor. The actual velocity field is described via the projection operator by

$$\dot{x}(t)=\Pi\big(U(x);C_x\big)\ \text{ for a.e. }\ t\in[0,T],\quad x(0)=x_0\in Q_0,$$

with spontaneous velocities $U(x)=(U_0(x_1),\ldots,U_0(x_n))$, $x\in Q_0$, satisfying

$$U(x)\in N_x+\dot{x}(t)\ \text{ for a.e. }\ t\in[0,T],\quad x(0)=x_0,$$

where N_x stands for the normal cone to Q_0 at x. Since all the participants exhibit the same behavior and want to reach the exit by the shortest path, their spontaneous velocities can be represented as

$$U(x)=\big(U_0(x_1),\ldots,U_0(x_n)\big)\ \text{ with }\ U_0(x)=-s\nabla D(x)$$

with $D(x)$ denoting the distance between the position $x=(x_1,\ldots,x_n)\in Q_0$ and the exit and with $s\ge 0$ standing for the speed. By taking this into account the aforementioned postulate that in the absence of other participants each participant tends to remain his/her spontaneous velocity until reaching the exit, the (uncontrolled) perturbations in this model are described by

$$f(x) = -(s_1, \ldots, s_n) \in \mathbb{R}^n \ \text{ for all } \ x = (x_1, \ldots, x_n) \in Q_0,$$

where s_i denotes the speed of the participant $i \in \{1, \ldots, n\}$. To control the actual speed of all the participants in the presence of the nonoverlapping condition (42), we suggest in [8] to involve control functions $a(\cdot) = (a_1(\cdot), \ldots, a_n(\cdot))$ into perturbations as follows:

$$f\big(x(t), a(t)\big) = \big(s_1 a_1(t), \ldots, s_n a_n(t)\big), \quad t \in [0, T].$$

To optimize the sweeping dynamics by using controls $a(\cdot)$, consider the cost functional

$$\text{minimize } \ J[x, a] := \frac{1}{2}\Big(\|x(T)\|^2 + \int_0^T \|a(t)\|^2 dt\Big) \tag{43}$$

the meaning of which is to *minimize the distance* of all the participants to the exit at the origin together with minimizing the *energy* of feasible controls $a(\cdot)$.

The given description of the controlled crowd motion model falls into the category of the sweeping optimal control problems (18)–(22) discussed in Subsection 3.2. Applying the necessary optimality conditions for such problems obtained in [8] allows us to develop an efficient algorithmic procedure to determine optimal controls and trajectories in the general case of finitely many participants and solve the problem analytically in the cases where $n = 2, 3$.

6.2 Optimal Control of Crowd Motions: Planar Model

The *planar* version of the microscopic crowd motion model [26, 38] is more realistic in practice while much more challenging mathematically in comparison with the corridor version of Subsection 6.1. In contrast to (42), the overlapping condition is not polyhedral anymore while being represented by

$$Q := \big\{x \in \mathbb{R}^{2n} \big|\ D_{ij}(x) \geq 0 \ \text{ for all } \ i \neq j\big\},$$

where $D_{ij}(x) := \|x_i - x_j\| - 2R$ is the signed distance between the disks i and j of the same radius R identified with $n \geq 2$ participants on the plane. The corresponding optimal control problem formulated and investigated in [10] is described via the sweeping dynamics as follows: minimize the cost functional (43) over the constrained controlled sweeping process

$$\begin{cases} -\dot{x}(t) \in N\big(x(t); C(t)\big) + f\big(x(t), a(t)\big) \ \text{ for a.e. } \ t \in [0, T], \\ C(t) := C + \bar{u}(t), \ \|\bar{u}(t)\| = r \in [r_1, r_2] \ \text{ on } \ [0, T], \ x(0) = x_0 \in C(0), \end{cases}$$

where the initial data and constraints are given by

$$f\big(x(t), a(t)\big) := \big(s_1 a_1(t) \cos\theta_1(t), s_1 a_1(t) \sin\theta_1(t), \ldots, s_n a_n(t) \cos\theta_n(t), s_n a_n(t) \sin\theta_n(t)\big),$$

$$\bar{u}_{i+1}(t) = \bar{u}_i(t) := \left(\frac{r}{\sqrt{2n}}, \frac{r}{\sqrt{2n}}\right), \quad i = 1, \ldots, n-1,$$

$$C := \big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n} \big|\ g_{ij}(x) \geq 0 \ \text{ for all } \ i \neq j \ \text{ as } \ i, j = 1, \ldots, n\big\}$$

with the functions $g_{ij}(x) := D_{ij}(x) = \|x_i - x_j\| - 2R$, and with

$$x(t) - \bar{u}(t) \in C \ \text{ for all } \ t \in [0, T].$$

This model falls into the category of optimizing controlled sweeping processes governed by prox-regular moving sets that is discussed in Subsection 4. Applying the necessary optimality conditions for such problems obtained in [9] leads us in [10] to a complete computation of optimal solutions in the case of two participants with establishing efficient relationships to determine optimal parameters in the general setting of finitely many participants in the crowd motion modeling.

6.3 Applications to Elastoplasticity and Hysteresis

Here we consider the model of this type discussed in [3], which can be described in the form of problem (P) from Section 5, where Z is a closed convex subset of the $\frac{1}{2}n(n+1)$-dimensional vector space E of symmetric tensors $n \times n$ with $\operatorname{int} Z \neq \emptyset$. Using the notation of [3], define the strain tensor $\epsilon = \{\epsilon\}_{i,j}$ by $\epsilon := \epsilon^e + \epsilon^p$, where ϵ^e is the elastic strain and ϵ^p is the plastic strain. The elastic strain ϵ^e depends on the stress tensor $\sigma = \{\sigma\}_{i,j}$ linearly, i.e., $\epsilon^e = A^2\sigma$, where A is a constant symmetric positive-definite matrix. The *principle of maximal dissipation* says that

$$\big\langle \dot{\epsilon}^p(t), z \big\rangle \le \big\langle \dot{\epsilon}^p(t), \sigma(t) \big\rangle \ \text{ for all } \ z \in Z. \tag{44}$$

It is shown in [3] that the variational inequality (44) is equivalent to the *sweeping process*

$$\dot{\zeta}(t) \in -N\big(\zeta(t); C(t)\big),\ \zeta(0) = A\sigma(0) - A^{-1}\epsilon(0) \in C(0), \tag{45}$$

where $\zeta(t) := A\sigma(t) - A^{-1}\epsilon(t)$ and $C(t) := -A^{-1}\epsilon(t) + AZ$. This model can be rewritten in the frame of our problem (P) with $x := \zeta$, $u := \epsilon$, $\psi(x,u) := x + A^{-1}u$, and $\Theta := AZ$. Thus we can apply Theorem 5.2 and Theorem 5.3 to this class of hysteresis operators for the general elasticity domain Z. Note that a similar model is considered in [21] but only for the von Mises yield criterion. The results obtained in [22] give us the flexibility of applications to many different elastoplasticity models including those with the Drucker-Prager, Mohr-Coulomb, Tresca, von Mises yield criteria, etc. More details with precise computations, examples, and discussions can be found in [22].

7 Concluding Remarks and Open Questions

The material presented in this paper aims to draw the reader's attention to a rather new and highly challenging class of optimal control problems governed by sweeping processes with controlled moving sets. The available results discussed above demonstrate that the method of discrete approximations, being combined with advanced tools of first-order and second-order variational analysis and generalized differentiation, provides efficient machineries to derive verifiable necessary conditions for optimal sweeping solutions and to apply them to various models of practical interest. From numerical viewpoints, our approach reduces complicated problems of infinite-dimensional optimization to much easier finite-dimensional ones. It opens the gate to employ known and develop new numerical algorithms in finite-dimensional optimization and operations research to solve optimal control problems for sweeping processes. Efficient *numerical implementations* of this approach to optimize important classes of problems with controlled sweeping dynamics constitute a *huge open area* of future research and applications.

Besides this, let us mention some other very challenging and significant *open questions* in sweeping optimal control and its applications. Needless to say that the given partial list is far from being complete.

• Investigate controlled sweeping processes of type (24) that contains, along with control actions in the moving set $C(t,u)$, *measurable* controls $v(\cdot)$ in f under the pointwise constraints $v(t) \in V$ for a.e. $t \in [0,T]$. This would allow us to unify the current sweeping control theory involving continuous controls with the conventional framework of optimal control, where control functions are merely measurable.

• Develop optimal control theory for sweeping processes with *infinite-dimensional* (mainly Hilbert) state spaces $x \in X$. A major goal here is to include into consideration evolution systems governed by *variational* and *quasi-variational inequalities* associated with *partial differential equations*.

• Investigate optimal control problems governed by *rate-independent operators* having the following description. Given two functionals $E\colon [0,T] \times Z \to \mathbb{R}$ and $\Psi\colon Z \times X \to [0,\infty)$ on a Banach (or finite-dimensional) space Z, consider the *doubly nonlinear evolution inclusion*

$$0 \in \partial_v \Psi\big(z(t), \dot{z}(t)\big) + \partial E\big(t, z(t)\big) \ \text{ a.e. } \ t \in [0,T].$$

If E is smooth, the inclusion above is equivalent to

$$\dot{z}(t) \in N_{C(z(t))}\big(\nabla E(t, z(t))\big) \text{ a.e. } \ t \in [0,T],$$

where $\{C(z)\}_{z\in Z}$ is the family of closed convex subsets of Z related to Ψ by the formula

$$\Psi(z,v) := \sup\left\{ (\sigma, v) \middle| \ \sigma \in C(z) \right\} \ \text{ for all } \ z, v \in Z.$$

Such problems are particularly important for applications to practical hysteresis models, especially those arising in problems of contact and nonsmooth mechanics.

• Develop applications of sweeping optimal control to *socioeconomic modeling*. The crowd motion model relates to problems of this type, but there are other important models in this area that can be investigated by using advanced tools of variational analysis and generalized differentiation.

• It has been recently realized that there are interesting models in *robotics*, which can be described as sweeping processes. Thus developing applications of the obtained and future results in sweeping optimal control to such models is a very challenging area of further research.

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