Abstract. In this paper, we introduce two-stage stochastic and distributionally robust $p$-order conic mixed integer programs (denoted by TSS-CMIPs and TSDR-CMIPs, respectively) in which the second stage problems have $p$-order conic constraints along with integer variables. First, we consider TSS-CMIPs and TSDR-CMIPs with structured $p$-order CMIPs in the second stage and derive classes of globally valid parametric (non)-linear cuts for them. These cuts provide convex programming equivalent, under certain conditions, for the second stage CMIPs. We then present conditions under which the addition of sparse nonlinear cuts in the extensive formulation of TSS-CMIPs is sufficient to relax the integrality restrictions on the second stage integer variables without effecting the integrality of the optimal solution of the TSS-CMIP. Our aforementioned cuts for TSS-CMIPs with $p = 1$ satisfy these conditions. We also perform extensive computational experiments by solving randomly generated structured TSS-CMIPs with polyhedral CMIPs and second-order CMIPs in the second stage, i.e. $p = 1$ and $p = 2$, respectively, and observe that there is a significant reduction in the total time taken to solve these problems after adding our sparse cuts. Finally, we demonstrate the significance of our results for TSS-CMIP by deriving (partial) convex hull for deterministic multi-constraint conic mixed integer sets with multiple integer variables. This study extends the result of Atamtürk and Narayanan [Math. Prog. 122: 1-20, 2008] for a simple polyhedral conic mixed integer set with single constraint and one integer variable.

1. Introduction

A $p$-order conic mixed integer program is a nonlinear nonconvex problem whose feasible region is defined by intersection of an affine set, $p$-order cones, and integrality constraints. It generalizes various optimization frameworks such as mixed integer program (MIP) and quadratically constrained convex quadratic MIP, and arises in applications ranging from portfolio optimization to machine learning [3, 33]. In this paper, we introduce two-stage stochastic $p$-order conic mixed integer programs (TSS-CMIPs) with integer variables and $p$-order conic constraints in the second stage. More specifically, the TSS-CMIP is defined as follows:

$$\min cx + \mathbb{E}_{\xi}[\Theta_\omega(x)]$$

$$\text{s.t. } Ax \geq b, x \in \mathbb{Z}^{n_1},$$

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where random variable $\xi_F$ follows a known probability distribution $P$ with finite sample space $\Omega$, and for scenario $\omega \in \Omega$ with $\bar{P}_\omega$ probability of occurrence:

$$\overline{Q}_\omega(x) := \min g_\omega y_\omega + \sum_{j \in J} \delta^j_\omega d^j_{\omega,0}$$

s.t. $W_\omega y_\omega \geq r_\omega - T_\omega x$,  

$$\|E^j_\omega y_\omega + F^j_\omega x - h^j_\omega\|_p \leq d^j_{\omega,0}, \quad j \in J,$$

$$y_\omega \in \mathbb{Z}^{n_1} \times \mathbb{R}^{q - q_1}, d^j_{\omega,0} \in \mathbb{R}_+, \quad j \in J.$$  

Here, $\|\cdot\|_p$ denotes $l_p$-norm, i.e. $\|y\|_p = (\sum_k |y_k|^p)^{1/p}$ for $p \geq 1$, $c \in \mathbb{R}^{n_1}$, $A \in \mathbb{R}^{m_1 \times n_1}$, $b \in \mathbb{R}^{m_1}$, and for each $\omega \in \Omega$ and $j \in J := \{1, \ldots, |J|\}$, $g_\omega \in \mathbb{R}^q$, $\delta^j_\omega \in \mathbb{R}$, $E^j_\omega \in \mathbb{R}^{m_2 \times q}$, $F^j_\omega \in \mathbb{R}^{m_2 \times n_1}$, $h^j_\omega \in \mathbb{R}^{m_2}$, $W_\omega \in \mathbb{R}^{m_3 \times q}$, $T_\omega \in \mathbb{R}^{m_3 \times n_1}$, and $r_\omega \in \mathbb{R}^{m_3}$. We refer to the formulation (2)-(5) and function $\overline{Q}_\omega(x)$ as the second stage subproblem and the recourse function, respectively. For TSS-CMIPs, we make following assumptions:

(A1) $T_\omega \in \mathbb{Z}^{m_3 \times n_1}$ and $F^j_\omega \in \mathbb{Z}^{m_2 \times n_1}$ for all $\omega \in \Omega$ and $j \in J$ (w.l.o.g.),

(A2) $X := \{x : Ax \geq b, x \in \mathbb{Z}^{n_1}\}$ is non-empty,

(A3) Relatively complete recourse, i.e. $\overline{K}_\omega(x) := \{y_\omega : (3)-(5) \text{ hold}\}$ is non-empty for all $\omega \in \Omega$ and $x \in X$.

Observe that the TSS-CMIPs generalize various classes of optimization problems studied in literature, which include: (a) two-stage stochastic mixed integer program (TSS-MIP), i.e., TSS-CMIP (1) with $J = \emptyset$, (b) TSS-MIP with quadratic objective function in the second stage, i.e.,

$$\overline{Q}_\omega(x) = \min \{y^T \Gamma_\omega y_\omega + \delta^T y_\omega : W_\omega y_\omega \geq r_\omega - T_\omega x, y_\omega \in \mathbb{Z}^{n_1} \times \mathbb{R}^{q - q_1}\}$$

where $\Gamma_\omega \in \mathbb{R}^{q \times q}$ and $\delta_\omega \in \mathbb{R}^q$, which is equivalent to TSS-CMIP (1) where

$$\overline{Q}_\omega(x) = \min \{d^1_{\omega,0} : W_\omega y_\omega \geq r_\omega - T_\omega x, \|\Gamma_\omega^{1/2} y_\omega + \delta_\omega\|_2 \leq d^1_{\omega,0},$$

$$\|\Gamma_\omega^{1/2} y_\omega + \delta_\omega\|_2 \leq (\delta^T \Gamma_\omega^{-1} \delta_\omega)^{1/2}, y_\omega \in \mathbb{Z}^{n_1} \times \mathbb{R}^{q - q_1}, d^1_{\omega,0} \in \mathbb{R}_+\};$$

(c) TSS-MIP with sum of $l_p$-norms in the objective function of the second stage, i.e., TSS-CMIP (1) where $g_\omega = 0$ and $\delta^j_\omega = 1$ for $j \in J$; (d) TSS-MIP with chance-constraints, defined using independent Gaussian random vectors with zero mean, in the second stage; and (e) TSS-MIP with robust MIP in the second stage where uncertainty set is ellipsoidal [10].

### 1.1 Two-stage distributionally robust $p$-order conic mixed integer program

We can further generalize TSS-CMIP (1) by integrating it with distributionally robust optimization framework [48] where we seek a solution that optimizes the expected value of the objective function for the worst case probability distribution within a prescribed (ambiguity) set of distributions that may be followed by the uncertain parameters. In particular, we introduce two-stage distributionally robust $p$-order conic mixed integer program (TSDR-CMIP), which is defined as follows:

$$\min \left\{ cx + \max_{P \in \mathcal{P}} \mathbb{E}_{\xi_F} [\overline{Q}_\omega(x)] \left| Ax \geq b, x \in \mathbb{Z}^{n_1} \right\} \right.,$$

(6)
where complete information about the probability distribution \( P \) followed by the random variable \( \xi \) is not known but it belongs to a set of distributions \( \mathcal{P} \) (referred to as the ambiguity or uncertainty set). In this paper, we consider a general ambiguity set which subsumes the ambiguity sets defined using: linear constraints on the first two moments of the distribution [13, 23, 46, 48], conic constraints to describe the set of distributions with moments [14, 20], Kantorovich distance or Wasserstein metric [35, 45, 44, 56], \( \zeta \)-structure metrics [58], and \( \chi^2 \) distance and Kullback-Leibler divergence [11, 16, 26, 34, 55, 57]. The TSDR-CMIPs generalize the two-stage distributionally robust mixed binary programs with general ambiguity set, studied by Bansal et al. [6], where both stages have linear constraints and integer variables are bounded between 0 and 1.

1.2 Contributions of this paper

As per our knowledge, the TSS-CMIP (1) and TSDR-CMIP (6) have not been studied in literature. In this paper, we extend the conic mixed integer rounding (CMIR) inequalities derived by Atamtürk and Narayanan [5] for describing the convex hull of a deterministic single constrained first-order (or polyhedral) and second-order conic mixed integer sets with one integer variable to TSS-CMIPs and TSDR-CMIPs. In particular, we introduce structured TSS-CMIPs and TSDR-CMIPs with multi-constraint \( p \)-order conic mixed integer sets having multiple integer variables in the second stage and derive classes of parametric (non)-linear cuts akin to parametric CMIR inequalities for them. We prove that under certain conditions, these cuts provide convex programming equivalent for the second stage CMIPs.

In addition, we present sufficient conditions under which by adding sparse nonlinear cuts in \((x, y_\omega, d_\omega)\) space to the deterministic equivalent (or extensive formulation) of TSS-CMIP (1), i.e.,

\[
\min cx + \sum_{\omega \in \Omega} \ell_\omega \left( g_\omega y_\omega + \sum_{j \in J} g^j_\omega d^j_\omega,0 \right)
\]

s.t. \( Ax \geq b \),
\( T_\omega x + W_\omega y_\omega \geq r_\omega \), \( \omega \in \Omega \),
\( \| P^j_\omega x + E^j_\omega y_\omega - h^j_\omega \|_p \leq d^j_\omega,0 \), \( j \in J, \omega \in \Omega \),
\( x \in \mathbb{Z}^{n_1}, y_\omega \in \mathbb{Z}^{q_1} \times \mathbb{R}^{q_2-q_1}, d^j_\omega,0 \in \mathbb{R}^+, j \in J, \omega \in \Omega \),

the integrality constraints on the \( y_\omega \) integer variables can be relaxed (without effecting the integrality of the optimal solution of the problem). It is well-known that MIP formulations for most problems arising in large-scale systems such as energy distribution, (air or ground) traffic control, and financial trading systems, have sparse constraint matrices [4, 21, 22, 54]. In fact, “most commercial MIP solvers consider sparsity of cuts as an important criterion for cutting-plane selection and use” [22]. As per our knowledge, deterministic large-scale CMIPs with sparse matrices, i.e., CMIPs defined by (7)-(11), have not been studied in the literature.

We also perform computational experiments to evaluate the effectiveness of our sparse cuts by solving the deterministic equivalent of randomly generated aforementioned structured TSS-CMIPs where \( p = 1 \) or \( p = 2 \), respectively. We observe that after adding our cuts, there is a significant reduction in the number of the second stage integer variables for instances where \( p = 1 \) and tightening of second stage feasible region for instances where \( p = 2 \), thereby leading to reduction in the total solution time taken to solve these problems. For instance, CPLEX 12.70 (with its default settings) could not solve 85 out of 290 TSS-CMIP instances within a time limit of 3 hours and the
allocated memory of 24 GB RAM. In contrast, after adding our parametric cuts at the root node, CPLEX could solve 82 out of these 85 (unsolvable) instances within 8.2 minutes (on average).

Furthermore, to demonstrate the significance of our results for TSS-CMIPs and TSDR-CMIPs, we introduce new deterministic multi-constraint conic mixed integer sets with multiple integer variables, and derive cuts which are sufficient to provide convex hull or partial convex hull of these sets. Note that these structured sets have not been studied even in the deterministic framework, and this study extends results of [5] for conic mixed integer sets with single constraint and one integer variable.

1.3 Literature review on special cases of TSS-CMIP

One of the most extensively studied special case of TSS-CMIP is the class of TSS-MIPs, i.e., (1) with $J = \emptyset$ (refer to [31] for comprehensive survey), which includes the problems whose second stage has pure integer programs [2, 30, 49], mixed binary programs [17, 24, 32, 42, 50, 52], or mixed integer programs [7, 50, 51, 53]. To solve TSS-MIP, many researchers have been using globally valid linear cuts in $(x, y)$ space, for each scenario $\omega \in \Omega$, to tighten the second stage problems with binary or integer variables, which are then embedded within Benders’ decomposition algorithm [12] to solve TSS-MIP. These cuts are of the form $\gamma_\omega y_\omega \geq \gamma_\omega,0 - \gamma_\omega,1 x$, where $x$ is a first-stage feasible solution, and $\gamma$, $\gamma_0$, and $\gamma_1$ are real-vectors, and are referred to as the “parametric” (linear) cuts. For instance, Sherali and Fraticelli [52] derive parametric linear cuts using the reformulation-linearization technique to solve TSS-MIP with $|\Omega| = 1$, only binary variables in the first-stage, and mixed binary programs in the second stage. Likewise, Gade et al. [24] utilize parametric Gomory fractional cuts for solving TSS-MIPs with only binary variables in the first stage and non-negative integer variables in the second stage. Similarly, Bodur et al. [15] use parametric cuts based on split disjunctions to solve TSS-MIP with mixed integer first stage and continuous second stage variables.

In the aforementioned studies, the parametric cuts are developed/added sequentially in the algorithms. Recently, Kim and Mehrotra [29] formulate an integrated staffing and scheduling problem under demand uncertainty as a TSS-MIP with the second stage MIPs having a certain structure and utilize parametric mixed integer rounding inequalities (added a priori) to obtain a linear programming equivalent of the second stage MIPs. Bansal et al. [7] generalize their observation to the general TSS-MIPs and show that under suitable conditions, the second stage MIPs can be convexified by adding parametric cuts a priori. As special cases, the authors consider structured parameterized mixed integer sets or convex objective integer program (COIP) in the second stage. In particular, they extend the results of Miller and Wolsey [38] for deterministic mixed integer sets and COIP to the two-stage stochastic framework, and consider TSS-MIPs with parametrized version of two special cases of the continuous multi-mixing set [8, 9] in the second stage. In this paper, we further generalize their results for TSS-CMIPs, which is equivalent to TSS-MIPs with parametric $p$-order conic constraints in the second stage. We utilize parametric (non)-linear cutting planes to obtain convex programming equivalent for some structured second stage CMIPs in TSS-CMIPs and TSDR-CMIPs.

Another special case of TSS-CMIP is two-stage stochastic quadratic integer program (TSS-QIP). In this direction, Özaltin et al. [43] study TSS-QIPs with only stochastic right-hand sides in the second stage along with deterministic quadratic objective functions, linear constraints, and only integer variables in both stages. They reformulate this problem using value functions for quadratic integer programs, and present a global branch-and-bound algorithm and a level-set approach to solve the problem. Lately, Mijangos [36, 37] develop a Branch and Fix Coordination based algorithm to
solve TSS-QIP with quadratic terms in the second stage objective function and two-stage stochastic convex problems where objective function and constraints are nonlinear; both of these problems have binary and continuous variables in the first stage and only continuous variables in the second stage. As per our knowledge, this paper is the first attempt to tackle TSS-CMIPs in its general form, i.e. integer variables in both stages and $p$-order conic constraints in the second stage.

1.4 Brief literature review on CMIPs

In the last two decades, researchers have extended various classes of cutting planes derived for MIPs to mixed integer nonlinear programs (MINLP). It includes extensions of Gomory mixed integer cuts, Mixed Integer Rounding (MIR) cuts, split cuts [19], and $n$-step MIR inequalities [27] for MIPs to solve MINLPs [18], second-order CMIPs [5], second-order conic mixed integer sets [39], and polyhedral CMIPs [47], respectively. Additionally, attempts have been made to derive convex hull description of (structured) conic mixed integer sets (see [5, 25, 28, 40] for few examples). In this paper, we consider TSS-CMIPs and TSDDR-CMIPs with (structured) $p$-order CMIPs in the second stage. Among all papers on deterministic CMIPs, the work of Atamtürk and Narayanan [5] is most closely related to our work.

Specifically, the authors [5] generalize the MIR inequalities [41] by studying a conic mixed integer set defined by a single second-order conic constraint and one integer variable. They introduce CMIR cuts which are non-linear in original space and linear in higher dimensional space. (See Section 2.1 for more details.) In this paper, we introduce TSS-CMIPs with multi-constraint $p$-order conic mixed integer sets having multiple integer variables in the second stage and describe the convex hull or tighter approximation of these sets using parametric CMIR inequalities. We would like to point out that these structured sets have not been studied even in the deterministic CMIP literature. In addition, we study large-scale CMIPs with sparse matrices and develop sparse nonlinear cuts which are sufficient to provide partial convex hull of the CMIPs, i.e., CMIPs with very few number of integer constraints.

1.5 Organization of this paper

In Section 2, we review the CMIR cuts [5] and provide a reformulation of the second stage CMIP using additional continuous variables. In Section 3, we introduce TSS-CMIPs and TSDDR-CMIPs with structured CMIPs in the second stage, derive globally valid parametric (non)-linear cuts for them, and prove that these cuts provide convex programming equivalent, under certain conditions, for the second-stage CMIPs. In Section 4, we present conditions under which the addition of sparse nonlinear cuts in the extensive formulation of TSS-CMIPs is sufficient to relax the integrality restrictions on the second stage integer variables without effecting the integrality of the optimal solution of the TSS-CMIP. We also provide special cases which satisfy these conditions. Thereafter, in Section 5, we explore the computational effectiveness of the sparse cuts in solving extensive formulation of the structured TSS-CMIPs. In Section 6, we study deterministic multi-constraint conic mixed integer sets with multiple integer variables and sparse matrices, and derive sparse cuts which are sufficient to provide convex hull or partial convex hull of these sets. We provide concluding remarks in Section 7.
2. Background and Reformulation of Second Stage CMIPs

In this section, we review the CMIR cuts for deterministic CMIPs \([5]\) to provide necessary background for the results in the following sections. We also present a reformulation of the second stage CMIPs using additional continuous variables.

2.1 Conic Mixed Integer Rounding (CMIR)

First we briefly review the CMIR cut generation procedure \([5]\) which we will use in the proofs of the subsequent theorems. Atamtürk and Narayanan \([5]\) generalize the well-known MIR procedure for MIPs to second-order CMIPs by studying a single-constraint conic mixed integer set with one integer variable, i.e.,

\[
Z := \{(\sigma, v, \rho_0) \in \mathbb{Z} \times \mathbb{R}_+^2 : \sqrt{(\sigma - \beta)^2 + v^2} \leq \rho_0\},
\]

where \(\beta \in \mathbb{R}\). They reformulated set \(Z\) using additional continuous variables to get

\[
Z := \{(\sigma, v, \rho_0, \rho_1, \rho_2) \in \mathbb{Z} \times \mathbb{R}_+^4 : |\sigma - \beta| \leq \rho_1, |v| \leq \rho_2, \sqrt{\rho_1^2 + \rho_2^2} \leq \rho_0\},
\]

and derived a facet-defining inequality for \(Z_1 := \{(\sigma, \rho_1) \in \mathbb{Z} \times \mathbb{R}_+ : |\sigma - \beta| \leq \rho_1\}\), referred to as the \textit{CMIR inequality},

\[
\left(1 - 2\beta^{(1)}\right)(\sigma - |\beta|) + \beta^{(1)} \leq \rho_1
\]

(12)

where \(\beta^{(1)} = \beta - |\beta|\). Atamtürk and Narayanan \([5]\) showed that the convex hull of \(Z_1\) can be obtained by adding (12) to the continuous relaxation of \(Z_1\). Likewise, they proved that the convex hull of \(Z\) is given by

\[
\left\{(\sigma, v, \rho_0) \in \mathbb{R} \times \mathbb{R}_+^2 : \sqrt{(\sigma - \beta)^2 + v^2} \leq \rho_0, \right. \\
\left. \sqrt{(1 - 2\beta^{(1)})(\sigma - |\beta|) + \beta^{(1)}}^2 + v^2 \leq \rho_0\}\right).
\]

2.2 Reformulation of Second Stage of TSS-CMIPs and TSDR-CMIPs

We reformulate the second stage problem \(Q_\omega(x)\) of the TSS-CMIP and TSDR-CMIP using additional continuous variables, as follows:

\[
Q_\omega(x) := \min g_\omega y_\omega + \sum_{j \in J} \tilde{g}_\omega^j d_\omega,0
\]

(13)

s.t. \(W_\omega y_\omega \geq r_\omega - T_\omega x\), \(v_\omega,i x + f_\omega,i x - h_\omega,i \leq d_\omega,i, \quad i = 1, \ldots, m_2, j \in J\), \(\|d_\omega^j\|_p \leq d_\omega,0, \quad j \in J\), \(y_\omega \in \mathbb{Z}^{q_1} \times \mathbb{R}^{q-q_1}, d_\omega,i \in \mathbb{R}_+, \quad i = 0, 1, \ldots, m_2, j \in J\),

(14) \hspace{1cm} (15) \hspace{1cm} (16) \hspace{1cm} (17)
where $d^j_\omega := (d^j_{\omega,1}, \ldots, d^j_{\omega,m_2}) \in \mathbb{R}^{m_2}$ for $j \in J$, and $c^j_\omega$ and $f^j_\omega$ denote the $i$th row of matrices $E^j_\omega$ and $F^j_\omega$, respectively. We define the feasible region of the reformulated second stage program by $K_\omega(x) := \{(y_\omega, d_\omega) : (14) - (17) \text{ hold}\}$ for $x \in X$ and $\omega \in \Omega$.

Next, we describe the feasible region of the reformulated second stage problem as intersection of a polyhedral conic mixed integer set and a set of $p$-order cones. Let $K_\omega(x) = K^1_\omega(x) \cap K^2_\omega(x)$ where $K^1_\omega(x) := \{(y_\omega, d_\omega) \in (\mathbb{Z}^n \times \mathbb{R}^{q-q^n}) \times \mathbb{R}^{m_2[|J|+|J]} : (14) - (15) \text{ hold}\}$ is a polyhedral conic mixed integer set and $K^2_\omega(x) := \{(y_\omega, d_\omega) \in \mathbb{R}^q \times \mathbb{R}^{m_2[|J|+|J]} : (16) \text{ holds}\}$ is an intersection of $|J|$ number of $p$-order cones. In Theorem 1, we provide a relation between the convex hull of $K_\omega(x)$ and the convex hull of $K^1_\omega(x)$.

**Theorem 1.** For each $x \in X$ and $\omega \in \Omega$, if $p = 1$,

$$\text{conv}(K_\omega(x)) = \text{conv}(K^1_\omega(x)) \cap K^2_\omega(x),$$

and if $p \geq 2$, $\text{conv}(K_\omega(x)) \subseteq \text{conv}(K^1_\omega(x)) \cap K^2_\omega(x)$.

**Proof.** For each $x \in X$ and $\omega \in \Omega$, suppose that a point $\hat{\eta}_\omega := (\hat{y}_\omega, \hat{d}_\omega) \in \text{conv}(K_\omega(x)) = \text{conv}(K^1_\omega(x) \cap K^2_\omega(x))$. Therefore, $\hat{\eta}_\omega$ can be written as convex combination of $q + 1 + (m_2 + 1)|J|$ points belonging to $K^1_\omega(x) \cap K^2_\omega(x)$, i.e., $\hat{\eta}_\omega = \sum \lambda_k \bar{\eta}_\omega^k$ where $\sum \lambda_k = 1$, $\lambda_k \in [0, 1]$, $\bar{\eta}_\omega^k \in K^1_\omega$, and $\bar{\eta}_\omega^k \in K^2_\omega$ for all $k = 1, \ldots, q + 1 + (m_2 + 1)|J|$. This also implies that $\hat{\eta}_\omega \in \text{conv}(K^1_\omega)$ and $\hat{\eta}_\omega \in \text{conv}(K^2_\omega) = K^2_\omega$. So, for $p \geq 1$,

$$\text{conv}(K_\omega(x)) \subseteq \text{conv}(K^1_\omega(x)) \cap K^2_\omega(x).$$

Now, for $p = 1$, assume that a point $\hat{\eta}_\omega = (\hat{y}_\omega, \{\hat{d}_\omega^j, \hat{d}_{\omega,0}^j\}_{j \in J})$ belongs to $\text{conv}(K^1_\omega(x)) \cap K^2_\omega(x)$, i.e. $\hat{\eta}_\omega \in \text{conv}(K^1_\omega(x))$ and $\hat{\eta}_\omega \in K^2_\omega(x)$, i.e., $\hat{d}_{\omega,0}^j - \sum^{m_2} \hat{d}_{\omega,i}^j \geq 0$. Since $\hat{\eta}_\omega \in \text{conv}(K^1_\omega(x))$, $\hat{\eta}_\omega$ can be written as convex combination of a finite number of points $\bar{\eta}_\omega^k = (\bar{y}_\omega^k, \{\bar{d}_\omega^k, \bar{d}_{\omega,0}^k\}_{j \in J})$, for $k \in \{1, 2, \ldots, q + 1 + (m_2 + 1)|J|\}$, belonging to the set $K^1_\omega$, i.e., for $\lambda_k \in [0, 1]$ and $\sum \lambda_k = 1$,

$$\sum \lambda_k \bar{\eta}_\omega^k = \hat{\eta}_\omega, \quad \text{where} \quad \bar{d}_{\omega,0}^k = \hat{d}_{\omega,0} + \sum^{m_2} \left( \bar{d}_{\omega,i}^k - \hat{d}_{\omega,i}^j \right) \geq 0 \quad \text{for all} \quad k.$$ 

Hence, $\bar{\eta}_\omega^k \in K^1_\omega(x) \cap K^2_\omega(x)$ for all $k$. This implies

$$\sum \lambda_k \bar{\eta}_\omega^k = \hat{\eta}_\omega \in \text{conv}(K^1_\omega(x) \cap K^2_\omega(x)) = \text{conv}(K_\omega(x)),$$

and therefore,

$$\text{conv}(K^1_\omega(x)) \cap K^2_\omega(x) \subseteq \text{conv}(K_\omega(x)).$$

Hence, for $p = 1$, we get $\text{conv}(K_\omega(x)) = \text{conv}(K^1_\omega(x)) \cap K^2_\omega(x)$. \hfill \square

### 3. Structured Two-Stage Stochastic and Distributionally Robust $p$-Order Conic Mixed Integer Programs

In this section, we introduce TSS-CMIPs and TSRD-CMIPs with structured $p$-order CMIPs in the second stage and derive classes of parametric (non)-linear inequalities to get convex programming equivalent for the second stage problems (Theorems 2-4). Specifically, we consider the following three structured CMIPs in the second stage, i.e. $Q_\omega(x)$ (or $Q_\omega(x)$) where
(a) \( E_\omega = I_2 \) (identity matrix of size 2 \( \times \) 2), \( F = [1, 0]^T \) for \( j \in J \), and \( W_\omega = 0 \) (Theorem 2);
(b) \( E_\omega^j = 1 \) (vector of all ones) for \( j \in J \) and \( W_\omega \) is a totally unimodular (TU) matrix (Theorem 3);
(c) \( E_\omega^j \) for all \( j \in J \), and \( W_\omega \) are TU matrices (Theorem 4).

It is important to note that for (a), we utilize the result of Atamtürk and Narayanan [5] for \( Z \) in Theorem 2; whereas for structures (b) and (c), no result is known for multi-constraint and multi-variable generalizations of \( Z \) or \( Z_1 \).

**Theorem 2.** In TSS-CMIP (1) and TSDR-CMIP (6), let

\[
\overline{Q}_\omega(x) := \min \ g^1_\omega y_{\omega,1} + g^2_\omega y_{\omega,2} + \hat{g}_\omega d_{\omega,0}
\]

\[
s.t. \quad \sqrt{(y_{\omega,1} + f_\omega x - h_\omega)^2 + (y_{\omega,2})^2} \leq d_{\omega,0}, \quad (22)
\]

\[
y_{\omega,1} \in Z, y_{\omega,2} \in \mathbb{R}_+, d_{\omega,0} \in \mathbb{R}_+. \quad (23)
\]

The convex hull of the feasible region of \( \overline{Q}_\omega(x) \) for all \( x \in X \) is given by

\[
\left\{(y_{\omega,1}, y_{\omega,2}, d_{\omega,0}) \in \mathbb{R} \times \mathbb{R}_+^2 : (22) \text{ and } \right. \]

\[
\sqrt{((1 - 2\mu_\omega)(y_{\omega,1} + f_\omega x - h_\omega) + \mu_\omega)^2 + (y_{\omega,2})^2} \leq d_{\omega,0}, \quad (24)
\]

where \( \mu_\omega = h_\omega - \lceil h_\omega \rceil \). Furthermore, in higher dimensional space \( \overline{Q}_\omega(x) \) can be reformulated as:

\[
Q_\omega(x) := \min \left\{ g^1_\omega y_{\omega,1} + g^2_\omega y_{\omega,2} + \hat{g}_\omega d_{\omega,0} : |y_{\omega,1} + f_\omega x - h_\omega| \leq d_{\omega,1}, \right.
\]

\[
|y_{\omega,2}| \leq d_{\omega,2}, \quad \sqrt{d_{\omega,1}^2 + d_{\omega,2}^2} \leq d_{\omega,0}, \quad (25)
\]

\[
y_{\omega,1} \in Z, y_{\omega,2} \in \mathbb{R}_+, d_{\omega,i} \in \mathbb{R}_+, i = 0, 1, 2 \}.
\]

Then, for all \( x \in X \), the convex hull of the feasible region of \( Q_\omega(x) \), denoted by \( K_\omega(x) \), is obtained by adding the following parametric linear inequality to the continuous relaxation of \( K_\omega(x) \):

\[
(1 - 2\mu_\omega)(y_{\omega,1} - \lceil h_\omega \rceil + f_\omega x) + \mu_\omega \leq d_{\omega,1}.
\]

**Proof.** For \( x \in X \) and \( \omega \in \Omega \), by substituting \( \sigma = y_{\omega,1} - f_\omega x \in Z \), \( \beta = h_\omega \), \( v = y_{\omega,2} \in \mathbb{R}_+ \) and \( \rho_0 = d_{\omega,0} \in \mathbb{R}_+ \) in the set \( Z \), we get \( K_\omega(x) \). Hence, the convex hull of the feasible region of \( \overline{Q}_\omega(x) \), i.e., \( \text{conv}(K_\omega(x)) \), for all \( x \in X \) can be obtained by adding inequality,

\[
\sqrt{((1 - 2\mu_\omega)(y_{\omega,1} + f_\omega x - h_\omega) + \mu_\omega)^2 + (y_{\omega,2})^2} \leq d_{\omega,0},
\]

to the continuous relaxation of \( K_\omega(x) \). Furthermore, we reformulate \( \text{conv}(K_\omega(x)) \) in higher dimensional space by adding variables \( d_{\omega,1} \) and \( d_{\omega,2} \), and obtain

\[
\left\{(y_{\omega,1}, y_{\omega,2}, d_{\omega,1}, d_{\omega,2}, d_{\omega,0}) \in \mathbb{R} \times \mathbb{R}_+^4 : \right.
\]

\[
|y_{\omega,1} + f_\omega x - h_\omega| \leq d_{\omega,1},
\]

\[
|y_{\omega,2}| \leq d_{\omega,2}, \quad \sqrt{d_{\omega,1}^2 + d_{\omega,2}^2} \leq d_{\omega,0},
\]

\[
(1 - 2\mu_\omega)(y_{\omega,1} - \lceil h_\omega \rceil + f_\omega x) + \mu_\omega \leq d_{\omega,1}
\}.
which is the convex hull of the feasible region of \( Q_\omega(x) \), i.e., \( \text{conv}(K_\omega(x)) \).

**Theorem 3.** In TSS-CMIP (1) and TSDR-CMIP (6), let

\[
Q_\omega(x) := \min g_\omega y_\omega + \sum_{j \in J} \tilde{g}_j^i d^i_{\omega,0}
\]

s.t. \( W_\omega y_\omega \geq r_\omega - T_\omega x \),

\[
\begin{align*}
\|1 y_\omega^j + F^j_\omega x - h^j_\omega\|_p & \leq d^j_{\omega,0}, & j \in J, \\
y_\omega^j & \in \mathbb{Z}, d^j_{\omega,0} \in \mathbb{R}_+, & j \in J,
\end{align*}
\]

where \( W_\omega \) is a TU matrix and \( r_\omega \) is integral. For \( p = 1 \) or \( p \geq 2 \), the convex hull or an approximation, respectively, of the feasible region of \( Q_\omega(x) \) for all \( x \in X \) are given by

\[
\begin{align*}
(y_\omega, d_{\omega,0}) & \in \mathbb{R}^{|J|} \times \mathbb{R}_+^{|J|} : (25),(26), \text{ and} \\
\|E^j_{\omega,i}y_\omega^j + F^j_{\omega,i}x - h^j_{\omega,i}\|_p & \leq d^{j}_{\omega,0}, & l \in \mathcal{L}, j \in J,
\end{align*}
\]

where the \( l \)th row of \( E^j_{\omega,i}y_\omega^j + F^j_{\omega,i}x - h^j_{\omega,i} \) is either \( y_\omega^j + f^j_{\omega,i}x - h^j_{\omega,i} \) or \( (1 - 2 \mu^j_{\omega,i}) \left( y_\omega^j - \left| h^j_{\omega,i} \right| + f^j_{\omega,i}x \right) + \mu^j_{\omega,i} \), and \( \mu^j_{\omega,i} = h^j_{\omega,i} - \left| h^j_{\omega,i} \right| \). Furthermore, in higher dimensional space \( Q_\omega(x) \) can be reformulated as:

\[
Q_\omega(x) := \min \left\{ g_\omega y_\omega + \sum_{j \in J} \tilde{g}_j^i d^i_{\omega,0} : (25), \right. \\
\left. |y_\omega^j + f^j_{\omega,i}x - h^j_{\omega,i}| \leq d^{j}_{\omega,i}, \quad i = 1, \ldots, m_2, \ j \in J, \right. \\
\left. \|d^j_{\omega,0}\|_p \leq d^{j}_{\omega,0}, \quad j \in J, \right. \\
\left. y_\omega^j \in \mathbb{Z}, d^j_{\omega,0} \in \mathbb{R}_+, d^j_{\omega} \in \mathbb{R}^{m_2} \right\}.
\]

Then, for all \( x \in X \), the convex hull (for \( p = 1 \)) and an approximation (for \( p \geq 2 \)) of the feasible region of \( Q_\omega(x) \), denoted by \( K_\omega(x) \), are obtained by adding \( m_2 \times |J| \) number of the following parametric linear inequalities (in the higher dimensional space) to the continuous relaxation of \( K_\omega(x) \):

\[
(1 - 2 \mu^j_{\omega,i}) \left( y_\omega^j - \left| h^j_{\omega,i} \right| + f^j_{\omega,i}x \right) + \mu^j_{\omega,i} \leq d^{j}_{\omega,i}, \quad i = 1, \ldots, m_2, \ j \in J.
\]

**Proof.** Let

\[
K^1_\omega(x) = \left\{(y_\omega, d_{\omega}) \in \mathbb{Z}^{|J|} \times \mathbb{R}_+^{m_2|J|+|J|} : \right. \\
W_\omega y_\omega \geq r_\omega - T_\omega x, \\
|y_\omega^j + f^j_{\omega,i}x - h^j_{\omega,i}| \leq d^{j}_{\omega,i}, \quad i = 1, \ldots, m_2, \ j \in J \right\}
\]

for all \( x \in X \). First we utilize the CMIR cut generation approach (discussed in Section 2.1) for each defining inequality (34) of \( K^1_\omega(x) \). In other words, we substitute \( \sigma = y_\omega^j, \beta = h^j_{\omega,i} - f^j_{\omega,i}x, \) and \( \rho_1 = d^j_{\omega,i} \) in \( Z_1 \) and get the following valid parametric CMIR inequalities (12) for \( K^1_\omega(x) \):

\[
(1 - 2 \mu^j_{\omega,i}) \left( y_\omega^j - \left| h^j_{\omega,i} - f^j_{\omega,i}x \right| \right) + \mu^j_{\omega,i} \leq d^{j}_{\omega,i}, \quad i = 1, \ldots, m_2, j \in J.
\]
Since \( f^j_{\omega,i} x \) is integral for all \( x \in X \), \( f^j_{\omega,i} x = f^j_{\omega,i} x \) and therefore, inequality (35) is equivalent to
\[
(1 - 2\mu^j_{\omega,i}) \left( y^j_{\omega} - \left[ h^j_{\omega,i} - f^j_{\omega,i} x \right] \right) + \mu^j_{\omega,i} \leq d^j_{\omega,i}, \quad i = 1, \ldots, m_2; j \in J. \tag{36}
\]
Thus,
\[
\text{conv} \left( K^1_\omega(x) \right) \subseteq K^3_\omega(x) := \left\{ (y, d, \omega) \in \mathbb{R}^{1|J|} \times \mathbb{R}^{m_2|J|+|J|} : (33), (34), (36) \text{ hold} \right\} \tag{37}
\]
for all \( x \in X \). Notice that we can rewrite the set \( K^3_\omega(x) \) as
\[
K^3_\omega(x) = \left\{ (y, d, \omega) \in \mathbb{R}^{1|J|} \times \mathbb{R}^{m_2|J|+|J|} : W \omega y \geq r_\omega - T_\omega x, \right.
\]
\[
d^j_{\omega,i} \geq y^j_{\omega} - \left( h^j_{\omega,i} - f^j_{\omega,i} x \right), \quad i = 1, \ldots, m_2, j \in J, \tag{38}
\]
\[
d^j_{\omega,i} \geq \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) - y^j_{\omega}, \quad i = 1, \ldots, m_2, j \in J, \tag{39}
\]
\[
d^j_{\omega,i} \geq (1 - 2\mu^j_{\omega,i}) \left( y^j_{\omega} - \left[ h^j_{\omega,i} + f^j_{\omega,i} x \right] \right) + \mu^j_{\omega,i}, \quad i = 1, \ldots, m_2, j \in J \right\}.
\]
Now let \( K^4_\omega(x) \) be a bounded face of \( K^3_\omega(x) \) and has maximum possible dimension. Since \( d^j_{\omega,i} \), for \( i = 1, \ldots, m_2 \) and \( j \in J \), must be minimal on this face, we have
\[
d^j_{\omega,i} = \max \left\{ y^j_{\omega} - \left( h^j_{\omega,i} - f^j_{\omega,i} x \right), \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) - y^j_{\omega}, \right.
\]
\[
(1 - 2\mu^j_{\omega,i}) \left( y^j_{\omega} - \left[ h^j_{\omega,i} + f^j_{\omega,i} x \right] \right) + \mu^j_{\omega,i} \right\}.
\]
Moreover, if \( h^j_{\omega,i} \in \mathbb{Z} \) for any \( j \in J \) then \( \mu^j_{\omega,i} = 0 \), and as a result inequality (36) reduces to inequality (38). However, when \( h^j_{\omega,i} \notin \mathbb{Z} \) for \( j \in J \), then there are three possible cases:

Case I. \( d^j_{\omega,i} = y^j_{\omega} - \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) \): This case will happen if and only if
\[
y^j_{\omega} - \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) \geq \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) - y^j_{\omega}
\]
and
\[
y^j_{\omega} - \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) \geq (1 - 2\mu^j_{\omega,i}) \left( y^j_{\omega} - \left[ h^j_{\omega,i} + f^j_{\omega,i} x \right] \right) + \mu^j_{\omega,i},
\]
which are equivalent to
\[
y^j_{\omega} \geq h^j_{\omega,i} - f^j_{\omega,i} x \quad \text{and} \quad y^j_{\omega} \geq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i} x,
\]
respectively. Note that the last inequality is stronger than the second last inequality. Therefore, we can claim that \( d^j_{\omega,i} = y^j_{\omega} - \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) \) if and only if \( y^j_{\omega} \geq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i} x \).

Case II. \( d^j_{\omega,i} = h^j_{\omega,i} - f^j_{\omega,i} x - y^j_{\omega} \): This case will happen if and only if \( h^j_{\omega,i} - f^j_{\omega,i} x - y^j_{\omega} \geq y^j_{\omega} - \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) \) and
\[
h^j_{\omega,i} - f^j_{\omega,i} x - y^j_{\omega} \geq (1 - 2\mu^j_{\omega,i}) \left( y^j_{\omega} - \left[ h^j_{\omega,i} + f^j_{\omega,i} x \right] \right) + \mu^j_{\omega,i},
\]
which are equivalent to \( y^j_\omega \leq h^j_{\omega,i} - f^j_{\omega,i}x \) and \( y^j_\omega \leq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x \), respectively, as \( \mu^j_{\omega,i} \leq 1 \). Again, note that the last inequality is stronger than the second last inequality. Therefore, \( d^j_{\omega,i} = h^j_{\omega,i} - f^j_{\omega,i}x - y^j_\omega \) if and only if \( y^j_\omega \leq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x \).

Case III. \( d^j_{\omega,i} = \left( 1 - 2\mu^j_{\omega,i} \right) \left( y^j_\omega - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i}x \right) + \mu^j_{\omega,i} \). This case will happen if and only if \( \left( 1 - 2\mu^j_{\omega,i} \right) \left( y^j_\omega - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i}x \right) + \mu^j_{\omega,i} \geq h^j_{\omega,i} - f^j_{\omega,i}x - y^j_\omega \) and \( \left( 1 - 2\mu^j_{\omega,i} \right) \left( y^j_\omega - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i}x \right) + \mu^j_{\omega,i} \leq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x \), respectively. Therefore, \( d^j_{\omega,i} = \left( 1 - 2\mu^j_{\omega,i} \right) \left( y^j_\omega - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i}x \right) + \mu^j_{\omega,i} \) if and only if \( \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x \leq y^j_\omega \leq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x \).

Next, for each \( j \in J \), we partition the set \( I := \{1, \ldots, m_2\} \) into the sets \( I^1_j, I^2_j, \) and \( I^3_j \), i.e. \( I = I^1_j \cup I^2_j \cup I^3_j \), such that

\[
\begin{align*}
I^1_j & := \{ i \in I : d^j_{\omega,i} = y^j_\omega - \left( h^j_{\omega,i} - f^j_{\omega,i}x \right) \}, \\
I^2_j & := \{ i \in I : d^j_{\omega,i} = \left( h^j_{\omega,i} - f^j_{\omega,i}x \right) - y^j_\omega \}, \quad \text{and} \\
I^3_j & := \{ i \in I : d^j_{\omega,i} = \left( 1 - 2\mu^j_{\omega,i} \right) \left( y^j_\omega - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i}x \right) + \mu^j_{\omega,i} \}. 
\end{align*}
\]

Therefore, in the light of the above discussed cases, we can rewrite \( K^4_\omega(x) \) as

\[
K^4_\omega(x) = \left\{ (y_\omega, d_\omega) \in \mathbb{R}^{|J|} \times \mathbb{R}^{m_2|J| + |J|} : W_\omega y_\omega \geq r_\omega - T_\omega x, \quad \begin{align*}
& y^j_\omega \geq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x, \quad d^j_{\omega,i} = y^j_\omega - \left( h^j_{\omega,i} - f^j_{\omega,i}x \right), \quad i \in I^1_j, j \in J, \\
& y^j_\omega \leq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x, \quad d^j_{\omega,i} = \left( h^j_{\omega,i} - f^j_{\omega,i}x \right) - y^j_\omega, \quad i \in I^2_j, j \in J, \\
& \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x \leq y^j_\omega \leq \left[ h^j_{\omega,i} \right] - f^j_{\omega,i}x, \quad i \in I^3_j, j \in J, \\
& d^j_{\omega,i} = \left( 1 - 2\mu^j_{\omega,i} \right) \left( y^j_\omega - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i}x \right) + \mu^j_{\omega,i}, \quad i \in I^3_j, j \in J \end{align*} \right\}.
\]

Interestingly, in a compact form \( K^4_\omega(x) \) can be written as

\[
K^4_\omega(x) = \left\{ (y_\omega, d_\omega) \in \mathbb{R}^{|J|} \times \mathbb{R}^{m_2|J| + |J|} : W_\omega y_\omega \geq r_\omega - T_\omega x, \quad \begin{align*}
& d^j_{\omega,i} = \theta^j_{\omega,i} y^j_\omega + \varphi^j_{\omega,i} x, \quad d^j_{\omega,i} \leq y^j_\omega \leq \overline{d}^j_{\omega,i}, \quad i = 1, \ldots, m_2, j \in J, \\
\end{align*} \right\},
\]

where \( \theta^j_{\omega,i}, \varphi^j_{\omega,i} \in \mathbb{R} \) and \( d^j_\omega, \overline{d}^j_\omega \in \mathbb{Z} \cup \{ -\infty, +\infty \} \) for \( j \in J \) and \( i = 1, \ldots, m_2 \). Since \( W_\omega \) is a TU matrix (because of assumption), transpose of \( (W_\omega^T, I, -I) \) is also a TU matrix. Therefore, each bounded face \( K^4_\omega(x) \) of \( K^3_\omega(x) \), for \( x \in X \), has extreme points with integral \( y^j_\omega \) as \( r_\omega - T_\omega x \) is integral. Since \( K^3_\omega(x) \) is a subset of the continuous relaxation of \( K^4_\omega(x) \), and all bounded faces of \( K^3_\omega(x) \) have extreme points with integral \( y_\omega \) components, \( K^3_\omega(x) \subseteq \text{conv} (K^1_\omega(x)) \). Hence, \( \text{conv} (K^1_\omega(x)) = K^3_\omega(x) \) for all \( x \in X \) because of (37).

Finally, because of Theorem 1, \( \text{conv} (K^1_\omega(x)) \cap K^2_\omega(x) = K^3_\omega(x) \cap K^2_\omega(x) \) where \( K^2_\omega(x) := \{ (y_\omega, d_\omega) \in \mathbb{R}^{|J|} \times \mathbb{R}^{m_2|J| + |J|} : d^j_\omega \leq d^j_{\omega,0}, j \in J \} \), provides the convex hull (for \( p = 1 \)) or an approximation
(for \( p \geq 2 \)) for \((\mathcal{K}_\omega(x))\). In other words, we obtain the convex hull (for \( p = 1 \)) or an approximation (for \( p \geq 2 \)) of \(\mathcal{K}_\omega(x)\) by adding \(m_2 \times |J|\) number of linear inequalities:

\[
d_{\omega,i}^j \geq (1 - 2\mu_{\omega,i}^j) \left( y_{\omega}^j - \left\lfloor h_{\omega,i}^j \right\rfloor + f_{\omega,i}^j \right) + \mu_{\omega,i}^j, \quad i = 1, \ldots, m_2, j \in J,
\]

to the continuous relaxation of \(\mathcal{K}_\omega(x)\). Moreover, \(\mathcal{K}_\omega^3(x) \cap \mathcal{K}_\omega^2(x)\) when projected to \((y_\omega, d_{\omega,0})\) space gives convex hull of the feasible region of \(\mathcal{Q}_\omega(x)\), i.e.

\[
\left\{(y_\omega, d_{\omega,0}) \in \mathbb{R}^{|\mathcal{J}|} \times \mathbb{R}_+^{|\mathcal{J}|} : \right. \quad (25), (26), \text{ and } \left. \right\|

FH_{\omega,j}^y + FH_{\omega,j}^x - \hat{\mathbf{r}}_{\omega,j}^l \right\|_p \leq d_{\omega,0}^i, \ l \in \mathcal{L}, j \in J \right\},
\]

where the \(i\)th row of \(FH_{\omega,j}^y + FH_{\omega,j}^x - \hat{\mathbf{r}}_{\omega,j}^l\) is either \(y_\omega^j + f_{\omega,i}^j x - h_{\omega,i}^j\) or \((1 - 2\mu_{\omega,i}^j) \left( y_{\omega}^j - \left\lfloor h_{\omega,i}^j \right\rfloor + f_{\omega,i}^j x \right) + \mu_{\omega,i}^j\). This completes the proof.

\[\square\]

**Theorem 4.** In TSS-CMIP (1) and TSDR-CMIP (6), let

\[
\mathcal{Q}_\omega(x) := \min g_\omega y_\omega + \sum_{j \in J} \hat{g}_\omega^j d_{\omega,0}^j\quad (44)
\]

**s.t.** \(W_{\omega} y_\omega \geq r_{\omega} - T_{\omega} x,\)

\[
\left\|

FH_{\omega,j}^y + FH_{\omega,j}^x - h_{\omega,j}^l \right\|_p \leq d_{\omega,0}^j, \quad j \in J, \quad \mathcal{I}_{\omega,j}^x \in \mathbb{Z}^q, d_{\omega,0}^j \in \mathbb{R}_+, j \in J
\]

where \(E_{\omega}^j, j \in J, \text{ and } W_{\omega} \) are TU matrices and \(r_{\omega}\) is integral. For \( p = 1 \) or \( p \geq 2 \), the convex hull or an approximation, respectively, of the feasible region of \(\mathcal{Q}_\omega(x)\) for all \(x \in \mathcal{X}\) are given by

\[
\left\{(y_\omega, d_{\omega,0}) \in \mathbb{R}^{|\mathcal{J}|} \times \mathbb{R}_+^{|\mathcal{J}|} : \right. \quad (45), (46), \text{ and } \left. \right\|

FH_{\omega,j}^y + FH_{\omega,j}^x - \hat{\mathbf{r}}_{\omega,j}^l \right\|_p \leq d_{\omega,0}^i, \ l \in \mathcal{L}, j \in J \right\},
\]

where the \(i\)th row of \(FH_{\omega,j}^y + FH_{\omega,j}^x - \hat{\mathbf{r}}_{\omega,j}^l\) is either \(e_{\omega,i}^j y_\omega^j + f_{\omega,i}^j x - h_{\omega,i}^j\) or \((1 - 2\mu_{\omega,i}^j) \left( e_{\omega,i}^j y_\omega^j - h_{\omega,i}^j \right) + f_{\omega,i}^j x + \mu_{\omega,i}^j\). Furthermore, in higher dimensional space \(\mathcal{Q}_\omega(x)\) can be reformulated as:

\[
\mathcal{Q}_\omega(x) := \min \left\{ g_\omega y_\omega + \sum_{j \in J} \hat{g}_\omega^j d_{\omega,0}^j : \right. \quad (45), \text{ and } \left. \right\|

\left| e_{\omega,i}^j y_\omega^j + f_{\omega,i}^j x - h_{\omega,i}^j \right| \leq d_{\omega,i}^j, \quad i = 1, \ldots, m_2, j \in J, \quad (49)
\]

\[
\left\|

\left| d_{\omega,i}^j \right|_p \leq d_{\omega,0}^j, \quad j \in J, \quad \mathcal{I}_{\omega,j}^x \in \mathbb{Z}^q, d_{\omega,0}^j \in \mathbb{R}_+, d_{\omega,i}^j \in \mathbb{R}^{m_2}_+, j \in J \right\}. \quad (51)
\]

Then, for all \(x \in \mathcal{X}\), the convex hull (for \( p = 1 \)) and an approximation (for \( p \geq 2 \)) of the feasible region of \(\mathcal{Q}_\omega(x)\), denoted by \(\mathcal{K}_\omega(x)\), is obtained by adding \(m_2 \times |J|\) number of the following linear inequalities (in the higher dimensional space) to the continuous relaxation of \(\mathcal{K}_\omega(x)\):

\[
(1 - 2\mu_{\omega,i}^j) \left( e_{\omega,i}^j y_\omega^j - h_{\omega,i}^j \right) + f_{\omega,i}^j x + \mu_{\omega,i}^j \leq d_{\omega,i}^j, \quad i = 1, \ldots, m_2, \quad j \in J.
\]
Proof. Let
\[ K^1_\omega(x) = \left\{ (y_\omega, d_\omega) \in \mathbb{R}^{|J|} \times \mathbb{R}^{|m_2|+|J|}_+ : W_\omega y_\omega \geq r_\omega - T_\omega x, \right. \]
\[ \left. |e^j_{\omega,i} y^j_\omega + f^j_{\omega,i} x - h^j_{\omega,i}| \leq d^j_{\omega,i}, \quad i = 1, \ldots, m_2, \quad j \in J \right\} \]
for all \( x \in X \). Since \( E^j_\omega \) is a TU matrix, \( e^j_{\omega,i} y^j_\omega \) is integral. Similar to the proof of the previous theorem, we again utilize the CMIR cut generation approach (discussed in Section 2.1) for each defining inequality (34) of \( K^1_\omega(x) \). However, in this case we substitute \( \sigma = e^j_{\omega,i} y^j_\omega \in \mathbb{Z} \), \( \beta = h^j_{\omega,i} - f^j_{\omega,i} x \), and \( \rho_1 = d^j_{\omega,i} \) in \( Z_1 \) and get the following valid parametric CMIR inequalities (12) for \( K^1_\omega(x) \):
\[ (1 - 2\mu^j_{\omega,i}) \left( e^j_{\omega,i} y^j_\omega - \left[ h^j_{\omega,i} - f^j_{\omega,i} x \right] \right) + \mu^j_{\omega,i} \leq d^j_{\omega,i}, \quad i = 1, \ldots, m_2, \quad j \in J, \]
which is equivalent to
\[ (1 - 2\mu^j_{\omega,i}) \left( e^j_{\omega,i} y^j_\omega - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i} x \right) + \mu^j_{\omega,i} \leq d^j_{\omega,i}, \]
for \( i = 1, \ldots, m_2 \) and \( j \in J \), as \( f^j_{\omega,i} x \) is integral for all \( x \in X \). Thus, \( \text{conv} \left( K^1_\omega(x) \right) \subseteq K^3_\omega(x) \) for all \( x \in X \), where

\[ K^3_\omega(x) = \left\{ (y_\omega, d_\omega) \in \mathbb{R}^{|J|} \times \mathbb{R}^{|m_2|+|J|}_+ : W_\omega y_\omega \geq r_\omega - T_\omega x, \right. \]
\[ \left. d^j_{\omega,i} \geq e^j_{\omega,i} y^j_\omega - \left( h^j_{\omega,i} - f^j_{\omega,i} x \right), \quad i = 1, \ldots, m_2, \quad j \in J, \right\} \]
\[ \left. d^j_{\omega,i} \geq \left( h^j_{\omega,i} - f^j_{\omega,i} x \right) - e^j_{\omega,i} y^j_\omega, \quad i = 1, \ldots, m_2, \quad j \in J, \right\} \]
\[ \left. d^j_{\omega,i} \geq (1 - 2\mu^j_{\omega,i}) \left( e^j_{\omega,i} y^j_\omega - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i} x \right) + \mu^j_{\omega,i}, \quad i = 1, \ldots, m_2, \quad j \in J \right\}. \]

Notice that \( K^3_\omega(x) \) is a subset of the continuous relaxation of \( K^1_\omega(x) \). Therefore, if we can prove that all bounded faces of \( K^3_\omega(x) \) have extreme points with integral \( y_\omega \) component, then it will imply that \( K^3_\omega(x) \subseteq \text{conv} \left( K^1_\omega(x) \right) \) for all \( x \in X \). So, we now consider a bounded face of \( K^3_\omega(x) \) with maximum possible dimension, denoted by \( K^3_\omega(x) \), and on this face, \( d^j_{\omega,i} \), \( i = 1, \ldots, m_2 \) and \( j \in J \), is minimal, i.e.,
\[ d^j_{\omega,i} = \max \left\{ z^j_{\omega,i} - h^j_{\omega,i}, h^j_{\omega,i} - z^j_{\omega,i}, (1 - 2\mu^j_{\omega,i}) \left( z^j_{\omega,i} - \left[ h^j_{\omega,i} \right] \right) + \mu^j_{\omega,i} \right\} \]
where \( z^j_{\omega,i} = e^j_{\omega,i} y^j_\omega + f^j_{\omega,i} x \in \mathbb{Z} \). Now if \( h^j_{\omega,i} \in \mathbb{Z} \) for any \( j \in J \), then \( \mu^j_{\omega,i} = 0 \) and as a result inequality (54) reduces to inequality (55). However, when \( h^j_{\omega,i} \notin \mathbb{Z} \) for \( j \in J \), then again there are three possible cases:

Case I. \( d^j_{\omega,i} = z^j_{\omega,i} - h^j_{\omega,i} \): This case will happen if and only if \( z^j_{\omega,i} - h^j_{\omega,i} \geq h^j_{\omega,i} - z^j_{\omega,i} \) and \( z^j_{\omega,i} - h^j_{\omega,i} \geq (1 - 2\mu^j_{\omega,i}) \left( z^j_{\omega,i} - \left[ h^j_{\omega,i} \right] \right) + \mu^j_{\omega,i} \), which are equivalent to \( z^j_{\omega,i} \geq h^j_{\omega,i} \) and \( z^j_{\omega,i} \geq \left[ h^j_{\omega,i} \right] \), respectively. Therefore, we can claim that \( d^j_{\omega,i} = z^j_{\omega,i} - h^j_{\omega,i} \) if and only if \( z^j_{\omega,i} \geq \left[ h^j_{\omega,i} \right] \).
Case II. $d_{j,0}^j = h_{j,0}^j - z_{j,0}^j$: This case will happen if and only if $h_{j,0}^j - z_{j,0}^j \geq z_{j,0}^j - h_{j,0}^j$ and $h_{j,0}^j - z_{j,0}^j \geq \left(1 - 2\mu_{j,0}^j\right)\left(z_{j,0}^j - \left[h_{j,0}^j\right]\right) + \mu_{j,0}^j$, which are equivalent to $z_{j,0}^j \leq h_{j,0}^j$ and $z_{j,0}^j \leq \left[h_{j,0}^j\right]$, respectively, as $\mu_{j,0}^j \leq 1$. Therefore, $d_{j,0}^j = h_{j,0}^j - z_{j,0}^j$ if and only if $z_{j,0}^j \leq \left[h_{j,0}^j\right]$.

Case III. $d_{j,0}^j = \left(1 - 2\mu_{j,0}^j\right)\left(z_{j,0}^j - \left[h_{j,0}^j\right]\right) + \mu_{j,0}^j$: This case will happen if and only if $\left(1 - 2\mu_{j,0}^j\right)\left(z_{j,0}^j - \left[h_{j,0}^j\right]\right) + \mu_{j,0}^j \geq h_{j,0}^j - z_{j,0}^j$ and $\left(1 - 2\mu_{j,0}^j\right)\left(z_{j,0}^j - \left[h_{j,0}^j\right]\right) + \mu_{j,0}^j \geq z_{j,0}^j - h_{j,0}^j$, which are equivalent to $z_{j,0}^j \geq \left[h_{j,0}^j\right]$ and $z_{j,0}^j \leq \left[h_{j,0}^j\right]$, respectively. Therefore, $d_{j,0}^j = \left(1 - 2\mu_{j,0}^j\right)\left(z_{j,0}^j - \left[h_{j,0}^j\right]\right) + \mu_{j,0}^j$ if and only if $z_{j,0}^j \leq \left[h_{j,0}^j\right]$.

For each $j \in J$, let $\mathcal{I} := \{1, \ldots, m_2\}$ be partitioned into the disjoint sets $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, i.e. $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$, such that $\mathcal{I}_1 := \{i \in I : d_{i,j}^j = z_{i,j}^j - h_{i,j}^j\}$, $\mathcal{I}_2 := \{i \in I : d_{i,j}^j = h_{i,j}^j - z_{i,j}^j\}$, and $\mathcal{I}_3 := \{i \in I : d_{i,j}^j = \left(1 - 2\mu_{j,0}^j\right)\left(z_{i,j}^j - \left[h_{i,j}^j\right]\right) + \mu_{j,0}^j\}$. Therefore, in the light of the above discussed cases, we can define $\mathcal{K}_\omega^4(x)$ as

$$\mathcal{K}_\omega^4(x) = \left\{(y_\omega, d_\omega) \in \mathbb{R}^{|J|} \times \mathbb{R}^{|m_2||J| + |J|} : W_\omega y_\omega \geq r_\omega - T_\omega x, \right.$$

$$e_{\omega,i}^j y_\omega \geq \left[h_{\omega,i}^j\right] - f_{\omega,i}^j x, \quad d_{\omega,i}^j = e_{\omega,i}^j y_\omega - \left(h_{\omega,i}^j - f_{\omega,i}^j x\right), \quad i \in \mathcal{I}_1, j \in J,$$

$$c_{\omega,i}^j y_\omega \leq \left[h_{\omega,i}^j\right] - f_{\omega,i}^j x, \quad d_{\omega,i}^j = f_{\omega,i}^j x - c_{\omega,i}^j y_\omega, \quad i \in \mathcal{I}_2, j \in J,$$

$$\left[h_{\omega,i}^j\right] - f_{\omega,i}^j x \leq e_{\omega,i}^j y_\omega \leq \left[h_{\omega,i}^j\right] - f_{\omega,i}^j x, \quad i \in \mathcal{I}_3, j \in J,$$

$$d_{\omega,i}^j = \left(1 - 2\mu_{j,0}^j\right)\left(e_{\omega,i}^j y_\omega - \left[h_{\omega,i}^j\right] + f_{\omega,i}^j x\right) + \mu_{j,0}^j, \quad i \in \mathcal{I}_3, j \in J \bigg\}.$$ 

Let $(\hat{y}_\omega, \hat{d}_\omega)$ be an extreme point of $\mathcal{K}_\omega^4(x)$ for a given $x \in X$. Then this point can be obtained by binding $|J|(|q + m_2|)$ constraints of $\mathcal{K}_\omega^4(x)$ such that the matrix defining these constraints is nonsingular. Since the polyhedron $\mathcal{K}_\omega^4(x)$ already has $m_2|J|$ equality constraints, we need a combination of at least $q|J|$ defining inequalities of $\mathcal{K}_\omega^4(x)$ to be binding at the point $(\hat{y}_\omega, \hat{d}_\omega)$. Let the constraints

$$u_{\omega,i} y_\omega \geq r_{\omega,i} - t_{\omega,i} x, \quad i \in \tau_0,$$

$$e_{\omega,i}^j y_\omega \geq \left[h_{\omega,i}^j\right] - f_{\omega,i}^j x, \quad i \in \tau_1, j \in J_1,$$

$$e_{\omega,i}^j y_\omega \leq \left[h_{\omega,i}^j\right] - f_{\omega,i}^j x, \quad i \in \tau_2, j \in J_2,$$

be binding at the point $(\hat{y}_\omega, \hat{d}_\omega)$ where $w_{\omega,i}$, $r_{\omega,i}$, and $t_{\omega,i}$ denote ith row of matrix/vector $W_\omega$, $r_\omega$, and $T_\omega$, respectively, and $\tau_0, \tau_1, j \in J_1 \subseteq J$, and $\tau_2, j \in J_1 \subseteq J$, are subsets of $\{1, \ldots, m_2\}$ such that $|\tau_0| + \sum_{j \in J_1} |\tau_1| + \sum_{j \in J_2} |\tau_2| = q|J|$. Then, $\hat{y}_\omega$ satisfies the following system of equations:

$$(57) \quad w_{\omega,i} \hat{y}_\omega = r_{\omega,i} - t_{\omega,i} x, \quad i \in \tau_0,$$

$$(58) \quad e_{\omega,i}^j \hat{y}_\omega = \left[h_{\omega,i}^j\right] - f_{\omega,i}^j x, \quad i \in \tau_1, j \in J_1,$$

$$(59) \quad e_{\omega,i}^j \hat{y}_\omega = \left[h_{\omega,i}^j\right] - f_{\omega,i}^j x, \quad i \in \tau_2, j \in J_2.$$
which in compact form is written as \( \Pi_\omega \hat{y}_\omega = \pi_\omega (x) \). Observe that \( \Pi_\omega \) is a totally unimodular matrix of size \( q |J| \times q |J| \) as \( W_\omega, E_\omega^f \) for \( j \in J \) are totally unimodular matrices. Also, since each component of the vector \( \pi_\omega (x) \), i.e., \( r_{\omega,i} - t_{\omega,i} x, \left[ h^{j}_{\omega,i} \right] - f^{j}_{\omega,i} x, \) or \( \left[ h^{j}_{\omega,i} \right] - f^{j}_{\omega,i} x, \) is integral, \( \hat{y}_\omega \) is integral. Hence, all bounded faces of \( \mathcal{K}^3_\omega (x) \) have extreme point \((\hat{y}_\omega, \hat{d}_\omega)\) with integral \( \hat{y}_\omega \) and since \( \mathcal{K}^3_\omega (x) \) is a subset of the continuous relaxation of \( \mathcal{K}^2_\omega (x) \), \( \mathcal{K}^3_\omega (x) \subseteq \text{conv} \left( \mathcal{K}^1_\omega (x) \right) \) for all \( x \in X \). Therefore, we have \( \mathcal{K}^3_\omega (x) = \text{conv} \left( \mathcal{K}^1_\omega (x) \right) \).

Finally because of Theorem 1, \( \text{conv} \left( \mathcal{K}^1_\omega (x) \right) \cap \mathcal{K}^2_\omega (x) = \mathcal{K}^3_\omega (x) \cap \mathcal{K}^2_\omega (x) \) where \( \mathcal{K}^2_\omega (x) := \{ (y_\omega, d_\omega) \in \mathbb{R}^{q |J|} \times \mathbb{R}_+^{m_2 |J| + |J|} \in \mathbb{R}^{q |J| (m_2 + 1)} : \left\| d^j_{\omega,0} \right\|_p \leq d^j_{\omega,0}, j \in J \}, \) provides the convex hull (for \( p = 1 \)) or an approximation (for \( p \geq 2 \)) for \( \mathcal{K}_\omega (x) \). In other words, we obtain the convex hull (for \( p = 1 \)) or an approximation (for \( p \geq 2 \)) of \( \mathcal{K}_\omega (x) \) by adding \( m_2 \times |J| \) number of linear inequalities:

\[
d^j_{\omega,i} \geq (1 - 2 \mu^j_{\omega,i}) \left( c^j_{\omega,i} y^j_{\omega} - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i} \right) + \mu^j_{\omega,i}, i = 1, \ldots, m_2, j \in J, \tag{60}
\]

to the continuous relaxation of \( \mathcal{K}_\omega (x) \). Moreover, \( \mathcal{K}^3_\omega (x) \cap \mathcal{K}^2_\omega (x) \) when projected to \((y_\omega, d_\omega,0)\) space gives the convex hull (for \( p = 1 \)) or an approximation (for \( p \geq 2 \)) of the feasible region of \( \Omega_\omega (x) \), i.e.

\[
\left\{ (y_\omega, d_\omega,0) \in \mathbb{R}^{q |J|} \times \mathbb{R}_+^{[J]} : \text{(45), (46), and} \right. \\
\left. \left\| E^j_{\omega, l} y^j_{\omega} + F^j_{\omega, l} x - \bar{h}^j_{\omega, l} \right\|_p \leq d^j_{\omega,0}, l \in L, j \in J \right\},
\]

where the \( i \)th row of \( E^j_{\omega, l} y^j_{\omega} + F^j_{\omega, l} x - \bar{h}^j_{\omega, l} \) is either \( c^j_{\omega,i} y^j_{\omega} + f^j_{\omega,i} x - h^j_{\omega,i} \) or \((1 - 2 \mu^j_{\omega,i}) \left( c^j_{\omega,i} y^j_{\omega} - \left[ h^j_{\omega,i} \right] + f^j_{\omega,i} \right) + \mu^j_{\omega,i} \). This completes the proof. \( \square \)

4. Sparse Cuts for Extensive Formulation of TSS-CMIPs

In this section, we present sufficient conditions under which the integrality restrictions on the second stage integer variables of the TSS-CMIPs can be relaxed (without effecting the integrality of the optimal solution) by adding sparse nonlinear inequalities in \((x, y_\omega, d_\omega)\) space to the extensive formulation of TSS-CMIPs. In other words, using these sparse cuts, we derive “partial convex hull(s)” for feasible region of the extensive formulation, denoted by \( \Omega = \{ (x, y_\omega, d_\omega)_{\omega \in \Omega} \} : \text{(8) - (11) hold} \}. \) Given a nonempty set \( \Gamma \subseteq \Omega \), we define a partial convex hull of \( \Omega \) by another conic mixed integer set,

\[
\Omega_{\text{ch}} := \left\{ T_{\omega} x + W_\omega y_\omega \geq r_\omega, \omega \in \Omega, \\
\left\| E^j_{\omega} y_\omega + F^j_{\omega} x - h^j_{\omega} \right\|_p \leq d^j_{\omega,0}, j \in J, \omega \in \Omega, \\
\left\| E^j_{\omega, l} y^j_{\omega} + F^j_{\omega, l} x - \bar{h}^j_{\omega, l} \right\|_p \leq d^j_{\omega,0}, l \in L, j \in J, \omega \in \Gamma, \\
x \in X, d \in \mathbb{R}^{q |J| + |I|}, y_\omega \in \mathbb{R}^q, \omega \in \Gamma, \\
y_\omega \in \mathbb{Z}^{m_1} \times \mathbb{R}^{q_\omega - q_1}, \omega \in \Omega \setminus \Gamma \},
\]

where \( E^j_{\omega, l}, F^j_{\omega, l}, \bar{h}^j_{\omega, l} \) are known matrices (or vectors) for all \( l \in L, j \in J, \) and \( \omega \in \Gamma \), such that

\[
\Omega \subseteq \Omega_{\text{ch}} \subseteq \text{conv} (\Omega) = \text{conv}(\Omega_{\text{ch}}).
\]
Note that $\mathcal{P}_{pch}$ has lesser number of integrality constraints (but possibly more linear or nonlinear inequalities) than $\mathcal{P}$. Whereas in comparison to $\text{conv}(\mathcal{P})$, $\mathcal{P}_{pch}$ might have lesser inequalities but more integrality constraints.

**Theorem 5.** Given a nonempty set $\Gamma \subseteq \Omega$, $\mathcal{P}_{pch}$ is a partial convex hull of $\mathcal{P}$ if $\text{conv}(\mathcal{K}_\omega(x)) = \mathcal{K}_\omega^{\text{tight}}(x)$ for all $x \in X$ and $\omega \in \Gamma$, where

$\mathcal{K}_\omega^{\text{tight}}(x) := \{(y_\omega,d_\omega) \in \mathbb{R}^q \times \mathbb{R}^{|J|}_+: W_\omega y_\omega \geq r_\omega - T_\omega x, \|E_j^j y_\omega + F_j^j x - h_j^\omega\|_p \leq d_{\omega,0}^j, j \in J, \|E_\omega^j y_\omega^j + F_\omega^j x - R_\omega^j\|_p \leq d_{\omega,0}^j, l \in \mathcal{L}, j \in J\}$.

**Proof.** Refer to Appendix 8.1

**Remark 1.** Theorem 5 for TSS-CMIP extends the results (Lemma 1 and Theorem 3) of [7] for TSS-MIP, i.e., TSS-CMIP with $|J| = 0$, with linear parametric cuts.

### 4.1 Sparse Cuts for Structured TSS-CMIPs

In the following theorems, we consider extensive formulations of structured TSS-CMIPs (introduced in Section 3), i.e., large-scale CMIPs with reformulation, and present partial convex hull for them, using inequalities derived in Theorems 2-4. Note that for $\hat{x} \in X$ and $\omega \in \Omega$, $\mathcal{K}_\omega(\hat{x}) = \text{Proj}_{x=\hat{x},y_\omega,d_\omega}(\mathcal{P})$ and $\mathcal{K}_\omega^{\text{tight}}(\hat{x}) = \text{Proj}_{x=\hat{x},y_\omega,d_\omega}(\mathcal{P}_{pch})$.

**Theorem 6.** Let

$\mathcal{P} := \left\{(y_{\omega,1} + f_\omega x - h_\omega)^2 + (y_{\omega,2})^2 \leq d_{\omega,0}, \omega \in \Omega \right\} \quad (61)$

$x \in X, y_{\omega,1} \in \mathbb{Z}, y_{\omega,2} \in \mathbb{R}_+, d_{\omega,0} \in \mathbb{R}_+, \omega \in \Omega$.

For each nonempty set $\Gamma \subseteq \Omega$, a partial convex hull of $\mathcal{P}$ is given by

$\mathcal{P}_{pch} := \left\{(x,y_{\omega,1},y_{\omega,2},d_{\omega,0}) \in X \times \mathbb{R} \times \mathbb{R}^2_+: (61) \text{ and } \sqrt{\left((1 - 2\mu_\omega)(y_{\omega,1} + f_\omega x - h_\omega) + \mu_\omega\right)^2 + (y_{\omega,2})^2} \leq d_{\omega,0}, \omega \in \Omega \right\}$,

where $\mu_\omega = h_\omega - \lfloor h_\omega \rfloor$.

**Proof.** For $x \in X$ and $\omega \in \Gamma$, let

$\mathcal{K}_\omega^{\text{tight}}(x) = \left\{(y_{\omega,1},y_{\omega,2},d_{\omega,0}) \in \mathbb{R} \times \mathbb{R}^2_+: (61) \text{ and } \sqrt{\left((1 - 2\mu_\omega)(y_{\omega,1} + f_\omega x - h_\omega) + \mu_\omega\right)^2 + (y_{\omega,2})^2} \leq d_{\omega,0} \right\}$.

From Theorem 2, we know that $\text{conv}(\mathcal{K}_\omega(x)) = \mathcal{K}_\omega^{\text{tight}}(x)$ for all $x \in X$ and $\omega \in \Gamma$. Hence, by utilizing Theorem 5, a partial convex hull of $\mathcal{P}$ is given by $\mathcal{P}_{pch}$.
Theorem 7. Let
\[ \mathcal{P} := \left\{ (x, \{y_{\omega}, d_{\omega,0}\} : \omega \in \Omega) \in X \times Z^{[\Omega]} \times \mathbb{R}^{[\Omega]} : T_{\omega}x + W_{\omega}y_{\omega} \geq r_{\omega}, \omega \in \Omega, \right\} \]
where \( W_{\omega} \) is a TU matrix. For each nonempty set \( \Gamma \subseteq \Omega \), a partial convex hull of \( \mathcal{P} \) with \( p = 1 \) is given by
\[ \mathcal{P}_{\text{pch}} := \left\{ (x, \{y_{\omega}, d_{\omega,0}\} : \omega \in \Omega) \in X \times Z^{[\Omega]} \times \mathbb{R}^{[\Omega]} : T_{\omega}x + W_{\omega}y_{\omega} \geq r_{\omega}, \omega \in \Omega, \right\} \]
where the \( i \)th row of \( F_{\omega,l}y_{\omega} + F_{\omega,l}x - h_{\omega,l} \) is either \( y_{\omega} + f_{\omega,i}x - h_{\omega,i} \) or \( (1 - 2\mu_{\omega,i}) \left( y_{\omega} - \left[ h_{\omega,i} \right] + f_{\omega,i}x \right) + \mu_{\omega,i}, \) and \( \mu_{\omega,i} = h_{\omega,i} - \left[ h_{\omega,i} \right] \).

Proof. For \( x \in X \) and \( \omega \in \Gamma \), let
\[ \mathcal{K}_{\text{tight}}(x) = \left\{ (y_{\omega}, d_{\omega,0}) : (25) \text{ and } (26) \right\} \]
where the \( i \)th row of \( E_{\omega,l}y_{\omega} + E_{\omega,l}x - h_{\omega,l} \) is either \( y_{\omega} + f_{\omega,i}x - h_{\omega,i} \) or \( (1 - 2\mu_{\omega,i}) \left( y_{\omega} - \left[ h_{\omega,i} \right] + f_{\omega,i}x \right) + \mu_{\omega,i} \), and \( \mu_{\omega,i} = h_{\omega,i} - \left[ h_{\omega,i} \right] \). From Theorem 3, we know that \( \text{conv}(\mathcal{K}_{\text{tight}}(x)) = \mathcal{K}_{\text{tight}}(x) \) for all \( x \in X \) and \( \omega \in \Gamma \) when \( p = 1 \). Hence, by utilizing Theorem 5, it is clear that \( \mathcal{P}_{\text{pch}} \) is a partial convex hull of \( \mathcal{P} \).

Theorem 8. Let
\[ \mathcal{P} := \left\{ (x, \{y_{\omega}, d_{\omega,0}\} : \omega \in \Omega) \in X \times Z^{[\Omega]} \times \mathbb{R}^{[\Omega]} : T_{\omega}x + W_{\omega}y_{\omega} \geq r_{\omega}, \omega \in \Omega, \right\} \]
where \( W_{\omega} \) and \( E_{\omega,j} \), \( j \in J \), are totally unimodular matrices. For each nonempty set \( \Gamma \subseteq \Omega \), a partial convex hull of \( \mathcal{P} \) with \( p = 1 \) is given by
\[ \mathcal{P}_{\text{pch}} := \left\{ (x, \{y_{\omega}, d_{\omega,0}\} : \omega \in \Omega) \in X \times Z^{[\Omega]} \times \mathbb{R}^{[\Omega]} : T_{\omega}x + W_{\omega}y_{\omega} \geq r_{\omega}, \omega \in \Omega, \right\} \]
where the $i$th row of $E_{j\omega,l}^j y_{j\omega} + F_{j\omega,l}^j x - h_{j\omega,l}^j$ is either $e_{j\omega,i}^j y_{j\omega} + f_{j\omega,i}^j x - h_{j\omega,i}^j$ or $(1 - 2\mu_{j\omega,i}^j) \left( e_{j\omega,i}^j y_{j\omega} + f_{j\omega,i}^j x + \mu_{j\omega,i}^j \right) + \mu_{j\omega,i}^j = h_{j\omega,i}^j - \left\lfloor h_{j\omega,i}^j \right\rfloor$.

Proof. For $x \in X$ and $\omega \in \Gamma$, let

$$\mathcal{K}_{\text{tight}}(x) = \left\{ (y_\omega, d_{\omega,0}) \in \mathbb{R}^q \times \mathbb{R}_+^{|J|} : (45) \text{ and } (46) \right\},$$

where the $i$th row of $E_{j\omega,l}^j y_{j\omega} + F_{j\omega,l}^j x - h_{j\omega,l}^j$ is either $e_{j\omega,i}^j y_{j\omega} + f_{j\omega,i}^j x - h_{j\omega,i}^j$ or $(1 - 2\mu_{j\omega,i}^j) \left( e_{j\omega,i}^j y_{j\omega} + f_{j\omega,i}^j x + \mu_{j\omega,i}^j \right) + \mu_{j\omega,i}^j = h_{j\omega,i}^j - \left\lfloor h_{j\omega,i}^j \right\rfloor$. From Theorem 4, we know $\text{conv}(\mathcal{K}_\omega(x)) = \mathcal{K}_{\text{tight}}(x)$ for all $x \in X$ and $\omega \in \Gamma$ when $p = 1$. Hence, by utilizing Theorem 5, we can claim that $\mathcal{P}_{\text{pch}}$ is a partial convex hull of $\mathcal{P}$. \hfill\square

5. Computational Experiments

We perform computational experiments to evaluate the effectiveness of adding our sparse cuts (a priori) for solving TSS-CMIP test instances. We first describe generation of these test problems in Section 5.1 and then present our computational results in Section 5.2.

5.1 Generation of TSS-CMIP test instances

For our computational experiments, we consider the extensive formulation of reformulated TSS-CMIP, i.e.,

$$\min cx + \sum_{\omega \in \Omega} p_\omega \left( g_\omega y_\omega + \sum_{j \in J} \tilde{g}_{j\omega} d_{j\omega,0}^j \right) \label{62}$$

subject to

$$Ax \geq b, \label{63}$$

$$T_\omega x + W_\omega y_\omega \geq r_\omega, \omega \in \Omega, \label{64}$$

$$|e_{j\omega,i}^j y_{j\omega} + f_{j\omega,i}^j x - h_{j\omega,i}^j| \leq d_{j\omega,i}^j, \ i = 1, \ldots, m_2, j \in J, \label{65}$$

$$\|d_{j\omega,0}^j\|_p \leq d_{j\omega,0}, \quad j \in J, \label{66}$$

$$x \in \mathbb{Z}^{n_1}, y_\omega \in \mathbb{Z}^{q_1} \times \mathbb{R}^{q_2 - q_1}, d_{j\omega,i}^j \in \mathbb{R}_+, i = 0, 1, \ldots, m_2, j \in J, \omega \in \Omega, \label{67}$$

with $p = 1$ or $p = 2$, and structured CMIPs in the second stage (as discussed in previous section), i.e. either $E_{j\omega} = I$ (an identity matrix) or $E_{j\omega}$ is a randomly generated deterministic TU matrix for $j \in J$, and $W_\omega$ is a deterministic TU matrix for all $\omega \in \Omega$. For each structured TSS-CMIP with $p = 2$ and only integer variables in the second stage, we generate two sets of random instances: instances from the first problem set are motivated from the Stochastic Integer Programming Library (SIPLIB) TSS-MIP instances [1], in particular stochastic server location problem (SSLP) and stochastic multiple
binary knapsack problem (SMBKP) instances, whereas for the second problem set, we consider instances with larger number of scenarios (up to 10,000). More specifically, in the first problem set, we generate random instances with similar problem size as of SSLP and SMBKP instances but with uncertain cost-coefficients, technology matrix, $F^j_\omega$, $h^j_\omega$ and right-hand-side.

Likewise for TSS-CMIPs with $p = 1$, we consider instances with TU $E^j_\omega$ matrix for all $j \in J$ and larger number of scenarios (up to 50,000). In Tables 1, 2, and 3, we provide details of problem categories for different types of structured second-stage CMIPs, i.e., $E^j_\omega$ is an identity matrix or any TU matrix, used for our experiments. We denote the number of linear constraints and number of integer variables by $\#L\text{Con}$ and $\#\text{IVar}$, respectively, in each stage. Also, we use $\#\text{PCCon}$ and $\#p\text{-OCon}$ to denote the number of polyhedral conic constraints (65) or number of rows in $E^j_\omega$, and number of $p$-order conic constraints (66), respectively, in the second stage.

Table 1: Details of TSS-CMIP Instances with $p = 1$ and $E^j_\omega$ is TU matrix

<table>
<thead>
<tr>
<th>Instance Category</th>
<th>Stage I</th>
<th>Stage II</th>
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<tr>
<td></td>
<td>$#L\text{Con}$</td>
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</table>

Table 2: Details of TSS-CMIP Instances with $p = 2$ and $E^j_\omega = I$

<table>
<thead>
<tr>
<th>Problem Set</th>
<th>Instance Category</th>
<th>Stage I</th>
<th>Stage II</th>
</tr>
</thead>
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<td>$#\text{IVar}$</td>
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</table>

Table 3: Details of TSS-CMIP Instances with $p = 2$ and $E^j_\omega$ is TU

<table>
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<tr>
<th>Problem Set</th>
<th>Instance Category</th>
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<th>Stage II</th>
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<tr>
<td></td>
<td>SCMIP.25.100.3</td>
<td>10</td>
<td>25</td>
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We use SCMIP.$\alpha$.\beta$.\lambda$ to denote our instance category in Tables 1, 2, and 3, where $\alpha$ is the number of integer variables in the first stage, $\beta$ is the number of integer variables in the second stage, and
Moreover, for instance categories in Table 2, we let \( g \) of extensive formulation without our sparse cuts, and extensive CMIP formulation, respectively. In value of continuous relaxation of extensive formulation with our sparse cuts, continuous relaxation from integer uniform impose an upper bound of 100 and a lower bound of \(-100\), we have unbounded solution values. Therefore, in order to obtain finite optimal solution values, we generate instances as follow: we draw \( \lambda \) from \( \text{uniform}[0, 10] \), \( A, b \) from \( \text{uniform}[0, 100] \), \( \tau, T \) from integer uniform \([-100, 100] \), \( F \) from integer uniform \([-10, 10] \), \( h \) from uniform \([-10, 10] \). Moreover, for instance categories in Table 2, we let \( g = 0 \) and \( \hat{g} = 1 \) for \( \omega \in \Omega \), \( j \in J \), while for instance categories in Tables 1 and 3, \( g \) is randomly generated from \( \text{uniform}[-1, 1] \) and \( \hat{g} \) is randomly generated from \( \text{uniform}[-10, 10] \). Since \( y \) is an unrestricted variable and \( g \in [-1, 1] \) for \( j \in J \) and \( \omega \in \Omega \), we observed that many randomly generated TSS-CMIP instances have unbounded solution values. Therefore, in order to obtain finite optimal solution values, we impose an upper bound of 100 and a lower bound of \(-100\) on each \( y \) variable. Note that when \( E = I \), the number of integer variables in the second stage, i.e., \( q \), is same as number of polyhedral conic constraints, i.e., \( m \). Whereas for instance categories in Table 1 and 3, \( E \) is not a square matrix. These instances will be available at computational Operations Research exchange (cORe) https://core.isrd.isi.edu/.

### 5.2 Computational Framework

In this section, we evaluate the effectiveness of our sparse cuts by performing computational experiments on instances belonging to the aforementioned instance categories with different number of scenarios. The results of our experiments are presented in Tables 4, 5, and 6. Each row in Tables 4 and 5 reports the average over five randomly generated instances corresponding to the instance category in Table 1 and 2, respectively, while each row in Table 6 reports the average over three randomly generated (relatively harder) instances belonging to instance category in Table 3. For instances in Table 1, we perform three experiments: NO-SCUT, WITH-SCUTS, and BD-WITH-SCUTS. While for instances in Table 2 and 3, we only perform the first two experiments. In NO-SCUT, we solve the reformulated extensive formulation of the problem using CPLEX 12.7.2 with its default settings, without adding our sparse cuts. Whereas in WITH-SCUTS, we add our sparse linear cuts (provided in Theorem 4 or derived through reformulation of \( P_{pc} \) in Theorem 8 for \( \Gamma = \Omega \)), a priori to the reformulated extensive formulation of the problem instance, relax the integrality constraints of second stage integer variables, and use CPLEX 12.70 with its default settings to solve it. In BD-WITH-SCUTS for TSS-CMIP with \( p = 1 \), we first convexify the second stage problem by adding our parametric cuts (provided in Theorem 4), and then solve it using Benders’ decomposition routine of CPLEX 12.70. For NO-SCUT and WITH-SCUTS, we allow presolve option in CPLEX 12.70; while for BD-WITH-SCUTS, we turn it off. All experiments are performed on a 8-core Xeon 2.4 GHz machine with 24 GB RAM running with Windows 10.

In Tables 4, 5 and 6, we report following statistics: number of integer variables in the extensive formulation with(out) sparse cuts (#IVar), number of linear constraints in the extensive formulation without our sparse cuts (#LCon), and the number of linear sparse cuts added in the the extensive formulation (#LCuts). Also, we denote the total time taken to solve TSS-CMIP instances without and with our sparse cuts in extensive formulation by T-EF and T-EFC, respectively, and the percentage of integrality gap closed by a priori addition of sparse cuts by \( G\% = 100 \times (V_{pcut} - V_{cp})/(V_{mip} - V_{cp}) \), where \( V_{pcut}, V_{cp}, \) and \( V_{mip} \) denote the optimal objective value of continuous relaxation of extensive formulation with our sparse cuts, continuous relaxation of extensive formulation without our sparse cuts, and extensive CMIP formulation, respectively.

\[ y_j^+ - y_j^- \] for \( j \in J \), \( \omega \in \Omega \).

\footnote{In order to improve the performance of CPLEX, we substitute unrestricted variable \( y \) by \( y_j^+ - y_j^- \) in our experiments, where \( y_j^+, y_j^- \geq 0 \) for \( j \in J \), \( \omega \in \Omega \).}
Table 4, T-BDC denotes the time taken to solve TSS-CMIP with \( p = 1 \) and linear programming equivalent of the second stage CMIP, using Benders’ decomposition routine of CPLEX. We use TL to notify that CPLEX cannot solve the corresponding instance within 3 hours time limit and OM to notify that our system ran out of 24 GB memory when solving this instance. Instances for which we could not obtain the optimal value due to TL or OM, we put - in column \( G\% \).

### 5.2.1 Computational results for TSS-CMIPs where \( p = 1 \) and \( E^j_\omega \) is TU

In Table 1, we observe that by adding sparse cuts a priori, the number of integer variables \( \#IVar \) is significantly reduced. Without our sparse cuts, CPLEX with its default settings took 800 seconds (on average) to solve the extensive formulation of the TSS-CMIP instances where \( p = 1 \) and \( E^j_\omega \), \( j \in J \), are TU matrices. However, by adding our cuts, CPLEX took 143 seconds (on average) to solve the extensive formulation of these instances, and reduced the time by up to 25 times and 4.6 times (on average). Our sparse cuts closed the integrality gap by almost 100%. It is worth noting that even though we disabled presolve when using Benders decomposition routine, for instance category SCMIP.5.10.5, T-BDC is on average 79.83% of T-EF.

<table>
<thead>
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<td>)</td>
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<td>#LCon</td>
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### 5.2.2 Computational results for TSS-CMIPs with \( p = 2 \) and \( E^j_\omega = I \)

For TSS-CMIP with \( p = 2 \), adding our sparse cuts in the extensive formulation and relaxing integrality restrictions on second stage integer variables provide an approximation of the problem. Let the approximation ratio be defined by \( R\% = 100 \times \frac{V_{approx}}{V_{mip}} \) where \( V_{approx} \) is the optimal objective value obtained from the experiment WITH-SCUTS (after adding sparse cuts in the extensive formulation and relaxing the integrality constraints on the second stage variables). We observe that the approximation ratio of all instances, with known \( V_{mip} \), in Table 5 is greater than
Table 5: Results of Computational Experiments for TSS-CMIPs with $p = 2$ and $E^j_I = I$

| Problem Set | Instance Category | $|\Omega|$ | NO-SCUT | WITH-SCUT | $\%$ |
|-------------|-------------------|----------|---------|----------|------|
|             |                   |          | #IVar   | #LCon    | T-EF | #IVar | #PCuts | T-EFC  | G%   |
| SCMIP.5.125.1 | 50                | 12505   | 14001   | 32.14   | 5    | 6250  | 5.43   | 35.42  |
| SCMIP.5.125.1 | 100               | 25005   | 28001   | 282.90  | 5    | 40501 | 91.52  | 15.90  |
| SCMIP.10.500.1 | 50                | 50010   | 53001   | TL      | 10   | 25000 | 148.22 | -      |
| SCMIP.10.500.1 | 100               | 100010  | 106001  | TL      | 10   | 50000 | 601.91 | -      |
| SCMIP.10.500.1 | 500               | 500010  | 530001  | TL      | 10   | 250000 | TL | -      |
| SCMIP.15.625.1 | 5                 | 6265    | 7051    | 27.41   | 15   | 3375  | 1.40   | 3.39   |
| SCMIP.15.625.1 | 10                | 12505   | 14101   | 56.19   | 15   | 6750  | 2.41   | 14.16  |
| SCMIP.15.625.1 | 15                | 18765   | 21151   | 115.88  | 15   | 10125 | 5.77   | 13.50  |
| SCMIP.15.625.1 | 50                | 62515   | 70501   | TL      | 15   | 33750 | 550.10 | -      |
| SCMIP.20.800.1 | 20                | 32020   | 33201   | TL      | 20   | 16000 | 500.88 | -      |
| SCMIP.20.800.1 | 50                | 80020   | 83001   | TL      | 20   | 40000 | 4532.02 | -      |
| SCMIP.240.120.1 | 20               | 5040    | 4905    | 9.29    | 240  | 2400  | 1.00   | 4.65   |
| SCMIP.240.120.1 | 50               | 12240   | 12255   | 51.94   | 240  | 6000  | 2.61   | 6.58   |
| SCMIP.5.10.1 | 50                | 1005    | 1255    | 4.26    | 5    | 500   | 0.67   | 6.92   |
| SCMIP.5.10.1 | 100               | 2005    | 2505    | 6.30    | 5    | 1000  | 1.26   | 27.68  |
| SCMIP.5.10.1 | 200               | 4005    | 5005    | 11.14   | 5    | 2000  | 2.07   | 27.69  |
| SCMIP.5.10.1 | 500               | 10005   | 12505   | 29.29   | 5    | 5000  | 4.81   | 43.49  |
| SCMIP.5.10.1 | 1000              | 20005   | 25005   | 67.35   | 5    | 10000 | 9.88   | 52.47  |
| SCMIP.5.10.1 | 5000              | 100005  | 125005  | 284.43  | 5    | 50000 | 82.74  | 72.91  |
| SCMIP.5.10.1 | 10000             | 200005  | 250005  | TL      | 5    | 100000 | 218.72 | -      |
| SCMIP.10.10.1 | 50                | 1010    | 2005    | 0.46    | 10   | 500   | 0.69   | 0.35   |
| SCMIP.10.10.1 | 100               | 2010    | 4005    | 0.79    | 10   | 1000  | 1.61   | 0.75   |
| SCMIP.10.10.1 | 200               | 4010    | 8005    | 2.75    | 10   | 2000  | 2.31   | 2.2    |
| SCMIP.10.10.1 | 500               | 10010   | 20005   | 4.34    | 10   | 5000  | 5.38   | 2.10   |
| SCMIP.10.10.1 | 1000              | 20010   | 40005   | 13.07   | 10   | 10000 | 14.62  | 10.57  |
| SCMIP.10.10.1 | 5000              | 100010  | 200005  | 502.44  | 10   | 50000 | 91.8   | 13.11  |
| SCMIP.10.10.1 | 10000             | 200010  | 400005  | 1490.35 | 10   | 100000 | 288.16 | 27.55  |

99.99%, thereby demonstrating the strength of our sparse cuts. In other words, these cuts provide near-optimal solution for TSS-CMIPs with $p = 2$ and $E^j_I = I$.

Now by comparing T-EF and T-EFC in Table 5, we observe that adding our sparse linear cuts (a priori) significantly reduces the time taken to solve the reformulated extensive formulation of the TSS-CMIP instances, using CPLEX with its default settings. More specifically, after adding our sparse cuts, CPLEX solved 132 out of 135 randomly generated instances within the time limit, except three instances of SCMIP.10.50.1 with 500 scenarios. Whereas, without our cuts, CPLEX could not solve 35 out of 135 TSS-CMIP instances within 3 hours time limit (32 instances) and allocated memory (3 instances), and took 148 seconds (on average) to solve the remaining 100 instances in comparison to 32 seconds (on average) after adding our cuts. Additionally, WITH-SCUTS took 22 minutes (on average) for 32 out of 35 (unsolvable) instances. Overall, for instances in Table 5, our sparse cuts closed the integrality gap by 19.1% (on average) for instances solved to optimality using NO-SCUT, and reduced the time taken to solve the TSS-CMIP instances by 5 times (on average).

In particular, for problem set I, WITH-SCUTS were performed at least 2 times and up to 19
times faster than NO-SCUT. Whereas for problem set II, WITH-SCUTS is 5 times (on average) and up to 39 times faster than NO-SCUT. It is interesting to observe that while by adding our sparse cuts, there is a trade-off between the decrease in the number of integer variables and increase in the number of constraints. For instances such as SCMIP 10.10.1 where $|\Omega| \in \{50, 100, 500, 1000\}$, WITH-SCUTS took marginally longer time that NO-SCUT. However, for instances with larger number of scenarios, WITH-SCUTS are notably faster than NO-SCUT.

5.2.3 Computational results for TSS-CMIPs with $p = 2$ where $E_\omega^J$ is TU

Notice that the problem instances considered in Table 6 are much harder than problem instances considered in Table 5, primarily because of multiple conic constraints (16) corresponding to each scenario (as $|J| = 3$) and multiple integer variables in each constraint (15). Nonetheless, we again observe that the approximation ratio $R\%$ for all problem instances, with known $V_{mip}$, in Table 6 is greater than 99.99%. It is evident from column T-EF in Table 6 which shows that CPLEX 12.70 with its default settings could not solve reformulated extensive formulation of 50 out of 75 TSS-CMIP instances (without our sparse cuts) within a time limit of 3 hours and the allocated 24 GB RAM. In contrast, after adding our sparse cuts, we solved all 75 instances in 449.64 seconds (on average). For instances solved to optimality using NO-SCUT, our cuts closed the integrality gap by 47.29% (on average) and reduced the time taken to solve the TSS-CMIP instances by 10.32 times (on average). It is worth to note that even though $G\%$ is small for some instances (i.e., SCMIP.15.625.3, SCMIP.240.120.3, and SCMIP.25.100.3 for $|\Omega| \in \{50, 200\}$), CPLEX with its default settings (without sparse cuts) took longer time, at least 2.4 times and up to 18 times, to solve these instances, in comparison to solving them using CPLEX with our sparse cuts.

6. Deterministic Sparse CMIPs

In this section, we introduce various new deterministic structured conic mixed integer sets with multiple integer variables and constraints, and derive their convex hull or partial convex hull. These sets generalize the set $Z_1 := \{(\sigma, \rho_1) \in \mathbb{Z} \times \mathbb{R}_+ : |\sigma - \beta| \leq \rho_1 \}$ studied by Atamtürk and Narayanan [5], as discussed in Section 2.1. More specifically, we study the following generalizations of $Z_1$:

(i) Set I:

$$P_K^m := \left\{ (\sigma, \rho, \rho_0) \in \mathbb{Z}^K \times \mathbb{R}_+^{mK} \times \mathbb{R}_+^K : A\sigma \geq b, \quad \left| \sigma_k - \beta_{ik} \right| \leq \rho_k^i, i = 1, \ldots, m, k = 1, \ldots, K \right\}$$

where $\beta \in \mathbb{R}_+^{mK}$, $A$ is a TU matrix, and $b \in \mathbb{Z}^{m3}$;

(ii) Set II:

$$S_K^{mn} := \left\{ (\sigma, \rho, \rho_0) \in \mathbb{Z}^{nK} \times \mathbb{R}_+^{mK} \times \mathbb{R}_+^K : A\sigma \geq b, \quad \sum_{t=1}^{n} g_{kt}^i \sigma_{kt} - \beta_{ik} \leq \rho_k^i, i = 1, \ldots, m, k = 1, \ldots, K \right\}$$

where $\beta \in \mathbb{R}_+^{mK}$, $b \in \mathbb{Z}^{m3}$, and matrices $A$ and $G^k = (g_{kt}^i), k = 1, \ldots, K$, are TU;
Table 6: Results of Computational Experiments for TSS-CMIPs where $p = 2$ and $E_{ij}^L$ is TU

| Problem Set | Instance Category | $|\Omega|$ | NO-SCUT | WITH-SCUTS |
|-------------|------------------|-----------|---------|------------|
|             |                  |           | #Ivar   | #LCon     | T-EF       | #Ivar | #PCuts | T-EFC | G% |
| SCMIP.5.125.3 | 50               | 37505    | 69001 TL | 5         | 15000      | 22.94 |-        |
| SCMIP.5.125.3 | 100              | 75005    | 138001 TL | 5         | 30000      | 73.51 |-        |
| SCMIP.10.500.3 | 50               | 150010   | 243001 TL | 10        | 45000      | 330.05 |-        |
| SCMIP.10.500.3 | 100              | 300010   | 486001 TL | 10        | 90000      | 1742.52 |-        |
| SCMIP.15.625.3 | 5                | 18765    | 31051 TL | 10        | 15000      | 56.05 0.13 |
| SCMIP.15.625.3 | 10               | 37515    | 62101 TL | 15        | 30000      | 162.69 |-        |
| SCMIP.15.625.3 | 15               | 56265    | 93151 TL | 15        | 90000      | 265.50 |-        |
| SCMIP.240.120.3 | 20            | 14640    | 26505 TL | 240        | 6000       | 46.33  |-        |
| SCMIP.240.120.3 | 50              | 36240    | 66255 TL | 240        | 15000      | 386.16 |-        |
| SCMIP.10.25.3  | 50               | 7510     | 11005 9.98 | 10        | 1500      | 1.82 71.42 |
| SCMIP.10.25.3  | 100              | 15010    | 22005 20.86 | 10        | 3000      | 4.44 63.89 |
| SCMIP.10.25.3  | 200              | 30010    | 44005 40.91 | 10        | 6000      | 10.33 97.25 |
| SCMIP.10.50.3  | 50               | 15010    | 23505 TL | 10        | 3750      | 6.39   |-        |
| SCMIP.10.50.3  | 100              | 30010    | 47005 OM | 10        | 7500      | 14.28  |-        |
| SCMIP.10.50.3  | 200              | 60010    | 94005 OM | 10        | 15000     | 54.38  |-        |
| SCMIP.10.75.3  | 50               | 22510    | 34005 OM | 10        | 5250      | 19.55  |-        |
| SCMIP.10.75.3  | 100              | 45010    | 68005 OM | 10        | 10500     | 41.16  |-        |
| SCMIP.10.75.3  | 200              | 90010    | 136005 OM | 10        | 21000     | 120.51 |-        |
| SCMIP.25.50.3  | 50               | 15025    | 23510 OM | 25        | 3750      | 5.89   |-        |
| SCMIP.25.50.3  | 100              | 30025    | 47010 OM | 25        | 7500      | 15.19  |-        |
| SCMIP.25.50.3  | 200              | 60025    | 94010 OM | 25        | 15000     | 54.75  |-        |
| SCMIP.25.50.3  | 500              | 150025   | 235010 OM | 25        | 37500     | 568.85 |-        |
| SCMIP.25.100.3 | 50               | 30025    | 55010 165.43 | 25        | 12000     | 49.12  3.74 |
| SCMIP.25.100.3 | 200              | 120025   | 220010 OM | 25        | 48000     | 2428.09 |-        |
| SCMIP.25.100.3 | 500              | 300025   | 550010 OM | 25        | 120000    | 4256.62 |-        |

(iii) Set III:

$$T_{m,u}^{K} := \left\{ (\eta, \sigma, \rho, \rho_0) \in \mathbb{Z}^u \times \mathbb{Z}^K \times \mathbb{R}_+^{mK} \times \mathbb{R}_+^K : A_1\eta + A_2\sigma \geq b, \right. \left. \|\rho\|_1 \leq \rho_0^k, \quad \left| \sum_{t=1}^u c_{kt}\eta_t + \sigma_k - \beta_{ik} \right| \leq \rho_0^k, i = 1, \ldots, m, k = 1, \ldots, K \right\}$$

where $\beta \in \mathbb{R}^{mK}$, $b \in \mathbb{Z}^{m3}$, and $A_2$ is a TU matrix;

(iv) Set IV:

$$U_{m,n,u}^{K} := \left\{ (\eta, \sigma, \rho, \rho_0) \in \mathbb{Z}^u \times \mathbb{Z}^{nK} \times \mathbb{R}_+^{mK} \times \mathbb{R}_+^K : A_1\eta + A_2\sigma \geq b, \right. \left. \|\rho\|_1 \leq \rho_0^k, \quad \left| \sum_{t=1}^u c_{kt}\eta_t + \sum_{t=1}^n g_{kt}\sigma_{kt} - \beta_{ik} \right| \leq \rho_0^k, i = 1, \ldots, m, k = 1, \ldots, K \right\}$$

where $\beta \in \mathbb{R}^{mK}$, $b \in \mathbb{Z}^{m3}$, and matrices $A_2$ and $G^k = (g_{kt}^k)$, $k = 1, \ldots, K$, are TU.

Next, in Theorems 9 and 10, we provide convex hull description of the sets $R_{K}^m$ and $S_{m,n}^m$, respectively; and in Theorems 11 and 12, we provide partial convex hull for the sets $T_{K}^{m,u}$ and
In the proof of Theorem 4, we substitute $K$ respectively. This reduces the set to the continuous relaxation of where $A$, $A_2$, and $b$ are zero vectors, and $g_{l1}^T = 1$. For the following theorems, we assume that the matrices/vectors $A_1$, $C^k = (c_{kt}^i)$ for $k = 1, \ldots, K$, and $b$ are integral, and denote the fractional part of $\beta_{ik}$ by $\beta_{ik}^{(1)} := \beta_{ik} - \lfloor \beta_{ik} \rfloor$.

**Theorem 9.** The convex hull of the set $R^m_K$ is given by

$$
\left\{ (\sigma, \rho, \rho_0) \in \mathbb{R}^K \times \mathbb{R}^{mK} \times \mathbb{R}^K : \right. \\
A \sigma \geq b, |\sigma_k - \beta_{ik}| \leq \rho_{ik}^k, \quad i = 1, \ldots, m, k = 1, \ldots, K, \\
\left. (1 - 2\beta_{ik}^{(1)}) (\sigma_k - |\beta_{ik}|) + \beta_{ik}^{(1)} \leq \rho_{ik}^k, \quad i = 1, \ldots, m, k = 1, \ldots, K \right\},
$$

and $\text{conv} (R^m_K \cap \Pi^m_K) = \text{conv} (R^m_K) \cap \Pi^m_K$, where

$$
\Pi^m_K := \left\{ (\sigma, \rho, \rho_0) \in \mathbb{R}^K \times \mathbb{R}^{mK} \times \mathbb{R}^K : \|\rho^k\|_1 \leq \rho_{0k}^k, k = 1, \ldots, K \right\}.
$$

Proof. In the proof of Theorem 3, by substituting $W_{\omega}$, $y_{\omega}$, $r_{\omega} - T_{\omega} x$, $f_{\omega,i}^j x - h_{\omega,i}^j$, $d_{\omega,1}, \ldots, d_{\omega,m}$, for $j = 1, \ldots, |J|$, by $A$, $\sigma$, $b$, $\beta_{ik}$, and $\rho_{ik}^k$, for $k = 1, \ldots, K$, respectively, the set $K^1_{\omega}(x)$ reduces to $R^m_K$. Additionally, $\text{conv}(K^1_{\omega}(x)) = \text{conv}(R^m_K)$ reduces to the set defined by (68) – (70). Now, by utilizing Theorem 1, we get $\text{conv} (R^m_K \cap \Pi^m_K) = \text{conv} (R^m_K) \cap \Pi^m_K$.

**Theorem 10.** The convex hull of the set $S^{m,n}_K$ is obtained by adding inequalities,

$$
(1 - 2\beta_{ik}^{(1)}) \left( \sum_{t=1}^n g_{ikt}^i \sigma_{kt} - |\beta_{ik}| \right) + \beta_{ik}^{(1)} \leq \rho_{ik}^k, \quad i = 1, \ldots, m, k = 1, \ldots, K,
$$

to the continuous relaxation of $S^{m,n}_K$, and

$$
\text{conv} \left( S^{m,n}_K \cap \Pi^{m,n}_K \right) = \text{conv} \left( S^{m,n}_K \right) \cap \Pi^{m,n}_K,
$$

where $\Pi^{m,n}_K := \left\{ (\sigma, \rho, \rho_0) \in \mathbb{R}^{nK} \times \mathbb{R}^{mK} \times \mathbb{R}^K : \|\rho^k\|_1 \leq \rho_{0k}^k, k = 1, \ldots, K \right\}$.

Proof. In the proof of Theorem 4, we substitute $W_{\omega}$, $y_{\omega}$, $r_{\omega} - T_{\omega} x$, $E_{\omega}^j$, $f_{\omega,i}^j x - h_{\omega,i}^j$, and $d_{\omega,i}$, for $i = 1, \ldots, m_2$ and $j = 1, \ldots, |J|$, by $A$, $\sigma$, $b$, $G_k$, $\beta_{ik}$, and $\rho_{ik}^k$, for $i = 1, \ldots, m$ and $k = 1, \ldots, K$, respectively. This reduces the set $K^1_{\omega}(x)$ to $S^{m,n}_K$ and $\text{conv}(K^1_{\omega}(x)) = \text{conv}(S^{m,n}_K)$ reduces to the set obtained by adding (71) to the continuous relaxation of $S^{m,n}_K$. Again, by utilizing Theorem 1, we have $\text{conv} \left( S^{m,n}_K \cap \Pi^{m,n}_K \right) = \text{conv} \left( S^{m,n}_K \right) \cap \Pi^{m,n}_K$.
Theorem 11. A partial convex hull of the set $T_{K,pch}^{m,u}$ is given by

$$T_{K,pch}^{m,u} := \left\{ (\eta, \sigma, \rho, \rho_0) \in \mathbb{Z}^u \times \mathbb{R}^K \times \mathbb{R}_+^m \times \mathbb{R}_+^K : A_1\eta + A_2\sigma \geq b, \right.$$ 

$$\|\rho^k\|_1 \leq \rho_0^k, \quad \sum_{t=1}^n c_{kt}^i \eta_t + \sigma_k - \beta_{ik} \leq \rho_i^k, \quad i = 1, \ldots, m, k = 1, \ldots, K,$$

$$\left(1 - 2\beta_{ik}^{(1)}\right) \left(\sum_{t=1}^n c_{kt}^i \eta_t + \sigma_k - [\beta_{ik}]\right) + \beta_{ik}^{(1)} \leq \rho_i^k, \quad i = 1, \ldots, m,$$

$$k = 1, \ldots, K\right\}.$$ 

Proof. This result can be easily proved using Theorem 7 in which we set $|\Omega| = |\Gamma| = 1$ and $X = \mathbb{Z}^u$, and substitute variables $x$, $y$, and $d^j_{\omega,0}$ for $j = 1, \ldots, |J|$, by $\eta$, $\sigma^k$, and $\rho_0^k$ for $k = 1, \ldots, K$, respectively, and parameters $T_\omega$, $W_\omega$, $r_\omega$, $E^j_\omega$, and $h^j_\omega$ by $A_1$, $A_2$, $b$, $C^k$, and $\beta_k$, respectively. Then, by reformulating $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}_{pch}$ using additional continuous variables $\rho_i^k$ for $i = 1, \ldots, m$, and $k = 1, \ldots, K$ (as discussed in Section 2), we get $T_{K,pch}^{m,u}$ and $T_{K,pch}^{m,u}$, respectively. Since $\overline{\mathcal{P}}_{pch}$ is a partial convex hull of $\overline{\mathcal{P}}$ according to Theorem 7, it is clear that $T_{K,pch}^{m,u}$ is a partial convex hull of $T_{K,pch}^{m,u}$. 

\[\square\]

Theorem 12. A partial convex hull of the set $U_{K,pch}^{m,n,u}$ is given by

$$U_{K,pch}^{m,n,u} := \left\{ (\eta, \sigma, \rho, \rho_0) \in \mathbb{Z}^u \times \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^K : A_1\eta + A_2\sigma \geq b, \right.$$ 

$$\|\rho^k\|_1 \leq \rho_0^k, \quad \sum_{t=1}^n c_{kt}^i \eta_t + \sum_{t=1}^n g_{kt}^i \sigma_k - \beta_{ik} \leq \rho_i^k, \quad i = 1, \ldots, m, k = 1, \ldots, K,$$

$$\left(1 - 2\beta_{ik}^{(1)}\right) \left(\sum_{t=1}^n c_{kt}^i \eta_t + \sum_{t=1}^n g_{kt}^i \sigma_k - [\beta_{ik}]\right) + \beta_{ik}^{(1)} \leq \rho_i^k, \quad i = 1, \ldots, m,$$

$$k = 1, \ldots, K\right\}.$$ 

Proof. We prove this result using Theorem 8 in which we set $|\Omega| = |\Gamma| = 1$ and $X = \mathbb{Z}^u$, and substitute variables $x$, $y$, and $d^j_{\omega,0}$ for $j = 1, \ldots, |J|$, by $\eta$, $\sigma^k$, and $\rho_0^k$ for $k = 1, \ldots, K$, respectively, and parameters $T_\omega$, $W_\omega$, $r_\omega$, $E^j_\omega$, $F^j_\omega$, and $h^j_\omega$ by $A_1$, $A_2$, $b$, $G^k$, $C^k$, and $\beta_k$, respectively. Then similar to the previous proof, by reformulating $\overline{\mathcal{P}}$ and $\overline{\mathcal{P}}_{pch}$ using additional continuous variables $\rho_i^k$ for $i = 1, \ldots, m$, and $k = 1, \ldots, K$ (as discussed in Section 2), we get $U_{K,pch}^{m,n,u}$ and $U_{K,pch}^{m,n,u}$, respectively. Since $\overline{\mathcal{P}}_{pch}$ is a partial convex hull of $\overline{\mathcal{P}}$ according to Theorem 8, it is clear that $U_{K,pch}^{m,n,u}$ is a partial convex hull of $U_{K,pch}^{m,n,u}$. 

\[\square\]

7. Conclusion

We derived parametric (non)-linear cuts, akin to conic mixed integer rounding cuts, for TSS-CMIPs and TSDR-CMIPs with structured $p$-order CMIPs in the second stage. These cuts provide convex programming equivalent or approximation for the second stage CMIPs with $p = 1$ or $p \geq 2$, respectively. We then presented conditions under which the addition of sparse nonlinear cuts in the extensive formulation of TSS-CMIPs is sufficient to relax the integrality restrictions on the second
stage integer variables without effecting the integrality of the optimal solution of the TSS-CMIP. For structured TSS-CMIPs with \( p = 1 \), our cuts satisfied these conditions. We also computationally evaluated the effectiveness of our sparse cuts by considering structured TSS-CMIPs with polyhedral CMIPs and second-order CMIPs in the second stage, i.e. \( p = 1 \) and \( p = 2 \), respectively. Our computational results showed that adding sparse cuts to the extensive formulation significantly reduces the time taken to solve extensive formulation of TSS-CMIPs compared to solving the same problem instances without these cuts, using CPLEX 12.70 with its default settings. Furthermore, we derived (partial) convex hull for deterministic multi-constraint polyhedral conic mixed integer sets with multiple integer variables.

References


8. APPENDIX

8.1 Proof of Theorem 5

Given \( \omega_1 \in \Omega \), let

\[
\text{conv} \left( \mathcal{K}_{\omega_1}(x) \right) = \mathcal{K}_{\text{tight}}^{\omega_1}(x)
\]

(72)

for all \( x \in X \). Suppose that a point \((\hat{x}, \hat{y}_{\omega_1}, \ldots, \hat{y}_{\omega_{|\Omega|}}, \hat{d}_{\omega_1,0}, \ldots, \hat{d}_{\omega_{|\Omega|},0}) \in \overline{P}\), which implies \((\hat{y}_{\omega_1}, \hat{d}_{\omega_1,0}) \in \mathcal{K}_{\omega_1}(\hat{x})\) and because of assumption (72), \((\hat{y}_{\omega_1}, \hat{d}_{\omega_1,0}) \in \mathcal{K}_{\text{tight}}^{\omega_1}(\hat{x})\). Since \((\hat{x}, \hat{y}_{\omega_1}, \ldots, \hat{y}_{\omega_{|\Omega|}}, \hat{d}_{\omega_1,0}, \ldots, \hat{d}_{\omega_{|\Omega|},0})\) satisfies all defining constraints of \( \overline{P}_{\text{pich,1}} \), defined by

\[
T_\omega x + W_\omega y_\omega \geq r_\omega, \\
\|E_j^j y_\omega + F_j^j x - h_j^j\|_p \leq d_{j,0}^j, \\
\|E_{\omega_1,l} y_{\omega_1} + F_{\omega_1,l} x - h_{\omega_1,l}\|_p \leq d_{\omega_1,0}^l, \\
x \in X, d \in \mathbb{R}_+^{\{|\Omega|\}}, y_\omega \in \mathbb{R}^q, \\
y_\omega \in \mathbb{Z}^{q_1} \times \mathbb{R}^{q-q_1}, \\
\omega \in \Omega \setminus \{\omega_1\},
\]

the point \((\hat{x}, \hat{y}_{\omega_1}, \ldots, \hat{y}_{\omega_{|\Omega|}}, \hat{d}_{\omega_1,0}, \ldots, \hat{d}_{\omega_{|\Omega|},0}) \in \overline{P}_{\text{pich,1}}\). Therefore,

\[
\overline{P} \subseteq \overline{P}_{\text{pich,1}} \text{ and } \text{conv}(\overline{P}) \subseteq \text{conv}(\overline{P}_{\text{pich,1}}).
\]

(73)

Now suppose that a point \((\hat{x}, \hat{y}_{\omega_1}, \ldots, \hat{y}_{\omega_{|\Omega|}}, \hat{d}_{\omega_1,0}, \ldots, \hat{d}_{\omega_{|\Omega|},0}) \in \overline{P}_{\text{pich,1}}\). Then point \((\hat{y}_{\omega_1}, \hat{d}_{\omega_1,0}) \in \mathcal{K}_{\text{tight}}^{\omega_1}(\hat{x})\). Also, because of assumption (72), \((\hat{y}_{\omega_1}, \hat{d}_{\omega_1,0}) \in \text{conv}(\mathcal{K}_{\omega_1}(\hat{x}))\), and hence this point can be written as convex combination of finite number of points, \(\bar{\eta}^k_{\omega_1} \in \mathbb{R}^{p+|J|}\) for \(k \in \{1, 2, \ldots\}\), belonging to \(\mathcal{K}_{\omega_1}(\hat{x})\), i.e.,

\[
(\hat{y}_{\omega_1}, \hat{d}_{\omega_1,0}) = \sum_k \lambda_k \bar{\eta}^k_{\omega_1}
\]

where \(\sum_k \lambda_k = 1\) and \(\lambda_k \geq 0\) for all \(k\). Since \((\hat{y}_{\omega_1}, \hat{d}_{\omega_1,0}) \in \mathcal{K}_{\omega}(\hat{x})\) for \(\omega \in \Omega \setminus \{\omega_1\}\) and \(\bar{\eta}^k_{\omega_1} \in \mathcal{K}_{\omega_1}(\hat{x})\),

\[
(\hat{x}, \bar{\eta}^k_{\omega_1}, \hat{y}_{\omega_2}, \hat{d}_{\omega_2,0}, \ldots, \hat{y}_{\omega_{|\Omega|}}, \hat{d}_{\omega_{|\Omega|},0}) \in \overline{P}
\]

for all \(k\) as \(\mathcal{K}_{\omega}(\hat{x}) = \text{Proj}_{x=\hat{x},y_\omega,d_\omega}(\overline{P})\) for all \(\omega \in \Omega\). This implies

\[
(\hat{x}, \sum_k \lambda_k \bar{\eta}^k_{\omega_1} = (\hat{y}_{\omega_1}, \hat{d}_{\omega_1,0}), \hat{y}_{\omega_2}, \hat{d}_{\omega_2,0}, \ldots, \hat{y}_{\omega_{|\Omega|}}, \hat{d}_{\omega_{|\Omega|},0}) \in \text{conv}(\overline{P}).
\]

Hence,

\[
\overline{P}_{\text{pich,1}} \subseteq \text{conv}(\overline{P}) \text{ and } \text{conv}(\overline{P}_{\text{pich,1}}) \subseteq \text{conv}(\overline{P}).
\]

(74)
From (73) and (74), we get
\[ \overline{P} \subseteq \overline{P}_{\text{pch},1} \subseteq \text{conv}(\overline{P}) = \text{conv}(\overline{P}_{\text{pch},1}) \]
which means \( \overline{P}_{\text{pch},1} \) is a partial convex hull of \( \overline{P} \).

Next, we assume that \( \text{conv} (K_{\omega_2}(x)) = K_{\text{tight}}(x) \) for all \( x \in X \). Replacing \( \overline{P} \) by \( \overline{P}_{\text{pch},1} \) and using the similar arguments above, we can prove that \( \overline{P}_{\text{pch},2} \) is a partial convex hull of \( \overline{P} \) where
\[
\begin{align*}
\overline{P}_{\text{pch},2} &:= \{ T_\omega x + W_\omega y_\omega \geq r_\omega, \quad \omega \in \Omega, \\
&\quad \| E^j_\omega y_\omega + F^j_\omega x - h^j_\omega \|_p \leq d^j_\omega, \quad j \in J, \omega \in \Omega \\
&\quad \| E^j_{\omega_1,l} y_{\omega_1} + F^j_{\omega_1,l} x - h^j_{\omega_1,l} \|_p \leq d^j_{\omega_1,0}, \quad l \in \mathcal{L}, j \in J, \\
&\quad \| E^j_{\omega_2,l} y_{\omega_2} + F^j_{\omega_2,l} x - h^j_{\omega_2,l} \|_p \leq d^j_{\omega_2,0}, \quad l \in \mathcal{L}, j \in J, \\
&\quad x \in X, y_{\omega_1} \in \mathbb{R}^q, y_{\omega_2} \in \mathbb{R}^q, y_\omega \in \mathbb{Z}^{q_1} \times \mathbb{R}^{q-q_1}, \omega \in \Omega \setminus \{\omega_1, \omega_2\} \}. 
\end{align*}
\]
We repeat the foregoing steps by replacing \( \overline{P} \) by \( \overline{P}_{\text{pch},i} \) for \( \omega_i, i = 2, \ldots, |\Omega| - 1 \). Meanwhile, for each step, we assume that \( \text{conv}(K_{\omega_i}^1(x)) = \text{Proj}_y(K_{\text{tight}}^{\omega_i}) \) for all \( x \in X \). Finally, we have \( \text{Proj}_{x,y,d}(P_{\text{tight},|\Omega|}) = \text{Proj}_{x,y,d}(P_{\text{tight}}) \) is a partial convex hull of \( P \). This completes the proof.