Data-Driven Chance Constrained Programs over Wasserstein Balls

Zhi Chen
Imperial College Business School, Imperial College London, London, United Kingdom,
zhi.chen@imperial.ac.uk

Daniel Kuhn
Risk Analytics and Optimization Chair, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland,
daniel.kuhn@epfl.ch

Wolfram Wiesemann
Imperial College Business School, Imperial College London, London, United Kingdom,
ww@imperial.ac.uk

We provide an exact deterministic reformulation for data-driven chance constrained programs over Wasserstein balls. For individual chance constraints as well as joint chance constraints with right-hand side uncertainty, our reformulation amounts to a mixed-integer conic program. In the special case of a Wasserstein ball with the 1-norm or the ∞-norm, the cone is the nonnegative orthant, and the chance constrained program can be reformulated as a mixed-integer linear program. Using our reformulation, we show that two popular approximation schemes based on the conditional-value-at-risk and the Bonferroni inequality can perform poorly in practice and that these two schemes are generally incomparable with each other.

Key words: Distributionally robust optimization; ambiguous chance constraints; Wasserstein distance.

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1. Introduction

Distributionally robust optimization is a powerful modeling paradigm for optimization under uncertainty, where the distribution of the uncertain problem parameters is itself uncertain, and where the performance of a decision is assessed in view of the worst-case distribution from a prescribed ambiguity set. The earlier literature on distributionally robust optimization has focused on moment ambiguity sets which contain all distributions that obey certain (standard or generalized) moment conditions; see, e.g., Delage and Ye (2010), Goh and Sim (2010) and Wiesemann et al. (2014). Pflug and Wozabal (2007) were the first to propose an ambiguity set of the form of a ball in the space of distributions with respect to the celebrated Wasserstein, Kantorovich or optimal transport distance. The type-1 Wasserstein distance $d_W(\mathbb{P}_1, \mathbb{P}_2)$ between two distributions $\mathbb{P}_1$ and $\mathbb{P}_2$ on $\mathbb{R}^K$, equipped with a general norm $\| \cdot \|$, is defined as the minimal transportation cost of moving
$P_1$ to $P_2$ under the premise that the cost of moving a Dirac point mass from \( \xi_1 \) to \( \xi_2 \) amounts to \( \| \xi_1 - \xi_2 \| \). Mathematically, this implies that

\[
d_W(P_1, P_2) = \inf_{P \in \mathcal{P}(P_1, P_2)} \mathbb{E}_P[\| \tilde{\xi}_1 - \tilde{\xi}_2 \|],
\]

where \( \tilde{\xi}_1 \sim P_1, \tilde{\xi}_2 \sim P_2 \), and \( \mathcal{P}(P_1, P_2) \) represents the set of all distributions on \( \mathbb{R}^K \times \mathbb{R}^K \) with marginals \( P_1 \) and \( P_2 \). The Wasserstein ambiguity set \( \mathcal{F}(\theta) \) is then defined as a ball of radius \( \theta \geq 0 \) with respect to the Wasserstein distance, centered at a prescribed reference distribution \( \hat{P} \):

\[
\mathcal{F}(\theta) = \{ P \in \mathcal{P}(\mathbb{R}^K) \mid d_W(P, \hat{P}) \leq \theta \}.
\]

One can think of the Wasserstein radius \( \theta \) as a budget on the transportation cost. Indeed, any member distribution in \( \mathcal{F}(\theta) \) can be obtained by rearranging the reference distribution \( \hat{P} \) at a transportation cost of at most \( \theta \). If only a finite training dataset \( \{ \hat{\xi}_i \}_{i \in [N]} \) is available, a natural choice for \( \hat{P} \) is the empirical distribution \( \hat{P} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\hat{\xi}_i} \), which represents the uniform distribution on the training samples. Throughout the paper, we will assume that \( \hat{P} \) is the empirical distribution.

While it has been recognized early on that Wasserstein ambiguity sets offer many conceptual advantages (e.g., their member distributions do not need to be absolutely continuous with respect to \( \hat{P} \) and, if properly calibrated, they constitute confidence regions for the unknown true data-generating distribution), it was believed that they almost invariably lead to hard global optimization problems. Recently, Mohajerin Esfahani and Kuhn (2018) and Zhao and Guan (2018) discovered that many interesting distributionally robust optimization problems over Wasserstein ambiguity sets can actually be reformulated as tractable convex programs—provided that \( \hat{P} \) is discrete and that the problem’s objective function satisfies certain convexity properties. These reformulations have subsequently been generalized to Polish spaces and non-discrete reference distributions by Blanchet and Murthy (2016) and Gao and Kleywegt (2016). Since then, distributionally robust optimization models over Wasserstein ambiguity sets have been proposed for many applications, including transportation (Carlsson et al. 2017) and machine learning (Blanchet et al. 2016, Gao et al. 2017, Shafieezadeh-Abadeh et al. 2017 and Sinha et al. 2017).

In this paper we study distributionally robust chance constrained programs of the form

\[
\min_{x \in \mathcal{X}} \quad c^\top x \\
\text{s.t.} \quad \mathbb{P}[\tilde{\xi} \in S(x)] \geq 1 - \varepsilon, \quad \forall P \in \mathcal{F}(\theta),
\]

where the goal is to find a decision \( x \) from within a compact polyhedron \( \mathcal{X} \subseteq \mathbb{R}^L \) that minimizes a linear cost function \( c^\top x \) and ensures that the exogenous random vector \( \tilde{\xi} \) falls within a decision-dependent safety set \( S(x) \subseteq \mathbb{R}^K \) with high probability \( 1 - \varepsilon \) under every distribution \( P \in \mathcal{F}(\theta) \).
Since the reference distribution \( \hat{\mathbb{P}} \) in (2) is the empirical distribution over the training dataset \( \{ \hat{\xi}_i \}_{i \in [N]} \), we refer to (2) as a data-driven chance constrained program.

If we set \( \theta = 0 \) in problem (2), then we recover a classical chance constrained program. This special case of (2) is NP-hard even if \( X \) is a polyhedron and the safety conditions describing \( S(x) \) are inequalities that are jointly affine in \( x \) and \( \xi \) (Luedtke et al. 2010, Theorem 1). Classical chance constrained programs have been studied intensively since the seminal works of Charnes et al. (1958) and Charnes and Cooper (1959), and they have found many applications, see, e.g., Shapiro et al. (2009), Birge and Louveaux (2011) and Prékopa (2013).

To date, the literature on distributionally robust optimization has focused primarily on variants of problem (2) where the safety set \( S(x) \) is described by a single linear inequality and the Wasserstein ambiguity set \( \mathcal{F}(\theta) \) is replaced with a set that bounds the support and certain moments of \( \hat{\xi} \). In this case, the distributionally robust chance constrained program can often be exactly reformulated or tightly approximated by a tractable conic optimization problem. This is achieved through the use of classical inequalities from probability theory, such as Hoeffding’s inequality (Ben-Tal and Nemirovski 2000 and Bertsimas and Sim 2004), Bernstein’s inequality (Nemirovski and Shapiro 2006) and the generalized Chebyshev inequality (Xu et al. 2012), or through tailored probability bounds on the shape of the distribution (Calafiore and El Ghaoui 2006), forward and backward deviations (Chen et al. 2007) or the mean-absolute deviation (Postek et al. 2018).

Alternatively, one can leverage duality results for moment problems to reformulate or approximate distributionally robust chance constraints over Chebyshev ambiguity sets, which stipulate bounds on the first- and second-order moments (El Ghaoui et al. 2003, Calafiore and El Ghaoui 2006 and Popescu 2007), over Chebyshev ambiguity sets with support information (Cheng et al. 2014) as well as Chebyshev ambiguity sets with unimodality constraints (Li et al. 2017).

Distributionally robust chance constrained programs become more involved if the safety set \( S(x) \) is described by multiple linear inequalities. For the special case where the inequalities in \( S(x) \) are jointly affine in \( x \) and \( \xi \) and the ambiguity set specifies the mean, support and an upper bound on the dispersion of \( \hat{\xi} \), Hanasusanto et al. (2017) provide an exact reformulation as a tractable conic optimization problem. The result has been extended by Hanasusanto et al. (2015) to ambiguity sets specifying structural properties, such as symmetry and unimodality, and to generic convex chance constraints and ambiguity sets involving convex moment constraints by Xie and Ahmed (2018b). In general, however, distributionally robust chance constrained programs with generic safety sets \( S(x) \) are approximated conservatively either by the Bonferroni approximation or the worst-case conditional value-at-risk (CVaR) approximation. The quality of the Bonferroni approximation crucially depends on certain Bonferroni weights. While Xie et al. (2017) show that the Bonferroni weights can be optimized efficiently under specific conditions, Chen et al. (2010)
show that the quality of the Bonferroni approximation can be poor even if the Bonferroni weights are chosen optimally. Chen et al. (2010) also show that the worst-case CVaR approximation can outperform the Bonferroni approximation with optimally chosen Bonferroni weights for Chebyshev ambiguity sets, provided that certain scaling factors in the worst-case CVaR approximation are selected judiciously. Zymler et al. (2013) show that the worst-case CVaR approximation is indeed exact for distributionally robust chance constrained programs over Chebyshev ambiguity sets if the scaling factors are selected optimally. This result has been extended to non-linear safety conditions by Yang and Xu (2016). Selecting the scaling factors optimally, however, amounts to solving a non-convex optimization problem. For further information, we refer the reader to the surveys by Ben-Tal and Nemirovski (2001), Nemirovski (2012) and Hanasusanto et al. (2015).

While the availability of further training data allows to refine the moment estimates in the aforementioned ambiguity sets, these sets remain conservative as they do not shrink to a singleton as \( N \) approaches infinity. In other words, distributionally robust chance constraints over moment-based ambiguity sets fail to tightly approximate classical chance constraints even if sufficient training data is available. This undesirable consequence of a moment-based description of ambiguity is alleviated by data-driven chance constraints, whose ambiguity sets contain all distributions that are close to the empirical distribution \( \hat{P} \) with respect to some distance measure. Popular choices of distance measures are the \( \phi \)-divergences (such as the Kullback-Leibler divergence or the \( \chi^2 \)-distance), which lead to ambiguity sets of the form

\[
G(\theta) = \left\{ P \in \mathcal{P}(\mathbb{R}^K) \mid \int_{\mathbb{R}^K} \phi \left( \frac{d\hat{P}(\xi)}{dP(\xi)} \right) d\hat{P}(\xi) \leq \theta \right\},
\]

where \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R} \) is the divergence function. Hu and Hong (2013) show that a distributionally robust chance constrained program over a Kullback-Leibler ambiguity set reduces to a classical chance constrained program over the reference distribution \( \hat{P} \) and an adjusted risk threshold \( \varepsilon' < \varepsilon \). While this result holds for any reference distribution, \( \phi \)-divergence ambiguity sets only contain distributions that are absolutely continuous with respect to \( \hat{P} \), that is, any distribution in \( G(\theta) \) only assigns positive probability to those measurable subsets \( A \subseteq \mathbb{R}^K \) for which \( \hat{P}(\hat{\xi} \in A) > 0 \). This is undesirable for problems with a large dimension \( K \) and/or few training data, where it is unlikely that every possible value of \( \hat{\xi} \) has been observed in \( \{\hat{\xi}_i\}_{i \in [N]} \). This shortcoming is addressed by Jiang and Guan (2016, 2018), who replace the reference distribution with a Kernel density estimator.

To our best knowledge, the paper of Xie and Ahmed (2018a) is the only previous work on data-driven chance constraints over Wasserstein ambiguity sets. The authors study the special class of covering problems, where the feasible region \( \mathcal{X} \) satisfies \( \eta \mathcal{X} \subseteq \mathcal{X} \) for every \( \eta \geq 1 \), and they prove that the corresponding variant of problem (2) is NP-hard. They also demonstrate that two popular approximation schemes, the CVaR approximation as well as the scenario approximation, can
perform arbitrarily poorly for classical chance constraints, that is, when the Wasserstein radius is $\theta = 0$. Based on this insight, the authors propose a bicriteria approximation scheme for covering problems with classical as well as distributionally robust chance constraints that determines solutions that trade off a higher risk threshold $\varepsilon' > \varepsilon$ in the chance constraint with a better objective value. This is achieved by solving a tractable convex relaxation of the chance constrained problem (using, e.g., a Markovian or Bernstein generator) and subsequently scaling the solution to this relaxation to be feasible for the chance constraint with the higher risk threshold $\varepsilon'$.

We note that there is also a related but distinct literature on safe tractable approximations to chance constrained programs. Using the training dataset $\{\hat{\xi}_i\}_{i \in [N]}$, this literature constructs uncertainty sets for classical robust optimization problems such that any solution to the robust optimization problem satisfies the chance constraint under the unknown true data-generating distribution with high confidence. In contrast to the aforementioned literature on data-driven chance constraints, the resulting optimization problems can be solved in polynomial time, but their feasible regions do not converge to the feasible region of the chance constrained program even if the number of samples approaches infinity. Safe tractable approximations can be constructed, amongst others, through the Strassen-Dudley representation theorem (Erdoğan and Iyengar 2006), goodness-of-fit statistics (Yanıkoğlu and den Hertog 2012) or statistical hypothesis tests (Bertsimas et al. 2018).

In this paper, we study distributionally robust chance constrained programs over the Wasserstein ambiguity set (1). We derive deterministic reformulations for individual chance constrained programs, where $\mathcal{S}(x) = \{\xi \in \mathbb{R}^K \mid a(\xi)^\top x < b(\xi)\}$ for affine functions $a(\cdot) : \mathbb{R}^K \to \mathbb{R}^L$ and $b(\cdot) : \mathbb{R}^K \to \mathbb{R}$, as well as for joint chance constrained programs with right-hand side uncertainty, where $\mathcal{S}(x) = \{\xi \in \mathbb{R}^K \mid Ax < b(\xi)\}$ for $A \in \mathbb{R}^{M \times L}$ and an affine function $b : \mathbb{R}^K \to \mathbb{R}^M$. Our reformulations are mixed-integer conic programs that reduce to mixed-integer linear programs when the norm $\|\cdot\|$ on $\mathbb{R}^K$ is the 1-norm or the $\infty$-norm. Using our reformulations, we study the properties of two popular approximation schemes based on the CVaR and the Bonferroni inequality.

The key contributions of our paper may be summarized as follows.

1. We derive deterministic reformulations for individual and joint data-driven chance constrained programs over Wasserstein ambiguity sets. Our reformulations reduce to mixed-integer programs that can be solved with off-the-shelf software.
2. We show that the CVaR offers a tight convex approximation to certain disjunctive constraints appearing in our reformulations. This provides a theoretical justification for the popularity of this approximation scheme in distributionally robust optimization.
3. We show that both the CVaR and the Bonferroni approximation may deliver solutions that are severely inferior to the optimal solution of our exact reformulation in data-driven settings. In addition, these two approximation schemes are generally incomparable with each other.
While preparing this paper for publication, we became aware of the independent work by Xie (2018) on distributionally robust chance constraints over Wasserstein ambiguity sets. Xie (2018) derives similar reformulations for different classes of individual and joint chance constraints. Since our models leverage the structural insights into the worst-case distributions, our reformulation for joint chance constraints employs fewer binary decision variables. Also, our reformulations allow us to study the approximations provided by CVaR and the Bonferroni inequality.

Notation. We use boldface uppercase and lowercase letters to denote matrices and vectors, respectively. Special vectors of appropriate dimensions include $0$ and $e$, which respectively correspond to the zero vector and the vector of all ones. We denote by $\| \cdot \|_*$ the dual norm of a general norm $\| \cdot \|$.

We use the shorthand $[N] = \{1, 2, \ldots, N\}$ to represent the set of all integers up to $N$. Given a (possibly fractional) real number $\ell \in [0, N]$, we define the partial sum of the $\ell$ first values in $\{k_i\}_{i \in [N]}$ as $\sum_{i=1}^\ell k_i = \sum_{i=1}^\lfloor \ell \rfloor k_i + (\ell - \lfloor \ell \rfloor)k_{\lfloor \ell \rfloor + 1}$. Random vectors are denoted by tilde signs (e.g., $\tilde{\xi}$), while their realizations are denoted by the same symbols without tildes (e.g., $\xi$). Given a random vector $\tilde{\xi}$ governed by a distribution $\mathbb{P}$, a measurable loss function $\ell(\xi)$ and a risk threshold $\varepsilon \in (0, 1)$, the value-at-risk (VaR) of $\ell(\xi)$ at level $\varepsilon$ is defined as $\mathbb{P}^{-}\text{VaR}_\varepsilon(\ell(\tilde{\xi})) = \inf\{\gamma \in \mathbb{R} | \mathbb{P}[\gamma \leq \ell(\tilde{\xi})] \leq \varepsilon\}$, and the CVaR of $\ell(\xi)$ at level $\varepsilon$ is defined as $\mathbb{P}^{-}\text{CVaR}_\varepsilon(\ell(\tilde{\xi})) = \inf\{\tau + \mathbb{E}_\mathbb{P}[(\ell(\tilde{\xi}) - \tau)^+] / \varepsilon | \tau \in \mathbb{R}\}$.

2. Exact Reformulation of Data-Driven Chance Constraints

Section 2.1 reviews a previously established result on the quantification of uncertainty over Wasserstein balls. We use this result to derive an exact reformulation of generic data-driven chance constrained programs in Section 2.2. We finally specialize this generic reformulation to the sub-classes of data-driven individual chance constrained programs as well as data-driven joint chance constrained programs with right-hand side uncertainty in Sections 2.3 and 2.4, respectively.

2.1. Uncertainty Quantification over Wasserstein Balls

Consider an open safety set $S \subseteq \mathbb{R}^K$, and denote by $\bar{S} = \mathbb{R}^K \setminus S$ its closed complement. The uncertainty quantification problem

$$\sup_{\mathbb{P} \in \mathcal{F}(\theta)} \mathbb{P}[\tilde{\xi} \notin S]$$

(3)

computes the worst (largest) probability of the system under consideration being unsafe, which is the case whenever the random vector $\tilde{\xi}$ attains a value in the unsafe set $\bar{S}$. Throughout the rest of the paper, we exclude trivial special cases and assume that $\theta > 0$ and $\varepsilon \in (0, 1)$.

To solve the uncertainty quantification problem (3), denote by $\text{dist}(\tilde{\xi}_i, \bar{S})$ the distance of the $i^{th}$ data point $\tilde{\xi}_i \in \mathbb{R}^K$ of the empirical distribution $\tilde{\mathbb{P}}$ to the unsafe set $\bar{S}$. This distance is based on a norm $\| \cdot \|$, which we keep generic at this stage. Without loss of generality, we assume that the
data points \( \{\xi_i\}_{i \in [N]} \) are ordered in increasing distance to \( \bar{S} \), that is, \( \text{dist}(\hat{\xi}, \bar{S}) \leq \text{dist}(\hat{\xi}_j, \bar{S}) \) for all \( 1 \leq i \leq j \leq N \). We also assume that \( \text{dist}(\xi_i, \bar{S}) = 0 \) (that is, the data point \( \xi_i \) is unsafe) if and only if \( i \in [I] \), where \( I = 0 \) if \( \text{dist}(\xi_i, \bar{S}) > 0 \) for all \( i \in [N] \). Finally, we denote by \( \xi_i \in \bar{S} \) an unsafe point that is closest to the data point \( \xi_i \), \( i \in [N] \), in terms of the distance \( \text{dist}(\xi_i, \bar{S}) \).

Blanchet and Murthy (2016) as well as Gao and Kleywegt (2016) have characterized the solution to the uncertainty quantification problem (3) in closed form. To keep our paper self-contained, we reproduce their findings without proof in Theorem 1 below.

**Theorem 1.** Let \( j^* = \max \{ j \in [N] \cup \{ 0 \} \mid \sum_{i=1}^j \text{dist}(\hat{\xi}_i, \bar{S}) \leq \theta N \} \). The uncertainty quantification problem (3) is solved by a worst-case distribution \( \mathbb{P}^* \in \mathcal{F}(\theta) \) that is characterized as follows:

(i) If \( j^* = N \), then \( \sup_{\mathbb{P} \in \mathcal{F}(\theta)} \mathbb{P}[\hat{\xi} \notin \bar{S}] = \mathbb{P}^*[\hat{\xi} \notin \bar{S}] = 1 \) for

\[
\mathbb{P}^* = \frac{1}{N} \sum_{i=1}^I \delta_{\xi_i} + \frac{1}{N} \sum_{i = I+1}^N \delta_{\xi_i^*}.
\]

(ii) If \( j^* < N \), then \( \sup_{\mathbb{P} \in \mathcal{F}(\theta)} \mathbb{P}[\hat{\xi} \notin \bar{S}] = \mathbb{P}^*[\hat{\xi} \notin \bar{S}] = (j^* + p^*)/N \) for

\[
\mathbb{P}^* = \frac{1}{N} \sum_{i=1}^I \delta_{\xi_i} + \frac{1}{N} \sum_{i = I+1}^{j^*} \delta_{\xi_i} + \frac{p^*}{N} \delta_{\xi_{j^*+1}} + \frac{1 - p^*}{N} \delta_{\xi_{j^*+1}} + \frac{1}{N} \sum_{i = j^*+2}^N \delta_{\xi_i},
\]

where \( p^* = (\theta N - \sum_{i=1}^{j^*} \text{dist}(\hat{\xi}_i, \bar{S}))/\text{dist}(\hat{\xi}_{j^*+1}, \bar{S}) \).

Intuitively speaking, the worst-case distribution \( \mathbb{P}^* \) in Theorem 1 transports the training dataset \( \{\xi_i\}_{i \in [N]} \) to the unsafe set \( \bar{S} \) in a greedy fashion, see Figure 1. The data points \( \xi_1, \ldots, \xi_I \) are already unsafe and hence do not need to be transported. The subsequent data points \( \xi_{I+1}, \ldots, \xi_{j^*+1} \) are closest to the unsafe set and are thus transported from \( S \) to \( \bar{S} \). Due to the limited transportation budget \( \theta \), the data point \( \xi_{j^*+1} \) is only partially transported. The safe samples \( \xi_{j^*+2}, \ldots, \xi_N \), finally, are too far away from the unsafe set \( \bar{S} \) and are thus left unchanged.

### 2.2. Reformulation of Generic Chance Constraints

We now develop deterministic reformulations for the distributionally robust chance constrained program (2). To this end, we focus on the ambiguous chance constraint

\[
\sup_{\mathbb{P} \in \mathcal{F}(\theta)} \mathbb{P}[\hat{\xi} \notin \bar{S}(x)] \leq \varepsilon.
\]  

For any fixed decision \( x \in \mathcal{X} \), we let \( \bar{S}(x) \) be an arbitrary open safety set, and we denote by \( \bar{S}(x) \) its closed complement, which comprises all unsafe scenarios. Every fixed training dataset \( \{\xi_i\}_{i \in [N]} \) then induces a (decision-dependent) permutation \( \pi(x) \) of \([N]\) that orders the training samples in increasing distance to the unsafe set, that is,

\[
\text{dist}(\hat{\xi}_{\pi_1(x)}, \bar{S}(x)) \leq \text{dist}(\hat{\xi}_{\pi_2(x)}, \bar{S}(x)) \leq \cdots \leq \text{dist}(\hat{\xi}_{\pi_N(x)}, \bar{S}(x)).
\]
Figure 1 Empirical and worst-case distributions. The left graph visualizes the empirical distribution $\hat{P}$, whose light grey (dark grey) data points are contained in (outside of) the safety set $S$ shown as an equilateral triangle (dashed lines). The right graph shows the corresponding worst-case distribution $P^*$, which moves the data points $\hat{\xi}_1$ and $\hat{\xi}_2$ entirely as well as the data point $\hat{\xi}_3$ partially to the unsafe set $\bar{S}$. Each transported data point is projected onto the boundary of the closest halfspace defining the safety set $S$.

We first show that a fixed decision $x$ satisfies the ambiguous chance constraint (4) over the Wasserstein ambiguity set (1) if and only if the partial sum of the $\varepsilon N$ smallest transportation distances to the unsafe set multiplied by the mass $1/N$ of a training sample exceeds $\theta$.

**Theorem 2.** For any fixed decision $x \in X$, the ambiguous chance constraint (4) over the Wasserstein ambiguity set (1) is equivalent to the deterministic inequality

$$\frac{1}{N} \sum_{i=1}^{\varepsilon N} \text{dist}(\hat{\xi}_{\pi_i(x)}, \bar{S}(x)) \geq \theta. \quad (5)$$

The left-hand side of (5) can be interpreted as the minimum cost of moving a fraction $\varepsilon$ of the training samples to the unsafe set. If this cost exceeds the prescribed transportation budget $\theta$, then no distribution in the Wasserstein ambiguity set can assign the unsafe set a probability of more than $\varepsilon$, which means that the distributionally robust chance constraint (4) is satisfied.

**Proof of Theorem 2.** From Theorem 1 we know that the ambiguous chance constraint (4) over the Wasserstein ambiguity set (1) is satisfied if and only if $P^*[\hat{\xi} \notin S(x)] \leq \varepsilon$ for the worst-case distribution $P^*$ defined in the statement of that theorem.

In case (i) of Theorem 1, the ambiguous chance constraint (4) is violated since $P^*[\hat{\xi} \notin S(x)] = 1$ while $\varepsilon < 1$ by assumption. At the same time, since $j^* = N$, we have $\frac{1}{N} \sum_{i=1}^{\varepsilon N} \text{dist}(\hat{\xi}_{\pi_i(x)}, \bar{S}(x)) \leq \theta$. If this inequality is strict, then (5) is violated as desired since $\frac{1}{N} \sum_{i=1}^{\varepsilon N} \text{dist}(\hat{\xi}_{\pi_i(x)}, \bar{S}(x)) \leq \frac{1}{N} \sum_{i=1}^{\varepsilon N} \text{dist}(\hat{\xi}_{\pi_i(x)}, \bar{S}(x))$. If the inequality is satisfied as an equality, on the other hand, we know that $\text{dist}(\hat{\xi}_{\pi_N(x)}, \bar{S}(x)) > 0$ since $\theta > 0$ by assumption and $\text{dist}(\hat{\xi}_{\pi_i(x)}, \bar{S}(x)) \leq \text{dist}(\hat{\xi}_{\pi_j(x)}, \bar{S}(x))$. 


for all \( i \leq j \) by construction of the re-ordering \( \pi(x) \). Thus, since \( \varepsilon < 1 \) by assumption, we have
\[
\frac{1}{N} \sum_{i=1}^{\varepsilon N} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)) < \frac{1}{N} \sum_{i=1}^{N} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)) = \theta
\]
and equation (5) is violated as desired.

In case \((ii)\) of Theorem 1, we have \( \mathbb{P}^\pi(\xi \notin S(x)) = (j^* + p^*)/N \) with \( j^* = \max \{j \in [N - 1] \cup \{0\} \mid \sum_{i=1}^{j} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)) \leq \theta N\} \) as well as \( p^* = (\theta N - \sum_{i=1}^{j^*} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)))/\text{dist}(\hat{\xi}_{\pi_{j^*+1}(x)}, \hat{S}(x)) \).

We claim that \((j^* + p^*)\) is the optimal value of the bivariate mixed-integer optimization problem
\[
\max_{j,p} j + p
\]
\[
\text{s.t. } \sum_{i=1}^{j} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)) + p \cdot \text{dist}(\hat{\xi}_{\pi_{j+1}(x)}, \hat{S}(x)) \leq \theta N
\]
\[
j \in [N - 1] \cup \{0\}, \quad 0 \leq p < 1.
\]
Indeed, the solution \((j, p) = (j^*, p^*)\) is feasible in (6) by definition of \( j^* \) and \( p^* \). Moreover, we have
\[
j + p < j^* + p^* \quad \text{for any other feasible solution } (j, p) \text{ that satisfies } j = j^* \quad \text{and } p \neq p^*.
\]
Assume now that the optimal solution \((j, p)\) to (6) would satisfy \( j > j^* \). Any such solution would violate the first constraint since
\[
\sum_{i=1}^{j} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)) > \theta N
\]
by definition of \( j^* \) while \( p \geq 0 \). Similarly, any solution \((j, p)\) with \( j < j^* \) cannot be optimal in (6) since \( j \leq j^* - 1 \) while \( p < p^* + 1 \).

We can re-express problem (6) as the univariate discrete optimization problem
\[
\max \left\{ j \in [0, N] \mid \sum_{i=1}^{j} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)) + (j - [j]) \cdot \text{dist}(\hat{\xi}_{\pi_{j+1}(x)}, \hat{S}(x)) \leq \theta N \right\}.
\]
Using our definition of partial sums, we observe that this problem is equivalent to
\[
\max \left\{ j \in [0, N] \mid \sum_{i=1}^{j} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)) \leq \theta N \right\}.
\]
By construction, the mapping \( \vartheta(j) = \sum_{i=1}^{j} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)), j \in [0, N], \) is continuous and monotonically nondecreasing. It therefore affords the right inverse \( \vartheta^{-1}(t) = \max\{j \in [0, N] \mid \vartheta(j) \leq t\} \) that satisfies \( \vartheta \circ \vartheta^{-1}(t) = t \) for all \( t \in [0, \vartheta(N)] \). Figure 2 visualizes the relationship between \( \vartheta \) and \( \vartheta^{-1} \). We thus conclude that the ambiguous chance constraint (4) is satisfied if and only if
\[
\max \left\{ j \in [0, N] \mid \sum_{i=1}^{j} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{S}(x)) \leq \theta N \right\} \leq \varepsilon N \quad \iff \quad \max\{j \in [0, N] \mid \vartheta(j) \leq \theta N\} \leq \varepsilon N
\]
\[
\iff \quad \vartheta^{-1}(\theta N) \leq \varepsilon N
\]
\[
\iff \quad \theta N \leq \vartheta(\varepsilon N),
\]
where the last equivalence follows from \( \vartheta \circ \vartheta^{-1}(\theta N) = \theta N \), which holds because \( \theta N \leq \vartheta(N) \) for \( j^* < N \), as well as the fact that \( \vartheta \) is monotonically nondecreasing. By definition, the right-hand side of the last equivalence holds if and only if (5) in the statement of the theorem is satisfied. \( \square \)
Remark 1. We emphasize that the inequality (5) fails to be equivalent to the ambiguous chance constraint (4) when \( \theta = 0 \), in which case the Wasserstein ball collapses to the singleton set \( \mathcal{F}(0) = \{ \hat{\mathcal{P}} \} \). To see this, suppose that \( \hat{\xi}_{\pi_i(x)} \in \hat{\mathcal{S}}(x) \) for all \( i = 1, \ldots, I \) and \( \hat{\xi}_{\pi_{i+1}(x)} \in \mathcal{S}(x) \) for all \( i = I + 1, \ldots, N \), where \( I \geq 1 \). If \( \varepsilon < I/N \), then the chance constraint (4) is violated because

\[
\hat{\mathbb{P}}[\hat{\xi} \notin \mathcal{S}(x)] = \frac{I}{N} > \varepsilon,
\]

while the inequality (5) holds trivially because \( \sum_{i=1}^{\varepsilon N} \text{dist}(\hat{\xi}_{\pi_i(x)}, \hat{\mathcal{S}}(x)) \geq 0 \).

Theorem 2 establishes that a decision \( x \in \mathcal{X} \) satisfies the ambiguous chance constraint (4) if and only if the sum of the \( \varepsilon N \) smallest distances of the training samples to the unsafe set \( \hat{\mathcal{S}}(x) \) weakly exceeds \( \theta N \). This result is of computational interest because the sum of the \( \varepsilon N \) smallest out of \( N \) real numbers is concave in those real numbers (while being convex in \( \varepsilon \)). This reveals that the constraint (5) is convex in the decision-dependent distances \( \{ \text{dist}(\hat{\xi}_i, \hat{\mathcal{S}}(x)) \}_{i \in [N]} \). In the remainder we develop an efficient reformulation of this convex constraint that does not require an enumeration of all possible sums of \( \varepsilon N \) different distances between the training samples and the unsafe set. This reformulation is based on the following auxiliary lemma.

Lemma 1. For any \( \varepsilon \in (0, 1) \), the sum of the \( \varepsilon N \) smallest out of \( N \) real numbers \( k_1, \ldots, k_N \) coincides with the optimal value of the linear program

\[
\max_{s, t} \varepsilon N t - e^\top s \\
\text{s.t. } k_i \geq t - s_i \quad \forall i \in [N] \\
s \geq 0.
\]
Proof of Lemma 1. By definition, the sum of the $\varepsilon N$ smallest elements of the set \{k_1, \ldots, k_N\} corresponds to the optimal value of the (manifestly feasible) linear program
\[
\min_v \sum_{i \in [N]} k_i v_i \\
s.t. \quad 0 \leq v \leq e, \quad e^\top v = \varepsilon N.
\]
The claim now follows from strong linear programming duality. \qed

Armed with Theorem 2 and Lemma 1, we are now ready to reformulate the chance constrained program (2) as a deterministic optimization problem.

**Theorem 3.** The chance constrained program (2) is equivalent to
\[
\min_{s, t, x} c^\top x \\
s.t. \quad \varepsilon N t - e^\top s \geq 0 N \\
\quad \text{dist}(\hat{\xi}_i, \bar{S}(x)) \geq t - s_i \quad \forall i \in [N] \\
\quad s \geq 0, \ x \in \mathcal{X}.
\]

**Proof of Theorem 3.** The claim follows immediately by using Theorem 2 to reformulate the chance constraint (4) as the inequality (5), using Lemma 1 to express the left-hand side of (5) as a linear maximization problem and substituting the resulting constraint back into (2). \qed

### 2.3. Reformulation of Individual Chance Constraints

Assume now that problem (2) accommodates an individual chance constraint defined through the safety set $S(x) = \{\xi \in \mathbb{R}^K \mid (A\xi + a)^\top x < b^\top \xi + b\}$. By Lemma 2 in the appendix, we have
\[
\text{dist}(\hat{\xi}_i, \bar{S}(x)) = \left(\frac{(b - A^\top x)^\top \hat{\xi}_i + b - a^\top x}{\|b - A^\top x\|_*}\right)^+ \quad \forall i \in [N],
\]
and thus Theorem 3 allows us to reformulate problem (7) as the deterministic optimization problem
\[
\min_{s, t, x} c^\top x \\
s.t. \quad \varepsilon N t - e^\top s \geq 0 N \\
\quad \left(\frac{(b - A^\top x)^\top \hat{\xi}_i + b - a^\top x}{\|b - A^\top x\|_*}\right)^+ \geq t - s_i \quad \forall i \in [N] \\
\quad s \geq 0, \ x \in \mathcal{X}.
\]

Unfortunately, problem (8) fails to be convex as its constraints involve fractions of convex functions. Below we show, however, that problem (8) can be reformulated as a mixed integer conic program.
PROPOSITION 1. Assume that $A^\top x \neq b$ for all $x \in \mathcal{X}$. For the safety set $S(x) = \{\xi \in \mathbb{R}^K \mid (A\xi + a)^\top x < b^\top \xi + b\}$, problem (2) is equivalent to the mixed integer conic program

$$Z_{\text{ICC}}^* = \min_{q,s,t,x} c^\top x$$

subject to

\begin{align*}
&\epsilon N t - e^\top s \geq \theta N\|b - A^\top x\|_s, \\
&(b - A^\top x)^\top \hat{\xi}_i + b - a^\top x + M q_i \geq t - s_i \quad \forall i \in [N] \\
&M(1 - q_i) \geq t - s_i \quad \forall i \in [N] \\
&q \in \{0,1\}^N, \ s \geq 0, \ x \in \mathcal{X},
\end{align*}

where $M$ is a suitably large (but finite) positive constant.

Proof of Proposition 1. We already know that the chance constrained program (2) is equivalent to the non-convex optimization problem (8). A complicating feature of this problem is the appearance of the maximum operator in the second constraint group, which evaluates the positive part of $(b - A^\top x)^\top \hat{\xi}_i + b - a^\top x$. To eliminate this maximum operator, for each $i \in [N]$ we introduce a binary variable $q_i \in \{0,1\}$, and we re-express the $i^{\text{th}}$ member of the second constraint group via the two auxiliary constraints

$$\frac{(b - A^\top x)^\top \hat{\xi}_i + b - a^\top x}{\|b - A^\top x\|_s} + M q_i \geq t - s_i \quad \text{and} \quad M(1 - q_i) \geq t - s_i. \quad (10)$$

Note that at optimality we have $q_i = 1$ if $(b - A^\top x)^\top \hat{\xi}_i + b - a^\top x$ is negative and $q_i = 0$ otherwise. Intuitively, $q_i$ thus activates the less restrictive one of the two auxiliary constraints in (10). Next, we apply the variable substitutions $t \leftarrow t/\|b - A^\top x\|_s$ and $s \leftarrow s/\|b - A^\top x\|_s$, which is admissible because $A^\top x \neq b$ for all $x \in \mathcal{X}$. This change of variables yields the postulated reformulation (9).

To see that a finite value of $M$ is sufficient for our reformulation to be exact, we show that the expression $((b - A^\top x)^\top \hat{\xi}_i + b - a^\top x)/\|b - A^\top x\|_s$ as well as the values of $t$ and $s$, $i \in [N]$, in (10) can all be bounded without affecting the optimal value of problem (9). This is clear for the fraction as $\mathcal{X}$ is compact and the denominator is non-zero for all $x \in \mathcal{X}$. Moreover, $t$ is nonnegative as otherwise the first constraint in (9) would be violated. For any fixed values of $x$ and $t$, an optimal value of $s_i$, $i \in [N]$, is given by $s_i^*(x,t) = t - ((b - A^\top x)^\top \hat{\xi}_i + b - a^\top x)/\|b - A^\top x\|_s)^+$. Since $\mathcal{X}$ is bounded, it thus remains to show that $t$ can be bounded from above. Indeed, for sufficiently large (but finite) $t$, the slope of $\epsilon N t - e^\top s^*(x,t)$ on the left-hand side of the first constraint in (9) is $-(1 - \epsilon)N$. Since $\epsilon < 1$, we thus conclude that this constraint is violated for large values of $t$. \hfill \Box

REMARK 2. The condition that $A^\top x \neq b$ for all $x \in \mathcal{X}$ does not restrict the generality of our formulation. Indeed, if an optimal solution $(q^*, s^*, t^*, x^*)$ to problem (9) satisfies $A^\top x^* \neq b$, then $x^*$ solves problem (2) since our argument in the proof of Proposition 1 applies to $x^*$ even if $A^\top x = b$ for some $x \in \mathcal{X}$. Assume now that an optimal solution $(q^*, s^*, t^*, x^*)$ to problem (9) satisfies $A^\top x^* = b$. 

Chen, Kuhn, and Wiesemann: Data-Driven Chance Constrained Programs over Wasserstein Balls
In that case, the ambiguous chance constraint in problem (2) requires that \( a^\top x^* < b \). If that is the case for \( x^* \), it is optimal in problem (2). If, finally, an optimal solution \((q^*, s^*, t^*, x^*)\) to problem (9) satisfies \( A^\top x^* = b \) and \( a^\top x^* \geq b \), then we can solve \( 2K + 1 \) variants of problem (9) that include exactly one of the constraints \([A^\top x]_k > [b]_k\), \([A^\top x]_k < [b]_k\), \( k \in [K] \), and \( a^\top x < b \). The solution that attains the least objective value amongst these problems is an optimal solution to problem (2).

**Remark 3.** The mixed-integer conic program (9) simplifies to a mixed-integer linear program whenever \( \|\cdot\| \) represents the 1-norm or the \( \infty \)-norm, and it can be reformulated as a mixed-integer second-order cone program whenever \( \|\cdot\| \) represents a \( p \)-norm for some \( p \in \mathbb{Q} \), \( p > 1 \), see Section 2.3.1 in Ben-Tal and Nemirovski (2001).

**Remark 4.** The deterministic reformulation (9) is remarkably parsimonious. For an \( L \)-dimensional feasible region \( X \subseteq \mathbb{R}^L \) and an empirical distribution \( \hat{P} \) with \( N \) data points, our reformulation (9) has \( N \) binary variables, \( L + N + 1 \) continuous decisions as well as \( 2N + 1 \) constraints (excluding those that describe \( X \)). In comparison, a classical chance constrained formulation, which is tantamount to setting the Wasserstein radius to \( \theta = 0 \) in problem (2), has \( N \) binary variables, \( L \) continuous decisions as well as \( N + 1 \) constraints. Thus, adding distributional robustness only requires an additional \( N + 1 \) continuous decisions as well as \( N \) further constraints.

### 2.4. Reformulation of Joint Chance Constraints with Right-Hand Side Uncertainty

Assume next that problem (2) accommodates a joint chance constraint defined through the safety set \( S(x) = \{ \xi \in \mathbb{R}^K \mid a^\top m x < b^\top m \xi + b_m \ \forall m \in [M] \} \), in which the uncertainty affects only the right-hand sides of the safety conditions. Without loss of generality, we may assume that \( b_m \neq 0 \) for all \( m \in [M] \). Indeed, if \( b_m = 0 \), then the \( m \)-th safety condition in the chance constraint becomes independent of the uncertainty and can thus be absorbed in \( X \). Observe that the complement of the safety set is now representable as \( \bar{S}(x) = \bigcup_{m \in [M]} H_m(x) \), where \( H_m(x) = \{ \xi \in \mathbb{R}^K \mid a^\top m x \geq b^\top m \xi + b_m \} \) is a closed halfspace for every \( m \in [M] \). By Lemma 2 in the appendix we have

\[
\text{dist}(\xi_i, S(x)) = \min_{m \in [M]} \left\{ \frac{(b^\top m \xi_i + b_m - a^\top m x)^+}{\|b_m\|_*} \right\} = \left( \min_{m \in [M]} \left\{ \frac{b^\top m \xi_i + b_m - a^\top m x}{\|b_m\|_*} \right\} \right)^+.
\]  

With this closed-form expression for the distance to the unsafe set, we can reformulate problem (2) as a mixed integer conic program.
Proposition 2. For the safety set \( S(x) = \{ \xi \in \mathbb{R}^K | a_m^\top x < b_m^\top \xi + b_m \forall m \in [M] \} \), where \( b_m \neq 0 \) for all \( m \in [M] \), the chance constrained program (2) is equivalent to the mixed integer conic program

\[
Z_{JCC}^* = \min_{p, q, s, t, x} \mathbf{c}^\top x \\
\text{s.t.} \quad \varepsilon N t - \mathbf{e}^\top s \geq \theta N \\
p_i + M q_i \geq t - s_i \quad \forall i \in [N] \\
M (1 - q_i) \geq t - s_i \quad \forall i \in [N] \\
\frac{b_m^\top \hat{\xi}_i + b_m - a_m^\top x}{\|b_m\|_*} \geq p_i \quad \forall m \in [M], i \in [N] \\
q \in \{0, 1\}^N, \ s \geq 0, \ x \in \mathcal{X},
\]

where \( M \) is a suitably large (but finite) positive constant.

Proof of Proposition 2. By Theorem 3, the chance constrained program (2) is equivalent to (7). Using (11), the \( i \)th member of the second constraint group in (7) can be reformulated as

\[
\begin{aligned}
\left( \min_{m \in [M]} \left\{ \frac{b_m^\top \hat{\xi}_i + b_m - a_m^\top x}{\|b_m\|_*} \right\} \right)^+ \geq t - s_i.
\end{aligned}
\]

To eliminate the maximum operator, we introduce a binary variable \( q_i \in \{0, 1\} \) as well as a continuous variable \( p_i \in \mathbb{R} \), which allow us to re-express the above constraint as

\[
\begin{cases}
p_i + M q_i \geq t - s_i \\
M (1 - q_i) \geq t - s_i \\
\frac{b_m^\top \hat{\xi}_i + b_m - a_m^\top x}{\|b_m\|_*} \geq p_i \quad \forall m \in [M], i \in [N].
\end{cases}
\]

A similar argument as in the proof of Proposition 1 shows that a finite value of \( M \) is sufficient for our reformulation to be exact. \( \square \)

Remark 5. The deterministic reformulation (12) has \( N \) binary variables, \( L + 2N + 1 \) continuous decisions as well as \( N(M + 2) + 1 \) constraints (excluding those that describe \( \mathcal{X} \)). In comparison, the corresponding classical chance constrained formulation has \( N \) binary variables, \( L \) continuous decisions as well as \( MN + 1 \) constraints. Thus, adding distributional robustness requires an additional \( 2N + 1 \) continuous decisions as well as \( 2N \) further (linear) constraints.

3. Tractable Safe Approximations

The exact mixed-integer conic programming reformulations of the distributionally robust chance constrained program (2) derived in Section 2 may become computationally prohibitive in the face of large problem dimensions or sample sizes. Thus, there is merit in studying safe tractable approximations with better scalability properties. Such approximations are obtained by constructing inner approximations to the exact feasible set of the chance constrained program (2). For example, safe
tractable approximations for individual chance constraints can be obtained via a CVaR approximation popularized by Nemirovski and Shapiro (2006), while joint chance constraints are often decomposed into a family of more tractable (and also more restrictive) individual chance constraints by using the Bonferroni inequality from probability theory. We remark that any joint chance constraint with $M$ safety conditions can be reformulated as an individual chance constraint with a single safety condition by aggregating the $M$ (suitably scaled) safety conditions, in which case it becomes susceptible to the CVaR approximation, too.

In Sections 3.1 and 3.2 below we propose a systematic approach to constructing safe convex approximations for the chance constrained program (2) and contrast it with the classical CVaR approximation. We also investigate several low-parametric classes of tractable safe approximations and discuss the complexity of finding the respective best-in-class approximations. Moreover, in Section 3.3 we demonstrate that the CVaR approximation is generally incomparable to the Bonferroni approximation for ambiguous joint chance constraints over Wasserstein balls.

### 3.1. Individual Chance Constraints

Consider an instance of problem (2) with an individual chance constraint corresponding to the safety set $S(x) = \{\xi \in \mathbb{R}^K \mid (A\xi + a)^\top x < b^\top \xi + b\}$. As in Section 2.3, we assume without much loss of generality that $A^\top x \neq b$ for all $x \in X$. By Proposition 1, the distributionally robust chance constrained program (2) is thus equivalent to the deterministic optimization problem

$$Z_{ICC}^* = \min_{(x,s,t) \in C_{ICC}} c^\top x;$$

whose feasible set is given by

$$C_{ICC} = \left\{ (x,s,t) \in X \times \mathbb{R}_+^N \times \mathbb{R} \left| \begin{array}{l} \varepsilon Nt - e^\top s \geq \theta N\|b - A^\top x\|, \\
(b - A^\top x)^\top \hat{\xi}_i + b - a^\top x \geq t - s_i \quad \forall i \in [N] \end{array} \right. \right\}.$$ 

As $C_{ICC}$ is non-convex, it is of interest to find tractable conservative (inner) approximations to problem (2). It turns out that any convex inner approximation of $C_{ICC}$ is dominated, in the sense of set inclusion, by a convex set of the form

$$C_{ICC}(\kappa) = \left\{ (x,s,t) \in X \times \mathbb{R}_+^N \times \mathbb{R} \left| \begin{array}{l} \varepsilon Nt - e^\top s \geq \theta N\|b - A^\top x\|, \\
\kappa_i((b - A^\top x)^\top \hat{\xi}_i + b - a^\top x) \geq t - s_i \quad \forall i \in [N] \end{array} \right. \right\}$$

parameterized by a vector of slope parameters $\kappa \in [0,1]^N$.

**Proposition 3.** For any convex set $W \subseteq C_{ICC}$, there exists $\kappa \in [0,1]^N$ with $W \subseteq C_{ICC}(\kappa) \subseteq C_{ICC}.$

**Proof of Proposition 3.** It is clear that $C_{ICC}(\kappa) \subseteq C_{ICC}$ for every $\kappa \in [0,1]^N$. Next, we show that for every $i \in [N]$ there exists $\kappa_i \in [0,1]$ such that the constraint $\kappa_i((b - A^\top x)^\top \hat{\xi}_i + b - a^\top x) \geq t - s_i$ is valid for $W$. The resulting set $C_{ICC}(\kappa)$ is thus a convex outer approximation of $W$. 
To determine $\kappa_i$, consider the sets $W_i = \{(x, s_i, t) \mid (x, s, t) \in W\}$ and $V_i = \{(x, s_i, t) \mid ((b - A^T x)^\top \hat{\xi}_i + b - a^\top x)^+ < t - s_i\}$. By construction, $W_i$ and $V_i$ are intersection-free and convex. Thus, they admit a separating hyperplane. The same holds true if we replace $W_i$ with

$$\overline{W}_i = \text{conv}(W_i \cup \{(x, s_i, t) \mid ((b - A^T x)^\top \hat{\xi}_i + b - a^\top x, t - s_i) = (0, 0)\}).$$

The separating hyperplane between $\overline{W}_i$ and $V_i$ must satisfy $t - s_i = 0$ whenever $(b - A^T x)^\top \hat{\xi}_i + b - a^\top x = 0$. In other words, the separating hyperplane must be of the form $\kappa_i((b - A^T x)^\top \hat{\xi}_i + b - a^\top x) = t - s_i$ for some $\kappa_i \in [0, 1]$. Thus, the claim follows.

Proposition 3 implies that amongst all convex conservative approximations to problem (13) it is sufficient to focus on those that are induced by a feasible set of the form $C_{ICC}(\kappa)$ for some $\kappa \in [0, 1]^N$. Thus, it is sufficient to focus on the family of approximate problems of the form

$$Z_{ICC}^*(\kappa) = \min_{(a, s, t) \in C_{ICC}(\kappa)} c^\top x$$

parameterized by $\kappa \in [0, 1]^N$. The following proposition asserts that the best approximation within this family is exact.

**Proposition 4.** We have $Z_{ICC}^* = \min_{\kappa \in [0, 1]^N} Z_{ICC}^*(\kappa)$.

**Proof of Proposition 4.** It follows from Proposition 1 that

$$\min_{\kappa \in [0, 1]^N} Z_{ICC}^*(\kappa) = \begin{cases} 
\min_{a, s, t, x, \kappa} c^\top x \\
\text{s.t. } \varepsilon N t - e^\top s \geq \theta N \|b - A^\top x\|, \\
\kappa_i((b - A^\top x)^\top \hat{\xi}_i + b - a^\top x) \geq t - s_i \forall i \in [N] \\
s \geq 0, \ x \in X, \ \kappa \in [0, 1]^N.
\end{cases}$$

For any fixed $(x, s, t)$, the optimal (that is, least restrictive) choice of $\kappa$ satisfies

$$\kappa_i = \begin{cases} 
1 & \text{if } (b - A^\top x)^\top \hat{\xi}_i + b - a^\top x \geq 0, \forall i \in [N], \\
0 & \text{otherwise}
\end{cases}$$

Eliminating $\kappa$ from (15) by substituting (16) into (15) converts the second constraint group to

$$(b - A^\top x)^\top \hat{\xi}_i + b - a^\top x)^+ \geq t - s_i \forall i \in [N],$$

which shows that (15) is equivalent to (13). Thus, the claim follows.

Proposition 4 implies that the family (14) of tractable upper bounding problems contains an instance $\kappa^* \in \arg\min_{\kappa \in [0, 1]^N} Z_{ICC}^*(\kappa)$ that recovers an optimal solution of the ambiguous chance constrained program (13), which is known to be NP-hard (Xie and Ahmed 2018a, Theorem 12). We may thus conclude that computing $\kappa^*$ is also NP-hard. The complexity of computing the best upper bound within the family (14) can be reduced by restricting attention to uniform slope parameters of the form $\kappa = \kappa e$ for some $\kappa \in [0, 1]$. Within this subset the choice $\kappa = e$ is optimal.
**Proposition 5.** We have \( \min_{\kappa \in [0,1]} Z^*_{\text{ICC}}(\kappa e) = Z^*_{\text{ICC}}(e) \).

**Proof of Proposition 5.** We first show that problem (14) is infeasible for \( \kappa = 0 \), that is, \( Z^*_\text{ICC}(0) = \infty \). Indeed, by the definition of \( C_{\text{ICC}}(0) \) we have

\[
Z^*_\text{ICC}(0) = \begin{cases} 
\min \limits_{s,t,x} c^\top x \\
\text{s.t. } \varepsilon N t - e^\top s \geq \theta N \| b - A^\top x \|_*, \\
\quad s \geq t e, \ s \geq 0, \ x \in X.
\end{cases}
\]

Any feasible solution of the above problem satisfies \( \varepsilon N t - e^\top s \leq \varepsilon N t - N \max\{t,0\} \leq 0 \), where the first inequality follows from the constraints \( s \geq t e \) and \( s \geq 0 \). As \( \theta > 0 \), the constraint \( \varepsilon N t - e^\top s \geq \theta N \| b - A^\top x \|_* \) is thus satisfied only if \( s = 0, \ t = 0 \) and \( A^\top x = b \). However, the last equality contradicts our standing assumption that \( A^\top x \neq b \) for all \( x \in X \), confirming that the above problem is infeasible and \( Z^*_\text{ICC}(0) = \infty \). Thus, \( Z^*_\text{ICC}(\kappa e) \) is minimized by some \( \kappa \in (0,1] \).

If \( \kappa \in (0,1] \), we can use the variable substitution \( t \leftarrow \kappa t \) and \( s \leftarrow \kappa s \) to re-express problem (14) as

\[
Z^*_\text{ICC}(\kappa e) = \begin{cases} 
\min \limits_{s,t,x} c^\top x \\
\text{s.t. } \varepsilon N t - e^\top s \geq \frac{\theta N}{\kappa} \| b - A^\top x \|_* \\
\quad (b - A^\top x)^\top \xi_i + b - a^\top x \geq t - s_i \ \forall i \in [N] \\
\quad s \geq 0, \ x \in X.
\end{cases}
\]

From this formulation it is evident that \( \kappa^* = 1 \) is the best (least restrictive) choice of \( \kappa \in (0,1] \). \( \square \)

Next, we demonstrate that the approximate problem (14) corresponding to \( \kappa = e \) can also be obtained by approximating the worst-case chance constraint in (2) with a worst-case CVaR constraint. To see this, note first that

\[
P[\xi \in S(x)] \geq 1 - \varepsilon \iff P[(A\xi + a)^\top x \geq b^\top \xi + b] \leq \varepsilon
\]

\[
\iff P-\text{VaR}_\varepsilon(a^\top x - b + (A^\top x - b)^\top \xi) \leq 0
\]

\[
\iff P-\text{CVaR}_\varepsilon(a^\top x - b + (A^\top x - b)^\top \xi) \leq 0
\]

for any \( P \in \mathcal{F}(\theta) \), where the first equivalence follows from the definition of the safety set, the second equivalence holds due to the definition of the VaR, and the last implication exploits the fact that the CVaR provides an upper bound on the VaR. Thus, the worst-case CVaR constrained program

\[
Z^*_\text{CVaR} = \begin{cases} 
\min \limits_{x \in X} c^\top x \\
\text{s.t. } \sup \limits_{P \in \mathcal{F}(\theta)} P-\text{CVaR}_\varepsilon(a^\top x - b + (A^\top x - b)^\top \xi) \leq 0
\end{cases}
\]

(17)

constitutes a conservative approximation for the worst-case chance constrained program (2), that is, \( Z^*_\text{ICC} \leq Z^*_\text{CVaR} \). We are now ready to prove that \( Z^*_\text{CVaR} = Z^*_\text{ICC}(e) \).
**Proposition 6.** We have $Z^*_{\text{CVaR}} = Z^*_{\text{ICC}}(e)$.

**Proof of Proposition 6.** Using now standard techniques, the worst-case CVaR in (17) can be re-expressed as the optimal value of a finite conic program,

$$
\sup_{P \in \mathcal{F}(\theta)} \mathbb{P}\text{-CVaR}_e(a^T x - b + (A^T x - b)^T \tilde{\xi}) = \begin{cases} 
\min_{\alpha, \beta, \tau} & \frac{1}{\varepsilon} \left( \theta \beta + \frac{1}{N} \sum_{i \in [N]} \alpha_i \right) \\
\text{s.t.} & \alpha_i \geq a^T x - b + (A^T x - b)^T \tilde{\xi}_i - \tau \quad \forall i \in [N] \\
& \alpha \geq 0, \beta \geq \|A^T x - b\|_* 
\end{cases}
$$

see Mohajerin Esfahani and Kuhn (2018, § 5.1 and § 7.1) for a detailed derivation. Substituting this reformulation into the worst-case CVaR constrained program (17) yields

$$
Z^*_{\text{CVaR}} = \begin{cases} 
\min_{x, \alpha, \beta, \tau} & c^T x \\
\text{s.t.} & \tau + \frac{1}{\varepsilon} \left( \theta \beta + \frac{1}{N} \sum_{i \in [N]} \alpha_i \right) \leq 0 \\
& \alpha_i \geq a^T x - b + (A^T x - b)^T \tilde{\xi}_i - \tau \quad \forall i \in [N] \\
& \alpha \geq 0, \beta \geq \|A^T x - b\|_* \quad x \in \mathcal{X}
\end{cases}
$$

As $\theta > 0$ and $\varepsilon > 0$, it is clear that $\beta = \|A^T x - b\|_*$ at optimality, and this insight allows us to eliminate $\beta$ from the above optimization problem. Multiplying the first constraint by the positive constant $\varepsilon N$ while renaming $\alpha$ as $s$ and $\tau$ as $-t$ then shows that $Z^*_{\text{CVaR}} = Z^*_{\text{ICC}}(e)$. \qed

**Remark 6.** Using similar arguments as in Proposition 6, one can show that problem (14) with $\kappa = \kappa e$ for any $\kappa \in (0, 1]$ is equivalent to a worst-case CVaR constrained program of the form (17), where the Wasserstein radius $\theta$ is inflated to $\theta/\kappa$. This observation reconfirms that $\kappa = 1$ is the least conservative choice amongst all uniform slope parameters in (14); see Proposition 5.

The intimate links between the worst-case CVaR approximation (17) and the worst-case chance constrained program (2) can also be studied through the lens of Theorem 2. To this end, recall that the ambiguous chance constraint (4) is equivalent to the constraint (5), which requires that

$$
\frac{1}{N} \sum_{i=1}^{\varepsilon N} \text{dist}(\tilde{\xi}_{\pi_i}(x), \hat{S}(x)) \geq \theta.
$$

We define the signed distance between a point $\xi$ and a closed set $\mathcal{C}$ as $\text{sgn-dist}(\xi, \mathcal{C}) = \text{dist}(\xi, \mathcal{C})$ if $\xi \in \mathcal{C}$ and $\text{sgn-dist}(\xi, \mathcal{C}) = -\text{dist}(\xi, \text{cl}(\mathcal{C}))$ otherwise. Here $\text{cl}(\mathcal{C})$ denotes the closure of the open set $\mathcal{C} = \mathbb{R}^K \setminus \mathcal{C}$. We then obtain the following result.

**Proposition 7.** For any fixed decision $x \in \mathcal{X}$, we have

$$
\sup_{P \in \mathcal{F}(\theta)} \mathbb{P}\text{-CVaR}_e(a^T x - b + (A^T x - b)^T \tilde{\xi}) \leq 0 \iff \frac{1}{N} \sum_{i=1}^{\varepsilon N} \text{sgn-dist}(\tilde{\xi}_{\pi_i}(x), \hat{S}(x)) \geq \theta,
$$

where $\pi(x)$ permutes the data points $\tilde{\xi}_i$ into ascending order of their signed distances to $\hat{S}(x)$. 

Proof of Proposition 7. It follows from the proof of Proposition 6 that the worst-case CVaR constraint \( \sup_{\tilde{x} \in \mathcal{X}(\theta)} \mathbb{P}(\text{CVaR}_\varepsilon(a^\top x - b + (A^\top x - b)^\top \hat{\xi}) \leq 0 \) holds if and only if

\[
\exists s \geq 0, t \in \mathbb{R} : \begin{cases}
\varepsilon N t - e^\top s \geq \theta N \|b - A^\top x\|_*, \\
(b - A^\top x)^\top \hat{\xi}_{\pi_i(x)} + b - a^\top x \geq t - s_i \quad \forall i \in [N].
\end{cases}
\]

(18)

This constraint system is satisfiable by \( t \in \mathbb{R} \) and some \( s \geq 0 \) if and only if it is satisfiable by \( t \) and \( s^*(t) \) defined by \( s_i^*(t) = (t - ((b - A^\top x)^\top \hat{\xi}_{\pi_i(x)} + b - a^\top x))^+ \), \( i \in [N] \). Since the second constraint in (18) is automatically satisfied by \( s^*(t) \), we thus conclude that (18) holds if and only if

\[
\exists t \in \mathbb{R} : \varepsilon N t - \sum_{i \in [N]} (t - ((b - A^\top x)^\top \hat{\xi}_{\pi_i(x)} + b - a^\top x))^+ \geq \theta N \|b - A^\top x\|_*
\]

\[
\iff \max_{t \in \mathbb{R}} \left\{ \varepsilon N t - \sum_{i \in [N]} (t - ((b - A^\top x)^\top \hat{\xi}_{\pi_i(x)} + b - a^\top x))^+ \right\} \geq \theta N \|b - A^\top x\|_*
\]

The first-order optimality condition for non-smooth optimization then implies that the maximum on the left-hand side of (19) is attained by \( t^* = (b - A^\top x)^\top \hat{\xi}_{\pi_{iN+1}(x)} + b - a^\top x \), which results in the equivalent constraint

\[
\sum_{i=1}^{\varepsilon N} ((b - A^\top x)^\top \hat{\xi}_{\pi_i(x)} + b - a^\top x) \geq \theta N \|b - A^\top x\|_*
\]

The result now follows if we divide both sides of the constraint by \( N \|b - A^\top x\|_* \). \( \square \)

Theorem 2 and Proposition 7 show that both the ambiguous chance constraint (4) and its worst-case CVaR approximation (17) impose lower bounds on the costs of moving a fraction \( \varepsilon \) of the training samples to the unsafe set. Moreover, since \( \text{sgn-dist}(\hat{\xi}_i, \bar{S}(x)) \leq \text{dist}(\hat{\xi}_i, \bar{S}(x)) \) by construction, the worst-case CVaR constraint conservatively approximates the ambiguous chance constraint. In fact, we have \( \text{sgn-dist}(\hat{\xi}_i, \bar{S}(x)) = \text{dist}(\hat{\xi}_i, \bar{S}(x)) \) for safe scenarios \( \hat{\xi}_i \in S(x) \), whereas \( \text{sgn-dist}(\hat{\xi}_j, \bar{S}(x)) < 0 \) even though \( \text{dist}(\hat{\xi}_j, \bar{S}(x)) = 0 \) for (strictly) unsafe scenarios \( \hat{\xi}_j \in \text{int}(\bar{S}(x)) \). In other words, the worst-case CVaR approximation (17) assigns fictitious transportation profits to training samples that are contained in the unsafe set. This leads to the following insight.

**Corollary 1.** The worst-case CVaR approximation is exact, that is, \( Z_{\text{CVaR}}^* = Z_{\text{ICC}}^* \), under either of the following conditions.

(i) We have \( \hat{\xi}_i \in S(x^*) \) for all \( i \in [N] \), where \( x^* \) is optimal in (2).

(ii) We have \( \varepsilon \leq 1/N \).
Proof of Corollary 1. The first condition immediately follows from Theorem 2 and Proposition 7 since $\text{sgn-dist}(\hat{\xi}_i, S(x)) = \text{dist}(\hat{\xi}_i, S(x))$ whenever $\hat{\xi}_i \in S(x)$. The second condition guarantees that $\hat{\xi}_i \in S(x)$, $i \in [N]$, for any solution $x \in X$ that satisfies the ambiguous chance constraint (4). This, in turn, implies that the first condition of the corollary is satisfied as well. \hfill \Box

3.2. Joint Chance Constraints with Right-Hand Side Uncertainty

Consider now an instance of problem (2) with a joint chance constraint corresponding to the safety set $S(x) = \{\xi \in \mathbb{R}^K | a_m^\top x < b_m^\top \xi + b_m \forall m \in [M]\}$. As in Section 2.4, we assume without loss of generality that $b_m \neq 0$ for all $m \in [M]$. By Proposition 2, the distributionally robust chance constrained program (2) is thus equivalent to the deterministic optimization problem

$$Z^*_{\text{JCC}} = \min_{(x, s, t) \in C_{\text{JCC}}} c^\top x,$$

whose feasible set is given by

$$C_{\text{JCC}} = \left\{ (x, s, t) \in X \times \mathbb{R}_+^N \times \mathbb{R} \left| \begin{array}{l}
\varepsilon N t - e^\top s \geq \theta N \\
\min_{m \in [M]} \left\{ \frac{b_m^\top \hat{\xi}_i + b_m - a_m^\top x}{\|b_m\|_*} \right\} \geq t - s_i \ \forall i \in [N] \end{array} \right. \right\}.$$ 

In analogy to Section 3.1, one can again show that any convex inner approximation of $C_{\text{JCC}}$ is weakly dominated by a polyhedron of the form

$$C_{\text{JCC}}(\kappa) = \left\{ (x, s, t) \in X \times \mathbb{R}_+^N \times \mathbb{R} \left| \begin{array}{l}
\varepsilon N t - e^\top s \geq \theta N \\
\kappa_i \left( \frac{b_m^\top \hat{\xi}_i + b_m - a_m^\top x}{\|b_m\|_*} \right) \geq t - s_i \ \forall m \in [M], \ i \in [N] \end{array} \right. \right\}$$

for some vector of slope parameters $\kappa \in [0,1]^N$. The following assertion akin to Proposition 3 formalizes this statement. Its proof is omitted for the sake of brevity.

**Proposition 8.** For any convex set $W \subseteq C_{\text{JCC}}$, there exists $\kappa \in [0,1]^N$ with $W \subseteq C_{\text{JCC}}(\kappa) \subseteq C_{\text{JCC}}$.

Proposition 8 implies that amongst all convex conservative approximations to problem (20) it is sufficient to consider the family of linear programs

$$Z^*_{\text{JCC}}(\kappa) = \min_{(x, s, t) \in C_{\text{JCC}}(\kappa)} c^\top x$$

parameterized by $\kappa \in [0,1]^N$. One can show that the best approximation within this family is exact. The proof of this result is similar to that of Proposition 4 and thus omitted.

**Proposition 9.** We have $Z^*_{\text{JCC}} = \min_{\kappa \in [0,1]^N} Z^*_{\text{JCC}}(\kappa)$.

Unfortunately, finding the best slope parameters $\kappa^* \in [0,1]^N$ is again NP-hard, but optimizing over the subclass of uniform slope parameters $\kappa = \kappa e$ for $\kappa \in [0,1]$ is easy, and $\kappa = e$ is optimal. This result is reminiscent of Proposition 5, and thus its proof is omitted for the sake of brevity.
Proposition 10. We have \( \min_{e \in [0,1]} Z^*_{\text{JCC}}(\kappa e) = Z^*_{\text{JCC}}(e) \).

We now demonstrate that \( Z^*_{\text{JCC}}(e) \) can again be interpreted as the feasible set of a worst-case CVaR constraint. To see this, denote by \( \Delta^M_{++} = \{ w \in (0,1)^M | e^T w = 1 \} \) the relative interior of the probability simplex and observe that for any vector of scaling factors \( w \), parameterized by \( e \), the quality of the CVaR approximation can be tuned by varying \( w \) for \( e \). Thus, the overall normalization \( e^T w = 1 \) is non-restrictive because the CVaR is positive homogeneous.

We now introduce a family of worst-case CVaR constrained programs

\[
Z^*_{\text{CVaR}}(w) = \left\{ \min_{x \in \mathcal{X}} c^T x \right. \\
\left. \text{s.t.} \sup_{P \in F(\theta)} \mathbb{P}\text{-CVaR}_\varepsilon \left( \max_{m \in [M]} \{ w_m (a_m^T x - b_m^T \xi - b_m) \} \right) \leq 0 \right\}
\]

parameterized by \( w \in \Delta^M_{++} \), all of which conservatively approximate the ambiguous chance constrained program (2), that is, \( Z^*_{\text{JCC}} = Z^*_{\text{CVaR}}(w) \). In fact, the family (22) contains an instance that is equivalent to the best bounding problem of the form (21) with uniform slope parameters.

Proposition 11. We have \( Z^*_{\text{CVaR}}(w^*) = Z^*_{\text{JCC}}(e) \) for \( w^* \in \Delta^M_{++} \) defined through

\[
w^*_m = \frac{\|b_m\|_1^{-1}}{\sum_{i \in [M] \|b_i\|_1^{-1}}} \quad \forall m \in [M].
\]

Proof of Proposition 11. Using techniques introduced by Mohajerin Esfahani and Kuhn (2018), the worst-case CVaR in (22) can be re-expressed as

\[
\sup_{P \in F(\theta)} \mathbb{P}\text{-CVaR}_\varepsilon \left( \max_{m \in [M]} \{ w_m (a_m^T x - b_m^T \xi - b_m) \} \right)
\]

\[
\left\{ \min_{\alpha, \beta, \tau} \tau + \frac{1}{\varepsilon} \left( \theta \beta + \frac{1}{N} \sum_{i \in [N]} \alpha_i \right) \right. \\
\left. \text{s.t.} \alpha_i \geq w_m (a_m^T x - b_m^T \xi_i - b_m) - \tau \quad \forall m \in [M], i \in [N] \right.
\]

\[
\beta \geq w_m \|b_m\|_*, \quad \forall m \in [M]
\]

\[
\alpha \geq 0.
\]
Substituting this reformulation into (22) yields

\[
Z_{\text{CVaR}}^*(w) = \begin{cases} 
\min_{x, \alpha, \beta, \tau} & c^T x \\
\text{s.t.} & \tau + \frac{1}{\varepsilon} \left( \theta \beta + \frac{1}{N} \sum_{i \in [N]} \alpha_i \right) \leq 0 \\
& \alpha_i \geq w_m (a_m^T x - b_m^T \xi_i - b_m) - \tau \quad \forall m \in [M], \ i \in [N] \\
& \beta \geq w_m \|b_m\|_* \quad \forall m \in [M] \\
& \alpha \geq 0, \ x \in \mathcal{X}.
\end{cases}
\] (23)

As \(\theta > 0\) and \(\varepsilon > 0\), it is clear that \(\beta = \max_{m \in [M]} \{ w_m \|b_m\|_* \} \) at optimality, and this insight allows us to eliminate \(\beta\) from the above optimization problem. Multiplying the first constraint by the positive constant \(\varepsilon N / \beta\) and the second constraint group by the positive constant \(1 / \beta\) while applying the variable substitutions \(s \leftarrow \alpha / \beta\) and \(-t \leftarrow \tau / \beta\), we obtain

\[
Z_{\text{CVaR}}^*(w) = \begin{cases} 
\min_{s, t, x} & c^T x \\
\text{s.t.} & \varepsilon N t - \theta N s \geq 0 \\
& \frac{w_m \|b_m\|_*}{\max_{m \in [M]} \{ w_m \|b_m\|_* \}} \left( \frac{b_m^T \hat{\xi}_i + b_m - a_m^T x}{\|b_m\|_*} \right) \geq t - s_i \quad \forall m \in [M], \ i \in [N] \\
& s \geq 0, \ x \in \mathcal{X}.
\end{cases}
\] (24)

Replacing \(w\) with \(w^*\), the second constraint group in problem (24) simplifies to

\[
\min_{m \in [M]} \left\{ \frac{(b_m^T \hat{\xi}_i + b_m - a_m^T x)}{\|b_m\|_*} \right\} \geq t - s_i \quad \forall i \in [N],
\]

which reveals that the feasible set of problem (24) coincides with \(C_{\mathcal{JCC}}(e)\). This observation implies the postulated assertion that \(Z_{\text{CVaR}}^*(w^*) = Z_{\mathcal{JCC}}^*(e)\). \(\square\)

As the quality of the CVaR approximation in (22) depends on the choice of \(w\), it would be desirable to identify the best (least conservative) approximation by solving \(\min_{w \in \Delta^M_{++}} Z_{\text{CVaR}}^*(w)\). This could be achieved, for instance, by treating \(w \in \Delta^M_{++}\) as an additional decision variable in (23). Unfortunately, the resulting optimization problem involves bilinear terms in \(x\) and \(w\) and is thus non-convex. Finding the best CVaR approximation therefore appears to be computationally challenging. Even if the optimal scaling parameters were known, we will see in Section 3.3 that the corresponding instance of problem (22) would generically provide a strict upper bound on \(Z_{\mathcal{JCC}}^*\).

The CVaR approximation (22) can again be interpreted as imposing a lower bound on the costs of moving training samples to the unsafe set. To see this, we define the minimum signed distance between a point \(\xi\) and a family of closed sets \(C_m, m \in [M]\), as \(\min\text{-dist}(\xi, \{C_m\}_{m \in [M]}) = \min_{m \in [M]} \text{sgn-dist}(\xi, C_m)\). We then obtain the following result, which is reminiscent of Theorem 2 and Proposition 7.
PROPOSITION 12. If \( \mathbf{w} \) is set to \( \mathbf{w}^* \) as defined in Proposition 11, then we have

\[
\sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \cdot \text{CVaR} \left( \max_{m \in [M]} \left\{ w_m^* (a_m^\top \mathbf{x} - b_m^\top \hat{\xi} - b_m) \right\} \right) \leq 0 \iff \frac{1}{N} \sum_{i=1}^{\varepsilon N} \min_{\mathcal{H}_m(\mathbf{x}) \in [M]} \text{dist}(\hat{\xi}_i(\mathbf{x})), \{ \mathcal{H}_m(\mathbf{x}) \}_{m \in [M]} \geq \varepsilon, \]

where \( \pi(\mathbf{x}) \) orders the data points \( \hat{\xi}_i \) by their minimum signed distances to \( \mathcal{H}_m(\mathbf{x}), m \in [M] \).

The proof of Proposition 12 closely resembles that of Proposition 7 and is therefore omitted.

COROLLARY 2. If \( \mathbf{w} \) is set to \( \mathbf{w}^* \) as defined in Proposition 11, then the worst-case CVaR approximation is exact, that is, \( Z_{\text{CVaR}}^{\mathbf{w}^*}(\mathbf{w}^*) = Z_{3\text{CC}}, \) under either of the following conditions.

(i) We have \( \hat{\xi}_i \in \mathcal{S}(\mathbf{x}^*) \) for all \( i \in \{N\} \), where \( \mathbf{x}^* \) is optimal in (2).

(ii) We have \( \varepsilon \leq 1/N \).

The proof is similar to that of Corollary 1 and is thus omitted.

3.3. Bonferroni Approximation

Consider again the joint chance constrained program with right-hand side uncertainty studied in Sections 2.4 and 3.2, and note that the Bonferroni inequality (or union bound) implies that

\[
\mathbb{P} \left[ \hat{\xi} \notin \mathcal{S}(\mathbf{x}) \right] = \mathbb{P} \left[ a_1^\top \mathbf{x} \geq b_1^\top \hat{\xi} + b_1 \right] \quad \text{or} \quad \cdots \quad \text{or} \quad a_M^\top \mathbf{x} \geq b_M^\top \hat{\xi} + b_M \leq \sum_{m \in [M]} \mathbb{P} \left[ a_m^\top \mathbf{x} \geq b_m^\top \hat{\xi} + b_m \right].
\]

Taking the supremum over all distributions in the Wasserstein ball then yields the estimate

\[
\sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \left[ \hat{\xi} \notin \mathcal{S}(\mathbf{x}) \right] \leq \sup_{\mathcal{P} \in \mathcal{F}(\theta)} \sum_{m \in [M]} \mathbb{P} \left[ a_m^\top \mathbf{x} \geq b_m^\top \hat{\xi} + b_m \right] \leq \sum_{m \in [M]} \sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \left[ a_m^\top \mathbf{x} \geq b_m^\top \hat{\xi} + b_m \right].
\]

For any collection of risk thresholds \( \varepsilon_m \geq 0, m \in [M] \), such that \( \sum_{m \in [M]} \varepsilon_m = \varepsilon \), the family of individual chance constraints

\[
\sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \left[ a_m^\top \mathbf{x} \geq b_m^\top \hat{\xi} + b_m \right] \leq \varepsilon_m \quad \forall m \in [M]
\]

thus provides a conservative approximation for the original joint chance constraint (4) because

\[
\sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \left[ \hat{\xi} \notin \mathcal{S}(\mathbf{x}) \right] \leq \sum_{m \in [M]} \sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \left[ a_m^\top \mathbf{x} \geq b_m^\top \hat{\xi} + b_m \right] \leq \sum_{m \in [M]} \varepsilon_m = \varepsilon,
\]

where the two inequalities follow from (25) and (26), respectively. We thus refer to (26) as the Bonferroni approximation of the original chance constraint (4). The Bonferroni approximation is attractive because the individual chance constraints in (26) are equivalent to simple linear inequalities. To see this, note that each individual chance constraint in (26) can be rewritten as

\[
\sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \left[ a_m^\top \mathbf{x} \geq b_m^\top \hat{\xi} + b_m \right] \leq \varepsilon_m \iff \sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \cdot \text{VaR}_{\varepsilon_m} \left( a_m^\top \mathbf{x} - b_m - b_m^\top \hat{\xi} \right) \leq 0
\]

\[
\iff \sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \cdot \text{VaR}_{\varepsilon_m} \left( -b_m^\top \hat{\xi} + a_m^\top \mathbf{x} - b_m \right) \leq 0
\]

\[
\iff \sup_{\mathcal{P} \in \mathcal{F}(\theta)} \mathbb{P} \cdot \text{VaR}_{\varepsilon_m} \left( -b_m^\top \hat{\xi} \right) \leq b_m - a_m^\top \mathbf{x},
\]
where the second equivalence holds because the value-at-risk is translation invariant. The $m^{th}$ individual chance constraint in (26) thus simplifies to the linear inequality $a^T_m x \leq b_m - \eta_m$, where the constant $\eta_m = \sup_{P \in \mathcal{F}(\theta)} \mathbb{P} \cdot \text{VaR}_{\varepsilon_m}(-b^T \xi)$ is independent of $x$ and can thus be computed offline. Specifically, by using Corollary 5.3 of Mohajerin Esfahani and Kuhn (2018), we can express $\eta_m$ as the optimal value of a deterministic optimization problem, that is,

$$
\eta_m = \begin{cases}
\min_{\alpha, \beta, w, \eta} & \eta \\
\text{s.t.} & \theta \beta + \frac{1}{N} \sum_{i \in [N]} \alpha_i \leq \varepsilon_m \\
& \alpha_i \geq 1 - w_i (\eta - b^T \hat{\xi}_i) \quad \forall i \in [N] \\
& \beta \geq w_i \|b\|_* \quad \forall i \in [N] \\
& \alpha \geq 0, \ w \geq 0.
\end{cases}
$$

The product of $\eta$ and $w_i$ in the second constraint group renders this problem non-convex. As the problem reduces to a linear program for any fixed value of the scalar decision variable $\eta$, however, $\eta_m$ can be computed efficiently to any accuracy by a line search along $\eta$. In summary, under the Bonferroni approximation the chance constrained program (2) thus reduces to a highly tractable linear program. However, the quality of the approximation relies on the choice of the individual risk thresholds $\{\varepsilon_m\}_{m \in [M]}$. It is often recommended to set $\varepsilon_m = \varepsilon / M$ for all $m \in [M]$, but Chen et al. (2007) have shown that this choice can be conservative when the safety conditions are positively correlated. Optimizing over all admissible choices of $\{\varepsilon_m\}_{m \in [M]}$ is impractical because $\eta_m$ generically displays a non-convex dependence on $\varepsilon_m$. Moreover, we will see that the Bonferroni approximation can be very conservative even if the risk thresholds $\{\varepsilon_m\}_{m \in [M]}$ are chosen optimally.

In the remainder of this section we compare the Bonferroni approximation with the worst-case CVaR approximation in the context of joint chance constrained programs with right-hand side uncertainty. While it is known that the worst-case CVaR approximation dominates the Bonferroni approximation for Chebyshev ambiguity sets that contain all distributions with given first- and second-order moments (see Chen et al. 2010 and Zymler et al. 2013), we will show that the two approximations are generally incomparable for Wasserstein ambiguity sets. To this end, we provide two examples where either of the two approximations is strictly less conservative than the other one.

**Example 1.** Consider the following instance of the distributionally robust problem (2):

$$
\begin{align*}
\min \ x_1 \\
\text{s.t.} \ & \mathbb{P}[x_1 > \tilde{\xi}_1, \ x_2 > \tilde{\xi}_2] \geq 1 - \varepsilon \quad \forall \mathbb{P} \in \mathcal{F}(\theta) \\
& \underline{x}_1 \leq x_1 \leq \overline{x}_1, \ x_2 \geq 0.
\end{align*}
$$

(27)

Here, we assume that $0 < \underline{x}_1 \leq \overline{x}_1 < 1$ and that the true data-generating distribution $\mathbb{P}_0$ is a two-point distribution which satisfies $\mathbb{P}_0[(\tilde{\xi}_1, \tilde{\xi}_2) = (1, 0)] = p$ and $\mathbb{P}_0(\tilde{\xi}_1, \tilde{\xi}_2) = (0, 0)] = 1 - p$ for $p \in (0, 1)$. 
Proposition 13. Let \( p \in (\mathcal{F}, \varepsilon, \varepsilon) \). As \( N \to \infty \), with probability going to 1, we have for any vanishing sequence of Wasserstein radii \( \theta(N) \) that

(i) the Bonferroni approximation to problem (27) that replaces the joint chance constraint with

\[
P[\xi_1 > \xi_1, \xi_2 > \xi_2] \geq 1 - \varepsilon_1 \forall \theta \in \mathcal{F}(\theta), \quad P[\xi_1 > \xi_2] \geq 1 - \varepsilon_2 \forall \theta \in \mathcal{F}(\theta)
\]

becomes exact if the risk thresholds \((\varepsilon_1, \varepsilon_2)\) are sufficiently close to \((\varepsilon, 0)\);

(ii) the worst-case CVaR approximation to (27) that replaces the joint chance constraint with

\[
\sup_{\theta \in \mathcal{F}(\theta)} \mathbb{P} - \text{CVaR}(\max \{ w_1(\xi_1 - x_1), w_2(\xi_2 - x_2) \}) \leq 0
\]

becomes infeasible for any choice of scaling factors \((w_1, w_2)\).

Proof of Proposition 13. We proceed in three steps. We first derive the optimal value of the classical chance constrained program associated with (27) under the true data-generating distribution \( \mathbb{P}_0 \) (Step 1). This value serves as a lower bound on the optimal value of problem (27). We then show that with probability going to 1 as \( N \to \infty \), the Bonferroni approximation achieves this bound (Step 2), whereas the worst-case CVaR approximation becomes infeasible (Step 3).

Step 1. Since \( p < \varepsilon \), the feasible region of the classical chance constrained program

\[
\begin{align*}
\min & \quad x_1 \\
\text{s.t.} & \quad P_0[\xi_1 > \xi_1, \xi_2 > \xi_2] \geq 1 - \varepsilon \\
& \quad x_1 \leq x_1 \leq x_1, \quad x_2 \geq 0
\end{align*}
\]

under the true data-generating distribution \( \mathbb{P}_0 \) is \( \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [\bar{x}_1, \bar{x}_1], x_2 > 0\} \). Hence, the optimal value of this problem is \( x_1 > 0 \), which is attained by any \( (x_1, x_2) \in \{\bar{x}_1\} \times (\mathbb{R}_+ \backslash \{0\}) \).

Step 2. Fix any \((x_1, x_2) \in [\bar{x}_1, \bar{x}_1] \times (\mathbb{R}_+ \backslash \{0\})\), and denote by \( S_1(x) = \{\xi \mid x_1 > \xi_1\} \) and \( S_2(x) = \{\xi \mid x_2 > \xi_2\} \) the two safety sets of the Bonferroni approximation. If \( \hat{\xi}_i = (1, 0)^\top \), then \( \hat{\xi}_i \in S_1(x) \cap S_2(x) \) with \( \text{dist}(\hat{\xi}_i, S_1(x)) = 0 \) and \( \text{dist}(\hat{\xi}_i, S_2(x)) = x_2 \). Likewise, if \( \hat{\xi}_i = (0, 0)^\top \), then \( \hat{\xi}_i \in S_1(x) \cap S_2(x) \) with \( \text{dist}(\hat{\xi}_i, S_1(x)) = x_1 \) and \( \text{dist}(\hat{\xi}_i, S_2(x)) = x_2 \). Under the appropriate permutations \( \pi^1(x) \) and \( \pi^2(x) \), Theorem 2 then implies that \( x \) satisfies both chance constraints of the Bonferroni approximation if and only if

\[
\frac{1}{N} \sum_{i=1}^{\varepsilon_1 N} \text{dist}(\hat{\xi}_i, S_1(x)) = \frac{1}{N} (\varepsilon_1 N - 1)^+ x_1 \geq \theta(N) \quad \text{and} \quad \frac{1}{N} \sum_{i=1}^{\varepsilon_2 N} \text{dist}(\hat{\xi}_i, S_2(x)) = \varepsilon_2 x_2 \geq \theta(N),
\]

(28)

where \( I \) denotes the number of samples \( \hat{\xi}_i, i \in [N], \) that satisfy \( \hat{\xi}_i = (1, 0)^\top \).

Choose \( \varepsilon_1 \in (p, \varepsilon) \) and \( \varepsilon_2 = \varepsilon - \varepsilon_1 \), as well as \( x_1 = \bar{x}_1 \) and any \( x_2 \geq \theta(N)/\varepsilon_2 \). This choice of \((\varepsilon_1, \varepsilon_2)\) and \( x \) satisfies the second constraint in (28) by construction. To see that the first constraint in (28)
is also satisfied with high probability as $N \to \infty$, we note that $\frac{1}{N}(\varepsilon_1 N - I)^+ x_1 = (\varepsilon_1 - I/N)^+ x_1 \to (\varepsilon_1 - p)^+ x_1$ almost surely as $N \to \infty$ due to the strong law of large numbers. We thus conclude that $\frac{1}{N}(\varepsilon_1 N - I)^+ x_1 \geq 0$ with high probability as $N \to \infty$, and thus this quantity will exceed $\theta(N)$, which goes to zero as $N$ approaches infinity.

**Step 3.** Using similar arguments as in the proofs of Propositions 7 and 11, one can show that the worst-case CVaR approximation is satisfied for a fixed decision $(x_1, x_2)$ if and only if

$$
\max_{t \in \mathbb{R}} \left\{ \varepsilon N t - \sum_{i \in [N]} \left( t - \min \left\{ \frac{w_1(x_1 - \hat{\xi}_{i,1})}{\max\{w_1, w_2\}}, \frac{w_2(x_2 - \hat{\xi}_{i,2})}{\max\{w_1, w_2\}} \right\} \right) \right\} \geq \theta(N) N. \quad (29)
$$

Here, $\hat{\xi}_{i,1}$ ($\hat{\xi}_{i,2}$) refers to the first (second) component of the vector $\hat{\xi}_i$. The first-order optimality condition for non-smooth optimization then implies that the maximum on the left-hand side of this constraint is attained by

$$
t^* = \min \left\{ \frac{w_1(x_1 - \hat{\xi}_{\pi[x_i] + 1}(w), 1)}{\max\{w_1, w_2\}}, \frac{w_2(x_2 - \hat{\xi}_{\pi[x_i] + 1}(w), 2)}{\max\{w_1, w_2\}} \right\},
$$

where we make use of the permutation $\pi(x)$ that orders the data points $\hat{\xi}_i$, $i \in [N]$ in ascending order of the expressions

$$
\min \left\{ \frac{w_1(x_1 - \hat{\xi}_{i,1})}{\max\{w_1, w_2\}}, \frac{w_2(x_2 - \hat{\xi}_{i,2})}{\max\{w_1, w_2\}} \right\}, \quad i \in [N].
$$

This implies that the worst-case CVaR constraint (29) holds if and only if

$$
\sum_{i=1}^{\varepsilon N} \min \left\{ \frac{w_1(x_1 - \hat{\xi}_{\pi_i}(w), 1)}{\max\{w_1, w_2\}}, \frac{w_2(x_2 - \hat{\xi}_{\pi_i}(w), 2)}{\max\{w_1, w_2\}} \right\} \geq \theta(N) N. \quad (30)
$$

Note that $w_1 / \max\{w_1, w_2\} \leq 1$ in the first term inside the minimum. Hence, a necessary condition for the inequality (30) to hold for any scaling factors $(w_1, w_2)$ is that $\sum_{i=1}^{\varepsilon N} (x_1 - \hat{\xi}_{\pi_i}(w), 1) \geq \theta(N) N$; otherwise, the sum of the first terms inside the minima is smaller than $\theta(N) N$. Note that for any permutation $\pi(x)$, the strong law of large numbers implies that $\frac{1}{N} \sum_{i=1}^{\varepsilon N} \hat{\xi}_{\pi_i}(w), 1$ converges to a number smaller than or equal to $p$ almost surely as $N$ approaches infinity. Since $\frac{1}{N} \sum_{i=1}^{\varepsilon N} x_1 = \varepsilon x_1$, we thus conclude that $\frac{1}{N} \sum_{i=1}^{\varepsilon N} (x_1 - \hat{\xi}_{\pi_i}(w), 1)$ converges to a number not exceeding $\varepsilon x_1 - p$ almost surely as $N$ approaches infinity. Since $\pi_1 \varepsilon < p$ by assumption, this implies that the inequality (30) is violated for all $x_1 \in [\tilde{x}_1, \pi_1]$ with high probability as $N$ approaches infinity. \hfill \square

**Example 2.** Consider the following instance of the distributionally robust problem (2):

$$
\min_{x_3} x_3
$$

s.t. $\mathbb{P}[x_1 > \tilde{\xi}, x_2 > \tilde{\xi}] \geq 1 - \varepsilon \quad \forall \mathbb{P} \in \mathcal{F}(\theta) \quad (31)$

$$
x_3 \leq x_1, x_2, x_3 \leq 1, \quad x_3 \geq x_1, \quad x_3 \geq x_2.
$$

Here, we assume $\frac{1}{2} < \varepsilon < 1$ and that the true data-generating distribution $\mathbb{P}_0$ is a two-point distribution which satisfies $\mathbb{P}_0[\tilde{\xi} = 1] = p$ and $\mathbb{P}_0[\tilde{\xi} = 0] = 1 - p$ for $p \in (0, 1)$. 

Chen, Kuhn, and Wiesemann: *Data-Driven Chance Constrained Programs over Wasserstein Balls*
Proposition 14. Let $p \in (\varepsilon/2, 2\varepsilon]$. As $N \to \infty$, with probability going to 1, we have for any vanishing sequence of Wasserstein radii $\theta(N)$ that

(i) the worst-case CVaR approximation to (31) that replaces the joint chance constraint with

$$\sup_{p \in \mathcal{P}} \mathbb{P}^{-\text{CVaR}} (\max \{ w_1(\hat{\xi} - x_1), w_2(\hat{\xi} - x_2) \} \leq 0)$$

becomes exact if the scaling factors $(w_1, w_2)$ are $(\frac{1}{2}, \frac{1}{2})$;

(ii) the Bonferroni approximation to problem (31) that replaces the joint chance constraint with

$$\mathbb{P}[x_1 > \hat{\xi}] \geq 1 - \varepsilon_1 \forall \mathbb{P} \in \mathcal{F}(\theta), \quad \mathbb{P}[x_2 > \hat{\xi}] \geq 1 - \varepsilon_2 \forall \mathbb{P} \in \mathcal{F}(\theta)$$

becomes infeasible for any choice of the risk thresholds $(\varepsilon_1, \varepsilon_2)$.

Proof of Proposition 14. We proceed in three steps. We first derive the optimal value of the classical chance constrained program associated with (31) under the true data-generating distribution $\mathbb{P}_0$ (Step 1). This value serves as a lower bound on the optimal value of problem (31). We then show that with probability going to 1 as $N \to \infty$, the worst-case CVaR approximation achieves this bound (Step 2), whereas the Bonferroni approximation becomes infeasible (Step 3).

Step 1. Since $p < \varepsilon$, a similar argument as in the proof of Proposition 13 allows us to conclude that the optimal value of the classical chance constrained program under the true data-generating distribution $\mathbb{P}_0$ is $\bar{x}$, which is attained by the solution $(x_1, x_2, x_3) = (\bar{x}, \bar{x}, \bar{x})$.

Step 2. By Proposition 12, the solution $x = (x_1, x_2, x_3) = (\bar{x}, \bar{x}, \bar{x})$ is feasible in the worst-case CVaR approximation with scaling factors $(w_1, w_2) = (\frac{1}{2}, \frac{1}{2})$ if and only if

$$\frac{1}{N} \sum_{i=1}^{I} \min_{\pi(x)} \{ \min_{\mathcal{H}_1(x), \mathcal{H}_2(x)} (\xi_{\pi(i)}, \mathcal{H}_1(x), \mathcal{H}_2(x)) \} \geq \theta(N), \quad (32)$$

where $\mathcal{H}_1(x) = \mathcal{H}_2(x) = \{ \xi \mid \xi \geq \bar{x} \}$, and the permutation $\pi(x)$ orders the data points $\hat{\xi}_i$ so that $\hat{\xi}_1, \ldots, \hat{\xi}_I = 1$, $I \in [N] \cup \{0\}$, and $\hat{\xi}_{I+1}, \ldots, \hat{\xi}_N = 0$. Since $\min_{\mathcal{H}_1(x), \mathcal{H}_2(x)} (\xi_{\pi(i)}, \mathcal{H}_1(x), \mathcal{H}_2(x)) = \bar{x} - 1$ for $i = 1, \ldots, I$ and $\min_{\mathcal{H}_1(x), \mathcal{H}_2(x)} (\xi_{\pi(i)}, \mathcal{H}_1(x), \mathcal{H}_2(x)) = \bar{x}$ for $i = I + 1, \ldots, N$, (32) holds if and only if

$$\frac{1}{N} \left( \min \{ \epsilon N, I \} (\bar{x} - 1) + (\epsilon N - I)^+ \bar{x} \right) \geq \theta(N).$$

Note that $I/N \to p$ as $N \to \infty$ by the strong law of large numbers. Since $p < \varepsilon$ and $\theta(N) \to 0$ as $N \to \infty$, the above inequality is thus satisfied with probability approaching 1 as $N \to \infty$ as long as $p(\bar{x} - 1) + (\varepsilon - p)\bar{x} = \varepsilon \bar{x} - \bar{x}p$ is strictly positive. This is the case, however, since $p < \bar{x} \varepsilon$ by assumption.

Step 3. Observe that the Bonferroni approximation is infeasible if $\varepsilon_1 = I/N$ because the first individual chance constraint $\mathbb{P}[x_1 > \hat{\xi}] \geq 1 - \varepsilon_1 \forall \mathbb{P} \in \mathcal{F}(\theta)$ is already violated under the empirical distribution. For the same reason, the Bonferroni approximation is infeasible if $\varepsilon_2 = I/N$. We next
show that when $N \to \infty$, with probability approaching to 1, any pair of Bonferroni weights $(\varepsilon_1, \varepsilon_2)$ satisfying $\varepsilon_1 + \varepsilon_2 = \varepsilon$ also satisfies $\min\{\varepsilon_1, \varepsilon_2\} \leq I/N$, that is, at least one of the two individual chance constraints is violated. Indeed, we have $\min\{\varepsilon_1, \varepsilon_2\} \leq \varepsilon/2$ and $p > \varepsilon/2$ by assumption, and $I/N \to p$ as $N \to \infty$ by the strong law of large numbers. □

4. Numerical Experiments

We compare our exact reformulation of the ambiguous chance constrained program (2) with the bicriteria approximation scheme of Xie and Ahmed (2018a) on a portfolio optimization problem in Section 4.1 as well as with a classical (non-ambiguous) chance constrained formulation on a transportation problem in Section 4.2. Our goal is to investigate the computational scalability of our reformulation as well as its out-of-sample performance in a data-driven setting. All results were produced on an Intel Xeon 2.66GHz processor with 8GB memory in single-core mode using Gurobi 8.0 (for the mixed-integer conic programs in Section 4.1) and CPLEX 12.8 (for the mixed-integer linear programs in Section 4.2).

4.1. Portfolio Optimization

We consider a portfolio optimization problem studied by Xie and Ahmed (2018a). The problem asks for the minimum-cost portfolio investment $\mathbf{x}$ into $K$ assets with random returns $\tilde{\xi}_1, \ldots, \tilde{\xi}_K$ that exceeds a pre-specified target return $w$ with high probability $1 - \varepsilon$. The problem can be cast as the following instance of the ambiguous chance constrained program (2):

$$\begin{align*}
\min_{\mathbf{x}} & \quad \mathbf{c}^\top \mathbf{x} \\
\text{s.t.} & \quad \mathbb{P}[\tilde{\mathbf{\xi}}^\top \mathbf{x} > w] \geq 1 - \varepsilon & \forall \mathbb{P} \in \mathcal{F}(\theta) \\
& \quad \mathbf{x} \geq \mathbf{0}.
\end{align*}$$

(33)

We compare our exact reformulation of problem (33) with the $(\sigma, \gamma)$-bicriteria approximation scheme of Xie and Ahmed (2018a), which produces solutions that satisfy the ambiguous chance constraint in (33) with probability $1 - \sigma \varepsilon$, $\sigma > 1$, and whose costs are guaranteed to exceed the optimal costs in (33) by a factor of at most $\gamma = \sigma/(\sigma - 1)$. Since the bicriteria approximation scheme can readily utilize support information for the random vector $\tilde{\mathbf{\xi}}$, we replace the ambiguity set $\mathcal{F}(\theta)$ with $\tilde{\mathcal{F}}(\theta) = \mathcal{F}(\theta) \cap \{\mathbb{P} | \mathbb{P}[\tilde{\mathbf{\xi}} \in \mathbb{R}_+^K] = 1\}$ in their approach. Contrary to the experiments conducted by Xie and Ahmed (2018a), we set $\sigma = 1$. This is to the disadvantage of their approach, as it does not provide any approximation guarantees in that case, but it allows us to compare the resulting portfolios as they provide the same return guarantees. For the performance of the bicriteria approximation scheme with $\sigma > 1$, we refer to Section 6.2 of Xie and Ahmed (2018a).

In our numerical experiments, we consider the same setting as Xie and Ahmed (2018a). We set $K = 50$, $w = 1$ and choose the cost coefficients $c_1, \ldots, c_{50}$ uniformly at random from $\{1, \ldots, 100\}$. 

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Table 1  Objective and runtime ratios of the bicriteria approximation scheme for different values of $\varepsilon$ and $\theta$. For each parameter setting, we report the 5%, 50% and 95% quantiles over 50 randomly generated instances.

<table>
<thead>
<tr>
<th>$(\varepsilon, \theta)$</th>
<th>Ratio of objective values</th>
<th>Ratio of runtimes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5%</td>
<td>50%</td>
</tr>
<tr>
<td>(0.05, 0.05)</td>
<td>1.6</td>
<td>2.4</td>
</tr>
<tr>
<td>(0.05, 0.10)</td>
<td>1.9</td>
<td>2.9</td>
</tr>
<tr>
<td>(0.05, 0.20)</td>
<td>2.3</td>
<td>2.8</td>
</tr>
<tr>
<td>(0.10, 0.05)</td>
<td>1.0</td>
<td>1.1</td>
</tr>
<tr>
<td>(0.10, 0.10)</td>
<td>1.5</td>
<td>2.3</td>
</tr>
<tr>
<td>(0.10, 0.20)</td>
<td>2.1</td>
<td>2.7</td>
</tr>
</tbody>
</table>

Each asset return $\hat{\xi}_i$ is governed by a uniform distribution on $[0.8, 1.5]$, and we assume that $N = 100$ training samples $\hat{\xi}_1, \ldots, \hat{\xi}_{100}$ are available. We use the 2-norm Wasserstein ambiguity set, which implies that our exact reformulation of problem (33) is a mixed-integer second-order cone program, and set the Wasserstein radius to $\theta \in \{0.05, 0.1, 0.2\}$. The risk threshold is set to $\varepsilon \in \{0.05, 0.1\}$.

Table 1 compares the objective values and runtimes of our exact reformulation and the bicriteria approximation scheme for various combinations of the risk threshold $\varepsilon$ and Wasserstein radius $\theta$. The table shows that despite incorporating additional support information, the bicriteria approximation scheme determines solutions whose costs significantly exceed those of the solutions found
by our exact reformulation. Perhaps more surprisingly, the bicriteria approximation scheme is also computationally more expensive. As Figure 3 shows, however, this is an artifact of the small sample size $N$ employed in the experiments of Xie and Ahmed (2018a), and the bicriteria approximation scheme is faster than our exact reformulation for larger samples sizes.

4.2. Transportation

We consider a probabilistic transportation problem studied by Luedtke et al. (2010) and Yanagisawa and Osogami (2013). The problem asks for the cost-optimal distribution of a single good from a set of factories $f \in [F]$ to a set of distribution centers $d \in [D]$. Each factory $f \in [F]$ has an individual production capacity $m_f$, and each distribution center $d \in [D]$ faces a random aggregate customer demand $\tilde{\xi}_d$. The cost of shipping one unit of the good from factory $f$ to distribution center $d$ is denoted by $c_{fd}$. We aim to find a transportation plan that minimizes the shipping costs, respects the production capacity of each factory and satisfies the demand at each distribution center with high probability. The problem can be cast as the following instance of problem (2):

$$\begin{aligned}
\min \quad & c^\top x \\
\text{s.t.} \quad & P \left[ \sum_{f \in [F]} x_{fd} \geq \tilde{\xi}_d \quad \forall d \in [D] \right] \geq 1 - \varepsilon \quad \forall \theta \in \mathcal{F}(\theta) \\
& \sum_{d \in [D]} x_{fd} \leq m_f \quad \forall f \in [F] \\
& x \geq 0.
\end{aligned}$$

Here, $x_{fd}$ denotes the quantity shipped from factory $f \in [F]$ to distribution center $d \in [D]$. Problem (34) is an ambiguous joint chance constrained program with right-hand side uncertainty. Since each safety condition in (34) contains a single random variable with coefficient 1 on the right-hand side, our exact reformulation reduces to the same mixed-integer linear program for any norm $\|\cdot\|$.

In our first experiment, we investigate the scalability of the exact reformulation of problem (34) that is offered by Proposition 2. To this end, we generate random test instances with 5 factories and 10, 20, ..., 50 distribution centers that are located uniformly at random on the Euclidean plane $[0, 10]^2$. We identify the transportation costs $c_{fd}$ with the Euclidean distances between the factories and distribution centers. The demand vector $\tilde{\xi}$ is described by 50, 100 or 150 samples from a uniform distribution that is supported on $[0.8\mu, 1.2\mu]$, where the expected demand $\mu_d$ at distribution center $d \in [D]$ is picked uniformly at random from the interval $[0, 10]$. The capacity of each factory is chosen uniformly at random, and the capacities are subsequently scaled so that the factories can jointly produce up to 150% of the maximum cumulative demand. For each instance, we choose 10 ascending Wasserstein radii $\theta_1 < \ldots < \theta_{10}$ uniformly so that $\theta_1 = 0.001$ and $\theta_{10}$ is the smallest radius for which the corresponding instance of problem (34) becomes infeasible. We fix $\varepsilon = 0.1$. 

Table 2  Solution times in seconds for $N = 50$ training samples. ‘CC’ and ‘$\theta_i$’ refer to problem (35) and problem (34) with different Wasserstein radii, respectively. We present median results over 100 random instances.

Where the median solution time exceeds 3,600s, we report the median optimality gap in brackets.

Tables 2–4 and Figure 4 compare the runtimes of our ambiguous chance constrained program with those of the classical chance constrained formulation of problem (34),

$$\min_{x, y} c^T x$$

s.t. $\sum_{f \in [F]} x_{fd} + My_i \geq \hat{\xi}_{id} \quad \forall d \in [D], \; i \in [N]$

$$e^T y \leq [\varepsilon N]$$

$$\sum_{d \in [D]} x_{fd} \leq m_f \quad \forall f \in [F]$$

$x \geq 0, \; y \in \{0, 1\}^N$,

(35)

where $M$ is a sufficiently large positive constant. The results show that for the smallest Wasserstein radius $\theta_1 = 0.001$, the ambiguous chance constrained program (34) is—as expected—more difficult to solve than the corresponding classical chance constrained program (35). Interestingly, the ambiguous chance constrained program becomes considerably easier to solve than the classical chance constrained program for the larger Wasserstein radii $\theta_2, \ldots, \theta_{10}$. This surprising result is explained in Figure 5, which shows that the feasible region of the ambiguous chance constrained program tends to convexify as the Wasserstein radius $\theta$ increases. In fact, one can show that the set of vectors $q \in \{0, 1\}^N$ that are feasible in the deterministic reformulation of problem (34) shrinks monotonically with $\theta$. Since it is the presence of these binary vectors that causes the non-convexity of problem (34), one can expect the problem to become better behaved as $\theta$ increases.

So far, the number of training samples $N$ in our experiments has been rather small. Indeed, while both the classical chance constrained formulation (35) and the ambiguous chance constrained problem (34) scale quite gracefully with the numbers of factories and distribution centers, both formulations are severely impacted by any increase in the number of training samples $N$. Based on
<table>
<thead>
<tr>
<th># of distribution centers</th>
<th>CC</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>$\theta_6$</th>
<th>$\theta_7$</th>
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<tr>
<td>10</td>
<td>16.3</td>
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<td>1.5</td>
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<td></td>
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<tr>
<td>20</td>
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<td>2.5</td>
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<td>30</td>
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<tr>
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<td>[0.8%]</td>
<td>20.3</td>
<td>6.5</td>
<td>5.6</td>
<td>5.5</td>
<td>5.4</td>
<td>5.4</td>
<td>5.7</td>
<td>6.2</td>
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Table 3  Solution times for $N = 100$ training samples. The table has the same interpretation as Table 2.

<table>
<thead>
<tr>
<th># of distribution centers</th>
<th>CC</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>$\theta_6$</th>
<th>$\theta_7$</th>
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<tr>
<td>10</td>
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<td>[0.7%]</td>
<td>85.6</td>
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<td>43.0</td>
<td>52.0</td>
<td>77.0</td>
</tr>
<tr>
<td>20</td>
<td>874.2</td>
<td>[1.9%]</td>
<td>143.9</td>
<td>90.5</td>
<td>76.3</td>
<td>75.6</td>
<td>72.8</td>
<td>72.5</td>
<td>73.2</td>
<td>85.7</td>
<td>112.4</td>
</tr>
<tr>
<td>30</td>
<td>[0.1%]</td>
<td>[3.2%]</td>
<td>213.8</td>
<td>126.4</td>
<td>113.0</td>
<td>109.5</td>
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<td>165.1</td>
</tr>
<tr>
<td>40</td>
<td>[0.3%]</td>
<td>[3.7%]</td>
<td>286.8</td>
<td>168.2</td>
<td>154.2</td>
<td>149.1</td>
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<td>182.8</td>
<td>231.5</td>
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<tr>
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<td>[3.0%]</td>
<td>324.6</td>
<td>207.0</td>
<td>189.3</td>
<td>190.9</td>
<td>190.0</td>
<td>190.4</td>
<td>191.8</td>
<td>233.0</td>
<td>294.4</td>
</tr>
</tbody>
</table>

Table 4  Solution times for $N = 150$ training samples. The table has the same interpretation as Table 2.

Figure 4  Median solution times (below dashed lines) and optimality gaps (above dashed lines) for $D = 10$ and $N = 50$ (left), $D = 30$ and $N = 100$ (middle) and $D = 50$ and $N = 150$ (right).

Based on this insight, we now compare the out-of-sample performance of the problems (34) and (35) when the solution of both problems is prematurely terminated after 120 seconds (60 seconds branch-and-
For a transportation problem with $F = 1$ factory, $D = 2$ distribution centers and $N = 10$ training samples, the graphs visualize the feasible regions of the classical chance constrained formulation (35) (left) and the ambiguous chance constrained problem (34) for a small (middle) and a large (right) value of $\theta$.

bound and subsequently 60 seconds ‘solution polishing’). To this end, we generate random problem instances with 5 factories, 20 distribution centers and 100, 200, ..., 1,000 training samples. We compare the out-of-sample performance of the classical chance constrained program (35) with a risk threshold of $\varepsilon = 0.1$ with the out-of-sample performance of the ambiguous chance constrained program (34) with $\varepsilon = 0.1$ and 10 different values of $\theta$, the best of which is selected using a 7-fold cross-validation on the training dataset. The results are shown in Figure 6. The figure shows that for $N < 800$ training samples, the classical chance constrained formulation produces solutions whose out-of-sample performance severely violates the target risk threshold of $\varepsilon = 0.1$. In contrast, the ambiguous chance constrained formulation produces solutions whose out-of-sample performance is consistently close to the target threshold, at a very modest increase of transportation costs (typically less than 2%).

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Figure 6 The left graph shows the out-of-sample performance of the classical chance constrained formulation (35) (blue) and the ambiguous chance constrained problem (34) (red). The right graph visualizes the increase in transportation costs if we implement the solution to problem (34) instead of the one to problem (35). In all cases, the shaded regions cover the 5% to 95% quantiles of 100 randomly generated instances, whereas the solid lines describe the median statistics.

References


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A. Distance to a Union of Half-Spaces

The distance of a point \( \hat{\xi} \in \mathbb{R}^K \) to a closed set \( C \subseteq \mathbb{R}^K \) with respect to a norm \( \| \cdot \| \) is defined as

\[
\text{dist}(\hat{\xi}, C) = \min\{\|\xi - \hat{\xi}\| \mid \xi \in C\}.
\]

Note that the minimum is always attained. In the following, we derive a closed-form expression for the distance of a point to the union of finitely many closed halfspaces.

**Lemma 2.** Let \( \mathcal{H}_m = \{ \xi \in \mathbb{R}^K \mid a_m \geq b_m^\top \xi \} \) be a closed halfspace for each \( m \in [M] \). If \( C = \bigcup_{m \in [M]} \mathcal{H}_m \) denotes the union of all halfspaces, then the distance of a point \( \hat{\xi} \) to \( C \) is given by

\[
\text{dist}(\hat{\xi}, C) = \min_{m \in [M]} \left\{ \frac{(b_m^\top \hat{\xi} - a_m)^+}{\|b_m\|_*} \right\} = \left( \min_{m \in [M]} \left\{ \frac{b_m^\top \hat{\xi} - a_m}{\|b_m\|_*} \right\} \right)^+.
\]

**Proof.** We first prove the assertion for \( M = 1 \), in which case \( C = \mathcal{H}_1 \). We thus have

\[
\text{dist}(\hat{\xi}, C) = \min_{\zeta} \{ \zeta \mid \zeta \geq \|\xi - \hat{\xi}\|, \ a_1 \geq b_1^\top \xi \}
\]

\[
= \max_{u,v,w} \left\{ v^\top \hat{\xi} - wa_1 \mid u = 1, \ v = b_1 w, \ u \geq \|v\|_*, \ w \geq 0 \right\}
\]

\[
= \max_w \left\{ (b_1^\top \hat{\xi} - a_1)w \mid w \leq 1/\|b_1\|_*, \ w \geq 0 \right\}
\]

\[
= \frac{(b_1^\top \hat{\xi} - a_1)^+}{\|b_1\|_*},
\]

where the second equality follows from strong conic duality, which holds because the primal minimization problem is strictly feasible. Similarly, for \( M \geq 1 \) we find

\[
\text{dist}(\hat{\xi}, C) = \min_{m \in [M]} \text{dist}(\hat{\xi}, \mathcal{H}_m) = \min_{m \in [M]} \left\{ \frac{(b_m^\top \hat{\xi} - a_m)^+}{\|b_m\|_*} \right\} = \left( \min_{m \in [M]} \left\{ \frac{b_m^\top \hat{\xi} - a_m}{\|b_m\|_*} \right\} \right)^+,
\]

where the second equality follows from the first part of the proof. \( \square \)