SPIDER: Near-Optimal Non-Convex Optimization via Stochastic Path Integrated Differential Estimator

Cong Fang * Chris Junchi Li † Zhouchen Lin * Tong Zhang †
July 4, 2018

Abstract

In this paper, we propose a new technique named Stochastic Path-Integrated Differential Estimator (SPIDER), which can be used to track many deterministic quantities of interest with significantly reduced computational cost. Combining SPIDER with the method of normalized gradient descent, we propose two new algorithms, namely SPIDER-SFO and SPIDER-SSO, that solve non-convex stochastic optimization problems using stochastic gradients only. We provide sharp error-bound results on their convergence rates. Specially, we prove that the SPIDER-SFO and SPIDER-SSO algorithms achieve a record-breaking $\tilde{O}(\epsilon^{-3})$ gradient computation cost to find an $\epsilon$-approximate first-order and $(\epsilon, \tilde{O}(\epsilon^{0.5}))$-approximate second-order stationary point, respectively. In addition, we prove that SPIDER-SFO nearly matches the algorithmic lower bound for finding stationary point under the gradient Lipschitz assumption in the finite-sum setting.

1 Introduction

In this paper, we study the optimization problem

$$\min_{x \in \mathbb{R}^d} f(x) \equiv \mathbb{E}[F(x; \zeta)]$$

(1.1)

where the stochastic component $F(x; \zeta)$, indexed by some random vector $\zeta$, is smooth and possibly non-convex. Non-convex optimization problem of form (1.1) contains many large-scale statistical learning tasks and is gaining tremendous popularity due to its favorable computational and statistical efficiency (Bottou, 2010; Bubeck et al., 2015; Bottou et al., 2016). Typical examples of form (1.1) includes principal component analysis, estimation of graphical models, as well as training deep neural networks (Goodfellow et al., 2016). The expectation-minimization structure of stochastic optimization problem (1.1) allows us to perform iterative updates and minimize the objective using its stochastic gradient $\nabla F(x; \zeta)$ as an estimator of its deterministic counterpart.

*Peking University; email: fangcong@pku.edu.cn; zlin@pku.edu.cn
†Tencent AI Lab; email: junchi.li.duke@gmail.com; tongzhang@tongzhang-ml.org
A special case of central interest is when the stochastic vector $\zeta$ is finitely sampled. In such finite-sum (or offline) case, we denote each component function as $f_i(x)$ and (1.1) can be restated as

$$\minimize_{x \in \mathbb{R}^d} \quad f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$$

where $n$ is the number of individual functions. Another case is when $n$ is reasonably large or even infinite, running across of the whole dataset is exhaustive or impossible. We refer it as the online (or streaming) case. For simplicity of notations we will study the optimization problem of form (1.2) in both finite-sum and online cases till the rest of this paper.

One important task for non-convex optimization is to search for, given the precision accuracy $\epsilon > 0$, an $\epsilon$-approximate first-order stationary point $x \in \mathbb{R}^d$ or $\|\nabla f(x)\| \leq \epsilon$. In this paper, we aim to propose a new technique, called the Stochastic Path-Integrated Differential EstimatoR (SPIDER), which enables us to construct an estimator that tracks a deterministic quantity with significantly lower sampling costs. As the readers will see, the SPIDER technique further allows us to design an algorithm with a faster rate of convergence for non-convex problem (1.2), in which we utilizes the idea of Normalized Gradient Descent (NGD) (Nesterov, 2004; Hazan et al., 2015). NGD is a variant of Gradient Descent (GD) where the stepsize is picked to be inverse-proportional to the norm of the full gradient. Compared to GD, NGD exemplifies faster convergence, especially in the neighborhood of the stationary points (Levy, 2016). However, NGD has been less popular due to its requirement of accessing the full gradient and its norm at each update. In this paper, we estimate and track the gradient and its norm via the SPIDER technique and then hybrid it with NGD. Measured by gradient cost which is the total number of computation of stochastic gradients, our proposed SPIDER-SFO algorithm achieves a faster rate of convergence of $O(\min(\frac{n^{1/2}}{\epsilon^2}, \epsilon^{-3}))$, which outperforms the best-known results prior to this work in both finite-sum (Allen-Zhu & Hazan, 2016)(Reddi et al., 2016) and online cases (Lei et al., 2017) by a factor of $O(\min(\frac{n^{1/6}}{\epsilon}, \epsilon^{-0.333}))$.

For the task of finding stationary points for which we already achieved a faster convergence rate via our proposed SPIDER-SFO algorithm, a follow-up question to ask is: is our proposed SPIDER-SFO algorithm optimal for an appropriate class of smooth functions? In this paper, we provide an affirmative answer to this question, yes, at least in the finite-sum case. To be specific, inspired by a counterexample proposed by Carmon et al. (2017b) we are able to prove that the gradient cost upper bound of SPIDER-SFO algorithm matches the algorithmic lower bound, up to a constant factor of relevant parameters. To put it differently, the gradient cost of SPIDER-SFO cannot be further improved for finding stationary points for some particular non-convex functions.

Nevertheless, it has been shown that for machine learning methods such as deep learning, an approximate stationary point that has at least one negative Hessian direction, including saddle points and local maximizers, are often not sufficient and needs to be avoided or escaped from (Dauphin et al., 2014; Ge et al., 2015). Specifically, under the smoothness condition for $f(x)$ and an additional Hessian-Lipschitz condition for $\nabla^2 f(x)$, we aim to find an $\epsilon$-approximate second-order stationary point which is a point $x \in \mathbb{R}^d$ satisfying $\|\nabla f(x)\| \leq \epsilon$ and $\lambda_{\min}(\nabla^2 f(x)) \geq -O(\epsilon^{0.5})$ (Nesterov & Polyak, 2006). As a side result, we propose a variant of our SPIDER-SFO algorithm, named SPIDER-SSO (Algorithm 2) for finding an approximate local minimizer, based a
so-called *Negative-Curvature-Search* method. Under an additional Hessian-Lipschitz assumption, SPIDER-SSO achieves an \((\epsilon, \mathcal{O}(\epsilon^{0.5}))\)-approximate second-order stationary point at a gradient cost of \(\tilde{O}(\min(n^{1/2}e^{-2}, \epsilon^{-3}))\). In the online case, this indicates that our SPIDER-SFO algorithm improves upon the best-known gradient cost in the online case by a factor of \(\tilde{O}(\epsilon^{-0.25})\) (Allen-Zhu & Li, 2017). For the finite-sum case, the gradient cost of SPIDER is sharper than that of the state-of-the-art NEON+FastCubic/CDHS algorithm in Agarwal et al. (2017); Carmon et al. (2016) by a factor of \(\tilde{O}(\frac{1}{\sqrt{\epsilon}})\) when \(n \geq \epsilon^{-1}\). \(^1\)

**Related Works.**\(^2\) In the recent five years, there has been a surge of literatures in machine learning community that analyze the convergence property of non-convex optimization algorithms. We refer the readers to the monograph by Jain et al. (2017) and the references therein on recent general and model-specific convergence rate results on non-convex optimization.

For the general problem of finding approximate stationary points, under the smoothness condition of \(f(x)\), it is known that vanilla Gradient Descent (GD) and Stochastic Gradient Descent (SGD), for finite-sum and online cases separately, achieve an \(\epsilon\)-approximate stationary point with a gradient cost of \(\tilde{O}(\min(n\epsilon^{-2}, \epsilon^{-4}))\) (Nesterov, 2004; Ghadimi & Lan, 2013). Such convergence rate has been recently improved by the variance-reduction type of algorithms (Johnson & Zhang, 2013), namely the finite-sum Stochastic Variance-Reduced Gradient (SVRG) and online Stochastically Controlled Stochastic Gradient (SCSG), to the gradient cost of \(\tilde{O}(\min(n^{2/3}\epsilon^{-2}, \epsilon^{-3.33}))\) (Allen-Zhu & Hazan, 2016; Reddi et al., 2016; Lei et al., 2017; Carmon et al., 2017a).

Recently, many literature study the problem of how to avoid or escape saddle points and achieve an approximate local minimizer at a polynomial gradient cost (Ge et al., 2015; Jin et al., 2017a; Xu et al., 2017; Allen-Zhu & Li, 2017; Hazan et al., 2015; Levy, 2016; Allen-Zhu, 2017; Reddi et al., 2018; Tripuraneni et al., 2017; Jin et al., 2017b; Lee et al., 2016; Agarwal et al., 2017; Carmon et al., 2016; Paquette et al., 2018). Among them, the group of authors Ge et al. (2015); Jin et al. (2017a) proposed the noise-perturbed variants of Gradient Descent (PGD) and Stochastic Gradient Descent (SGD) that escapes from all saddle points and achieves an \(\epsilon\)-approximate second-order stationary point in gradient cost of \(\tilde{O}(\min(n\epsilon^{-2}, \text{poly}(d)\epsilon^{-4}))\) stochastic gradients. Levy (2016) proposed the noise-perturbed variant of NGD, which suggests that NGD escapes saddle points at a faster rate than GD. Allen-Zhu (2017) proposed the Natasha2 algorithm that uses Negative-Curvature-Search to avoids saddle points achieves such approximate local minimizer at a \(\tilde{O}(\epsilon^{-3.25})\) gradient cost. In late last year, two groups Xu et al. (2017); Allen-Zhu & Li (2017) proposed a generic saddle-point-escaping method called *NEON*, a Negative-Curvature-Search method using stochastic gradients. Using such *NEON* method, one can convert a series of stationary-points-finding algorithms whose update rules use stochastic gradients and Hessian-vector products (GD, SVRG, FastCubic/CDHS(Agarwal et al., 2017; Carmon et al., 2016), SGD, SCSG, Natasha2, etc.) to the ones using only stochastic gradients to find approximate local minimizers without increasing the gradient cost. \(^3\)

---

\(^1\) In the finite-sum case, when \(n \leq \epsilon^{-1}\) SPIDER-SFO has a slower rate of \(\tilde{O}(\epsilon^{-2.5})\) than the state-of-art \(\tilde{O}(n^{3/4}\epsilon^{-1.75})\) rate achieved by *NEON*+FastCubic/CDHS (Allen-Zhu & Li, 2017). *NEON*+FastCubic/CDHS has exploited appropriate acceleration techniques, which has *not* been considered for SPIDER.

\(^2\) Limited by space and our knowledge, we have listed all literatures that we believe are mostly related to this work, and it is very likely to be inexhaustive – any inputs on missing literatures are highly welcome.

\(^3\) Here the gradient cost also includes the number of stochastic Hessian-vector product accesses, which has similar
work, the best-known gradient cost for finding an \((\epsilon, O(\epsilon^{0.5}))\)-approximate second-order stationary point is \(\tilde{O}(\min(ne^{-1.5} + n^{3/4}e^{-1.75}, \epsilon^{-3.5}))\) achieved by FastCubic/CDHS (Agarwal et al., 2017) and \(\text{NEON} + \text{SVRG}\) (Allen-Zhu & Li, 2017) in the finite-sum case and \(\text{NEON} + \text{Natasha2}\) (Allen-Zhu, 2017) in the online case, separately.\(^4\) See also recent works by Reddi et al. (2018); Tripuraneni et al. (2017) for algorithms that achieve similar rates for finding approximate local minimizers.

As the current work is carried out in its final phase, the authors were notified that an idea of resemblance was earlier presented by Nguyen et al. (2017b) named the stochastic recursive gradient (Nguyen et al., 2017a,b). Nevertheless, our SPIDER-type of algorithms differ from theirs in many essential aspects. The original goal of Nguyen et al. (2017a,b) is to design a memory-saving variant of SAGA (Defazio et al., 2014) which seems to accommodate larger step size than SVRG (Johnson & Zhang, 2013). However, their error bounds in both convex and non-convex worlds do not improve over the classical results. In contrast, our idea of marrying the SPIDER technique with a stochastic version of NGD that uses sub-sampled stochastic gradients allows us to track the gradient at a significantly reduced gradient cost.

**Organization.** The rest of this paper is organized as followed. §2 briefs the main results of this paper and details a comparison with concurrent works. §3 presents the core idea of stochastic path-integrated differential estimator that can track certain quantities with much reduced computational costs. §4 provides the error-bound theorems for finding an approximate stationary point and local minimizer, and §5 concludes the paper with future directions. All the detailed proofs are deferred to the appendix in their order of appearance.

**Notation.** Throughout this paper, we treat the parameters \(L, \Delta, \sigma, \) and \(\rho\), to be specified in §2, as global constants. Let \(\| \cdot \|\) denote the Euclidean norm of a vector or spectral norm of a square matrix. Denote \(p_n = O(q_n)\) for a sequence of vectors \(p_n\) and positive scalars \(q_n\) if there is a global constant \(C\) such that \(\|p_n\| \leq Cq_n\), and \(p_n = \tilde{O}(q_n)\) such \(C\) hides a poly-logarithmic factor of the parameters. Denote \(p_n = \Omega(q_n)\) if there is a global constant \(C\) such that \(\|p_n\| \geq Cq_n\). Let \(\lambda_{\min}(A)\) denote the least eigenvalue of a real symmetric matrix \(A\). For fixed \(K \geq k \geq 0\), let \(x_k, \ldots, x_K\) denote the sequence \(\{x^k, \ldots, x^K\}\). Let \([n] = \{1, \ldots, n\}\) and \(S\) denote the cardinality of a multi-set \(S \subset [n]\) of samples (a generic set that allows elements of multiple instances). For simplicity, we further denote the averaged sub-sampled stochastic estimator \(A_S := (1/S) \sum_{i \in S} A_i\) and averaged sub-sampled gradient \(\nabla f_S := (1/S) \sum_{i \in S} \nabla f_i\). Other notations are explained at their first appearance.

## 2 Informal Main Result

This section briefs the main results of this paper. We first introduce the formal definition of approximate stationary point or local minimizer, as follows.

**Definition 1.** We call \(x \in \mathbb{R}^d\) an \(\epsilon\)-approximate first-order stationary point, or simply an approx-

---

\(^4\)Allen-Zhu (2017) also obtains a gradient cost of \(\tilde{O}(\epsilon^{-3.25})\) to achieve a (modified and weakened) \((\epsilon, O(\epsilon^{0.25}))\)-approximate second-order stationary point.
imate stationary point, if
\[ \|\nabla f(x)\| \leq \epsilon. \quad (2.1) \]

Also, call \( x \) an \((\epsilon, \delta)\)-approximate second-order stationary point, or simply an approximate local minimizer, if
\[ \|\nabla f(x)\| \leq \epsilon, \quad \lambda_{\text{min}}(\nabla^2 f(x)) \geq -\delta. \quad (2.2) \]

The definition of an \((\epsilon, \delta)\)-approximate second-order stationary point generalizes the classical version where \( \delta = \sqrt{\rho \epsilon} \), see e.g. Nesterov & Polyak (2006). For our purpose of analysis, we also pose the following assumption:

**Assumption 1.** We assume the following

(i) The \( \Delta := f(x^0) - f(x^*) < \infty \) where \( x^* = \arg\min_{x \in \mathbb{R}^d} f(x) \) is a global minimizer of \( f(x) \);

(ii) The component function \( f_i(x) \) has an averaged \( L \)-Lipschitz gradient, i.e. for all \( x, y \),
\[ \mathbb{E}\|\nabla f_i(x) - \nabla f_i(y)\|^2 \leq L^2\|x - y\|^2; \]

(iii) (For online case only) the stochastic gradient has a finite variance bounded by \( \sigma^2 < \infty \), i.e.
\[ \mathbb{E}\|\nabla f_i(x) - \nabla f(x)\|^2 \leq \sigma^2. \]

For the problem of finding an \((\epsilon, \delta)\)-approximate second-order stationary point, we pose in addition to Assumption 1 the following assumption:

**Assumption 2.** Each component function \( f_i(x) \) has \( L \)-Lipschitz continuous gradient and \( \rho \)-Lipschitz continuous Hessian, i.e. for all \( i, x, y \),
\[ \|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|, \]
and
\[ \|\nabla^2 f_i(x) - \nabla^2 f_i(y)\| \leq \rho\|x - y\|. \]

Assumptions 1 and 2 are standard for non-convex stochastic optimization (Agarwal et al., 2017; Carmon et al., 2017b; Jin et al., 2017a; Xu et al., 2017; Allen-Zhu & Li, 2017). Our main result is informally listed as follows:

(i) For the problem of finding an \( \epsilon \)-approximate first-order stationary point, under Assumption 1 our results indicate a gradient cost of \( \mathcal{O}(\min(\epsilon^{-3}, n^{1/2}\epsilon^{-2})) \) which supersedes the best-known convergence rate results for stochastic optimization problem (1.2) [Theorems 1 and 2]. Before this work, the best-known result is \( \mathcal{O} \left( \min(\epsilon^{-3.333}, n^{2/3}\epsilon^{-2}) \right) \), achieved by Allen-Zhu & Hazan (2016); Reddi et al. (2016) in the finite-sum case and Lei et al. (2017) in the online case,
<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Online</th>
<th>Finite-Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stationary Point</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GD / SGD</td>
<td>(Nesterov, 2004)</td>
<td>$\epsilon^{-4}$</td>
</tr>
<tr>
<td>SVRG / SCSG</td>
<td>(Allen-Zhu &amp; Hazan, 2016) (Reddi et al., 2016) (Lei et al., 2017)</td>
<td>$\epsilon^{-3.333}$ $n + n^{2/3}\epsilon^{-2}$</td>
</tr>
<tr>
<td>SPIDER-SFO (this work)</td>
<td>$\epsilon^{-3}$</td>
<td>$n + n^{1/2}\epsilon^{-2}$ $\Delta$</td>
</tr>
<tr>
<td><strong>Local Minimizer</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Perturbed GD / SGD</td>
<td>(Ge et al., 2015) (Jin et al., 2017a)</td>
<td>poly($d$)$\epsilon^{-4}$ $n\epsilon^{-2}$</td>
</tr>
<tr>
<td>NEON+GD / NEON+SGD</td>
<td>(Xu et al., 2017) (Allen-Zhu &amp; Li, 2017)</td>
<td>$\epsilon^{-4}$ $n\epsilon^{-2}$</td>
</tr>
<tr>
<td>AGD (Jin et al., 2017b)</td>
<td>N/A</td>
<td>$n\epsilon^{-1.75}$</td>
</tr>
<tr>
<td>NEON+SVRG / NEON+SCSG</td>
<td>(Allen-Zhu &amp; Hazan, 2016) (Reddi et al., 2016) (Lei et al., 2017)</td>
<td>$\epsilon^{-3.5}$ ($\epsilon^{-3.333}$) $n\epsilon^{-1.5} + n^{2/3}\epsilon^{-2}$</td>
</tr>
<tr>
<td>NEON+FastCubic/CDHS</td>
<td>(Agarwal et al., 2017) (Carmon et al., 2016) (Tripuraneni et al., 2017)</td>
<td>$\epsilon^{-3.5}$ $n\epsilon^{-1.5} + n^{3/4}\epsilon^{-1.75}$</td>
</tr>
<tr>
<td>NEON+Natasha2</td>
<td>(Allen-Zhu, 2017) (Xu et al., 2017) (Allen-Zhu &amp; Li, 2017)</td>
<td>$\epsilon^{-3.5}$ ($\epsilon^{-3.25}$) $n\epsilon^{-1.5} + n^{2/3}\epsilon^{-2}$</td>
</tr>
<tr>
<td>SPIDER-SSO (this work)</td>
<td>$\epsilon^{-3}$</td>
<td>$n^{1/2}\epsilon^{-2}$ $\Theta$</td>
</tr>
</tbody>
</table>

Table 1: Comparable results on the gradient cost for nonconvex optimization algorithms that uses only individual (or stochastic) gradients. Note that the gradient cost hides a poly-logarithmic factors of $d$, $n$, $\epsilon$. For clarity and brevity purposes, we record for most algorithms the gradient cost for finding an $(\epsilon, O(\epsilon^{0.5}))$-approximate second-order stationary point. For some algorithms we added in a bracket underneath the best gradient cost for finding an $(\epsilon, \epsilon^\alpha)$-approximate second-order stationary point among $\alpha \in (0, 1]$, for the fairness of comparison.

\(\Delta\): we provide lower bound for this gradient cost entry.

\(\Theta\): this entry is for $n \geq \Omega(\epsilon^{-1})$ only, in which case SPIDER-SSO outperforms NEON+FastCubic/CDHS.

We summarize the comparison with concurrent works that solve (1.2) under similar assumptions in Table 1. In addition, we provide Figure 1 which draws the gradient cost against the magnitude of $n$ for both an approximate stationary point and local minimizer.\(^5\) For simplicity, we leave out the

\(^5\) One of the results not included in this table is Carmon et al. (2017a), which finds an $\epsilon$-approximate stationary...
complexities of the algorithms that has Hessian-vector product access and only records algorithms that uses stochastic gradients only. Specifically, the yellow-boxed complexity $O(n_{\epsilon}^{-1.5} + n_{\epsilon}^{3/4} \epsilon^{-1.75})$ in Table 1, which was achieved by NEON+FastCubic/CDHS (Allen-Zhu & Li, 2017) for finding an approximate local minimizer in the finite-sum case, is the only result that has not been outperformed by our SPIDER-SSO algorithm in some parameter regime when $n \leq O(\epsilon^{-1}).$

3 Stochastic Path-Integrated Differential Estimator: Core Idea

In this section, we present in detail the underlying idea of our Stochastic Path-Integrated Differential Estimator (SPIDER) technique behind our algorithm design. As the readers will see, such technique significantly avoids excessive access of stochastic gradients and reduces the gradient cost, which is of independent interest and has potential applications in many stochastic estimation problems.

Let us consider an arbitrary deterministic vector quantity $Q(x)$. Assume that we observe a sequence $\hat{x}_{0:K}$, and we want to dynamically track $Q(\hat{x}^{k})$ for $k = 0, 1, \ldots, K$. Assume further that we have an initial estimate $\hat{Q}(\hat{x}^{0}) \approx Q(\hat{x}^{0})$, and an unbiased estimate $\xi_{k}(\hat{x}_{0:k})$ of $Q(\hat{x}^{k}) - Q(\hat{x}^{k-1})$ such that for each $k = 1, \ldots, K$

$$\mathbb{E} [\xi_{k}(\hat{x}_{0:k}) | \hat{x}_{0:k}] = Q(\hat{x}^{k}) - Q(\hat{x}^{k-1}).$$

point in $O(n_{\epsilon}^{-1.75})$ gradient evaluations. However, their result relies on a more stringent Hessian-Lipschitz condition, in which case a second-order stationary point can be found in similar gradient cost (Jin et al., 2017b).

6 Due to the NEON method (Xu et al., 2017; Allen-Zhu & Li, 2017), nearly all existing Hessian-vector product based algorithms in stochastic optimization can be converted to ones that uses stochastic gradients only.
Then we can integrate (in the discrete sense) the stochastic differential estimate as
\[
\tilde{Q}(\hat{x}_{0:K}) := \tilde{Q}(\hat{x}^0) + \sum_{k=1}^{K} \xi_k(\hat{x}_{0:k}).
\] (3.1)

We call estimator \(\tilde{Q}(\hat{x}_{0:K})\) the Stochastic Path-Integrated Differential EstimatoR, or SPIDER for brevity. We conclude the following proposition which bounds the second moment of the error of our estimator \(\|\tilde{Q}(\hat{x}_{0:K}) - Q(\hat{x}^K)\|\):

**Proposition 1.** We have
\[
E\|\tilde{Q}(\hat{x}_{0:K}) - Q(\hat{x}^K)\|^2 = E\|\tilde{Q}(\hat{x}^0) - Q(\hat{x}^0)\|^2 + \sum_{k=1}^{K} E\|\xi_k(\hat{x}_{0:k}) - (Q(\hat{x}^k) - Q(\hat{x}^{k-1}))\|^2.
\] (3.2)

Proposition 1 can be easily concluded using the property of square-integrable martingales. Now, let \(\mathcal{A}\) map any \(x \in \mathbb{R}^d\) to a random estimate \(\mathcal{A}_i(x)\) such that, conditioning on the observed sequence \(x_{0:k}\), we have for each \(k = 1, \ldots, K\),
\[
E\left[\mathcal{A}_i(x^k) - \mathcal{A}_i(x^{k-1}) \mid x_{0:k}\right] = \mathcal{V}^k - \mathcal{V}^{k-1}.
\] (3.3)

At each step \(k\) let \(S_s\) be a subset that samples \(S_s\) elements in \([n]\) with replacement, and let the stochastic estimator \(\mathcal{A}_{S_s} = (1/S_s) \sum_{i \in S_s} \mathcal{A}_i\) satisfy
\[
E\|\mathcal{A}_i(x) - \mathcal{A}_i(y)\|^2 \leq L^2\|x - y\|^2,
\] (3.4)

and \(\|x^k - x^{k-1}\| \leq \epsilon_1\) for all \(k = 1, \ldots, K\). Finally, we set our estimator \(\mathcal{V}^k\) of \(\mathcal{A}(x^k)\) as
\[
\mathcal{V}^k = \mathcal{A}_{S_s}(x^k) - \mathcal{A}_{S_s}(x^{k-1}) + \mathcal{V}^{k-1}.
\]

Applying Proposition 1 immediately concludes the following lemma, which gives an error bound of the estimator \(\mathcal{V}^k\) in terms of the second moment of \(\|\mathcal{V}^k - \mathcal{A}(x^k)\|\):

**Lemma 1.** We have under the condition (3.4) that for all \(k = 1, \ldots, K\),
\[
E\|\mathcal{V}^k - \mathcal{A}(x^k)\|^2 \leq \frac{kL^2\epsilon_1^2}{S_s} + E\|\mathcal{V}^0 - \mathcal{A}(x^0)\|^2.
\] (3.5)

In this paper, our proposed algorithms based on SPIDER takes \(\mathcal{A}_i\) as the stochastic gradient \(\nabla f_i\). In general, the stochastic mapping \(\mathcal{A}\) is not restricted to gradient mapping. It turns out that one can use SPIDER to track many quantities of interest, such as function values, zero-order estimate gradient, functionals of Hessian matrices, etc.
Algorithm 1 SPIDER-SFO: Input $x^0$, $q$, $S_1$, $S_2$, $\eta$ (For finding first-order stationary point)

1: for $k = 0$ to $K$ do
2:    if mod $(k, q) = 0$ then
3:        Draw $S_1$ samples (or compute the full gradient for the finite-sum case), let $v^k = \nabla f_{S_1}(x^k)$
4:    else
5:        Draw $S_2$ samples, and let $v^k = \nabla f_{S_2}(x^k) - \nabla f_{S_2}(x^{k-1}) + v^{k-1}$
6:    end if
7:    if $\|v^k\| \leq 2\epsilon$ then
8:        return $x^k$
9:    else
10:       $x^{k+1} = x^k - \eta \cdot (v^k/\|v^k\|)$
11:    end if
12: end for

4 SPIDER for Searching Stationary Points

In this section, we apply SPIDER to the task of Searching First-Order (SFO) stationary point for non-convex stochastic optimization. The main advantage for SPIDER-SFO lies on the marriage of our SPIDER technique to the idea motivated by Normalized Gradient Descent (NGD), which is crucial for us to establish a sharp bound on the gradient cost for generic non-convex functions. We propose the main error-bound theorems for finding approximate stationary points and local minimizers, separately in §4.1 and §4.2.

4.1 First-Order Stationary Point

Recall that NGD has iteration update rule

$$x^{k+1} = x^k - \eta \frac{\nabla f(x^k)}{\|\nabla f(x^k)\|}, \quad (4.1)$$

where $\eta$ is a constant step size. The NGD update rule (4.1) ensures $\|x^{k+1} - x^k\|$ being constantly equal to the stepsize $\eta$, and is known to fastly escape from saddle points and converges to local minimizers (Levy, 2016). Nevertheless, obtaining the full gradient at each step is either very costly (finite-sum case) or impossible (online case), and hence NGD is not scalable to large datasets. To conquer this problem, we propose SPIDER-SFO in Algorithm 1, which is a stochastic version of NGD (Algorithm 1) and we apply the SPIDER technique to maintain an estimator in each epoch $\nabla f(x^k)$ at a higher accuracy under limited gradient budgets.

To analyze the convergence rate of SPIDER-SFO, let us first consider the online case for Algorithm 1. We let the input parameters be

$$S_1 = \frac{2\sigma^2}{\epsilon^2}, \quad S_2 = \frac{2\sigma}{\epsilon n_0}, \quad \eta = \frac{\epsilon}{Ln_0}, \quad q = \frac{\sigma n_0}{\epsilon}, \quad (4.2)$$
where \( n_0 \in [1, 2\sigma/\epsilon] \) is a free parameter to choose. \(^7\) In this case, \( v^k \) in Line 5 of Algorithm 1 is a SPIDER for \( \nabla f(x^k) \). To see this, recall \( \nabla f_{i_k}(x^{k-1}) \) is the stochastic gradient drawn at step \( k \) and

\[
\mathbb{E} \left[ \nabla f_i(x^k) - \nabla f_{i_k}(x^{k-1}) \right] = \nabla f(x^k) - \nabla f(x^{k-1}). \tag{4.3}
\]

Plugging in \( v^k = v^k \) and \( A_i = \nabla f_i \) in Lemma 1 of §3, we can use \( v^k \) in Algorithm 1 as the SPIDER and conclude the following lemma that is pivotal to our analysis.

**Lemma 2.** Set the parameters \( S_1, S_2, \eta, \) and \( q \) as in (4.2), and \( k_0 = \lceil k/q \rceil \cdot q \). Then under the Assumption 1, we have

\[
\mathbb{E} \left[ \|v^k - \nabla f(x^k)\|^2 \mid x_{0:k_0} \right] \leq \epsilon^2.
\]

Here we compute the conditional expectation over the randomness of \( x_{(k_0+1):k} \).

Lemma 2 shows that SPIDER maintains an error of \( \|v^k - \nabla f(x^k)\| = \mathcal{O}(\epsilon) \). In other words, \( v^k \) is a sufficiently accurate estimator of \( \nabla f(x^k) \) whenever \( \|\nabla f(x^k)\| = \Omega(\epsilon) \). Using this lemma, we are ready to present the following theorem for Searching First-Order (SFO) stationary point of (1.2):

**Theorem 1** (First-Order Stationary Point, online setting). For the online case, set the parameters \( S_1, S_2, \eta, \) and \( q \) as in (4.2). Then under the Assumption 1, Algorithm 1 terminates in an average of \( 4L\Delta n_0/\epsilon^2 + 1 \) iterations and outputs an \( x^k \) satisfying

\[
\|v^k\| \leq 2\epsilon \quad \text{and} \quad \mathbb{E}\|\nabla f(x^k)\| \leq 3\epsilon.
\]

The expected gradient cost is bounded by \( 16L\Delta \sigma \cdot \epsilon^{-3} + 2\sigma^2 \epsilon^{-2} \) for any choice of \( n_0 \in [1, 2\sigma/\epsilon] \). Treating \( \Delta, L \) and \( \sigma \) as constants, the stochastic gradient complexity is \( \mathcal{O}(\epsilon^{-3}) \).

Let us compare the sampling efficiency among SGD, SCSG and SPIDER-SFO. Simple analysis indicates that to ensure convergence, the choice of mini-batch size for SGD should at least be \( \mathcal{O}(\sigma^2 L^{-1} \cdot n_0^{-1} \epsilon^{-2}) \). For SCSG (Lei et al., 2017) and Natasha2 (Allen-Zhu, 2017) the size should be \( \mathcal{O}(\sigma \cdot n_0^{-1} \epsilon^{-4/3}) \). In contrast, our SPIDER-SFO only needs a reduced mini-batch size of \( \mathcal{O}(\sigma \cdot n_0^{-1} \epsilon^{-1}) \), which is the key ingredient for its superior performance.

Turning to the finite-sum case, analogous to the online case we let

\[
S_2 = \frac{n^{1/2}}{n_0}, \quad \eta = \frac{\epsilon}{Ln_0}, \quad q = n_0 n^{1/2}, \tag{4.4}
\]

where \( n_0 \in [1, n^{1/2}] \). In this case, one computes the full gradient \( v^k = \nabla f_{S_1}(x^k) \) in Line 3 of Algorithm 1. We conclude our second upper-bound result:

**Theorem 2** (First-Order Stationary Point, finite-sum setting). In the finite-sum case, set the parameters \( S_2, \eta, \) and \( q \) as in (4.4), and let \( S_1 = [n] \), i.e. we obtain the full gradient in Line 3.

\(^7\) When \( n_0 = 1 \), the mini-batch size is \( 2\sigma/\epsilon \), which is the largest mini-batch size that Algorithm 1 is allowed to choose.
Then under Assumption 1, Algorithm 1 terminates in an average of $4L\Delta n_0/\epsilon^2 + 1$ iterations and outputs an $x^k$ satisfying

$$
\|v^k\| \leq 2\epsilon \quad \text{and} \quad \mathbb{E}\|\nabla f(x^k)\| \leq 3\epsilon.
$$

The expected gradient cost is bounded by $n + 8(L\Delta) \cdot n^{1/2} \epsilon^{-2}$ for any choice of $n_0 \in [1, n^{1/2}]$. Treating $\Delta, L$ and $\sigma$ as constants, the stochastic gradient complexity is $\mathcal{O}(n + n^{1/2}\epsilon^{-2})$.

**Lower Bound for Finding First-Order Stationary Points**

To conclude the optimality of our algorithm we need an algorithmic lower bound result (Carmon et al., 2017b; Woodworth & Srebro, 2016). Consider the finite-sum case and random algorithm $F$ that maps functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ to a sequence of iterates in $\mathbb{R}^d$, with

$$
x^k = F^{k-1}((\xi, \nabla f_{F_{0}}(x^0), \nabla f_{F_{1}}(x^1), \ldots, \nabla f_{F_{k-1}}(x^{k-1})), \quad (4.5)
$$

where $F^k$ are measure mapping into $\mathbb{R}^d$, $F_{ik}$ is the individual function chosen by $F$ at iteration $k$, and $\xi$ is a uniform random number in $[0, 1]$. The lower-bound result for solving (1.2) is stated as follows:

**Theorem 3** (Lower bound for SFO for the finite-sum setting). For any $L > 0, \Delta > 0$, and $2 \leq n \leq O(\Delta^2 L^2 \cdot \epsilon^{-4})$, for any algorithm $A$ satisfying (4.5), there exists a dimension $d = \tilde{O}(\Delta^2 L^2 \cdot n\epsilon^{-4})$, and a function $f$ satisfies Assumption 1 in the finite-sum case, such that in order to find a point $\hat{x}$ for which $\|\nabla f(\hat{x})\| \leq \epsilon$, $A$ must cost at least $\Omega(L\Delta \cdot n^{1/2}\epsilon^{-2})$ stochastic gradient accesses.

Note the condition $n \leq O(\epsilon^{-4})$ in Theorem 3 ensures that our lower bound $\Omega(n^{1/2}\epsilon^{-2}) = \Omega(n + n^{1/2}\epsilon^{-2})$, and hence our upper bound in Theorem 1 matches the lower bound in Theorem 3 up to a constant factor of relevant parameters, and is hence near-optimal. Inspired by Carmon et al. (2017b), our proof of Theorem 3 utilizes a specific counterexample function that requires at least $\Omega(n^{1/2}\epsilon^{-2})$ stochastic gradient accesses. Note Carmon et al. (2017b) only analyzed such counterexample in the deterministic case $n = 1$ and we generalize such analysis to the finite-sum case $n \geq 1$.

**Remark 1.** Note by setting $n = O(\epsilon^{-4})$ the lower bound complexity in Theorem 3 becomes $O(\epsilon^{-4})$. We emphasize that this does not violate the $O(\epsilon^{-3})$ upper bound in the online case (Theorem 1), since the counterexample established in the lower bound depends not on the stochastic gradient variance $\sigma^2$ specified in Assumption 1(iii), but on the component number $n$. To obtain the lower bound result for the online case with the additional Assumption 1(iii), with more efforts one might be able to construct a second counterexample that requires $\Omega(\epsilon^{-3})$ stochastic gradient accesses with the knowledge of $\sigma$ instead of $n$. We leave this as a future work.

**4.2 Second-Order Stationary Point**

To find a second-order stationary point with (2.1), we can fuse our SPIDER-SFO in Algorithm 1 with a Negative-Curvature-Search (NC-Search) iteration that solves the following task: given a point $x \in \mathbb{R}^d$, decide if $\lambda_{\min}(\nabla^2 f(x)) \geq -\delta$ or find a unit vector $w_1$ such that $w_1^T \nabla^2 f(x) w_1 \leq -\delta/2$.  

11
For the online case, NC-Search can be solved by Oja’s algorithm (Oja, 1982; Allen-Zhu, 2017) via Hessian-vector products and also first-order algorithm such as NEON (Allen-Zhu & Li, 2017; Xu et al., 2017) with the gradient cost of $\tilde{O}(\delta^{-2})$.

8 When $w_1$ is found, one can set $w_2 = \pm(\delta/\rho)w_1$ where $\pm$ is a random sign. Then under under Assumption 2, Taylor’s expansion implies that (Allen-Zhu & Li, 2017)

$$f(x + w_2) \leq f(x) + [\nabla f(x)]^T w_2 + \frac{1}{2} w_2^T [\nabla^2 f(x)] w_2 + \frac{\rho}{6} \|w_2\|^3. \quad (4.6)$$

Taking expectation, one has $\mathbb{E}f(x + w_2) \leq f(x) - \delta^3/(2\rho^2) + \delta^3/(6\rho^2) = f(x) - \delta^3/(3\rho^2)$. This indicates that when we find a direction of negative curvature or Hessian, updating $x \leftarrow x + w_2$ decreases the function value by $\Omega(\delta^3)$ in expectation. Our SPIDER-SFO algorithm fused with NC-Search is described in the following steps:

Step 1. Run SPIDER-SFO to find an $\epsilon$-approximate first-order stationary point;

Step 2. Run an efficient NC-Search iteration to find an $O(\delta)$-approximate negative Hessian direction $\pm w_1$ using stochastic gradients, e.g. NEON2 (Allen-Zhu & Li, 2017). If such direction $w_1$ cannot be found, STOP and return the current iterate;

Step 3. Else, we update $x \leftarrow x + (\delta/\rho)w_1$ in $\delta/(\rho\eta)$ mini-steps and goto Step 1 (without recomputing an approximate full-gradient as if the algorithm is restarted).

The formal pseudocode of the algorithm described above, which we refer to as SPIDER-SSO, is detailed in Algorithm 2. The core reason that SPIDER-SSO enjoys a highly competitive convergence rate is that, instead of performing a single large step $\delta/\rho$ at the approximate direction of negative curvature as in NEON2 (Allen-Zhu & Li, 2017), we split such one large step into $\delta/(\rho\eta)$ small, equal-length mini-steps in Step 3, where each mini-step moves the iteration by an $\eta$ distance. This allows the algorithm to successively maintain the SPIDER estimate of the current gradient in Step 3 and avoid re-computing the gradient in Step 1.

Our final results on the convergence rate of Algorithm 2 is stated as:

**Theorem 4** (Second-Order Stationary Point). Let Assumptions 1 and 2 hold. For the online case, under the same setting of Theorem 1, for any $p > 0$ there exists an event $\mathcal{H}_p$ with $\mathbb{P}(\mathcal{H}_p) \geq 1 - p$ such that on $\mathcal{H}_p$, Algorithm 2 outputs an $x^k$ satisfying

$$\mathbb{E}[\|\nabla f(x^k)\mid \mathcal{H}_p] \leq \epsilon \quad \text{and} \quad \lambda_{\min}(\nabla^2 f(x^k)) \geq -\delta, \quad (4.7)$$

with a gradient cost in expectation

$$\tilde{O}\left(\frac{\Delta L\sigma}{\epsilon^3} + \frac{\Delta \sigma L\rho}{\epsilon^2 \delta^2} + \frac{\Delta L^2 \rho^2}{\delta^5} + \frac{\sigma^2}{\epsilon^2} + \frac{L^2}{\delta^2}\right).$$

8Recall that the NEgative-curvature-Originated-from-Noise method (or NEON method for short) proposed independently by Allen-Zhu & Li (2017); Xu et al. (2017) is a generic procedure that convert an algorithm that finds approximate stationary points to the one that finds approximate local minimizers.
Algorithm 2 SPIDER-SSO: Input $x^0$, $q$, $S_1$, $S_2$, $\eta$ (For finding second-order stationary point)

1: for $k = 0$ to $K$ do
2: $j = 0$
3: if mod($k$, $q$) = 0 then
4: Draw $S_1$ samples, $v^k = \nabla f_{S_1}(x^k)$
5: else
6: Draw $S_2$ samples, $v^k = \nabla f_{S_2}(x^k) - \nabla f_{S_2}(x^{k-1}) + v^{k-1}$
7: if $j = 0$ then
8: if $\|v^k\| \leq 2\epsilon$ then
9: Run a parallel efficient NC-search iteration, e.g. NEON2, to find $w_1$ satisfying $\|w_1\| = 1$
10: and $w_1^\top \nabla^2 f(x)w_1 \leq -\delta/2$
11: if $w_1$ is found then
12: Randomly flip a sign, and set $w_2 = \pm \eta w_1$ and $j = \delta/(\rho \eta) - 1$
13: $x^{k+1} = x^k - w_2$
14: else
15: return $x^k$ ◦ found second-order stationary point
16: end if
17: else
18: $x^{k+1} = x^k - \eta \cdot (v^k/\|v^k\|)$
19: end if
20: else
21: $x^{k+1} = x^k - w_2; j = j - 1$
22: end if
23: end if

Analogously for the finite-sum case, under the same setting of Theorem 2, for any $p > 0$ there exists an event $H_p$ with $\mathbb{P}(H_p) \geq 1 - p$ such that on $H_p$, Algorithm 2 outputs an $x^k$ satisfying (4.7) with a gradient cost in expectation

$$\tilde{O} \left( \frac{\Delta L n^{1/2}}{\epsilon^2} + \frac{\Delta \rho L n^{1/2}}{\epsilon \delta^2} + \frac{\Delta^2 \rho^2}{\delta^3} + n + \frac{L^2}{\delta^2} \right).$$

In the gradient costs above, both $\tilde{O}(\cdot)$ terms hide a polylogarithmic factor of $p$.

**Corollary 5.** Treating $\Delta$, $L$, $\sigma$, and $\rho$ as constants, with high probability the gradient cost for finding an $(\epsilon, \delta)$-approximate second-order stationary point is $\tilde{O}(\epsilon^{-3} + \delta^{-2} \epsilon^{-2} + \delta^{-5})$ for the online case and $\tilde{O}(n^{1/2} \epsilon^{-2} + n^{1/2} \delta^{-2} \epsilon^{-1} + \delta^{-5})$ for the finite-sum case, respectively. When $\delta = O(\epsilon^{1/2})$, the gradient cost is $O(\min(n^{1/2} \epsilon^{-2}, \epsilon^{-3}))$ in the regime $n \geq \Omega(\epsilon^{-1})$.

Notice that one may directly apply an online variant of the NEON method to the SPIDER-SFO Algorithm 1. Simple analysis suggests that the NEON+ SPIDER-SFO algorithm achieves a gradient cost of $\tilde{O}(\epsilon^{-3} + \epsilon^{-2} \delta^{-3} + \delta^{-5})$ for the online case and $\tilde{O}(n^{1/2} \epsilon^{-2} + n \delta^{-3} + \delta^{-5})$ for the finite-sum case (Allen-Zhu & Li, 2017; Xu et al., 2017). We discuss the differences in detail.

- The dominate term in the gradient cost of NEON+ SPIDER-SFO is the so-called coupling
term in the regime of interest: $\epsilon^{-2}\delta^{-3}$ for the online case and $n^{1/2}\epsilon^{-1}\delta^{-3}$ for the finite-sum case, separately. Due to this term, most convergence rate results in concurrent works such as Agarwal et al. (2017); Tripuraneni et al. (2017); Xu et al. (2017); Allen-Zhu & Li (2017) has a gradient cost that cannot break through the $\mathcal{O}(\epsilon^{-3.5})$ barrier when $\delta$ is chosen to be $\mathcal{O}(\epsilon^{0.5})$. Observe that we always need to run SPIDER-SFO first before going into Step 2 of NC-Search, which at least costs $\mathcal{O}(\min(\epsilon^{-2}, n))$ stochastic gradient accesses (Carmon et al., 2017b).

- Our analysis sharpens the seemingly non-improvable coupling term by modifying the single large NEON step to many mini-steps. Such modification enables us to maintain the SPIDER estimator and obtain a coupling term $\mathcal{O}(\min(n, \epsilon^{-2})\delta^{-2})$ of SPIDER-SSO, which improves upon the NEON coupling term $\mathcal{O}(\min(n, \epsilon^{-2})\delta^{-3})$ by a factor of $\delta$.

- For the finite-sum case, SPIDER-SSO enjoys a convergence rate that is faster than existing methods only in the regime $n = \Omega(\epsilon^{-1})$ [Table 1]. For the case of $n = \mathcal{O}(\epsilon^{-1})$, using SPIDER to track the gradient in the NEON procedure can be more costly than applying appropriate acceleration techniques (Agarwal et al., 2017; Carmon et al., 2016).

Then one may simply use in different parameter regimes SPIDER-SSO or NEON+FastCubic/CDHS (Xu et al., 2017; Allen-Zhu & Li, 2017) and achieve a gradient cost of

$$\mathcal{O} \left( \min\left(n\epsilon^{-1.5}, \epsilon^{-1.75}, n^{1/2} \epsilon^{-2}, n^{1/2} \epsilon^{-1} \delta^{-2}\right) + \min\left(n + n^{3/4} \delta^{-0.5}, \delta^{-2}\right) \delta^{-3} \right).$$

At the time of submission of this paper, the authors of this paper are carrying out an ongoing project to design a unified algorithm to achieve the gradient cost upper bound above.

5 Future Directions

Despite the theoretical success of our SPIDER-type algorithms for non-convex stochastic optimization, several important questions are left:

(i) The lower bound results for finding stationary points and local minimizers are not complete. Specially, it is not yet clear if our $\mathcal{O}(\epsilon^{-3})$ for the online case and $\mathcal{O}(n^{1/2} \epsilon^{-2})$ for the finite-sum case gradient cost upper bound for finding a local minimizer (when $n \geq \Omega(\epsilon^{-1})$) is optimal or the gradient cost can be further improved, assuming both Lipschitz gradient and Lipschitz Hessian.

(ii) Empirical studies of our SPIDER-type algorithms that support the theoretical results are missing, which we leave for research in immediate future.

---

9 SPIDER-SSO enjoys a faster rate than NEON+SPIDER-SFO where computing the “full” gradient dominates the gradient cost, namely $\delta = \mathcal{O}(1)$ in the online case and $\delta = \mathcal{O}(n^{1/2} \epsilon)$ for the finite-sum case.
References


A Deferred Proofs in §3

In this and next section, we sometimes denotes for brevity that $\mathbb{E}_k[\cdot] = \mathbb{E}[\cdot | x_{0:k}]$, the expectation operator conditional on $x_{0:k}$, for an arbitrary $k \geq 0$.

A.1 Proof of Proposition 1

Proof of Proposition 1. It is straightforward to verify from the definition of $\tilde{Q}$ in (3.1) that

$$\tilde{Q}(\hat{x}_{0:K}) - Q(\hat{x}^0) = \tilde{Q}(\hat{x}^0) - Q(\hat{x}^0) + \sum_{k=1}^K \xi_k(\hat{x}_{0:k}) - (Q(x^k) - Q(x^{k-1}))$$

is a martingale, and hence (3.2) follows from the property of $L^2$ martingales (Durrett, 2010).

A.2 Proof of Lemma 1

Proof of Lemma 1. For any $k > 0$, we have from Proposition 1 (by applying $\tilde{Q} = \mathcal{V}$)

$$\mathbb{E}_k\|\mathcal{V}^k - \mathcal{A}(x^k)\|^2 = \mathbb{E}_k\|A_{S_*}(x^k) - \mathcal{A}(x^k) - A_{S_*}(x^{k-1}) + \mathcal{A}(x^{k-1})\|^2 + \|\mathcal{V}^{k-1} - \mathcal{A}(x^{k-1})\|^2. \quad (A.1)$$

Then

$$\mathbb{E}_k\|A_{S_*}(x^k) - \mathcal{A}(x^k) - A_{S_*}(x^{k-1}) + \mathcal{A}(x^{k-1})\|^2 \leq \frac{1}{S_*} \mathbb{E}\|A_i(x^k) - \mathcal{A}(x^k) - A_i(x^{k-1}) + \mathcal{A}(x^{k-1})\|^2 \quad (A.2)$$

where in $\overset{a}{=} \overset{b}{\leq}$ we use Eq (3.3), and $S_*$ are random sampled from $[n]$ with replacement. Combining (A.1) and (A.2), we have

$$\mathbb{E}_k\|\mathcal{V}^k - \mathcal{A}(x^k)\|^2 \leq \frac{L^2 \xi^2}{S_*} + \mathbb{E}\|\mathcal{V}^{k-1} - \mathcal{A}(x^{k-1})\|^2. \quad (A.3)$$

Telescoping the above display for $k' = k - 1, \ldots, 0$ and using the iterated law of expectation, we have

$$\mathbb{E}\|\mathcal{V}^k - \mathcal{A}(x^k)\|^2 \leq \frac{kL^2 \xi^2}{S_*} + \mathbb{E}\|\mathcal{V}^0 - \mathcal{A}(x^0)\|^2. \quad (A.4)$$
B Deferred Proofs in §4

B.1 Proof of Lemma 2

Proof of Lemma 2. For \( k = k_0 \), we have

\[
E_{k_0} \| v^{k_0} - \nabla f(x^{k_0}) \|^2 = E_{k_0} \| \nabla f_{S_1}(x^{k_0}) - \nabla f(x^{k_0}) \|^2 = \frac{\sigma^2}{S_1} = \frac{\epsilon^2}{2}.
\]  (B.1)

From Line 10 of Algorithm 1 we have for all \( k \geq 0 \),

\[
\| x^{k+1} - x^k \| = \eta = \frac{\epsilon}{L n_0}.
\]  (B.2)

Applying Lemma 1 with \( \epsilon_1 = \epsilon/(L n_0) \), \( S_1 = 2\sigma/(\epsilon n_0) \), \( K = k - k_0 \leq q = \sigma n_0/\epsilon \), we have

\[
E_{k_0} \| v^k - \nabla f(x^k) \|^2 \leq \frac{\sigma n_0 L^2}{\epsilon} \cdot \frac{\epsilon^2}{2} \cdot \frac{\epsilon n_0}{2\sigma} + E_{k_0} \| v^{k_0} - \nabla f(x^{k_0}) \|^2 = \epsilon^2,
\]  (B.3)

completing the proof.

B.2 Proof of Theorem 1

The rest of this section devotes to the proofs of Theorems 1, 2, 3 and 4. To prepare for them, we first conclude via standard analysis the following

Lemma 3. Setting \( k_0 = \lceil k/q \rceil \cdot q \), we have under Assumption 1 that if \( \| v^k \| \geq 2\epsilon \), then

\[
E_{k_0} \left[ f(x^{k+1}) - f(x^k) \right] \leq -\frac{\epsilon^2}{4 L n_0}.
\]  (B.4)

Proof of Lemma 3. Let \( \eta^k := \eta/\| v^k \|. \) Since \( f \) has \( L \)-Lipschitz continuous gradient [Assumption 1], we have

\[
f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \| x^{k+1} - x^k \|^2 \]  (B.5)

\[
= f(x^k) - \eta^k \langle \nabla f(x^k), v^k \rangle + \frac{L(\eta^k)^2}{2} \| v^k \|^2
\]

\[
= f(x^k) - \eta^k \left( 1 - \frac{\eta^k L}{2} \right) \| v^k \|^2 - \frac{\eta^k}{2} \langle \nabla f(x^k) - v^k, v^k \rangle
\]

\[
\leq f(x^k) - \frac{\eta^k}{2} \left( 1 - \frac{\eta^k L}{2} \right) \| v^k \|^2 + \eta^k \| v^k - \nabla f(x^k) \|^2,
\]  (B.6)
where in \( \leq \), we applied Cauchy-Schwarz inequality. Since \( \eta^k = \frac{\epsilon}{L\eta_0} \|v^k\| \leq \frac{1}{2L\eta_0} \leq \frac{1}{2\epsilon} \), we have

\[
\eta^k \left( \frac{1}{2} - \frac{\eta^k L}{2} \right) \|v^k\|^2 \geq \frac{1}{4} \cdot \frac{\epsilon}{L\eta_0} \|v^k\|^2 \|v^k\|^2 \geq \frac{\epsilon^2}{2L\eta_0}.
\]

(B.7)

Hence

\[
f(x^{k+1}) \leq f(x^k) - \frac{\epsilon^2}{2L\eta_0} + \frac{\eta^k}{2} \|v^k - \nabla f(x^k)\|^2.
\]

(B.8)

Taking expectation on the above display and using Lemma 2, and \( \eta^k \leq \frac{1}{2L\eta_0} \), we have

\[
\mathbb{E}_{k_0} f(x^{k+1}) - \mathbb{E}_{k_0} f(x^k) \leq -\frac{\epsilon^2}{4L\eta_0}.
\]

(B.9)

The proof is done via the following lemma:

\[\square\]

**Lemma 4.** Under Assumption 1, we have if \( \|v^k\| \leq 2\epsilon \), then

\[
\mathbb{E}\|v^k\| \leq 3\epsilon.
\]

(B.10)

**Proof.** By taking a total expectation in Lemma 2, we have

\[
\mathbb{E}\|v^k - \nabla f(x^k)\|^2 \leq \epsilon^2.
\]

(B.11)

Then by Jensen’s inequality

\[
\left( \mathbb{E}\|v^k - \nabla f(x^k)\| \right)^2 \leq \mathbb{E}\|v^k - \nabla f(x^k)\|^2 \leq \epsilon^2.
\]

So using triangle inequality

\[
\mathbb{E}\|\nabla f(x^k)\| = \mathbb{E}\|v^k - (v^k - \nabla f(x^k))\| \\
\leq \mathbb{E}\|v^k\| + \mathbb{E}\|v^k - \nabla f(x^k)\| \leq 3\epsilon.
\]

(B.12)

This completes our proof.

Now, we are ready to prove Theorem 1.

**Proof of Theorem 1.** If \( \|v^k\| \geq 2\epsilon \) in all \( 4L\Delta n_0/\epsilon^2 + 1 \) iterations, at each iteration the function value descends by at least \( \epsilon^2/(4L\eta_0) \) in expectation. We thus have

\[
-\Delta \leq f(x^*) - f(x^0) = \mathbb{E}f(x^*) - f(x^0) \leq \mathbb{E}f(x^K) - f(x^0) \leq -\left( \Delta + \frac{\epsilon^2}{4L\eta_0} \right).
\]

where \( K = 4L\Delta/\epsilon^2 + 1 \), contradicting the fact that \( -\Delta > -\left( \Delta + \frac{\epsilon^2}{4L\eta_0} \right) \).
To compute the gradient cost, note in each \( q \) iterations we access for one time \( S_1 \) stochastic gradients and for \( q \) times of \( S_2 \) stochastic gradients, and hence the cost is no greater than

\[
\left[ \frac{\Delta}{\epsilon^2/(4Ln_0)} \cdot \frac{1}{q} \right] S_1 + \frac{\Delta}{\epsilon^2/(4Ln_0)} S_2 \leq 2 \frac{\Delta}{\epsilon^2/(4Ln_0)} \cdot S_2 + S_1 \\
= 2 \left( \frac{4Ln_0 \Delta}{\epsilon^2} \right) \cdot \frac{2\sigma}{\epsilon n_0} + \frac{2\sigma^2}{\epsilon^2} \\
= \frac{16L\sigma \Delta}{\epsilon^3} + \frac{2\sigma^2}{\epsilon^2}. \tag{B.13}
\]

This concludes a gradient cost of \( 16L\Delta \sigma \epsilon^{-3} + 2\sigma^2 \epsilon^{-2} \).

\[ \square \]

### B.3 Proof of Theorem 2

**Proof of Theorem 2.** We can find that (B.2) is still hold. For Lemma 2, we have

\[
\mathbb{E}_{k_0} \| v_{k_0} - \nabla f(x_{k_0}) \|^2 = \mathbb{E}_{k_0} \| \nabla f(x_{k_0}) - \nabla f(x_{k_0}) \|^2 = 0. \tag{B.14}
\]

With the above display, applying Lemma 1 with \( \epsilon_1 = \frac{\epsilon}{L n_0} \), and \( S_1 = \frac{n_1 / 2}{\epsilon n_0} \), \( K = k - k_0 \leq q = n_0 n_1 / 2 \), we have

\[
\mathbb{E}_{k_0} \| v_{k_0} - \nabla f(x_{k_0}) \|^2 \leq n_0 n_1 / 2 L^2 \cdot \frac{\epsilon^2}{L^2 n_0^2} \cdot \frac{\epsilon n_0}{n_1 / 2} + \mathbb{E}_{k_0} \| v_{k_0} - \nabla f(x_{k_0}) \|^2 \tag{B.1} = \epsilon^2. \tag{B.15}
\]

The gradient cost analysis is similar to the online case and is bounded by

\[
\left[ \frac{\Delta}{\epsilon^2/(4Ln_0)} \cdot \frac{1}{q} \right] S_1 + \frac{\Delta}{\epsilon^2/(4Ln_0)} S_2 \leq 2 \frac{\Delta}{\epsilon^2/(4Ln_0)} \cdot S_2 + S_1 \\
= 2 \left( \frac{4Ln_0 \Delta}{\epsilon^2} \right) \frac{n_1 / 2}{n_0} + n \\
= \frac{8(L\Delta) \cdot n_1 / 2}{\epsilon^2} + n. \tag{B.16}
\]

This concludes a gradient cost of \( n + 8(L\Delta) \cdot n_1 / 2 \epsilon^{-2} \) which is independent of \( n_0 \).

\[ \square \]

### B.4 Proof of Theorem 3

Our proof is a direct extension of Carmon et al. (2017b). Before we drill into the proof of Theorem 3, we first introduce the hard instance \( \tilde{f}_K \) with \( K \geq 1 \) constructed by Carmon et al. (2017b).

\[
\tilde{f}_K(x) := -\Psi(1)\Phi(x_1) + \sum_{i=2}^{K} \left[ \Psi(-x_{i-1})\Phi(-x_i) - \Psi(x_{i-1})\Phi(x_i) \right], \tag{B.17}
\]

21
where the component functions are

\[ \Psi(x) := \begin{cases} 
0 & x \leq \frac{1}{2} \\
\exp\left(1 - \frac{1}{(2x-1)^2}\right) & x > \frac{1}{2} 
\end{cases} \]  

(B.18)

and

\[ \Phi(x) := \sqrt{e} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \] 

(B.19)

where \( x_i \) denote the \( i \)-th coordinate of \( x \), with \( i \in [d] \). \( \hat{f}_K(x) \) constructed by Carmon et al. (2017b) is a zero-chain function, that is for every \( i \in [d] \), \( \nabla_i \hat{f}(x) = 0 \) whenever \( x_{i-1} = x_i = x_{i+1} \). So any deterministic algorithm can only recover “one” dimension in each iteration (Carmon et al., 2017b).

In addition, it satisfies that: If \( |x_i| \leq 1 \) for any \( i \leq K \),

\[ \| \nabla \hat{f}_K(x) \| \geq 1. \]  

(B.20)

Then to handle random algorithms, Carmon et al. (2017b) further consider the following extensions:

\[ \tilde{f}_{K,B}(x) = \hat{f}_K \left( (B^K)^T \rho(x) \right) + \frac{1}{10} \| x \|^2 = \hat{f}_K \left( \langle b^{(1)}, \rho(x) \rangle, \ldots, \langle b^{(K)}, \rho(x) \rangle \right) + \frac{1}{10} \| x \|^2, \]  

(B.21)

where \( \rho(x) = \frac{x}{\sqrt{1 + \| x \|^2 / R^2}} \) and \( R = 230\sqrt{K} \). \( B^K \) is chosen uniformly at random from the space of orthogonal matrices \( O(d, K) = \{ D \in \mathbb{R}^{d \times K} | D^\top D = I_K \} \). The function \( \tilde{f}_{K,B}(x) \) satisfies the following:

(i)

\[ \tilde{f}_{K,B}(x) - \inf_x \tilde{f}_{K,B}(x) \leq 12K. \]  

(B.22)

(ii) \( \tilde{f}_{K,B}(x) \) has \( l \) (independent of \( K \) and \( d \)) Lipschitz continuous gradient.

(iii) if \( d \geq 52 \cdot 230^2 K^2 \log(\frac{2K^2}{\delta}) \), for any algorithm \( A \) solving (1.2) with \( n = 1 \), and \( f(x) = \tilde{f}_{K,B}(x) \), then with probability \( 1 - \delta \),

\[ \| \nabla \tilde{f}_{K,B}(x^k) \| \geq \frac{1}{2}, \]  

for every \( k \leq K \).  

(B.23)

The above properties found by Carmon et al. (2017b) is very technical. One can refer to Carmon et al. (2017b) for more details.

**Proof of Theorem 3.** Our lower bound theorem proof is as follows. The proof mirrors Theorem 2
in Carmon et al. (2017b) by further taking the number of individual function \( n \) into account. Set
\[
 f_i(x) := \frac{\ln^{1/2} \epsilon^2}{L} \tilde{f}_{K,B_i^K}(C_i^T x/b) = \frac{\ln^{1/2} \epsilon^2}{L} \left( \tilde{f}_K \left( (B_i^K)^T \rho(C_i^T x/b) \right) + \frac{1}{10} \|C_i^T x/b\|^2 \right),
\]
and
\[
 f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x).
\]

where \( B_i^{nK} = [B_1^K, \ldots, B_n^K] \) is chosen uniformly at random from the space of orthogonal matrices \( O(d, K) = \{ D \in \mathbb{R}^{(d/n) \times nK} | D^T D = I_{nK} \} \), with each \( B_i^K \in \{ D \in \mathbb{R}^{(d/n) \times K} | D^T D = I_K \} \), \( i \in [n] \), \( C_i = [C_{ij}] \) is chosen uniformly at random from the space of orthogonal matrices \( O(d, K) = \{ D \in \mathbb{R}^{d \times d} | D^T D = I_d \} \), with each \( C_i^K \in \{ D \in \mathbb{R}^{d \times (d/n)} | D^T D = I_{(d/n)} \} \), \( i \in [n] \).

\( K = \frac{\Delta L}{12 \ln^{1/2} \epsilon^2} \), with \( n \leq \frac{14 \Delta^2 L^2}{K (K - 1)} \) and \( b = \frac{\ell}{L} \). We first verify that \( f(x) \) satisfies Assumption 1. For Assumption 1i, from (B.22), we have
\[
 f(x) - \min_x f(x) \leq \frac{1}{n} \sum_{i=1}^n (f_i(x) - \min_x f_i(x)) \leq \frac{l_2 \Delta^2 L}{12 \ln^{1/2} \epsilon^2} = \frac{12 \Delta L}{L} = \Delta.
\]

For 1ii, for any \( i \), using the \( \tilde{f}_{K,B_i^K} \) has l-Lipschitz continuous gradient, we have
\[
 \left\| \nabla \tilde{f}_{K,B_i^K}(C_i^T x/b) - \nabla \tilde{f}_{K,B_i^K}(C_i^T y/b) \right\|^2 \leq l^2 \left\| C_i^T (x - y)/b \right\|^2,
\]
Because \( \| \nabla f_i(x) - \nabla f_i(y) \|^2 = \left\| \frac{\ln^{1/2} \epsilon^2}{L} \right\| \nabla \tilde{f}_{K,B_i^K}(C_i^T x/b) - \nabla \tilde{f}_{K,B_i^K}(C_i^T y/b) \right\|^2 \), and using \( C_i^T C_i = I_{d/n} \), we have
\[
 \left\| \nabla f_i(x) - \nabla f_i(y) \right\|^2 \leq \left( \frac{\ln^{1/2} \epsilon^2}{L} \right)^2 \frac{l^2}{b^2} \left\| C_i^T (x - y) \right\|^2 = nL^2 \left\| C_i^T (x - y) \right\|^2,
\]
where we use \( b = \frac{\ell}{L} \). Summing \( i = 1, \ldots, n \) and using each \( C_i \) are orthogonal matrix, we have
\[
 \mathbb{E} \| \nabla f_i(x) - \nabla f_i(y) \|^2 \leq L^2 \| x - y \|^2.
\]
Set \( d/n \geq 52nK(R)^2 \log \frac{2(nK)^2}{\delta} \), with \( R = 230 \sqrt{K} \), we have
\[
 d \geq 52nK(R)^2 \log \frac{2(nK)^2}{\delta} \sim \mathcal{O}\left( \frac{n \Delta^2 L^2}{\epsilon^4} \log \left( \frac{n \Delta^2 L^2}{\epsilon^4 \delta} \right) \right),
\]
from Lemma 2 of Carmon et al. (2017b) (or similarly Lemma 7 of Woodworth & Srebro (2016)), with probability at least \( 1 - \delta \), after \( T \) iterations, if \( k_i \leq K \), for any \( j_i \in \{k_i, \ldots, K\} \), we have
\[
 \langle B_{ij_i}^K, \rho(C_{ij} x^T/b) \rangle \leq \frac{1}{2},
\]
where \( k_i \) denotes that \( x^T \) has recovered \( k_i \) coordinates for \( f_i \) in iteration \( T \), \( B_{ij_i}^K \) denotes the \( ji \)-th column of \( B_i^K \) and \( \sum_{i=1}^n k_i = T \). So \( f_i \) can be solved only after \( K \) times
calling it. Otherwise, from (B.23), for any \( x^T \) generated from \( A \), we have \( \| \nabla \tilde{f}_{K,B_i^K}(C_i^T x^T / b) \| \geq \frac{1}{2} \).

From the above result, for any algorithm \( F \), after running \( T = \frac{nK}{2} = \frac{\Delta L n^{1/2}}{2\Delta e^2} \) iterations, at least \( \frac{n}{2} \) functions cannot be solved (the worst case is when \( F \) exactly solves \( \frac{n}{2} \) functions), so

\[
\left\| \nabla f(x^{nK/2}) \right\|^2 = \frac{1}{n^2} \sum_{i \text{ not solved}} \frac{ln^{1/2} \epsilon^2}{Lb} \| C_i \nabla \tilde{f}_{K,B_i^K}(C_i^T x^{nK/2} / b) \|^2 \geq \frac{1}{8}, \tag{B.29}
\]

where in \( a \), we use \( C_i^T C_j = 0_{d/n} \) when \( i \neq j \), and \( C_i^T C_i = I_{d/n} \). \( \square \)

### B.5 Proof of Theorem 4

To prove Theorem 4, we first verify that \( v^k \) in Algorithm 2 is analogously a SPIDER estimator of the true gradient \( \nabla f(x^k) \) with an \( \epsilon^2 \) error in second moments.

**Lemma 5.** Under the same setting of Theorem 4, for any \( k \geq 0 \) we have by setting \( k_0 = \lfloor k/q \rfloor \cdot q \) in Algorithm 2 that \( \mathbb{E}_{k_0} \| v^k - \nabla f(x^k) \|^2 \leq \epsilon^2 \).

**Proof.** \( x^k \) is updated only in Line 12, 17, 20. In either of these updates, we have

\[
\| x^{k+1} - x^k \| = \eta = \frac{\epsilon}{Ln_0}. \tag{B.30}
\]

Using the same proof method as in Lemma 2 (which relies heavily on Lemma 1) we immediately obtain the result. \( \square \)

**Proof of Theorem 4.** We conclude from Lemma 5 that, when performing Line 17 of Algorithm 2 the one-step update has

\[
\mathbb{E}[f(x_1) - f(x_0)] \leq -\frac{\epsilon^2}{4Ln_0}.
\]

In addition, performing the negative curvature updates in Line 20 of Algorithm 2 achieves a function increment of, in \( \delta/(\rho \eta) = \delta/(\rho \frac{\epsilon}{Ln_0}) \) iterations (taking expectation in (4.6)),

\[
\mathbb{E}[f(x_{\delta/(\rho \eta)}) - f(x_0)] \leq -\frac{\delta^3}{2\rho^2} + \frac{\delta^3}{6\rho^2} = -\frac{\delta^3}{3\rho^2}.
\]

Therefore on average the function increment at each iteration is upper bounded by

\[- \min \left( \frac{\epsilon^2}{4Ln_0}, \frac{\delta^3}{3\rho^2}, \frac{\rho \frac{\epsilon}{Ln_0}}{3L_0} \right) = - \min \left( \frac{\epsilon^2}{4Ln_0}, \frac{\epsilon(\delta^2/\rho)}{3L_0} \right). \]

Thus the total number of iteration is bound by:

\[
\frac{\Delta}{\min} \leq \Delta \cdot \frac{4Ln_0}{\epsilon^2} + \Delta \cdot \frac{3Ln_0}{\epsilon(\delta^2/\rho)} \tag{B.31}
\]

24
One can observe that the total stochastic gradient complexity consists of two parts: the SPIDER maintenance cost and NC-Search cost. We carefully estimate them as follows.

(i) We first analyze online case. From (B.31), the number of stochastic gradient accesses in the SPIDER maintenance is bounded by

\[
\left\lceil \frac{\Delta}{\min \cdot q} \right\rceil S_1 + \frac{\Delta}{\min} s_{1=q}S_2 \leq 2 \frac{\Delta}{\min} S_2 + S_1
\]

\[
\leq 2 \left( \frac{4Ln_0}{e^2} + \frac{3\Delta Ln_0}{e(\delta^2/\rho)} \right) \frac{2\sigma}{en_0} + \frac{2\sigma^2}{e^2}
\]

\[
= \frac{16L\sigma \Delta}{e^3} + \frac{12L\sigma \rho \Delta}{\delta^2 e^2} + \frac{2\sigma^2}{e^2}.
\]

(B.32)

The NEON2 (Allen-Zhu & Li, 2017) step has a complexity of \( C \cdot L^2 \delta^{-2} \) with probability \( 1 - p \), where \( C \) hides a polylog factor of relevant parameters and \( p \) (Allen-Zhu & Li, 2017). So the total number of NC-search is bounded by \( \left\lceil \Delta \cdot \frac{3\rho^2}{\delta^2} \right\rceil \), and thus the stochastic gradient access for NC-Search is less than

\[
\left\lceil \Delta \cdot \frac{3\rho^2}{\delta^2} \right\rceil \cdot C \cdot L^2 \delta^{-2} \leq C \cdot \frac{3\Delta \rho^2 L^2}{\delta^5} + C \cdot \frac{L^2}{\delta^2}.
\]

In total, stochastic gradient access for online case is bounded by

\[
\frac{16L\sigma \Delta}{e^3} + \frac{12L\sigma \rho \Delta}{\delta^2 e^2} + C \cdot \frac{3\Delta \rho^2 L^2}{\delta^5} + \frac{2\sigma^2}{e^2} + C \cdot \frac{L^2}{\delta^2}.
\]

(ii) For the offline case, similarly, gradient costs taken by SPIDER maintenance satisfies:

\[
\left\lceil \frac{\Delta}{\min \cdot q} \right\rceil S_1 + \frac{\Delta}{\min} s_{1=q}S_2 \leq 2 \frac{\Delta}{\min} S_2 + S_1
\]

\[
\leq 2 \left( \frac{4Ln_0}{e^2} + \frac{3\Delta Ln_0}{e(\delta^2/\rho)} \right) \frac{n^{1/2}}{n_0} + n
\]

\[
= \frac{8L\Delta n^{1/2}}{e^2} + \frac{6L\rho \Delta n^{1/2}}{\delta^2 e^2} + n.
\]

(B.33)

The NEON2 step analysis is exactly the same as in the online case. Therefore, the total stochastic gradient accesses for offline case is bounded by

\[
\frac{8L\Delta n^{1/2}}{e^2} + \frac{6L\rho \Delta n^{1/2}}{\delta^2 e^2} + C \cdot \frac{3\Delta \rho^2 L^2}{\delta^5} + n + C \cdot \frac{L^2}{\delta^2}.
\]

(i) and (ii) complete the proof of the whole theorem.

\[\square\]