A Branch-and-Cut Algorithm for Two-Echelon Inventory Routing Problems

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Abstract

Research Works identified in the literature address the classic version of the Inventory Routing Problem where a supplier manages the stock level of his customers and defines when and how much product to supply and how to combine customers in routes while minimizing storage and transportation costs. We present a new version of the problem which considers a two-echelon system with indirect deliveries. In this variant, the products are delivered to customers through distribution centers to meet their deterministic demands with minimum total cost. We consider order costs, fixed costs and unit transportation costs. We propose a mathematical formulation and a branch-and-cut algorithm to solve several classes of Two-Echelon Inventory Routing Problems with different inventory policies and route configurations. New valid inequalities for a two-echelon version are introduced. We analyze and discuss the impact of the new, as well as the already known valid inequalities and the efficiency of the proposed algorithm. Computational experiments are presented for a set of randomly generated instances. The results show that, for the simplest variant of the problem, the proposed method is able to solve to optimality up to medium instances.

Keywords: Distribution, Inventory Routing, Two-Echelon, Branch-and-Cut, Valid Inequalities

1. Introduction

Large companies tend to manage their activities in a coordinated way. Accordingly, the integrated supply chain has proven to have a major impact on overall success in terms of cost, quality and service time. The vast majority of a company’s activities are interconnected and impact the operation of the others that belong to the same supply chain. In this context, two logistics operations are often referred in the literature as keys to achieving an efficient

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supply chain: inventory management and transportation. The coordination of these two operations is often known as the Inventory Routing Problem (IRP) (see [1]). The IRP aims at determining simultaneously the delivery quantities and the routes in order to minimize the inventory holding and transportation costs.

More complex systems have been studied to take into account current configurations such as urban distribution. In most cities, urban freight accounts for a large part of traffic, occupation of roads and pollution. Decentralized models of local distribution tend to replace centralized models in order to ensure a sustainable and efficient distribution. According to [2], city logistic appears to be one of the principal keys for managing the urban distribution, supporting economic, environmental and social aspects. Thus, consolidation facilities, called Distribution Centers (DCs), are introduced to coordinate freight flow without and within the urban areas. The DCs, located on the outskirts of the city, receive products from a supplier through large vehicles, stock these products and deliver them with smaller vehicles to the final customers located in urban areas. Indeed, a two-echelon distribution system can provide an efficient way to reduce vehicle traffic, diminish the air pollution in cities and noises.

We present a Two-Echelon IRP (2E-IRP), which is a new extension of the IRP with a finite horizon to the best of our knowledge. In this problem, the customers must be served by the supplier strictly through DCs and routes must be defined in both echelons over a given time horizon. We address some variants of 2E-IRP in the context of replenishment policies and routing configuration. Three replenishment policies are modeled: Maximum Level (ML) and Order-Up-To Level (OU) policies, often considered in the IRPs; and a reorder point/reorder quantity policy \((R,Q)\), often studied for multi-echelon systems (see [3]), but apparently not yet considered for this class of problems. We also considered two other routing configurations in addition to the multi-vehicle case: the single vehicle configuration; and when multi-tours are allowed.

We proposed a Mixed Integer Linear Programming (MILP) formulation for 2E-IRPs. In addition, we analyze several valid inequalities available for IRPs and we introduced new ones inherent to the 2E-IRP. An exact Branch-and-Cut (B&C) algorithm is developed to solve the problem. We ran experiments on a set of randomly generated instances. The paper is organized as follows. Section 2 presents a review of relevant works in the context of the 2E-IRP. 2E-IRP are described and modeled in Section 3 while Section 4 states adapted and new valid inequalities. The B&C algorithm is introduced in Section 5. The description of the instances generation and the discussion about the computational results obtained are shown in Section 6. Finally, Section 7 provides conclusions.

2. Literature Review

The literature review is made along four directions: IRPs with focus on classical versions of the problem; Two-Echelon Capacitated Vehicle Routing Problems (2E-CVRPs), considered as a subproblem of the 2E-IRP; and inventory management on multi-echelon systems, in order to identify and analyze replenishment policies.
Several heuristics are available for a classical IRP under an OU policy with deterministic demands and a finite time horizon. In the OU policy, whenever a DC/customer is replenished its inventory level reaches the allowed capacity. [4] proposed a two-step heuristic for this problem, respecting a minimum inventory level for each customer. [5] studied a multi-vehicle IRP (MIRP) considering consistency requirement related to the quality of service. They proposed a matheuristic that combines an Adaptive Large Neighborhood Search with MILP formulations to solve subproblems.

[6] studied a MIRP under a ML policy, in which there are no constraints on delivery quantities, provided that the capacities are respected. They presented a hybrid algorithm based on an Iterated Local Search meta-heuristic with a randomized Variable Neighborhood Descent. In the proposed constructive heuristic, an initial solution is created based on replenishment services to prevent stockout. Then, quantities delivered are increased to expand the number of periods served by delivery, as long as the capacities are respected. To decrease inventory costs, they combined the proposed hybrid algorithm with an exact formulation to solve an inventory management subproblem.

[7] are the first to exactly solve an IRP. They propose a B&C algorithm and some valid inequalities to the IRP under an OU policy. [8] reformulated this problem using a shortest-path network representation for decisions of customers inventory replenishment and solved the problem by a B&C. They proposed a priori tour heuristic to generate an initial solution, where a route is predetermined solving a TSP and combined to a partial formulation without routing decisions. [9] compared several formulations for a MIRP, some of them proposed by the authors. The models were compared by B&C algorithms with inequalities added at the root node. [10] proposed a B&C algorithm including new valid inequalities to solve a MIRP under a ML policy. For this problem, [11] proposed a branch-price-and-cut with valid inequalities to strengthen the linear relaxation.

IRP variants are often addressed in the literature (see survey [1]). [12] proposed a B&C algorithm to solve an IRP with pickups and deliveries, where a product has to be picked up from multiple origins and delivered to the customers over a planning horizon. [13] studied an infinite horizon IRP in two levels, but considering a system with one supplier, only one DC and a set of customers. They proposed a decomposition solution approach, where customers are initially partitioned into disjoint sets by a genetic algorithm and subproblems are solved separately. However, the IRP with a finite horizon apparently has not yet been addressed in the literature in the context of the two-echelon. To analyze two-echelon systems, we extended the literature review on 2E-CVRPs and multi-echelon inventory control problems.

In particular, 2E-CVRP have been considerably studied in the recent years, as evidenced in [14]. According to the authors, the 2E-CVRP involves tactical planning decisions as the routing of freight and the assignment of customers to DCs with the objective of minimizing routing and handling costs. Thus, in a classical version of this problem, demands of each customer must be served by one depot through DCs. Direct shippings and split deliveries on the second echelon are not allowed. Two different homogeneous fleets of capacitated vehicles are available, one at each echelon. Each DC has a storage capacity and a maximum number of vehicles that can be routed through it. Most of these characteristics are observed in the routing subproblem of the 2E-IRP.
introduced a formal definition of a classical version of 2E-CVRP. The authors proposed a flow-based model, valid inequalities derived from VRP formulations, two matheuristics and a B&C to solve the problem. separated the problem into subproblems, one for each echelon, linking by an adjustment of DC loadings. Handling costs were not considered. They proposed a hybrid heuristic that combines a Greedy Randomized Adaptive Search Procedure (GRASP) and a Path Relinking. The GRASP is used to generate solutions, which are improved by a local search procedure. Then, a path between the current and the best solution is built by a Path Relinking procedure. indicated that the formulation proposed by may not provide correct upper bounds when the solution contains more than two DCs. Thus, they design a relaxation for 2E-CVRP and proposed a B&C to solve the problem.

presented a branch-and-cut-and-price which relies on a reformulation that overcomes symmetry issues. The formulation is strengthened with several classes of valid inequalities. More recently, studied a variant of 2E-CVRP that considers grouping constraints in the second echelon. In this case, the customers are divided into several disjoint groups and customers from the same group are served by vehicles from the same DC. The authors solved the problem by a B&C with several families of valid inequalities.

summarizes some of the first works on multi-echelon inventory systems. He described some structures and classified inventory control problems into distinguishing features such as deterministic/stochastic demands, stationary/nonstationary model, continuous/periodic review and backlog/no backlog. He explained techniques used to analyze multi-echelon inventory control problems for some inventory policies, such as analytic models of expected cost and Dynamic Programming. A considerable number of research works consider (R,Q) policies to manage inventory in two-echelon distribution systems. In a (R,Q) policy, when the inventories reach a level R, it must be replenished with a quantity Q. Often, these works deal with a basic configuration of stochastic demands and lead times, backorders, no split deliveries at DC and the order quantity at DC being an integer multiple of the order quantity at customers. Some of them are discussed below.

studied a system with one supplier, one DC and a set of customers. They proposed a cost structure and an iterative procedure to find a stationary solution for the case where each customer i has a reorder point Ri and the (R,Q) at the DC is a multiple of the common reorder quantity of customers. proposed an analytic model to a two-echelon inventory control problem with multiple identical suppliers, considering ordering, holding and transportation cost and a quantity order at DC split equally between suppliers.

formulated a multi-product inventory control problem under (R,Q) dependent policies and identical customers. They decomposed the problem by echelon and derived expressions for inventory policy parameters. An iterative heuristic was developed to solve the problem. addressed a supplier selection and inventory control problem with identical customers to maximize the total expected profit, in which a stochastic demand must be satisfied through a DC by a subset of potential suppliers with different characteristics. They proposed a Mixed Integer Nonlinear Programming formulation that was decomposed into two sub-models.

The 2E-IRP proposed in this work contains main characteristics of problems addressed
previously. For each DC, an IRP is solved to meet the demands of its customers. On the first echelon, an IRP is managed by the supplier, which supply the DCs. We consider a two-echelon supply chain similar to the system described for 2E-CVRPs. Since \((R,Q)\) policies are often used in supply chains, we also studied the case that considers this replenishment policy, in addition to the cases with classical inventory policies for IRPs.

3. Problem Description and Mathematical Formulation

2E-IRP is an extension of the classical IRP, where there are two echelons \(e \in \{1, 2\}\). The problem can be shown to be NP-hard via reduction to IRP, which is a special case of 2E-IRP when just one DC is considered. Each DC must meet the demands of a single product of a subset of customers. Direct deliveries from the supplier to a customer are not allowed. The subset of customers assigned to each DC is given. We consider a multi-vehicle configuration \((m\text{-vehicle})\), where there are two different fleets of \(m^e\) homogeneous vehicles, one for each echelon. Vehicles capacity and inventory capacities at DCs and customers must be respected. Figure 1 illustrates the 2E-IRP.

The principal supplier located at the first echelon is represented by the vertex 0 \(M^1 = \{0\}\) and supplies a set of DCs \(N^1 = N^1_0\). In turn, each DC \(u \in M^2 \ (M^2 = N^1)\) supplies a subset of customers \(N^2_u \subset N^2\) on the second echelon. The problem is defined on an undirected graph \(G = (V, A)\), where \(V = M^1 \cup N^1(= M^2) \cup N^2\) is a vertex set and \(A = \{(i, j) : i, j \in N^e \cup \{u\}, j < i, u \in M^e, e \in \{1, 2\}\}\) is the edge set. The edge set \(A\) connects supplier and DCs on the first echelon and each DC and its customers on the second echelon. An initial inventory level \(I^0_i\) is associated to each vertex \(i \in V\). Inventory holding capacities \(C_i\) of DCs and customers \(i \in N^e\) must be respected. The inventory capacity of the supplier is considered to be unlimited. Each customer \(i \in N^2\) has a final demand \(d^f_i\). A quantity of
product $r^t$ is made available at the supplier at each time period $t \in T = \{1, \ldots, |T|\}$. A homogeneous fleet of $m^1$ primary vehicles with capacity $B^1$ is available from the supplier. A different homogeneous fleet of $m^2$ secondary vehicles with capacity $B^2$ is available for DCs. We consider that routes managed by the supplier and by each DC are independents of each other.

Each vertex $(i, j) \in A$ selected for the solution incurs a transportation cost $c_{ij}$. An unit of inventory held by the supplier, a DC or a customer $i \in V$ incurs a cost $h_i$ per time period $t \in T'$, where $T' = T \cup \{0\}$. In addition, we consider an unit transportation cost $\alpha^e$ and a fixed transportation cost $f^e$ per vehicle routed for each echelon $e \in \{1, 2\}$. We also consider an order cost $g_i$ related to each DC and customer $i \in N^1 \cup N^2$. A primary vehicle can visit several DCs on a route starting and ending at the supplier, as long as its capacity is respected. Similarly, a second vehicle can visit several customers of a DC on a route starting and ending at the DC, respecting its capacity. The maximum inventory level at DCs and customers must be respected and no stockout is allowed. The objective of this problem is to minimize the total cost so that the final customer demands are met during each time period. Therefore, delivery quantities and routes at each time period should be defined in both echelons.

In this paper, we provide an extension of the mathematical formulation proposed by [7, 25]. For each depot (supplier or DC) $u \in M^e$, there is a maximum number of routes $K^e_u \subset K^e$ allowed, where $K^e$ is the total number of routes allowed per echelon $e \in \{1, 2\}$ at each time period. However, the number of vehicles available must also be respected. At first, we consider the ML replenishment policy, where there are no constraints for the quantity deliveries, as long as capacities are respected. The main difference from classical IRPs is that in our problem there are additional variables and constraints to link the supplier, DCs and customers inventories. It is assumed that the time periods have the same duration at both echelons and routes started and finished in the same period.

The route index formulation uses the undirected routing variables $x_{i,j}^{k,t}$ equal to the number of the edges $(i, j)$ used on the route $k$ at time period $t$. The binary variables $y_{i,j}^{k,t}$ is equal to 1 when the vertex $i$ is present on the route $k$ at time period $t$. The inventory level of the vertex $i$ at the end of the time period $t$ is denoted by $I^t_i$. The variables $q_{i,j}^{k,t}$ denote the quantity delivered to $i$ via a route $k$ at time period $t$. A basic version of the 2E-IRP can be formulated as follows.

$$
\min \sum_{e \in \{1,2\}} \sum_{u \in M^e} \left[ \sum_{t \in T} \sum_{k \in K^e_u} \left( f^e y_{i,j}^{k,t} + \sum_{i \in N^e_u \cup \{u\}} \sum_{j < i} \alpha^e c_{i,j} x_{i,j}^{k,t} \right) + \sum_{t \in T'} \sum_{i \in N^e_u \cup \{u\}} h_i I^t_i + \sum_{t \in T} \sum_{k \in K^e_u} \sum_{i \in N^e_u} g_i q_{i,j}^{k,t} \right] \quad (1)
$$

subject to

$$
I^t_0 = I^{t-1}_0 + r^t - \sum_{k \in K^e_u} \sum_{u \in N^1} q_{i,j}^{k,t} \quad \forall t \in T \quad (2)
$$
Constraints (6) impose that the quantities delivered to a DC or to a customer not exceed
the quantities provided by the supplier and quantities delivered to the subset of customers assigned to
the DC. Constraints (4) define the inventory level of the customers at each time period.
Constraints (3) determine for each DC the inventory over the time horizon, based on the products pro-
duced by the supplier and quantities delivered to the subset of customers assigned to the
DC. Similarly, constraints (2) define the inventory level at the supplier at each time period, taking into account quanti-
ties delivered to the subset of customers assigned to the DC. Constraints (1) define the inventory level of the customers at each time period, taking into account quantities delivered to the subset of customers assigned to the DC.

\[
\begin{align*}
I_u^t &= I_u^{t-1} + \sum_{k \in K_u^e} q_{k,i}^{t,t} - \sum_{k \in K_u^e} \sum_{i \in N_u^e} q_{i}^{k,t} \\
I_i^t &= I_i^{t-1} + \sum_{k \in K_u^e} q_{i}^{k,t} - d_i^t \\
I_i^t &\leq C_i \\
\sum_{k \in K_u^e} q_{i}^{k,t} &\leq C_i - I_i^{t-1} \\
q_{i}^{k,t} &\leq C_i y_{i}^{k,t} \\
\sum_{i \in N_u^e} q_{i}^{k,t} &\leq B_{s}^{e} y_{i}^{k,t} \\
\sum_{j \in N_u^e \cup \{u\}} x_{i,j}^{k,t} + \sum_{j \in N_u^e} x_{j,i}^{k,t} &\leq 2y_{i}^{k,t} \\
\sum_{i \in S} \sum_{j \in S} x_{i,j}^{k,t} &\leq \sum_{i \in S} y_{i}^{k,t} - y_{m}^{k,t} \\
\sum_{k \in K_u^e} q_{i}^{k,t} &\leq 1 \\
\sum_{u \in M^e} \sum_{k \in K_u^e} y_{i}^{k,t} &\leq m^e \\
x_{i,j}^{k,t} &\in \{0, 1\} \\
x_{i,u}^{k,t} &\in \{0, 1, 2\} \\
y_{i}^{k,t} &\in \{0, 1\} \\
q_{i}^{k,t} &\geq 0 \\
I_i^t &\geq 0 \\
\forall u \in M^2, t \in T \\
\forall u \in M^2, i \in N_u^2, t \in T \\
\forall e \in \{1, 2\}, i \in N_u^e, t \in T \\
\forall e \in \{1, 2\}, u \in M^e, i \in N_u^e, t \in T \\
\forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, t \in T \\
\forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, t \in T \\
\forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, t \in T \\
S \subseteq N_u^e, t \in T, m \in S \\
\forall e \in \{1, 2\}, u \in M^e, i \in N_u^e, t \in T \\
\forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, j < i, t \in T \\
\forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, t \in T \\
\forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, t \in T \\
\forall e \in \{1, 2\}, i \in N_u^e \cup \{0\}, t \in T' \\
\end{align*}
\]

The objective function (1) is to minimize respectively the sum of fixed vehicle costs, transportation costs, inventory holding costs and order costs at both echelons. Constraints (2) define the inventory level at the supplier at each time period, taking into account quantities delivered to DCs and the products made available at the supplier. Similarly, constraints (3) determine for each DC the inventory over the time horizon, based on the products provided by the supplier and quantities delivered to the subset of customers assigned to the DC. Constraints (4) define the inventory level of the customers at each time period.
Constraints (5) ensure that inventory capacities of DCs and customers are respected. Constraints (6) impose that the quantities delivered to a DC or to a customer not exceed
its remaining capacity at each time period. Constraints (7) link the quantity delivery to routing variables in both echelons. Constraints (8) enforce that a quantity delivery by route respects the vehicle capacity. Constraints (9) and (10) are respectively degree and subtour elimination constraints. Constraints (11) ensure that each DC and customer is visited at most once per time period. The number of vehicles available per echelon is respected through constraints (12). Constraints (13)–(15) and (16)–(17) oblige the integrality and non-negative conditions of the variables.

Some variants of the 2E-IRP can be obtained by considering particular replenishment policies and route configurations. In addition to the ML policy, we consider the OU and \((R, Q)\) policies. For the OU policy, constraints (18) must be included in the model. As no delays or lead-times are considered on the problem, for the \((R, Q)\) policy, we assume that \(R\) is equal to zero. Thus, we called it Order Fixed Quantity (OFQ), in which quantities delivered must be fixed by echelon \((Q_e)\). To apply the OFQ policy, constraints (19)–(21) must be added to the formulation.

Regarding the number and use of vehicles, two other cases can be considered: the single vehicle configuration (1-vehicle) and when multi-tours are allowed (\(\infty^2\)-vehicle). In both cases, constraints (12) are removed from the model. In the first case, one vehicle is available from the supplier and for each DC, each one being able to perform at most one tour per time period (\(|K_u^1| = 1\) and \(|K_u^2| = 1\), \(\forall u \in M^2\)). The second case is similar to the first, but allows multi-tours at a single time period on the second echelon. [26] studied a similar configuration to a long-term IRP, where the duration of all tours made by a vehicle must respect the cycle time.

\[
\begin{align*}
\sum_{k \in K_u^e} q_{i,k}^{k,t} &\geq \sum_{k \in K_u^e} (C_i y_{i,k}^{k,t}) - I_{i}^{t-1} & \forall e \in \{1, 2\}, u \in M^e, i \in N_u^e, t \in T \quad (18) \\
\sum_{k \in K_u^e} q_{i,k}^{k,t} &\leq Q_e & \forall e \in \{1, 2\}, u \in M^e, i \in N_u^e, t \in T \quad (19) \\
\sum_{k \in K_u^e} q_{i,k}^{k,t} - Q_e &\geq C_i (\sum_{k \in K_u^e} y_{i,k}^{k,t} - 1) & \forall e \in \{1, 2\}, u \in M^e, i \in N_u^e, t \in T \quad (20) \\
Q_e &\geq 0 & \forall e \in \{1, 2\} \quad (21)
\end{align*}
\]

4. Valid Inequalities

We adapted some valid inequalities available in the literature for IRP. We applied to our problem the inequalities (22)–(24) proposed by [7] and (25)–(26) presented in [6].

\[
\begin{align*}
y_{i}^{k,t} &\leq y_{i}^{k,t} & \forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, t \in T \quad (22) \\
x_{i,u}^{k,t} &\leq 2y_{i}^{k,t} & \forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, t \in T \quad (23) \\
x_{i,j}^{k,t} &\leq y_{i}^{k,t} & \forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i, j \in N_u^e, j < i, t \in T \quad (24) \\
x_{i,u}^{k,t} &\leq 2y_{u}^{k,t} & \forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i \in N_u^e, t \in T \quad (25) \\
x_{i,j}^{k,t} &\leq y_{j}^{k,t} & \forall e \in \{1, 2\}, u \in M^e, k \in K_u^e, i, j \in N_u^e, j < i, t \in T \quad (26)
\end{align*}
\]
Constraints (22) force a DC (the supplier) to be part of a route at a time period if at least one of its customers (a DC) is visited in the same route and period. Constraints (23)–(26) strengthen the link between route and assignment variables. Constraints (23) guarantee that if a customer (a DC) is the first or last in a route of its DC (the supplier) for a time period, the assignment variable of the customers (DC) for this configuration must have a value of one. Constraints (25) are similar but applied to assignment variables of depots, i.e., the respective DC on the second echelon and the supplier on the first echelon. Constraints (24) and (26) link route and assignment variables for two DCs or customers.

For vehicle/route index formulations with a homogeneous fleet, \[25\] explained the following symmetry breaking constraints, which can be used in the 2E-IRP. Let \(k_{\min} = \min\{k \in K_u\}\).

\[
y_u^{k,t} \leq y_{u^{k-1},t} \quad \forall e \in \{1, 2\}, u \in M^e, k \in K_u^e \setminus \{k_{\min}\}, t \in T \tag{27}
\]

\[
y_i^{k,t} \leq \sum_{j \in N_u^i} y_{j^{k-1},t} \quad \forall e \in \{1, 2\}, u \in M^e, i \in N_u^e, k \in K_u^e \setminus \{k_{\min}\}, t \in T \tag{28}
\]

Constraints (27) impose that a route \(k\) of a depot (supplier or DC) is used only if all possible routes with indexes less than \(k\) are used in the solution. Similarly, constraints (28) are applied to DCs and customers. Note that constraints (22)-(28) can be directly applied to the 2E-IRP.

\[10\] presented constraints to impose a minimum number of necessary deliveries to meet demands of a time window, which are valid for the second echelon in our problem. An adjustment was made in constraints (29) and (30). We used the inventory variables at time \(t-1\) instead of time \(t\).

\[
\sum_{t_1 \leq t \leq t_2} \sum_{k \in K_u^e} y_u^{k,t} \geq \frac{\sum_{t_1 \leq t \leq t_2} d_i^e - C_i}{\min\{B^e, C_i\}} \quad \forall u \in M^2, i \in N_u^2, t_1, t_2 \in T, t_2 \geq t_1 \tag{29}
\]

\[
\sum_{t_1 \leq t \leq t_2} \sum_{k \in K_u^e} y_i^{k,t} \geq \frac{\sum_{t_1 \leq t \leq t_2} d_i^e - I_{t_1} - I_{t_2}}{\min\{B^e, C_i\}} \quad \forall u \in M^2, i \in N_u^2, t_1, t_2 \in T, t_2 \geq t_1 \tag{30}
\]

\[
\sum_{t_1 \leq t \leq t_2} \sum_{k \in K_u^e} y_i^{k,t} \geq 1 - \frac{I_{t_2} - I_{t_1}}{\sum_{t_1 \leq t \leq t_2} d_i^e} \quad \forall u \in M^2, i \in N_u^2, t_1, t_2 \in T, t_2 \geq t_1 \tag{31}
\]

Constraints (29) impose a minimum number of deliveries requested in a timely window by the number of times that the inventory capacity must be raised to the maximum level allowed to meet the total demand in this time interval, limited by the minimum between the inventory and vehicle capacity. In constraints (30), the numerator is tightened by considering the inventory level at the instant of the preceding time. Constraints (31) define the minimum number of deliveries by evaluating if the inventory at the preceding time is sufficient for the whole period. Only constraints (29) can be rounded because there are no variables on the right-hand side. We adapted constraints (30) and (31) to the first echelon, which are explained in Section 4.1.
4.1. New Valid Inequalities for the 2E-IRP

In this section we present valid inequalities, exploring features of the 2E-IRP. Constraints (32), (33) and (34) are respectively equivalent to (29), (30) and (31), but applied to the first echelon. Therefore, we replaced the customers demand by delivery quantities to customers, that represents DCs demand. Because constraints (34) are nonlinear, we do not consider them in our experiment tests.

\[
\sum_{t_1 \leq t \leq t_2} \sum_{k \in K_0} y_{k,t}^u \geq t_1 \leq t_2 \sum_{k \in K_2} \sum_{i \in N_u^2} q_{i}^{k,t} - C_u \min\{B^1_u, C_u\} \quad \forall u \in M^2, t_1, t_2 \in T, t_2 \geq t_1 (32)
\]

\[
\sum_{t_1 \leq t \leq t_2} \sum_{k \in K_0} y_{k,t}^u \geq t_1 \leq t_2 \sum_{k \in K_2} \sum_{i \in N_u^2} q_{i}^{k,t} - I_{u}^{t_1 - 1} \min\{B^1_u, C_u\} \quad \forall u \in M^2, t_1, t_2 \in T, t_2 \geq t_1 (33)
\]

\[
\sum_{t_1 \leq t \leq t_2} \sum_{k \in K_0} y_{k,t}^u \geq 1 - \sum_{t_1 \leq t \leq t_2} \sum_{k \in K_2} \sum_{i \in N_u^2} q_{i}^{k,t} \frac{I_{u}^{t_1 - 1}} \quad \forall u \in M^2, t_1, t_2 \in T, t_2 \geq t_1 (34)
\]

We introduce a new group of valid inequalities in the context of the two-echelon. Due to our instance configuration described in Section 6.1, a route is traveled on the first echelon solely if the inventory level at one or more DCs it is not enough to meet the demands of its customers. This is justified, among other factors, by the cost configuration used to generate the instances set. We assumed that inventory holding from the upper echelons to lower echelons is a value-added operation. Thus, the inventory holding cost of the supplier is always lower than inventory costs at DCs that are lower than at customers. Constraints (35) impose that, if a delivery is made from the supplier to a DC, at least one delivery will be made to one of its customers at a subsequent time period. Constraints (36) apply the same logic to the quantities shipped when ML policy is used.

\[
\sum_{k \in K_0^1} y_{k,t}^u \leq \sum_{t_1 \in T} \sum_{t \geq t_1} \sum_{k \in K_2^1} \sum_{i \in N_u^2} y_{i,t}^{k,t_1} \quad \forall u \in M^2, t \in T (35)
\]

\[
\sum_{k \in K_0^1} q_{u}^{k,t} \leq \sum_{t_1 \in T} \sum_{t \geq t_1} \sum_{k \in K_2^1} \sum_{i \in N_u^2} q_{i}^{k,t_1} \quad \forall u \in M^2, t \in T (36)
\]

5. Branch-and-Cut Algorithm

We developed a Branch-and-Cut algorithm based on the proposed MILP to solve the 2E-IRP. Initially, at each node of the Branch-and-Bound (B&B) tree, a Linear Program (LP) is solved. The LP is a relaxation of the MILP, which is defined by (1)–(17) and, depending on the inventory policy and route configuration considered, some of constraints (18)–(21), without integralities and subtour elimination constraints (10), but with a subset of valid inequalities among (22)–(36). Then, violated subtour elimination constraints with \( m = \arg \max_{j \in S} \{y_{j}^{k,t}\} \) are iteratively identified, by using a separation algorithm [27], and
added to the model which is reoptimized until no violated constraints are found. If the current solution is non-integer, then branching occurs, otherwise a feasible solution is found. The process iterates until the upper bound is equal to the lower bound. The initial valid inequalities in the LP are chosen based on preliminary experiments presented below with the proposed MILP. All valid inequalities are included in the root node.

We have proposed a two-phase heuristic to generate an initial upper bound for the B&C. Similarly to [8], the main idea is to simplify routing decisions. The first phase, we optimally solve a partial problem that does not consider route decisions but is penalized if a route is used in the solution. Thus, knowing delivery quantities and time periods for each customer and DC, we solve several TSPs to define routes in the second phase. This Partial Formulation (PF) defines how much and at what time period products will be delivered from a DC (supplier) to customers (DCs) to meet final demands, respecting inventory capacities, vehicle capacities and all conditions imposed. Some of the constraints (18)–(21) can be added to the model depending of the inventory policy and route configuration considered.

In addition, to minimize inventory and order costs, we penalize the use of routes to reduce the number of routes in the solution. Whenever a route is allocated to the solution, a cost equal to the highest inventory capacity among the ones of the DCs (max\(u \in \{0\} \ C_u\)) is added to the total cost. This is justified by the impact of new routes on the total cost. Preliminary experiments showed that penalize all routes with max\(u \in \{0\} \ C_u\) generate on average better results than if only routes on the first echelon are penalized or if only fixed transportation costs are considered. Then, a TSP is solved for each situation where a route is required involving deliveries from one DC (supplier) to a subset of customers (DCs) on a specific time period.

\[
(PF) \quad \min \sum_{e \in \{1,2\}} \sum_{u \in M^e} \left( \sum_{k \in T^e} \sum_{i \in N^e_u \cup \{u\}} h_i I_i^e + \sum_{t \in T} \sum_{k \in K_u^e} \sum_{i \in N_u^e} g_i y^{k,t} + \sum_{k \in K_u^e} \max_{u \in \{0\}} C_u y^{k,t} \right)
\]

subject to \(2 - (8), (11), (12), (15) - (17)\)

### 6. Computational Experiments

As benchmark instances for the 2E-IRP was not identified, we generated a set of instances based on the parameters proposed by [7]. However, it was necessary to adapt some parameters of the instance generation to apply the OFQ inventory policy. Moreover, as in our problem, there are two echelons with dependent demands, we considered different parameters and data scales. We suppose the first echelon is characterized by larger distances and DCs with higher capacities than customers. This configuration is present in, for example, supply chains where a supplier is located far away from its customers. In order to reduce
costs, there are intermediate depots (DCs) to store products. The instance generation is described in details in Section 6.1.

All experiments were conducted on an Intel Core i5, 2.40 GHz and 8 GB RAM personal computer, with a maximum running time of one hour. For the two-phase heuristic, we impose a maximum running time of five minutes. The methods are implemented in C++ language using CPLEX 12.7.1 and one thread. For the B&C, we use the default setting in CPLEX for branching variable selection and for the node selection rule (best-bound-first strategy). We use Concorde [28] to solve TSP instances to optimality. A computational experiment is conducted to analyze the effectiveness of inequalities in Section 6.2. Computational results for each replenishment policy are presented in Section 6.3.

6.1. Generation of Instances

A set of 5 test instances was randomly generated for each combination of time horizon \( T \in \{3, 6, 10\} \) and number of DCs \(|N^1| = |M^2| \in \{3, 4, 5\}\) yielding a total of 45 instances. Instances with a time horizon equal to 3, 6 and 10 are respectively called small, medium and big instances. For each instance, the number of customers per DC \(|N^3_u|, u \in M^2\), was randomly generated in the interval \([5, 10]\). The demand \(d_i^u\) per customer \(i \in N^2\) is constant over time, i.e., \(d_i^u = d_i\), and was randomly generated in the interval \([50, 100]\). The quantity \(r_i^u\) made available at the supplier at each time period \(t \in T\) is equal to the sum of customers demands \((\sum_{i \in N^3} d_i)\).

We generated the maximum inventory level \(C_i\) at a customer \(i\) multiplying this demand \(d_i\) for a randomly selected number from a set \(\{2, 3\}\) in addition to a fixed quantity \(\max(d_i, 2)\), where \(\max(d_i)\) is the biggest demand among all customers. This margin is added to reduce the probability of non-existence of feasible solutions when the policy OFQ is considered. The starting inventory level of a customer \(i\) is defined as \(I_i^0 = (C_i - d_i)\) and the starting inventory level at the supplier \(I_u^0 = \sum_{i \in N^2} C_i\). The maximum inventory level \(C_u\) for each DC \(u \in M^2\) is equal to \(\sum_{i \in N^3} C_i\). We defined \(I_u^0 = [C_u/l_u]\), where \(l_u\) is randomly selected from \(\{3, 6\}\).

For the multi-vehicle case, we first consider the number of secondary vehicles equal to the number of DCs and that there is only one primary vehicle available. As originally proposed, the primary vehicle capacity \(B^1\) is equal to \([3/2 \sum_{i \in N^2} d_i]\). However, on the second echelon, we consider a homogeneous fleet with \(B^2 = [3/2 \max_{u \in N^2} \sum_{i \in N_3} d_i]\). When multi-tours are allowed, the vehicle capacity on the second echelon is set to one-third of the initially generated capacity.

We opted for a spatial configuration where the supplier is located far away from the cities and each city has a DC. The cities are not overlapped. Therefore, the supplier coordinates \((X_0, Y_0)\) are randomly generated in the interval \([0, 10000]\). The DCs coordinates \((X_u, Y_u), \forall u \in M^2\), are generated in the interval \([500, 9500]\). The coordinates \((X_i, Y_i)\) of each customer \(i\) serviced by the DC \(u\) are generated in the interval \([X_u - 500, X_u + 500]\) and \([Y_u - 500, Y_u + 500]\) for \(X_i\) and \(Y_i\), respectively. We consider Euclidean distances \(c_{ij} = NINT(\sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2})\). We consider that the unit transportation cost is lower on the first echelon. Thus, the unit transportation cost for the first \(\alpha^1\) and second \(\alpha^2\) echelon are generated respectively in the intervals \([0.01, 0.05]\) and \([0.06, 0.10]\). However,
the fixed transportation costs on the two echelons \((f^1 \text{ and } f^2)\) are proportional to vehicles’ capacity: \(f^1 = [50, 100]\) and \(f^2 = [10, 50]\).

Due to the added value of the product as it approaches the end customer, we consider that the inventory holding cost of the supplier is always less than the costs of DCs, which in turn is always less than of customers: \(h_0 < h_u < h_i, \forall u \in M^2, i \in N^2\). Thus, \(h_0 = 0.01, h_u = [0.02, 0.10]\) and \(h_i = [0.11, 0.20]\). In contrast, the order fixed costs are higher on the first echelon. The order cost \(g_u\) for each DC \(u\) and \(g_i\) for each customer \(i\) were generated as follows: \(g_u = 5l_u\) and \(g_i = 5l_i\), where \(l_u\) and \(l_i\) were randomly selected from \([6, 7, 8, 9, 10]\) and \([1, 2, 3, 4]\), respectively. The generated instances are available at [https://sites.google.com/view/katyannefarias/](https://sites.google.com/view/katyannefarias/).

### 6.2. Analysis of Valid Inequalities

To analyze the effectiveness of each family of valid inequalities, we present a detailed experiment on a subset of 10 instances with 3 DCs and time horizon \(T = \{3, 6\}\). For each instance, tests were made to all 9 combinations of the 3 inventory policies (ML, OU and OFQ) and the 3 route configurations (1, m and \(\infty^2\)-vehicle) with some exceptions. Table 1 shows a comparison between the model, including one family of valid inequalities and the model without valid inequalities in terms of upper bound (UB), lower bound (LB), linear relaxation (or lower bound of the root node, LB\(_{\text{ROOT}}\)) and running time. For each of these factors, we give the number of times that the factor obtained by the model with a set of inequalities is better (# improved), equal (# equaled) or worst (# worsened) than the factor got by the basic model. If an optimal solution is found by both models, the LBs are considered equals.

| #UB improved | 19 | 21 | 1 | 19 | 1 | 8 | 5 | 14 | 23 | 21 | 5 |
| #UB equaled  | 52 | 50 | 86 | 56 | 84 | 31 | 28 | 55 | 50 | 51 | 20 |
| #UB worst    | 19 | 19 | 3 | 15 | 5 | 21 | 27 | 21 | 17 | 18 | 5 |
| AM of UB Gap (%) | 0.1 | 0.3 | -1.1 | -1.3 | -1.1 | -22.2 | -9.6 | -7.7 | 0.8 | -0.6 | 1.1 |
| #LB improved | 30 | 24 | 12 | 35 | 15 | 18 | 12 | 18 | 27 | 27 | 6 |
| #LB equaled  | 38 | 36 | 68 | 38 | 65 | 17 | 15 | 35 | 36 | 35 | 20 |
| #LB worst    | 22 | 30 | 10 | 17 | 10 | 25 | 33 | 37 | 27 | 28 | 4 |
| AM of LB Gap (%) | 0.2 | -0.2 | -0.1 | 0.3 | -0.1 | -0.7 | -0.9 | -0.3 | 0.2 | 0.2 | 0.0 |
| #LB\(_{\text{R}}\) improved | 62 | 43 | 0 | 39 | 1 | 39 | 36 | 37 | 31 | 53 | 18 |
| #LB\(_{\text{R}}\) equaled  | 3 | 3 | 88 | 10 | 86 | 1 | 1 | 5 | 0 | 6 | 4 |
| #LB\(_{\text{R}}\) worst    | 25 | 44 | 2 | 41 | 3 | 20 | 23 | 48 | 59 | 31 | 8 |
| AM of LB\(_{\text{R}}\) Gap (%) | 0.7 | 0.1 | 0.0 | 0.0 | 0.0 | 0.3 | 0.4 | -0.3 | -0.8 | 0.6 | 0.3 |
| #Time improved | 26 | 20 | 26 | 24 | 25 | 4 | 5 | 12 | 15 | 13 | 8 |
| #Time equaled  | 48 | 51 | 52 | 51 | 52 | 42 | 42 | 52 | 51 | 52 | 10 |
| #Time worst    | 16 | 19 | 12 | 15 | 13 | 14 | 13 | 26 | 24 | 25 | 12 |
| PM of Time Gap (%) | 0.5 | -3.1 | 0.0 | -1.2 | 0.0 | -3.0 | -4.8 | -4.8 | -1.2 | -1.6 | -1.0 |

To observe relatively how much the factor has been improved or worsened, we calculated the mean of gaps between the best bounds of the model including inequalities and of the
basic model. We consider a positive gap when the result of the model with inequalities is improved and a negative gap otherwise. When an execution of the model with inequalities is interrupted by memory usage and not by the running time limit, we consider a negative time gap of 100%. In this case, the running time of model with inequalities is considered worse than the running time of the basic model. In contrast, if an execution only of the basic model is interrupted by memory usage, we consider a positive time gap of 100%. As costs have approximately the same order of magnitude on the tests, we calculated arithmetic means (AMs) to UB, LB and $\text{LB}_\text{ROOT}$ gaps. We opted to a pondered mean (PM) of time gaps to give more importance to percentages of improvement or not longer running times. Valid inequalities (27) and (28) are applied to $m$-vehicle and $\infty^2$-vehicle cases and inequalities (26) are restricted to ML policy.

Valid inequalities (22) are effective in all items. We consider that inequalities (23) are effective because they improved and achieved positive gaps of UB and also $\text{LB}_\text{ROOT}$ to a considerable number of instances, despite not having a positive mean of the time gap. This value can be justified by a set of six tests that vary in inventory policy and route configuration. Even in a longer time, the model, including constraints (23) found optimal solutions for five of these six cases. If we disregard these cases, the mean of time gap becomes 0.20%. Valid inequalities (24) and (26) have approximately the same results compared to the model without inequalities. Therefore, we will not insert them in the proposed algorithm. The use of constraints (25) does not improve UBs on average. A supplementary test was performed by adding to the model inequalities (23) and (25), since they are similar constraints. However, the use of these two constraint families together presented a bad average result for UBs.

Constraints (27) and (28) will not be considered because they have a negative impact on UB, LB and execution time. Inequalities (29) and (32) perform the same function as (30) and (33). As the first two are not effective in all evaluated items, only (30) and (33) will be used. These inequalities also presented negative means of $\text{LB}_\text{ROOT}$ and time gaps, but this is less relevant than the improvement in UB and LB. A negative mean of time gaps also occurs with constraints (36). On the other hand, they present in general good results of UB, LB and $\text{LB}_\text{ROOT}$. Inequalities (35) have a great negative effect on UB gaps of four tests. In fact, for two tests the basic model found a feasible solution, but the model, including constraints (35) was not able to find a feasible solution within one hour. The mean of UB gap is equal to 4.00% if these cases are excluded.

Accordingly, we insert at the root node of the proposed B&C algorithm valid inequalities (22), (23), (30), (33), (35) and (36). The last four groups of constraints present a negative mean of the time gap, which is relatively negligible compared to improvements in UB and LB.

6.3. Computational Results

Figures 2–4 present a comparative between gaps obtained by the B&C algorithm and the complete formulation for each combination of the replenishment policies and route configurations. In particular, Figure 2 shows average gaps of small instances.
Both methods solved optimally all small instances when 1-vehicle configuration with ML and OU policy and \( m \)-vehicle with ML policy are considered. However, in these cases, the B&C algorithm was almost always faster than the model. Only when 1-vehicle and OQF policy are considered, the complete formulation presents a better average gap. In all other cases, the proposed algorithm presents lower gaps. When a more complicated replenishment policy and route configuration are applied, gaps achieved by both methods have a higher difference.

Average gaps of medium instances are shown in Figure 3. On the 1-vehicle configuration, the formulation presents a slightly better performance compared to other route configurations. In other cases, the B&C given gaps of higher quality. As presents in Figure 4, on large instances, the B&C algorithm always presents better average gaps.
For each replenishment policy, we opted to present the results for the \( m \)-vehicle and \( \infty \)-vehicle cases, which are more difficult to solve than the 1-vehicle case due to a greater number of variables and constraints. Due to the experiments were performed with the same set of instances for both route configurations, several optimal solutions of the 1-vehicle and \( m \)-vehicle cases have equal costs. However, all results are available at [https://sites.google.com/view/katyannefarias/](https://sites.google.com/view/katyannefarias/). A summary of the computational results for the ML, OU and OFQ policies are presented respectively in Tables 2, 3 and 4. We provide a comparison between the proposed B&C and the MILP (1)–(17), including some of constraints (18)–(21).

We show for each route configuration and time horizon: the number of instances solved to optimality (#Optimal); the number instances where a feasible solution was found (#Feasible); the number of instances solved to optimality only by the B&C (#New optimal) and with a feasible solution found only by the B&C (#New feasible); the number of instances that the B&C did not prove or find an optimal solution proven by the formulation (#Optimal not proven/found); the number of UB, LB and time improved, equaled or worst compared to the formulation; and means of gaps between B&C and formulation results, as explained in Section 6.2, but also with arithmetic means of the time gap, since these analyzes are performed by instance size with running times in the same order of magnitude.

Table 2 shows results for the ML policy. When the time horizon is equal to 3 for the \( m \)-vehicle case, the proposed algorithm achieves the same results as the full formulation in a shorter time. For the \( \infty \)-vehicle case, in small instances, the B&C improved some UBs. However, most of LBs are slightly worse. In general, the B&C reaches better results when the time horizon is equal to 6 and 10, finding new feasible solutions and improving UBs and LBs for both route configurations. Negative means of time gaps and the majority of worst running times were mainly due to overuse memory.

When replenishments respect an OU policy, for a time horizon equal to 3, the UB and LB were improved on several instances, as shown by Table 3. On these instances, running
times were also in average improved for the $m$-vehicle case and, in the $\infty$-vehicle case, the mean of the time gap is negative due to the overuse of memory. Similarly to the ML policy, for a time horizon of 6 and 10, feasible solutions not identified by the complete model were found by the B&C and, in general, better bounds were achieved.

On the $m$-vehicle case, when an optimal solution was found by both methods the proposed B&C was always faster. Only in one instance with 6 and one with 10 time periods, the full model presents a better gap. On the $\infty$-vehicle configuration, the execution of almost all instances with 3 time periods was limited by the use of memory. However, except in two instances, including instances with 6 and 10 time periods, the B&C presents better gaps.

It is important to emphasize that if we compare the performance of the B&C with that of the complete formulation when adding to the problem the OU replenishment policy, we verify that the increase in the gaps obtained by the formulation occurs in a greater proportion than the gaps reached by the B&C algorithm. In other words, the impact on the performance of the proposed algorithm when we insert the OU policy is less than the impact on the complete formulation. The same is true when, given an inventory policy, we compare the $m$-vehicle with the $\infty$-vehicle case.

Table 4 allows us to observe that when the OFQ is considered the problem seems to be more difficult. With this policy, no optimal solutions were found or proven by the B&C algorithm. However, in all groups of instances, the proposed algorithm was able to find new feasible solutions and to considerably improve the UB of several instances. Most parts of the best LBs were also improved on the $\infty$-vehicle configuration. However, on the $m$-vehicle case, the best LBs have on average a lower quality. Even in this configuration, the B&C was not able to find/prove 4 optimal solutions and, in some small instances, the execution was interrupted by the overuse of memory, which justifies the negative mean of the time gap. In all other cases, both methods have reached the running time limit. Similarly to the OU

<table>
<thead>
<tr>
<th>#Optimal</th>
<th>$T = 3$</th>
<th>$T = 6$</th>
<th>$T = 10$</th>
<th>$T = 3$</th>
<th>$T = 6$</th>
<th>$T = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>#New optimal</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>#Optimal not proved</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>#Feasible</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>#New feasible</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>#Feasible not found</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

| #UB improved | 0 | 10 | 11 |
| #UB equaled | 15 | 4 | 0 |
| #UB worst | 0 | 1 | 4 |
| AM of UB Gap (%) | 0.00 | 14.52 | 40.57 |

| #LB improved | 0 | 10 | 7 |
| #LB equaled | 15 | 2 | 0 |
| #LB worst | 0 | 3 | 8 |
| AM of LB Gap (%) | 0.00 | 1.36 | 1.42 |

| #Time improved | 15 | 2 | 0 |
| #Time equaled | 0 | 8 | 15 |
| #Time worst | 0 | 5 | 0 |
| AM of Time Gap (%) | 86.01 | -57.84 | 0.00 | -267.21 | -46.67 | 0.00 |

---

Table 2: Summary for ML policy of the B&C compared to the complete formulation
Table 3: Summary for OU policy of the B&C compared to the complete formulation

<table>
<thead>
<tr>
<th></th>
<th>m-vehicle</th>
<th>s-vehicle</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>T = 3</td>
<td>T = 6</td>
</tr>
<tr>
<td>#Optimal</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>#Optimal not proved</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>#New optimal</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>#New feasible</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>#Optimal not proved</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>#feasible</td>
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<td>15</td>
</tr>
<tr>
<td>#feasible not found</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>#UB improved</td>
<td>4</td>
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<tr>
<td>#UB equaled</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td>#UB worst</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Arith. Mean of UB Gap (%)</td>
<td>1.07</td>
<td>25.13</td>
</tr>
<tr>
<td>#LB improved</td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>#LB equaled</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>#LB worst</td>
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<td>1</td>
</tr>
<tr>
<td>Arith. Mean of LB Gap (%)</td>
<td>3.81</td>
<td>4.70</td>
</tr>
<tr>
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<td>9</td>
<td>0</td>
</tr>
<tr>
<td>#Time equaled</td>
<td>14</td>
<td>15</td>
</tr>
<tr>
<td>#Time worst</td>
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<td>1</td>
</tr>
<tr>
<td>Arith. Mean of Time Gap (%)</td>
<td>14.72</td>
<td>-6.67</td>
</tr>
</tbody>
</table>

Policy, the OFQ policy has a lower impact on the performance of the B&C than on the full formulation.

7. Conclusions and Future Work

We have presented a B&C algorithm to solve several variants of the 2E-IRP which can be considered as new problems. The proposed algorithm is able to solve the 2E-IRP with one, several vehicles or allowing multi-tours in the second echelon, and for an ML, OU or OFQ policy. We have adapted valid inequalities available for the classical IRP and we have also proposed new ones. To analyze the effectiveness of the valid inequalities and the performance of the algorithm, we generated a set of random instances. We observe that if replenishment policies or more complex route configuration are included in the problem, the impact on the performance of the algorithm is lower in the B&C than in the complete formulation. In addition, the B&C algorithm has presented in generally better results in terms of cost and running time.

The OU policy turned out to be very strictly, which implies high costs. Depending on the company needs, an intermediate option is the OFQ policy. Substantial savings can be achieved using the ML policy [7]. As regards the routing configurations, the multi-tours presented the highest costs on our experiment instances. However, it is important to consider this configuration because it is the case of urban areas that only allow small vehicles which can make multi-tours in a single period due to short distances. In the next steps, we want to use column generation to solve the LP at each node of the B&B three (Branch-and-Price) and heuristic methods to solve large instances. Another important direction is to add new features in the problem, such as transshipment between DCs or customers and uncertain data, to better fit with industrial company assumptions and constraints.
Table 4: Summary for OFQ policy of the B&C compared to the complete formulation

<table>
<thead>
<tr>
<th></th>
<th>m-vehicle</th>
<th>∞-vehicle</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = 3$</td>
<td>$T = 6$</td>
</tr>
<tr>
<td>#Optimal</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>#New optimal</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>#Optimal not proved</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>#Feasible</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>#New feasible</td>
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<td>12</td>
</tr>
<tr>
<td>#Feasible not found</td>
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<td>0</td>
</tr>
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</tr>
<tr>
<td>#UB equaled</td>
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<tr>
<td>#UB worst</td>
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<td>8.64</td>
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<tr>
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<td>-2.31</td>
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<td>Arith. Mean of Time Gap (%)</td>
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<td>0.00</td>
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Acknowledgement

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References


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