Greedy Systems of Linear Inequalities and Lexicographically Optimal Solutions

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Abstract

The present note reveals the role of the concept of *greedy system of linear inequalities* played in connection with *lexicographically optimal solutions* on convex polyhedra and *discrete convexity*. The lexicographically optimal solutions on convex polyhedra represented by a greedy system of linear inequalities can be obtained by a *greedy procedure*, a special form of which is the greedy algorithm of J. Edmonds for polymatroids. We also examine when the lexicographically optimal solutions become integral. By means of the Fourier-Motzkin elimination K. Murota and A. Tamura have recently shown the existence of integral points in a polyhedron arising as a subdifferential of an integer-valued, integrally convex function due to P. Favati and F. Tardella (K. Murota and A. Tamura: Integrality of subgradients and biconjugates of integrally convex functions. arXiv:1806.00992v1 [math.CO] 4 June 2018; revised, 7 September 2018), which can be explained by our present result. A characterization of integrally convex functions is also given.

Keywords: Greedy system, lexicographic optimality, discrete convexity, integrally convex functions

1. Introduction

The present work was motivated by the recent paper [13] by K. Murota and A. Tamura about the existence of integral points in a polyhedron arising as the subdifferential of what is called an integrally convex function, due to P. Favati and F. Tardella [6], when the function is integer-valued (also see [10, 11, 12]). Murota and Tamura [13] showed the existence of integral points in the polyhedron by means of the Fourier-Motzkin elimination, which is itself an interesting application of the Fourier-Motzkin elimination.

The present note reveals the role of the concept of *greedy system of linear inequalities* played in connection with *lexicographically optimal solutions* on convex polyhedra and discrete convexity. The lexicographically optimal solutions on convex polyhedra represented by a greedy system of linear inequalities can be obtained by a *greedy procedure*, a special form of which is the greedy algorithm of J. Edmonds for polymatroids (see, e.g., [7]). We also examine when the lexicographically optimal solutions become integral. The polyhedron considered by Murota and Tamura [13] has *integral extreme points* that can be obtained by a "greedy procedure" for a greedy system of linear inequalities.

The present note is organized as follows. In Section 2 we give some definitions and preliminaries on signed sets, lexicographic optimality, and integral convexity required in this note. In Section 3 we examine the relation between lexicographically optimal solutions and greedy systems of linear inequalities, and give a signed greedy procedure for such systems and polyhedra. We also discuss, in Section 4, relations of the present results to other polyhedra expressed by signed-set functions, especially the subdifferentials of integrally convex functions considered by Murota and Tamura. Section 5 gives concluding remarks.

2. Definitions and Preliminaries

We give some definitions and preliminaries that will be used in the following arguments.

2.1. Signed sets

Let V be a nonempty finite set with $|V| = n$ and define

$$3^V = \{(X, Y) \mid X, Y \subseteq V,\ X \cap Y = \emptyset\}. \tag{2.1}$$

Each $(X, Y) \in 3^V$ is made to correspond to the $\{0, \pm 1\}$-vector $\chi_{(X,Y)} \in \mathbb{R}^V$ defined by

$$\chi_{(X,Y)}(v) = \begin{cases} 1 & \text{if } v \in X \\ -1 & \text{if } v \in Y \\ 0 & \text{otherwise} \end{cases} \qquad (v \in V). \tag{2.2}$$

We call every $(X, Y) \in 3^V$ a *signed set*. For any $(X_1, Y_1), (X_2, Y_2) \in 3^V$, we write $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. The binary relation $\sqsubseteq$ on 3^V is a partial order with the minimum element $(\emptyset, \emptyset)$. Maximal elements are given by signed sets $(S, T) \in 3^V$ such that $S \cup T = V$. Each such maximal (S, T) is called an *orthant*, whose meaning will be made clear later.

Let $L = (v_1, \cdots, v_n)$ be a linear ordering of V and $\sigma : V \to \{+, -\}$ be a sign function. The pair (L, σ) is called a *signed linear ordering* of V. For a given orthant $(S, T) \in \mathcal{T}$ consider a signed linear ordering $(L = (v_1, \cdots, v_n), \sigma)$ such that $\sigma(v) = +$ if $v \in S$ and $\sigma(v) = -$ if $v \in T$. We call such a signed linear ordering a *linear ordering of* (S, T). For each $i = 1, \cdots, n$ define $L_i = \{v_1, \cdots, v_i\}$ and $L_0 = \emptyset$. For any $X \subseteq V$ define $X^\sigma = (X^\sigma_+, X^\sigma_-) \in 3^V$ with $X^\sigma_+ = \{v \in X \mid \sigma(v) = +\}$ and $X^\sigma_- = \{v \in X \mid \sigma(v) = -\}$. We say that X^σ is the *set* X *signed by* σ.

For any vector $x \in \mathbb{R}^V$ define $\mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\}$ and $\mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\}$, and then define the *signed support* $\mathrm{supp}(x) \in 3^V$ of x by $\mathrm{supp}(x) = (\mathrm{supp}^+(x), \mathrm{supp}^-(x))$.

For any vector $x \in \mathbb{R}^V$ and any set $U \subseteq V$ define x^U to be the vector in $\mathbb{R}^U$ such that $x^U(v) = x(v)$ for all $v \in U$. For any nonempty proper subset U of V and for any $x \in \mathbb{R}^U$ and $y \in \mathbb{R}^{V \setminus U}$ the *direct sum* $x \oplus y \in \mathbb{R}^V$ of x and y is defined by $(x \oplus y)(v) = x(v)$ for $v \in U$ and $(x \oplus y)(v) = y(v)$ for $v \in V \setminus U$.

2.2. Convex polyhedra and lexicographically optimal solutions

Let P be a bounded convex polyhedron in $\mathbb{R}^V$. Choose a signed linear ordering $(L = (v_1, \cdots, v_n), \sigma)$ of V. We call $x \in P$ the *lexicographically maximum solution* in P with respect to signed linear ordering (L, σ) of V if x lexicographically maximizes the sequence $(\sigma(v_1)x(v_1), \cdots, \sigma(v_n)x(v_n))$ over P.

For a weight function $w : V \to \mathbb{R}$ consider the following linear optimization problem over P.

$$\begin{aligned} \mathbf{P} : \text{Maximize} \quad & \sum_{v \in V} w(v)x(v) \\ \text{subject to} \quad & x \in P. \end{aligned} \tag{2.3}$$

Suppose that for a signed linear ordering $(L = (v_1, \cdots, v_n), \sigma)$ of V the weight function w satisfies

$$\sigma(v_1)w(v_1) \gg \cdots \gg \sigma(v_n)w(v_n) > 0. \tag{2.4}$$

Here $\alpha \gg \beta$ for $\alpha, \beta \in \mathbb{R}$ means that $\alpha - \beta > 0$ is sufficiently large (as large as possible whenever these parameters appear in any arguments). Then the optimal solution of Problem $\mathbf{P}$ is the *lexicographically maximum solution* in P with respect to signed linear ordering $(L = (v_1, \cdots, v_n), \sigma)$. Define a sequence of (possibly repeated) faces F_i of P for $i = 1, \cdots, n$ by

$$F_i = \{x \in F_{i-1} \mid \sigma(v_i)x(v_i) = \max\{\sigma(v_i)y(v_i) \mid y \in F_{i-1}\}\}, \tag{2.5}$$

where $F_0 = P$. We see that for each $i = 1, \cdots, n$ the dimension of face F_i is at most $n - i$ and face F_n is a vertex of P, which gives the optimal solution of $\mathbf{P}$ under (2.4), i.e., the lexicographically maximum solution in P with respect to signed linear ordering $(L = (v_1, \cdots, v_n), \sigma)$.

2.3. Integrally convex functions

In this subsection we give the definition of integrally convex function introduced by Favati and Tardella [6] and show a characterization of integral convexity.

For any two integral vectors $a, b \in \mathbb{Z}^V$ with $a \leq b$ define *integral standard boxes* $[a, b]_{\mathbb{R}} = \{z \in \mathbb{R}^V \mid a \leq z \leq b\}$ in $\mathbb{R}^V$ and $[a, b]_{\mathbb{Z}} = [a, b]_{\mathbb{R}} \cap \mathbb{Z}^V$ in $\mathbb{Z}^V$.

Consider a function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ on integer lattice $\mathbb{Z}^V$ such that its effective domain $\mathrm{dom}(f) \equiv \{x \in \mathbb{Z}^V \mid f(x) < +\infty\}$ is nonempty. Define a function $\bar{f} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ in such a way that the epigraph of $\bar{f}$ is obtained as the convex hull of the set of halflines $\{(x, \alpha) \mid \alpha \geq f(x)\}$ for all $x \in \mathrm{dom}(f)$. Such a function $\bar{f}$ is called the *lower envelope* of f. If we have $\bar{f}(x) = f(x)$ for all $x \in \mathrm{dom}(f)$, then we say f is extensible to the convex function $\bar{f} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ and f is called *discrete convex*. A convex polyhedron $D \subseteq \mathbb{R}^V$ is called a *linearity domain* of $\bar{f}$ if there exists an affine function $y(x) = \langle c, x \rangle + \alpha$ in $x \in \mathbb{R}^V$ such that $\bar{f}(x) \geq y(x)$ for all $x \in \mathbb{R}^V$ and $D = \{x \in \mathbb{R}^V \mid \bar{f}(x) = y(x)\}$, where $\langle c, x \rangle + \alpha = \sum_{v \in V} c(v)x(v) + \alpha$ for a constant vector $c \in (\mathbb{R}^V)^*$ and a real α.

For a function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}(f) \neq \emptyset$ and its lower envelope $\bar{f} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ suppose that for every integral standard box $[a, b]_{\mathbb{Z}}$ in $\mathbb{Z}^V$ with $\max\{b(v) - a(v) \mid v \in V\} \leq 1$ and $[a, b]_{\mathbb{Z}} \cap \mathrm{dom}(\bar{f}) \neq \emptyset$ the following (†) holds:

(†) the lower envelope of the restriction of f on $[a, b]_{\mathbb{Z}}$ coincides with the restriction of $\bar{f}$ on $[a, b]_{\mathbb{R}}$.

(Here, the restriction of f on $[a, b]_{\mathbb{Z}}$ should be defined on $\mathbb{Z}^V$ while its effective domain is within $[a, b]_{\mathbb{Z}}$. We consider the restriction of $\bar{f}$ on $[a, b]_{\mathbb{R}}$ similarly in $\mathbb{R}^V$.) Then f is called an *integrally convex function* ([6]) and can easily be shown to be discrete convex.[1] (This definition of integral convexity is slightly different from the original one in [6] but we can easily see the equivalence.) See [10] for more details about integral convexity and for a class of integrally convex functions appearing as M-convex functions, L-convex functions, and others.

For any subset $U \subseteq V$ and any vector $a \in \mathbb{R}^V$ define $\mathbf{A}(U, a) = \{z \in \mathbb{R}^V \mid z^{V \setminus U} = a^{V \setminus U}\}$. We call $\mathbf{A}(U, a)$ a *coordinate affine subspace* of $\mathbb{R}^V$ (associated with a coordinate set U and a vector $a \in \mathbb{R}^V$) and if $a^{V \setminus U}$ is an integral vector, we call it an *integral coordinate affine subspace* of $\mathbb{R}^V$.

We have a characterization of integral convexity as follows.

Theorem 2.1: *Let $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be a discrete convex function on $\mathbb{Z}^V$ and let $\bar{f} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ be its lower envelope. Then f is integrally convex if and only if*

[1] It should be noted that for a discrete convex function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ we may have a hole in $\mathrm{dom}(f)$, i.e., possibly $\mathrm{dom}(f) \neq \{x \in \mathbb{Z}^V \mid \bar{f}(x) < +\infty\}$. But there is no hole for any integrally convex function.

for every integral coordinate affine subspace $\mathbf{A}$ *of* $\mathbb{R}^V$ *with* $\mathbf{A} \cap \mathrm{dom}(\bar{f}) \neq \emptyset$ *we have the following:*

($\ddagger$) *the lower envelope of the restriction of* f *on* $\mathbf{A} \cap \mathbb{Z}^V$ *coincides with the restriction of* $\bar{f}$ *on* $\mathbf{A}$.

(Proof) By the definition of integral convexity the only-if part is clear. We show the if part by induction on the size of V.

When $|V| = 1$, the present theorem holds due to the assumption.

Now suppose that the present theorem holds when $|V| = k$ for some integer $k \geq 1$. We show the if part when $|V| = k + 1$.

By the induction hypothesis, for every k-dimensional, integral coordinate affine subspace $\mathbf{A}$ of $\mathbb{R}^V$ with $\mathbf{A} \cap \mathrm{dom}(\bar{f}) \neq \emptyset$ the restriction of f on $\mathbf{A} \cap \mathbb{Z}^V$ is integrally convex. Hence it suffices to show that Condition (†) holds for any integral standard box $[a, b]_{\mathbb{Z}}$ with (1) $[a, b]_{\mathbb{Z}} \cap \mathrm{dom}(\bar{f}) \neq \emptyset$ and (2) $b(v) - a(v) = 1$ for all $v \in V$. Let D be any linearity domain of $\bar{f}$ such that D has a nonempty intersection with $[a, b]_{\mathbb{R}}$. Then for any facet F of the box $[a, b]_{\mathbb{R}}$ there exists a unique k-dimensional, integral coordinate affine subspace $\mathbf{A}$ of $\mathbb{R}^V$ that includes F. Because of the induction hypothesis the intersection $F \cap D$, if nonempty, is a linearity domain of the extension of the restriction of f on $\mathbf{A} \cap \mathbb{Z}^V$. This implies that $F \cap D$ is a convex hull of a set of extreme points of F. Since this holds for every facet F of the box $[a, b]_{\mathbb{R}}$, it follows that $D \cap [a, b]_{\mathbb{R}}$ is a convex hull of a set of extreme points of $[a, b]_{\mathbb{R}}$ and hence is a linearity domain of the extension of the restriction of f on $[a, b]_{\mathbb{Z}}$. Hence Condition (†) holds.

This completes the proof by induction. □

It should be noted that Condition ($\ddagger$) implies that there is no hole in $\mathrm{dom}(f)$, which can be seen by considering a hole $z \in \mathrm{dom}(f)$ (if any exists), for which the zero-dimensional integral affine subspace $\mathbf{A} = \{z\}$ does not satisfy ($\ddagger$) since the restriction of f on $\mathbf{A} \cap \mathbb{Z}^V$ has the empty effective domain.

A careful examination of the proof of Theorem 2.1 leads us to the following.[2]

Theorem 2.2: *Let* $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ *be a discrete convex function on* $\mathbb{Z}^V$ *and let* $\bar{f} : \mathbb{R}^V \to \mathbb{R} \cup \{+\infty\}$ *be its lower envelope. Then* f *is integrally convex if and only if for every integral coordinate affine subspace* $\mathbf{A}$ *of* $\mathbb{R}^V$*, of dimension zero or one, with* $\mathbf{A} \cap \mathrm{dom}(\bar{f}) \neq \emptyset$ *the following holds:*

($\ddagger$) *the lower envelope of the restriction of* f *on* $\mathbf{A} \cap \mathbb{Z}^V$ *coincides with the restriction of* $\bar{f}$ *on* $\mathbf{A}$.

[2]This is a strengthening of Theorem 2.1, suggested by Fabio Tardella.

3. Convex Polyhedra and Greedy Systems of Linear Inequalities

Let P be a bounded convex polyhedron in $\mathbb{R}^V$ and $h_P : (\mathbb{R}^V)^* \to \mathbb{R}$ be the *support function* of P, i.e.,

$$h_P(z) = \max\{\langle z, x\rangle \mid x \in P\} \qquad (\forall z \in (\mathbb{R}^V)^*), \tag{3.1}$$

where $\langle z, x\rangle = \sum_{v\in V} z(v)x(v)$. Suppose that for some finite set $Q \subset (\mathbb{R}^V)^*$ the polyhedron P is expressed by the following system of linear inequalities

$$\langle z, x\rangle \le h_P(z) \qquad (\forall z \in Q). \tag{3.2}$$

Here it should be noted that each inequality in (3.2) gives a hyperplane $\langle z, x\rangle = h_P(z)$ supporting P, due to the definition of support function h_P. As is well known, there exists a unique minimal such set Q composed of those corresponding to facets of P when P is full-dimensional. But such a minimal set Q is not what we want to keep. Instead, we impose the following additional condition (A*). For any $U \subseteq V$ and $(X, Y) \in 3^V$ put

$$Q(U) = \{z \in Q \mid \mathrm{supp}(z) \in 3^U\}, \quad Q(X, Y) = \{z \in Q \mid \mathrm{supp}(z) \sqsubseteq (X, Y)\}. \tag{3.3}$$

(A*) *If for coefficients $\lambda_z \ge 0$ for $z \in Q$ we have*

$$\mathrm{supp}(\sum_{z\in Q} \lambda_z z) = (X, Y) \in 3^V, \tag{3.4}$$

then there exist some coefficients $\mu_z \ge 0$ for $z \in Q(X, Y)$ such that

$$\sum_{z\in Q} \lambda_z z = \sum_{z\in Q(X,Y)} \mu_z z, \qquad \sum_{z\in Q} \lambda_z h_P(z) \ge \sum_{z\in Q(X,Y)} \mu_z h_P(z). \tag{3.5}$$

When the system of linear inequalities (3.2) for a bounded P satisfies Assumption (A*), we call it a *greedy system of linear inequalities*. The term, greedy, can be understood through a procedure to find a lexicographically optimal solution in P.

It should be noted here that any bounded convex polyhedron P has a representation by a greedy system of linear inequalities.[3] Also note that since P is bounded, the conical hull of Q becomes $(\mathbb{R}^V)^*$, so that Assumption (A*) implies that every $z \in (\mathbb{R}^V)^*$ with $\mathrm{supp}(z) = (X, Y)$ belongs to the conical hull of $Q(X, Y)$.

[3] A direct construction of such a (not necessarily minimal) representation is given as follows. For each nonempty coordinate subset $W \subseteq V$ let P_W be the projection of P into the coordinate subspace $\mathbb{R}^W$ and then for each facet F of P_W, considering a normal vector z of F as a vector in $\mathbb{R}^V$, add z to Q.

Now, consider the following procedure, Signed_Greedy_Procedure. Recall that $L_i = \{v_1, \cdots, v_i\}$ for $i = 1, \cdots, n$ and L_i^σ is the set L_i signed by σ.

Algorithm Signed_Greedy_Procedure
Input: A signed linear ordering $(L = (v_1, \cdots, v_n), \sigma)$ of V.
Output: A vector $x \in P$.
Step 1: For each $i = 1, \cdots, n$ do the following:

1. If $\sigma(v_i) = +$, then compute $x(v_i)$ by
   $x(v_i) = \max\{y(v_i) \mid \langle z, y\rangle \leq h_P(z),\ \forall k \in \{1, \cdots, i-1\} : y(v_k) = x(v_k),\ z \in Q(L_i^\sigma),\ z(v_i) > 0\}$.
2. If $\sigma(v_i) = -$, then compute $x(v_i)$ by
   $x(v_i) = \min\{y(v_i) \mid \langle z, y\rangle \leq h_P(z),\ \forall k \in \{1, \cdots, i-1\} : y(v_k) = x(v_k),\ z \in Q(L_i^\sigma),\ z(v_i) < 0\}$.

Step 2: Return x.

To prove the validity of Algorithm Signed_Greedy_Procedure we first show the following lemma, which easily follows from the Farkas Lemma for systems of linear inequalities.

Lemma 3.1: *Let P be a bounded polyhedron expressed by* (3.2)*. Under Assumption* (A*)*, for any nonempty $U \subseteq V$ and any $x \in \mathbb{R}^U$ satisfying*

$$\langle z^U, x\rangle \leq h_P(z) \qquad (\forall z \in Q(U)), \tag{3.6}$$

there exists a vector $y \in P$ such that $y^U = x$.

(Proof) Let $x \in \mathbb{R}^U$ be a vector satisfying (3.6). Consider the following system of linear inequalities in $y \in \mathbb{R}^{V \setminus U}$.

$$\langle z, x \oplus y\rangle \leq h_P(z) \qquad (\forall z \in Q \setminus Q(U)). \tag{3.7}$$

It follows from the Farkas Lemma that (3.7) has a feasible solution y if and only if there exist no coefficients $\lambda_z \geq 0$ for $z \in Q \setminus Q(U)$ such that

$$\sum_{z \in Q\setminus Q(U)} \lambda_z z^{V\setminus U} = \mathbf{0}, \qquad \sum_{z \in Q\setminus Q(U)} \lambda_z (h_P(z) - \langle z^U, x\rangle) < 0. \tag{3.8}$$

If $\lambda_z \geq 0$ ($z \in Q \setminus Q(U)$) satisfy the first equation of (3.8), then suppose that the signed support of $\sum_{z\in Q\setminus Q(U)} \lambda_z z$ is equal to $(W, Z) \in 3^U$. It follows from Assumption (A2) that there exist $\mu_z \geq 0$ ($z \in Q(W, Z)$) such that

$$\sum_{z \in Q\setminus Q(U)} \lambda_z z = \sum_{z \in Q(W,Z)} \mu_z z, \qquad \sum_{z \in Q\setminus Q(U)} \lambda_z h_P(z) \geq \sum_{z \in Q(W,Z)} \mu_z h_P(z). \tag{3.9}$$

It follows from (3.6) and (3.9) that

$$\sum_{z\in Q\setminus Q(U)} \lambda_z(h_P(z) - \langle z^U, x\rangle) \geq \sum_{z\in Q(W,Z)} \mu_z(h_P(z) - \langle z^U, x\rangle) \geq 0. \tag{3.10}$$

Hence (3.7) has a feasible solution y and we have $x \oplus y \in P$. □

Then we have the following theorem under Assumption (A*).

Theorem 3.2: *The output x of Algorithm* Signed_Greedy_Procedure *is the lexicographically maximum solution in P with respect to signed linear ordering $(L = (v_1, \cdots, v_n), \sigma)$ of V and hence is an extreme point of P.*

(Proof) Consider the monotone increasing sequence of subsets $L_i = \{v_1, \cdots, v_i\}$ of V for $i = 1, \cdots, n$. By repeatedly using Lemma 3.1 for $U = L_i$ from $i = 1$ till $i = n$ we can see that the finally obtained x by Algorithm Signed_Greedy_Procedure is the lexicographically maximum solution in P with respect to signed linear ordering $(L = (v_1, \cdots, v_n), \sigma)$ of V and hence is an extreme point of P (also see the arguments in Section 2.2). □

4. Signed-set functions and polyhedra

Before considering a general class of signed-set functions $f : 3^V \to \mathbb{R}$ let us begin with a special class of signed-set functions called *bisubmodular* functions.

A signed-set function $f : 3^V \to \mathbb{R}^V$ is called a *bisubmodular function* if it satisfies the following inequalities:

$$f(X,Y) + f(U,W) \geq f((X\cup U)\setminus(Y\cup W), (Y\cup W)\setminus(X\cup U)) + f(X\cap U, Y\cap W)$$

for all $(X,Y), (U,W) \in 3^V$ (see, e.g., [1, 2, 3, 7]). It is known that every bisubmodular function f with $f(\emptyset,\emptyset) = 0$ is a tight function for the associated *bisubmodular polyhedron*

$$\mathrm{P}_*(f) = \{x \in \mathbb{R}^V \mid \forall (X,Y) \in 3^V : x(X) - x(Y) \leq f(X,Y)\}, \tag{4.1}$$

where for any $X \subseteq V$ define $x(X) = \sum_{v\in X} x(v)$ with $x(\emptyset) = 0$, and note that $x(X) - x(Y) = \langle \chi_{(X,Y)}, x\rangle$. Also, *every* extreme point of the bisubmodular polyhedron $\mathrm{P}_*(f)$ can be computed by a signed greedy procedure for an appropriate signed linear ordering of V, which is exactly the specialization of Algorithm Signed_Greedy_Procedure given in Section 3 to $P = \mathrm{P}_*(f)$ in (4.1) and $Q = 3^V$. In this case, because of the bisubmodularity, every extreme point can be computed by $\mathrm{O}(n)$ calls for the function evaluation of f. It should also be noted that every extreme point of a convex polyhedron $P \subset \mathbb{R}^V$ is lexicographically optimal with respect to a signed linear ordering of V if and only if P is a bisubmodular polyhedron ([5, 1, 3, 7]).

Now consider an integer-valued signed-set function $f : 3^V \to \mathbb{Z} \cup \{+\infty\}$ with $f(\emptyset, \emptyset) = 0$, where f is not necessarily bisubmodular. We regard f as a function on $\mathbb{Z}^V$ with $f(\chi_{(X,Y)}) = f(X, Y)$ for all $(X, Y) \in 3^V$. Define $P = \mathrm{P}_*(f)$ by (4.1) as well. We assume that $\mathrm{P}_*(f)$ is nonempty and bounded, and define

$$Q = \{(X, Y) \in 3^V \mid h_P(\chi_{(X,Y)}) = f(X, Y) < +\infty\}, \tag{4.2}$$

i.e., Q is the set of tight signed sets for $P = \mathrm{P}_*(f)$.

Then for the present integer-valued signed set function f we have the following.

Theorem 4.1: *Suppose that Assumption* (A*) *with Q given by* (4.2) *hold. Then the output x of Algorithm* Signed_Greedy_Procedure *is an integral extreme point of $P = \mathrm{P}_*(f)$.*

(Proof) All the coefficients of the linear inequalities in (4.1) belong to $\{+1, -1, 0\}$. Since f is integer-valued, the output x of Signed_Greedy_Procedure is integral and is an extreme point of $P = \mathrm{P}_*(f)$, due to Theorem 3.2. □

Murota and Tamura [13] have recently shown the existence of integral points in a polyhedron arising as a subdifferential of an integer-valued, integrally convex function ([6]), by means of the Fourier-Motzkin elimination. It should be noted that the Fourier-Motzkin elimination process proceeds from the coordinate subspaces of dimension from n down to 1 while the signed greedy procedure proceeds in a reversed order from 1 to n. When the subdifferential is bounded, the integrality result [13] can be explained by the above theorem. (Note that when the subdifferential is unbounded, it always contains an integral vector.)

Suppose that an integer-valued integrally convex function $f : \mathbb{Z}^V \to \mathbb{Z} \cup \{+\infty\}$ has a bounded subdifferential at $x = \mathbf{0}$ and $f(\mathbf{0}) = 0$. Also suppose for simplicity that $\mathrm{dom}(f) = 3^V$. Now f is a function on 3^V, where recall that $f(X, Y) = f(\chi_{(X,Y)})$ for $(X, Y) \in 3^V$. Then the subdifferential of f at $x = \mathbf{0}$ is given by (4.1) and let Q be given by (4.2). Since by the assumption f is integrally convex and the subdifferential $\mathrm{P}_*(f)$ is bounded, we have $(\{v\}, \emptyset), (\emptyset, \{v\}) \in Q \ \ (\forall v \in V)$ and every extreme point of the subdifferential $\mathrm{P}_*(f)$ is determined by the lower envelope of f restricted on the unit hypercube $\{\chi_{(X,Y)} \mid (X, Y) \sqsubseteq (S, T)\}$ for each orthant (S, T). Hence Assumption (A*) with Q given by (4.2) hold. It follows from Theorem 4.1 that the subdifferential $\mathrm{P}_*(f)$ has an integral extreme point.

5. Concluding Remarks

We have shown the role of the concept of greedy system of linear inequalities played in connection with lexicographically optimal solutions on convex polyhedra and discrete convexity. Our results here give fundamental and useful views on the greedy structures of systems of inequalities and associated polyhedra (Theorem 3.2) and on discrete convexity (Theorems 2.1, 2.2, and 4.1), which explains the integrality result of Murota and Tamura [13] in particular. It is interesting to find other systems and polyhedra to which the present results are applicable. It is also interesting to investigate the possibility of extending the present framework to some of more general discrete convexity (cf. [8, 11]). Finally it should be noted that we have treated only bounded convex polyhedra. We can consider extension of our results to those for unbounded convex polyhedra. Here it should be noted that for an unbounded pointed convex polyhedron P it may happen that no extreme point of P is obtained in a greedy way by Signed_Greedy_Procedure (see, e.g., an example in [13, Remark 3.1]).

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References

[1] K. Ando and S. Fujishige: On structures of bisubmodular polyhedra. *Mathematical Programming*, Ser. A **74** (1996) 293–317.

[2] K. Ando, S. Fujishige and T. Naitoh: A characterization of bisubmodular functions. *Discrete Mathematics* **148** (1996) 299–303.

[3] A. Bouchet and W. H. Cunningham: Delta-matroids, jump systems, and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* **8** (1995) 17–32.

[4] R. Chandrasekaran and S. N. Kabadi: Pseudomatroids. *Discrete Mathematics* **71** (1988) 205–217.

[5] F. D. J. Dunstan and D. J. A. Welsh: A greedy algorithm for solving a certain class of linear programmes. *Mathematical Programming* **62** (1973) 338–353.

[6] P. Favati and F. Tardella: Convexity in nonlinear integer programming. *Ricerca Operativa* **53** (1990) 3–44.

[7] S. Fujishige: *Submodular Functions and Optimization* (North-Holland, 1991).

[8] H. Hirai: Discrete convex functions on graphs and their algorithmic applications. In: Fukunaga T., Kawarabayashi K. (eds) *Combinatorial Optimization and Graph Algorithms* (Springer, Singapore, 2017) 67–100.

[9] S. T. McCormick and S. Fujishige: Strongly polynomial and fully combinatorial algorithms for bisubmodular function minimization. *Mathematical Programming*, Ser. A **122** (2010) 87–120.

[10] K. Murota: *Discrete Convex Analysis* (SIAM, 2003).

[11] K. Murota: Discrete convex analysis: A tool for economics and game theory. *Journal of Mechanism and Institution Design* **1** (2016) 151–273.

[12] K. Murota and A. Shioura: Relationship of M-/L-convex functions with discrete convex functions by Miller and by Favati-Tardella. *Discrete Applied Mathematics* **115** (2001) 151–176.

[13] K. Murota and A. Tamura: Integrality of subgradients and biconjugates of integrally convex functions. arXiv:1806.00992v1 [math.CO] 4 June 2018; revised, 7 September 2018.