Analysis of Process Flexibility Designs under Disruptions

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September, 2018

Abstract. Most of the previous studies of process flexibility designs have focused on expected sales and demand uncertainty. In this paper, we examine the worst-case performance of flexibility designs in the case of demand and supply uncertainties, where the latter can be in the form of either plant or arc disruptions. We define the Plant Cover Index under Disruptions (DPCI) as the minimum required plants’ capacity to supply fixed number of products after the disruptions. By exploiting DPCI, we establish that under symmetric uncertainty sets the worst-case performance can be expressed in terms of DPCI, supply and demand uncertainties. Additionally, DPCI enables us to make meaningful comparisons of different designs. In particular, we demonstrate that under disruptions the 2-long chain design is superior to a broad class of designs. Moreover, we identify a condition wherein both $Q$-short and $Q$-long chain designs have the same worst-case performance. We also discuss the notion of fragility that quantifies the impact of disruptions in the worst case and compare fragilities of $Q$-short and $Q$-long chain designs under different types of disruptions. Finally, by employing DPCI, we develop an algorithm to generate designs that perform well under supply and demand uncertainties in both the worst case and in expectation.

Key words: Worst-case analysis, Process flexibility design, Disruptions, Fragility, Networks

1 Introduction

The predicaments of a contemporary market cause many companies to face heightened levels of uncertainties. This issue has been an area of growing concern and the subject of numerous discussions in the supply chain literature (Chou et al. 2008, Buzacott and Mandelbaum 2008). Flexibility is necessary to respond to such uncertainties (Simchi-Levi 2010). Flexibility enables an expeditious response to changing demands, without raising inventory costs or increasing storage capacity. However, firms can also be exposed to a variety of low-probability high-impact risks that can disrupt operations. Therefore, the challenge is to maximize the benefits of flexibility despite supply and demand uncertainties.

Process flexibility, or the ability of each plant to produce multiple products, is one of the key strategies employed in modern industrial practice to respond to such uncertainties, see Jordan and Graves (1995) and Simchi-Levi (2010). In simpler terms, a design is more flexible if it responds to changes in supply and demand in an efficacious and cost-effective manner (Upton 1994).

Many authors have related accounts of both the successes of flexibility and the failures of inflexibility. For example, the paper by Biller et al. (2006) reports that a failure by Chrysler to keep up with...
the demand, despite having the underutilized capacity at some plants, resulted in an estimated loss of $240M in pretax profit. Mak and Shen (2009) relay media accounts stating that the Ford Motor Company made a $485M investment in 2002 to boost flexibility at their engine and transmission plants worldwide; Chou et al. (2010) report that both GM and Nissan undertake similar initiatives. Process flexibility is an active research area as it has garnered notable attention in industries such as the automobile, textile, and electronics (Chou et al. 2010). We refer the reader to Chou et al. (2008) and Buzacott and Mandelbaum (2008) for an overview of process flexibility designs.

In process flexibility, designs are modeled as bipartite graphs wherein the vertex partitions correspond to plants and products. An arc links a plant to a product if the latter can be produced by the aforementioned plant. Supply-related uncertainty can manifest itself in the form of either arc or plant disruptions, or both. An (plant-to-product) arc disruption occurs when a plant can no longer produce a specific product or its production witnesses a reduced capacity; for example, due to the failure of suppliers and/or machines. Similarly, a plant disruption - for instance, a worker’s strike or a natural disaster - forces a plant to shut down partially or completely and causes massive damage to its production. Most related literature have taken cognizance of uncertainty in flexibility designs without paying enough attention to supply uncertainties. We refer the reader to, for instance, Jordan and Graves (1995), Chou et al. (2011), Simchi-Levi and Wei (2012), Désir et al. (2016), and our own brief literature review in §1.2.

The primary endeavor of this paper is to address this limitation and provide an analytical study to understand the worst-case performance of flexibility designs susceptible to supply and demand uncertainties simultaneously. We specifically focus on the worst-case analysis as it provides insights on the effectiveness of flexibility designs and inspires a new method of generating flexibility designs that are effective with respect to both the worst-case and the expected-case performances. Studying the worst-case performance is particularly helpful whenever companies need to protect themselves in the face of extreme events. Moreover, our method requires very little information about customer demand and supply uncertainties, which makes it effective from the practical perspective as manufacturers often cannot accurately estimate such types of uncertainties.

1.1 Results and Contributions

The main contributions of the paper can be summarized as follows. In §3, we define the notion of the Plant Cover Index under Disruptions (DPCI) that allows us to characterize the worst-case performance of flexibility designs with both demand and supply uncertainties simultaneously.

In §4, we exploit DPCI to show that the 2-long chain is optimal for a wide-range of flexibility designs under the assumption of symmetric uncertainties for plant and arc disruptions as well as demand. Furthermore, we make meaningful comparisons between the performance of $Q$-long chain against $Q$-short chains for any $Q \geq 2$. We demonstrate that the worst-case performances of $Q$-short chain and $Q$-long chain designs coincide whenever the number of complete arc disruptions is sufficiently large, regardless of plant disruptions.
In §5, we extend the notion fragility – previously defined for the expected performance – in order to quantify the effect of disruptions on the worst-case performance of a flexibility design. In particular, we show that $Q$-long chain design is less fragile (sensitive) than $Q$-short chains in the case of a complete plant disruption for any $Q \geq 2$. In contrast, $Q$-short chain designs are less fragile than $Q$-long chain if the number of complete arc disruptions is sufficiently large regardless of other disruption parameters.

In §6, we employ DPCI to develop a heuristic algorithm for constructing flexibility designs that outperform designs generated by other algorithms in the literature, with respect to the worst-case performance while maintaining a similar expected performance in both balanced and unbalanced as well as homogenous and non-homogenous systems. Finally, we note that most of our proofs are relegated to Appendix A.

1.2 Literature Review

Many firms have integrated flexibility into their operation strategies; see, e.g., Simchi-Levi (2010), Fine and Freund (1990), Li and Tirupati (1994, 1997) and Mieghem (1998). Flexibility enables firms to increase the diversification of their product assortment in order to outperform competitors. Furthermore, it empowers them to shift between products at manufacturing facilities as a quick response to heightened variation in demand for a broader array of products. However, this flexibility comes at a cost since a plant is often less costly to be designed for production of only one product than multiple ones (Feng et al. 2017). Thus, the firms must decide upon a wide array of possible flexibilities that they may employ.

Jordan and Graves (1995) observed that the 2-long chain design on its own can offer numerous flexibility-related benefits. Motivated by this work, the concept of the 2-long chain and other sparse designs have found applications in multistage supply chains (Graves and Tomlin 2003), serial production lines (Hopp et al. 2004), and queueing networks (Iravani et al. 2005), among others.

These applications prompted a number of subsequent works that analytically explore the effectiveness of the 2-long chain design from the expected performance point of view. In particular, Chou et al. (2010) are the first to provide a theoretical justification that the performance of 2-long chain is comparable to the full flexibility design. They also show that under some general conditions, the performance of a sparse design can be within $(1-\epsilon)$-optimality of the full flexibility design. Simchi-Levi and Wei (2012) establish the optimality of the 2-long chain among all 2-flexibility designs in the expected performance. However, by relaxing the 2-flexibility restriction Désir et al. (2016) show that the 2-long chain is not optimal over all designs with $2n$ arcs. Specifically, they provide a class of disconnected instances with $2n$ arcs that perform better than the 2-long chain. Wang and Zhang (2015) obtain a distribution-free lower-bound for the ratio of the expected performance of $Q$-long chain over that with full flexibility. Similar in spirit results are shown by Bidkhori et al. (2016) for unbalanced designs.

In general, designing optimal flexibility is challenging due to the combinatorial nature of the underlying problem. For this reason, various heuristics and guidelines have been proposed to construct effective sparse flexibility designs, we refer to, e.g., Chou et al. (2010, 2011).
Design indices can be used as an efficient way to compare the performance of different designs without complex simulations and need for the detailed information on demand uncertainties. The reader can refer to Deng and Shen (2013) who provide a list of indices from the related literature. In particular, the plant cover index (PCI) is proposed by Simchi-Levi and Wei (2015). They illustrate the relationship between PCI and JG index (Jordan and Graves 1995) as well as graph expanders from Chou et al. (2011). The paper by Simchi-Levi and Wei (2015) can be viewed as the most related work to ours in the sense that it also evaluates the performance of flexibility designs facing demand uncertainty by examining the worst-case scenario. However, neither this work nor the aforementioned papers take into account supply uncertainty.

To the best of our knowledge, Lim et al. (2011) is the only paper that studies flexibility designs under supply uncertainty. They introduce the concept of fragility to quantify the change of expected performance of the 2-long and 2-short chain designs resulting from a disruption. Using an approximation scheme, they support their simulation results and numerical study. In particular, they observe that for a single arc disruption, the expected fragility decreases as the size of chains decreases. On the other hand, in the case of a single plant disruption, the expected fragility decreases as the size of chains increases.

Our work is different from Lim et al. (2011) in several important aspects. First, we investigate the fragility with respect to the worst-case performance instead of the expected performance. Second, we consider general $Q$-long and $Q$-short chains with any degree $Q \geq 2$, while Lim et al. (2011) focus on 2-long and 2-short chains. Third, we take into consideration both single and multiple disruptions. Moreover, to develop our results we employ DPCI that does not need any information about the demand. Finally, DCPI is also exploited to provide an algorithm for constructing sparse designs that perform well under disruptions.

2 Definitions

Let sets $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ define plants and products, respectively, where $m$ is the number of plants and $n$ is the number of products. The process flexibility design $D$ is the set of arcs that form a bipartite graph over the sets $A$ and $B$, i.e., $D \subseteq A \times B$. We use $i$, $j$ and $r$ to represent plant, product and arc indices, respectively. The set of neighbors for any $u \in A \cup B$ is denoted by $N(u, D)$, i.e., $N(u, D) = \{v \mid (u, v) \text{ or } (v, u) \in D\}$. Let $\deg_D(u)$ represent the degree of vertex $u \in A \cup B$ in design $D$, i.e., $\deg_D(u) = |N(u, D)|$.

Plant capacities are denoted by vector $c^{(p)} \in \mathbb{R}_+^m$. Similarly, set $C^{(a),D}$ denotes arc capacities of design $D$, i.e., $C^{(a),D} = \{c^{(a),D}_{ij} \mid (a_i, b_j) \in D\}$. For the sake of simplicity, we use $C^{(a)}$ as opposed to $C^{(a),D}$ whenever design $D$ is specified. We say process flexibility design $D$ is homogenous if all plant capacities are the same and non-homogenous when they are not.

Process flexibility design $D$ is balanced if $m = n$ and it is unbalanced, otherwise. It is also assumed that there are no isolated vertices in $D$, that is, $|N(u, D)| \geq 1$ for all $u \in A \cup B$. A balanced design
The set \( \sum \rho \) and flexibility design – such design corresponds to a 2-long chain design, respectively. For any \( Q \), let \( \sum Q \) be a balanced bipartite graph in which all plant and product vertices have the same degree of \( Q \). The most popular example of \( Q \)-long chains is 2-long chain design which is defined as a cycle that connects plants and products in a balanced graph (Jordan and Graves 1995). Other well-known examples of \( Q \)-long chains include the dedicated (\( Q = 1 \)) and full-flexibility (\( Q = n \)) designs (Feng and Shen 2018). See Figures 1b, 1c, and 1d for examples of dedicated, full flexibility, and 2-long chain designs, respectively.

- A \( Q \)-long chain design has an arc set represented by \( \mathcal{LC}_Q \) in which for all \( a_i \in A \), \( \mathcal{N}(a_i, \mathcal{LC}_Q) = \{b_j \mid j = i, i + 1, \ldots, i + Q - 1, \) take \( j - n \) whenever \( j > n \} \). Note that a \( Q \)-long chain design is always connected for \( Q \geq 2 \). The most popular example of \( Q \)-long chains is 2-long chain design which is defined as a cycle that connects plants and products in a balanced graph (Jordan and Graves 1995). Other well-known examples of \( Q \)-long chains include the dedicated (\( Q = 1 \)) and full-flexibility (\( Q = n \)) designs (Feng and Shen 2018). See Figures 1b, 1c, and 1d for examples of dedicated, full flexibility, and 2-long chain designs, respectively.

- A \( Q \)-short chain design is a disconnected graph that comprises of \( c, c \geq 2 \), connected components with sizes \( z_w, w \in \{1, \ldots, c\} \), such that each component is a \( Q \)-long chain design and \( \sum_{w=1}^{c} z_w = n \). An arc set of a \( Q \)-short chain is denoted by \( \mathcal{SC}_Q \). It should be noted that \( Q \leq \min\{z_1, \ldots, z_c\} \), and we assume there are no components of size one. For given \( n \) and \( Q \) we can define a family of \( Q \)-short chain designs represented by \( \{\mathcal{SC}_Q\} \) including all \( \mathcal{SC}_Q \) with different numbers of connected components or components’ sizes; we use \( \mathcal{SC}_Q \) to denote any member of \( \{\mathcal{SC}_Q\} \). Figures 1e and 1f show two members of \( \{\mathcal{SC}_2\} \).

In the remainder of the paper, in any comparison of \( \mathcal{SC}_Q \) and \( \mathcal{LC}_Q \) we assume that both have the same number of plants and products.

Vectors \( \mathbf{d} \in \mathbb{R}_+^n \), \( \mathbf{g} \in [0, 1]^m \) and \( \mathbf{h} \in [0, 1]^{|\mathcal{D}|} \) denote product demands, plant disruptions and arc disruptions, respectively. The set \( \sum([n]) \) includes all permutations for the set \( \{1, \ldots, n\} \) and \( \sigma \in \sum([n]) \). Similarly, \( \sum(\mathcal{D}) \) is the set of all permutations for the index set of design \( \mathcal{D} \), i.e., \( \{\langle i, j \rangle \mid (a_i, b_j) \in \mathcal{D}\} \), and \( \rho \in \sum(\mathcal{D}) \).

The operators \( \min^j(\mathbf{x}) \) and \( \max^j(\mathbf{x}) \) return the \( j \)-th smallest and the \( j \)-th largest elements of vector \( \mathbf{x} = [x_1, x_2, \ldots, x_n] \), respectively. For any \( t \in \mathbb{R} \) define \( t^+ = \max\{0, t\} \). The symbol \( \mathbf{1} \) denotes the vector of all ones of appropriate dimension. Finally, \( \mathbf{1} \) refers to an indicator function.
2.1 Demand and Disruption Uncertainty Sets

One of the major modeling decisions that we have deferred until this point is the selection of appropriate uncertainty sets of demand and disruption scenarios. There are a variety of viable choices for selecting an uncertainty set; see, e.g., Bertsimas and Sim (2004), Bertsimas and Brown (2009), and Bertsimas et al. (2013).

As emphasized by Iravani et al. (2005), exact information in related applications is usually unavailable. Thus, this paper makes the assumption that the uncertainty sets are symmetric, which implies that the worst-case performance will not be altered if products are relabeled. With respect to disruptions, this assumption implies that all arcs and all plants face the same disruption risks.

We note that symmetric demand uncertainty sets are frequently used in the flexibility design literature; see a recent work by Simchi-Levi and Wei (2015). Symmetric uncertainty sets in the worst-case performance correspond to the common assumption of symmetrical designs in the expected performance. Symmetrical designs are designs with exchangeable or identical demand distributions and identical plant capacities; see, e.g., Chou et al. (2010), Chen et al. (2015), Désir et al. (2016), Feng et al. (2017), and Feng and Shen (2018).

It should be mentioned that our results are valid for the entire class of symmetric demand uncertainty sets and not just one. In addition, the analysis for symmetric uncertainty sets provides us insights for problems with asymmetric ones (Deng and Shen 2013). In particular, with the results given in §3 we identify the key feature of the flexibility designs that perform well under any type of uncertainties. This leads to the development of an algorithm in §6 to generate flexibility designs robust against arbitrary types of supply and demand uncertainties.

Formally, given $x \in \mathbb{R}^n$ and permutation $\sigma \in \Sigma([n])$, let $x_\sigma$ be the rearrangement of the elements of $x$ according to permutation $\sigma$, i.e., $x_\sigma = [x_{\sigma(1)}\ x_{\sigma(2)}\ \ldots\ x_{\sigma(n)}]$. Then set $\mathcal{U}$ is symmetric if for any $u \in \mathcal{U}$, $u_\sigma \in \mathcal{U}$ for any permutation of the index set of $\mathcal{U}$. In the remainder of this paper, we use the following demand, arc disruption, and plant disruption uncertainty sets.

**Demand uncertainty set:** Set $\mathcal{U}_d$ denotes the symmetric uncertainty set associated with demands, where $d \in \mathcal{U}_d$ indicates a sample of this set. Examples of symmetric demand uncertainty sets include:

- budgeted uncertainty: $\mathcal{U}_d = \{d \mid d_j = a + b z_j\ \forall j \in \{1,\ldots,n\}, \|z\|_1 \leq \Gamma, \|z\|_\infty \leq 1\}$ for some $a, b, \Gamma \in \mathbb{R}_+$, where $\Gamma$ is known as the **budget of uncertainty**;

- triangle uncertainty : $\mathcal{U}_d = \{d \mid \sum_{j=1}^{n} d_j = t, d_j \geq 0, \forall j \in \{1,\ldots,n\}\}$ for some $t \in \mathbb{R}_+$;

- box uncertainty: $\mathcal{U}_d = \{d \mid \ell \leq d_j \leq u, \forall j \in \{1,\ldots,n\}\}$ for some $\ell, u \in \mathbb{R}_+$;

- ellipsoidal uncertainty: $\mathcal{U}_d = \{d \mid \sum_{j=1}^{n} (d_j - z)^2 \leq t, \forall j \in \{1,\ldots,n\}\}$ for some $z, t \in \mathbb{R}_+$;

or the intersection of any symmetric uncertainty sets.
**Arc disruption uncertainty set:** We use $U_{a}^{\alpha,\beta}$ to denote the symmetric uncertainty set associated with arc disruptions such that at most $\alpha$ arcs can be completely disrupted and at most $|D| - \alpha$ arcs can be partially disrupted proportionally to $\beta \in (0,1]$, i.e., no more than $(1 - \beta)$ fraction of the arc capacity can be lost. Specifically, $U_{a}^{\alpha,\beta}$ is given by

$$U_{a}^{\alpha,\beta} = \left\{ h \in [0,1]^{|D|} | h_r \in \{0,1\} \ \forall r \in R_1 \text{ and } \beta \leq h_r \leq 1, \forall r \in R_2; \right.$$  

$$\beta \in (0,1], \ \ R_1, R_2 \subseteq \{1, \ldots, |D|\}, \ |R_1| \leq \alpha, \ |R_2| \leq |D| - \alpha, \ R_1 \cap R_2 = \varnothing \}$$

where $h_r$ represents the available portion of the capacity of arc $r$ after disruptions. This uncertainty set is similar to the budgeted uncertainty sets commonly used in the robust optimization literature, see, e.g., Bertsimas and Sim (2004); here $\alpha$ and $1 - \beta$ determine the budget of uncertainty and the level of complete and partial disruptions.

**Plant disruption uncertainty set:** We denote by $U_{p}^{\gamma,\lambda}$ the symmetric uncertainty set associated with plant disruptions such that at most $\gamma$ plants can be completely disrupted and at most $m - \gamma$ plants can be partially disrupted proportionally to $\lambda \in (0,1]$, i.e., no more than $(1 - \lambda)$ fraction of the plant capacity can be lost. Specifically, $U_{p}^{\gamma,\lambda}$ is given by

$$U_{p}^{\gamma,\lambda} = \left\{ g \in [0,1]^{m} | g_i \in \{0,1\} \ \forall i \in I_1 \text{ and } \lambda \leq g_i \leq 1, \forall i \in I_2; \right.$$  

$$\lambda \in (0,1], \ I_1, I_2 \subseteq \{1, \ldots, m\}, \ |I_1| \leq \gamma, \ |I_2| \leq m - \gamma, \ I_1 \cap I_2 = \varnothing \}$$

where $g_i$ denotes the available portion of the capacity of plant $i$ after disruptions. Similar to the uncertainty set associated with arc disruptions, $\gamma$ and $1 - \lambda$ determine the budget of uncertainty and the level of complete and partial plant disruptions.

For the sake of simplicity, we use $U_{a}$ and $U_{p}$ wherever there is no ambiguity with respect to disruption parameters $\alpha$, $\beta$ and $\gamma$, $\lambda$, respectively. If $h_r < 1$ ($g_i < 1$), then the corresponding arc (plant) is disrupted. When $\beta = 1$ ($\lambda = 1$) we have only complete arc (plant) disruptions. Conversely, if $\alpha = 0$ ($\gamma = 0$), then there are no complete arc (plant) disruptions.

**Remark 1** Recall that we seek for an optimal solution in the worst case. Clearly, this occurs when the most number of plants and arcs are disrupted. Thus, in order to evaluate any design in the worst case, we can assume that there are exactly $\gamma$ plants and $\alpha$ arcs that are completely disrupted, and exactly $m - \gamma$ plants and $|D| - \alpha$ arcs that lose $(1 - \lambda)$ and $(1 - \beta)$ fractions of their capacities, respectively. □

**Remark 2** In the worst case, the complete arc disruptions occur only for those arcs that are incident to plants which are not completely disrupted. Specifically, in the worst-case scenario the budget of the complete arc disruptions, $\alpha$, is used to disrupt arcs that are still supplied by plants. For example, if for $\gamma = 1$ plant $a_i$ with $\text{deg}_{D}(a_i) = 2$ is completely disrupted and $\alpha = 3$, then in total 5 arcs are completely inactive. □
3 Robust Measure and Plant Cover Index under Disruption

In this section, we formulate the robust measure to evaluate the worst-case performance of general flexibility designs with any plant and arc capacities under supply and demand uncertainties. To this end, first in §3.1 we define the plant cover index under disruptions (DPCI). Then in §3.2, we establish a relationship between DPCI and the robust measure for flexibility designs.

Given vectors \( d \in U_d, g \in U_p \) and \( h \in U_a \), let \( P(d, g, h, D) \) denote the performance of design \( D \). The performance is measured by the maximum possible demand that can be supplied through \( D \) under plant and arc disruptions. Denote by \( f_{ij} \) the amount of demand for product \( j \) satisfied by plant \( i \), i.e., the product flow from \( i \) to \( j \). Then \( P(d, g, h, D) \) can be obtained by solving the following maximum flow problem:

\[
P(d, g, h, D) = \max \left\{ \sum_{(a_i, b_j) \in D} f_{ij} \right\}
\]

s.t. \[ \sum_{b_j \in N(a_i, D)} f_{ij} \leq c^{(p)}_i g_i \quad \forall a_i \in A, \]  
[ \sum_{a_i \in N(b_j, D)} f_{ij} \leq d_j \quad \forall b_j \in B, ]  
[ 0 \leq f_{ij} \leq c^{(a)}_{ij} h_{ij} \quad \forall (a_i, b_j) \in D, ]

where constraint (1b) provides an upper bound on the demand that can be satisfied by plant \( i \) under disruption \( g_i \). Constraint (1c) enforces that the total production of product \( j \) does not exceed its demand. Finally, constraint (1d) ensures that the flow of product \( j \) from plant \( i \) does not exceed the arc capacity under disruption \( h_{ij} \).

From the max-flow min-cut theorem, by taking the dual of (1) we obtain:

\[
P(d, g, h, D) = \min \left\{ \sum_{i=1}^{m} c^{(p)}_i p_i + \sum_{j=1}^{n} d_j q_j + \sum_{(a_i, b_j) \in D} c^{(a)}_{ij} t_{ij} \right\}
\]

s.t. \[ p_i + q_j + t_{ij} \geq 1 \quad \forall (a_i, b_j) \in D, \]  
[ \mathbf{p} \in \{0, 1\}^m, \mathbf{q} \in \{0, 1\}^n, ]  
[ t_{ij} \in \{0, 1\} \quad \forall (a_i, b_j) \in D, ]

where dual variables \( p_i, q_j \) and \( t_{ij} \) correspond to constraints (1b), (1c) and (1d), respectively. Furthermore, by using the total unimodularity property of the constraint matrix (Wolsey and Nemhauser 1999, Corollary 2.8 and Proposition 2.1) it can be shown that the linear programming relaxation of (2) has a binary optimal solution.

Finally, for design \( D \) given uncertainty sets \( U_d, U_p \) and \( U_a \), denote by \( R(U_d, U_p, U_a, D) \) the optimal objective function value of the following problem:
\[ R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) = \min_{d \in \mathcal{U}_d, g \in \mathcal{U}_p, h \in \mathcal{U}_a} P(d, g, h, \mathcal{D}). \]

Simply speaking, \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) \) provides the worst-case performance of design \( \mathcal{D} \) under supply and demand uncertainties.

### 3.1 Plant Cover Index under Disruptions (DPCI)

A subset of plant and product vertices forms a vertex cover if every arc in the design has at least one of its endpoints in this subset. For any integers \( k \) and \( \ell \), where \( 0 \leq k \leq n \) and \( 0 \leq \ell \leq |\mathcal{D}| \), and any vector \( g \in \mathcal{U}_p \) we define the plant cover index under disruptions (DPCI) at \( k \) and \( \ell \) as the minimum plant capacity that is required to create a vertex cover on \( \mathcal{D} \), given that the vertex cover contains exactly \( k \) products, exactly \( \ell \) arcs are ignored (i.e., not required to be covered), and plants are disrupted according to \( g \).

Denote by \( \delta^{k,\ell}(g, \mathcal{D}) \) DPCI at \( k \), \( \ell \) and \( g \). Based on its definition, DPCI can be computed as the objective function value of the following linear 0–1 program for any given \( k \), \( \ell \) and \( g \):

\[
\text{(DPCI)} : \quad \delta^{k,\ell}(g, \mathcal{D}) = \min_{p, q, t} \sum_{i=1}^{m} c_i^{(p)} g_i p_i \\
\text{s.t.} \quad \sum_{j=1}^{n} q_j = k, \\
\sum_{(a_i, b_j) \in \mathcal{D}} t_{ij} = \ell, \\
p_i + q_j + t_{ij} \geq 1 \quad \forall (a_i, b_j) \in \mathcal{D}, \\
p \in \{0, 1\}^m, q \in \{0, 1\}^n, \\
t_{ij} \in \{0, 1\} \quad \forall (a_i, b_j) \in \mathcal{D}. 
\]

Next, we provide some basic properties of DPCI.

**Remark 3** For any design \( \mathcal{D} \) and all \( k \in \{0, \ldots, n\}, \ell \in \{0, \ldots, |\mathcal{D}|\} \), and \( g \in \mathcal{U}_p \) we have:

\( (i) \) \( \delta^{0,0}(g, \mathcal{D}) = \sum_{i=1}^{m} c_i^{(p)} g_i \)

\( (ii) \) \( \delta^{k,|\mathcal{D}|}(g, \mathcal{D}) = 0 \)

\( (iii) \) \( \delta^{n,\ell}(g, \mathcal{D}) = 0 \)

\( (iv) \) \( \delta^{k+1,\ell}(g, \mathcal{D}) \leq \delta^{k,\ell}(g, \mathcal{D}) \)

\( (v) \) \( \delta^{k,\ell+1}(g, \mathcal{D}) \leq \delta^{k,\ell}(g, \mathcal{D}) \)

\( (vi) \) \( \delta^{k,\ell}(g, \mathcal{D}) = \min_{S \subseteq B, |S| = k, E \subseteq D, |E| = \ell} \sum_{a_i \in \mathcal{N}(B \setminus S, D \setminus E)} c_i^{(p)} g_i \)

In particular, Equality \( (vi) \) illustrates that \( \delta^{k,\ell}(g, \mathcal{D}) \) can be expressed as the minimum disrupted capacity of plants incident to \( \mathcal{N}(B \setminus S, D \setminus E) \) for any \( S \subseteq B \) and \( E \subseteq D \) such that \( |S| = k \), \( |E| = \ell \) and \( g \in \mathcal{U}_p \), i.e., subset \( S \cup \mathcal{N}(B \setminus S, D \setminus E) \) creates a vertex cover on design \( \mathcal{D} \setminus E \) such that \( \mathcal{N}(B \setminus S, D \setminus E) \) has the minimum disrupted capacity. \( \square \)
If \( \ell = 0 \) and \( g = e \), then DPCI reduces to PCI proposed by Simchi-Levi and Wei (2015) that corresponds to \( \delta^{k,0}(e, D) \). The authors prove that the problem of computing PCI is \( NP \)-hard. Thus, computing DPCI is also \( NP \)-hard. PCI is employed to characterize the worst-case performance under only demand uncertainties, while by defining DPCI we attempt to take into account possible plant and arc disruptions in addition to demand uncertainties. Specifically, in the next subsection, we show that DPCI provides a convenient tool for evaluating \( R(U_d, U_p, U_a, D) \) and comparing the worst-case performance of different designs.

### 3.2 Robust Measure

Our aim in this subsection is to provide an explicit representation of \( R(U_d, U_p, U_a, D) \) under symmetric uncertainty sets by employing DPCI given by (3). Thereafter, we exploit this representation to compare the worst-case performances of different designs. To this end, Lemma 1 gives us an upper-bound for \( R(U_d, U_p, U_a, D) \) when vectors \( d, g \) and \( h \) are fixed.

**Lemma 1** Given design \( D \), for any \( d \in U_d, g \in U_p, h \in U_a \) and any \( k \in \{0, \ldots, n\} \), \( \ell \in \{0, \ldots, |D|\} \)

\[
R(U_d, U_p, U_a, D) \leq \delta^{k, \ell}(g, D) + \sum_{j=1}^{k} \min^{j}(d) + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^{r}(C^{(a)}). \tag{4}
\]

**Proof.** Let vectors \( p' \in \{0,1\}^m \), \( q' \in \{0,1\}^n \) and \( t'_{ij} \in \{0,1\} \) for \((a_i, b_j) \in D\) be an optimal solution of (3). Thus, \( \sum_{i=1}^{m} c_i^{(p)} g_i p'_i = \delta^{k, \ell}(g, D) \), \( \sum_{j=1}^{n} q'_j = k \) and \( \sum_{(a_i, b_j) \in D} t'_{ij} = \ell \). Note that \( p', q' \) and \( t' \) is also a feasible solution for (2).

Let \( \sigma \) be a permutation in \( \sum([n]) \) such that \( q'_j = 1 \) if and only if \( d_{\sigma(j)} \in \{\min^{z}(d) \mid 1 \leq z \leq k\} \). Similarly, let \( \rho \) be a permutation in \( \sum([D]) \) such that \( t'_{ij} = 1 \) if and only if \( c^{(a)}_{\rho(i,j)} \in \{\min^{z}(C^{(a)}) \mid 1 \leq z \leq \ell, (a_i, b_j) \in D\} \). Moreover, if \( c^{(a)}_{\rho(i,j)} \in \{\min^{z}(C^{(a)}) \mid 1 \leq z \leq \ell - \alpha, (a_i, b_j) \in D\} \), then \( h_{\rho(i,j)} = \beta \), and if \( c^{(a)}_{\rho(i,j)} \in \{\min^{z}(C^{(a)}) \mid \ell - \alpha < z \leq \ell, (a_i, b_j) \in D\} \), then \( h_{\rho(i,j)} = 0 \). Hence, we get

\[
\sum_{i=1}^{m} c_i^{(p)} g_i p'_i + \sum_{j=1}^{n} d_{\sigma(j)} q'_j + \sum_{(a_i, b_j) \in D} c^{(a)}_{ij} h_{\rho(i,j)} t'_{ij} = \delta^{k, \ell}(g, D) + \sum_{j=1}^{k} \min^{j}(d) + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^{r}(C^{(a)}).
\]

Thus, \( P(d_{\sigma}, g, h_{\rho}, D) \leq \delta^{k, \ell}(g, D) + \sum_{j=1}^{k} \min^{j}(d) + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^{r}(C^{(a)}) \). Recall that \( U_d, U_p \) and \( U_a \) are symmetric. Therefore,

\[
R(U_d, U_p, U_a, D) \leq P(d_{\sigma}, g, h_{\rho}, D) \leq \delta^{k, \ell}(g, D) + \sum_{j=1}^{k} \min^{j}(d) + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^{r}(C^{(a)}). \tag*{□}
\]
Next, we provide an explicit formula for \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) \). Specifically, we show that there always exist some integers \( k \) and \( \ell \) as well as vectors \( \mathbf{d}^* \in \mathcal{U}_d \), \( \mathbf{g}^* \in \mathcal{U}_p \), and \( \mathbf{h}^* \in \mathcal{U}_a \) such that Lemma 1 holds at equality.

**Proposition 1** Let \((\mathbf{d}^*, \mathbf{g}^*, \mathbf{h}^*) \in \arg\min_{\mathbf{d} \in \mathcal{U}_d, \mathbf{g} \in \mathcal{U}_p, \mathbf{h} \in \mathcal{U}_a} P(\mathbf{d}, \mathbf{g}, \mathbf{h}, \mathcal{D}) \), then there exist some integers \( 0 \leq k \leq n \) and \( 0 \leq \ell \leq |\mathcal{D}| \) such that

\[
R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) = \delta^{k,\ell}(\mathbf{g}^*, \mathcal{D}) + \sum_{j=1}^{k} \min_j(d^*) + \beta \cdot \sum_{r=1}^{(\ell-\alpha)+} \min_r(C^{(a)}).
\]  

**(5)**

**Proof.** The max-flow problem (1) is always feasible. Since the min-cut problem (2) is the dual of (1) by the strong duality property of linear programs their optimal solutions coincide and we have

\[
P(\mathbf{d}^*, \mathbf{g}^*, \mathbf{h}^*, \mathcal{D}) = \min_{\mathbf{p}, \mathbf{q}, \mathbf{t}} \left\{ \sum_{i=1}^{m} c_i^{(p)} g_i^* p_i + \sum_{j=1}^{n} d_j^* q_j + \sum_{(a_i, b_j) \in \mathcal{D}} c_{ij}^{(a)} h_{ij}^* t_{ij} \right\}
\]

s.t. \( p_i + q_j + t_{ij} \geq 1 \quad \forall (a_i, b_j) \in \mathcal{D}, \)

\( p \in \{0, 1\}^m, q \in \{0, 1\}^n, \)

\( t_{ij} \in \{0, 1\} \quad \forall (a_i, b_j) \in \mathcal{D}. \)

Let \( \mathbf{p}^*, \mathbf{q}^*, \) and \( \mathbf{t}^* \) denote an optimal solution to the optimization problem above, and let also

\( k := \sum_{j=1}^{n} q_j^* \) and \( \ell := \sum_{(a_i, b_j) \in \mathcal{D}} t_{ij}^*. \)

Then we have that \( \sum_{i=1}^{m} c_i^{(p)} g_i^* p_i^* \geq \delta^{k,\ell}(\mathbf{g}^*, \mathcal{D}), \)

\( \sum_{j=1}^{n} d_j^* q_j^* \geq \sum_{j=1}^{k} \min_j(d^*), \)

\( \text{and} \sum_{(a_i, b_j) \in \mathcal{D}} c_{ij}^{(a)} h_{ij}^* t_{ij}^* \geq \beta \cdot \sum_{r=1}^{(\ell-\alpha)+} \min_r(C^{(a)}). \)

Thus, we get

\[
R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) = P(\mathbf{d}^*, \mathbf{g}^*, \mathbf{h}^*, \mathcal{D}) = \sum_{i=1}^{m} c_i^{(p)} g_i^* p_i^* + \sum_{j=1}^{n} d_j^* q_j^* + \sum_{(a_i, b_j) \in \mathcal{D}} c_{ij}^{(a)} h_{ij}^* t_{ij}^*
\]

\[
\geq \delta^{k,\ell}(\mathbf{g}^*, \mathcal{D}) + \sum_{j=1}^{k} \min_j(d^*) + \beta \cdot \sum_{r=1}^{(\ell-\alpha)+} \min_r(C^{(a)}).
\]  

**(6)**

Therefore, by Lemma 1 and Equation (6), Proposition 1 holds for some integers \( 0 \leq k \leq n \) and \( 0 \leq \ell \leq |\mathcal{D}| \).

Lemma 1 and Proposition 1 are extensions of the results derived in Simchi-Levi and Wei (2015), where the latter assumes that there are no disruptions and arcs are uncapacitated. More specifically, due to possible disruptions, our results are different in the following two aspects. First, in (4) and (5) we use DPCI instead of PCI. Second, in (4) and (5) we have an additional third term that is associated with arc disruptions. Next, we exploit Lemma 1 and Proposition 1 to provide an explicit representation of the worst-case performance.
Proposition 2 The worst-case performance of flexibility design $D$ under uncertainty sets $U_d, U_p$, and $U_a$ is given by

$$R(U_d, U_p, U_a, D) = \min_{0 \leq k \leq n, 0 \leq \ell \leq |D|} \left\{ \delta^{k, \ell}(g, D) + \sum_{j=1}^{k} d_j + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^{r}(C^{(a)}) \right\}. \tag{7}$$

Proof. From Lemma 1 and Proposition 1 we get

$$R(U_d, U_p, U_a, D) = \min_{0 \leq k \leq n, 0 \leq \ell \leq |D|} \left\{ \min_{g \in U_p} \delta^{k, \ell}(g, D) + \min_{d \in U_d} \sum_{j=1}^{k} \min^{j}(d) + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^{r}(C^{(a)}) \right\}. \tag{8}$$

The symmetric property of $U_d$ implies that

$$\min_{d \in U_d} \sum_{j=1}^{k} \min^{j}(d) = \min_{d \in U_d} \sum_{j=1}^{k} d_j. \tag{8}$$

Thus,

$$R(U_d, U_p, U_a, D) = \min_{0 \leq k \leq n, 0 \leq \ell \leq |D|} \left\{ \delta^{k, \ell}(g, D) + \sum_{j=1}^{k} d_j + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^{r}(C^{(a)}) \right\}. \tag{8}$$

It is worth mentioning that Equation (7) is valid for both balanced and unbalanced designs with either homogenous or non-homogenous plant capacities wherein arc capacities can be different and limited or uncapacitated.

Next, our aim is to establish conditions for comparison of the worst-case performance of different flexibility designs. For this purpose, we first consider the following definitions.

Definition 1 Design $D_1$ is more symmetrically robust than design $D_2$ if and only if

$$R(U_d, U_p, U_a, D_1) \geq R(U_d, U_p, U_a, D_2),$$

for any symmetric uncertainty sets $U_d, U_p$, and $U_a$ defined in §3.1.

Definition 2 Two different designs $D_1$ and $D_2$ are configured in equal conditions if they have the same number of plants, products and arcs, and their plant and arc capacities belong to the same vector $c^{(p)}$ and set $C^{(a)}$, respectively.

In the following result, we show that the performance of different designs configured in equal conditions can be compared by examining their DPCIs.

Theorem 1 For any designs $D_1$ and $D_2$ configured in equal conditions, design $D_1$ is more symmetrically robust than $D_2$, i.e., $R(U_d, U_p, U_a, D_1) \geq R(U_d, U_p, U_a, D_2)$ for any symmetric uncertainty sets $U_d, U_p$ and $U_a$ if and only if $\min_{g \in U_p} \delta^{k, \ell}(g, D_1) \geq \min_{g \in U_p} \delta^{k, \ell}(g, D_2)$ for any $0 \leq k \leq n$ and $0 \leq \ell \leq |D_1| = |D_2|$. 

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Proof. From Equation (7), we have \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}_t) = \min_{0 \leq j \leq n, 0 \leq \ell \leq |\mathcal{D}_t|} \left\{ \min_{g \in \mathcal{U}_p} \delta_{k, \ell}^k(g, \mathcal{D}_t) + \sum_{j=1}^k d_j + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min_r^r(C^{(a)}, \mathcal{D}_t) \right\} \forall t \in \{1, 2\}. \) (9)

If \( \min_{g \in \mathcal{U}_p} \delta_{k, \ell}^k(g, \mathcal{D}_1) \geq \min_{g \in \mathcal{U}_p} \delta_{k, \ell}^k(g, \mathcal{D}_2) \) for any \( \mathcal{U}_p, 0 \leq k \leq n \) and \( 0 \leq \ell \leq |\mathcal{D}_1| \), then by Equation (9), we get \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}_1) \geq R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}_2) \).

Conversely, if there exist \( \hat{k}, \hat{\ell} \), and \( \mathcal{U}_p \) such that \( \min_{g \in \mathcal{U}_p} \delta_{k, \ell}^k(g, \mathcal{D}_1) < \min_{g \in \mathcal{U}_p} \delta_{k, \ell}^k(g, \mathcal{D}_2) \), then we construct an example to show that \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}_1) < R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}_2) \) for some uncertainty sets \( \mathcal{U}_d \) and \( \mathcal{U}_a \). To this end, define \( C = \sum_{i=1}^m c_i^{(p)} \), and let \( \mathbf{d} \) be the vector such that \( \hat{d}_j = 0 \) for \( 1 \leq j \leq \hat{k} \), and \( \hat{d}_j = C \) for \( \hat{k} < j \leq n \). Moreover, let \( \mathcal{U}_d \) be the set of all permutations of vector \( \mathbf{d} \), i.e., \( \mathcal{U}_d = \sum(<\mathbf{d}) \).

Additionally, let \( \mathcal{U}_a \) be the arc disruption uncertainty set wherein \( \alpha = \hat{\ell} \) and let \( \beta \) be sufficiently large such that \( \beta \cdot C^{(a)}_{t, j} > \lambda \cdot c_i^{(p)} \), for all \( (a_i, b_j) \in \mathcal{D}_t \), \( t \in \{1, 2\} \). Then based on Remark 3 parts \( iv \) and \( v \), we have that \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) = \min_{g \in \mathcal{U}_p} \delta_{k, \ell}^k(g, \mathcal{D}) \). Thus, we get \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}_1) < R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}_2) \). □

Theorem 1 implies that in order to compare the performance of two different designs configured in equal conditions under uncertainties we only need to obtain the information about minimums of their DPCIIs over \( g \in \mathcal{U}_p \), i.e., \( \min_{g \in \mathcal{U}_p} \delta_{k, \ell}^k(g, \mathcal{D}) \) at any \( 0 \leq k \leq n \) and \( 0 \leq \ell \leq |\mathcal{D}| \).

One should note that most of the related literature is mostly focused on the performance of 2-long chain design, \( \mathcal{LC}_2 \). Primarily, it has been observed that fewer, and longer chains are preferred for increasing the expected performance and \( \mathcal{LC}_2 \) has almost the same expected performance as the full flexibility design (Jordan and Graves 1995). However, in Appendix B, we show that these observations cannot be extended to the worst-case performance under disruptions. In particular, we note that \( \mathcal{LC}_2 \) does not have the same worst-case performance as the full flexibility design. Moreover, \( \mathcal{LC}_2 \) and \( \mathcal{SC}_2 \) have the same worst-case performance whenever there is at least one complete arc disruption. These findings encourage us to explore the worst-case performance of chains with degrees higher than 2 and short-chain designs under supply uncertainty in the next section.

4 Worst-Case Performance of Chaining

Herein, we apply the results of the previous section to analyze the worst-case performance of flexibility designs under disruptions and demand uncertainty. In particular, we study the performance of \( \mathcal{LC}_Q \) and any \( \mathcal{SC}_Q \) in \( \{\mathcal{SC}_Q\} \). In §4.1, we show that \( \mathcal{LC}_2 \) is superior to a wide-range of designs, configured in equal conditions, for any type of disruptions. In §4.2, we demonstrate that the worst-case performance of \( \mathcal{LC}_Q \) is the same as that of any \( \mathcal{SC}_Q \) in the presence of a sufficiently large number of complete arc disruptions.

Hereafter, throughout the paper, we assume that designs are homogenous, i.e., all plants have equal capacity and without loss of generality \( c^{(p)} = e \), which is a common assumption in the related literature; see, e.g., Désir et al. (2016), Deng and Shen (2013), Chen et al. (2015), and Wang and Zhang (2015). Before we proceed with the discussion we need the following two technical results.
Lemma 2  For any design $\mathcal{D}$, $0 \leq k \leq n$ and $0 \leq \ell \leq |\mathcal{D}|$, we have

$$\min_{g \in U_p} \delta^{k,\ell}(g, \mathcal{D}) = \lambda \cdot \left( \delta^{k,\ell}(e, \mathcal{D}) - \gamma \right)^+. $$

Indeed, $\min_{g \in U_p} \delta^{k,\ell}(g, \mathcal{D})$ is a quadratic problem; nevertheless, Lemma 2 demonstrates that it can still be solved as linear integer program (3). It also should be noted that by using Lemma 2 we can rewrite Equation (7) as

$$R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) = \min_{0 \leq k \leq n, \ 0 \leq \ell \leq |\mathcal{D}|} \left\{ \lambda \cdot \left( \delta^{k,\ell}(e, \mathcal{D}) - \gamma \right)^+ + \sum_{j=1}^{k} d_j + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^{r}(C^{(a)}) \right\} . $$

Therefore, the necessary and sufficient condition of Theorem 1 reduces to $\delta^{k,\ell}(e, \mathcal{D}_1) \geq \delta^{k,\ell}(e, \mathcal{D}_2)$ for any $0 \leq k \leq n$ and $0 \leq \ell \leq |\mathcal{D}|$. Throughout the paper, $(k^*, \ell^*, d^*)$ is used to denote an optimal solution of (10).

Next, we evaluate $\min_{g \in U_p} \delta^{k,\ell}(g, \mathcal{L}C_Q)$, which can be done via the following technical lemma.

Lemma 3  For $\mathcal{L}C_Q$ and any $0 \leq k \leq n$ and $0 \leq \ell \leq n \cdot Q$, we have

$$\min_{g \in U_p} \delta^{k,\ell}(g, \mathcal{L}C_Q) \geq \lambda \cdot (n - k - \lfloor \frac{\ell}{Q} \rfloor - \gamma)^+. $$

In particular, for any $0 \leq k \leq n - 1$ and $\ell = 0$,

$$\min_{g \in U_p} \delta^{k,0}(g, \mathcal{L}C_Q) = \lambda \cdot \left( \min\{n, n - k + Q - 1\} - \gamma \right)^+. $$

Furthermore, for any $0 \leq k \leq n$ and $(Q - 1)^2 \leq \ell \leq n \cdot Q$, Inequality (11) holds as equality,

$$\min_{g \in U_p} \delta^{k,\ell}(g, \mathcal{L}C_Q) = \lambda \cdot (n - k - \lfloor \frac{\ell}{Q} \rfloor - \gamma)^+. $$

4.1 Superiority of 2-Long Chain

We show that under plant and arc disruptions the 2-long chain design is superior to a broad class of designs in the same configuration. In particular, we establish the following two results. We first demonstrate the superiority of $\mathcal{L}C_2$ over all designs where each product vertex has degree of two. Then we show that $\mathcal{L}C_2$ outperforms all connected designs with $2n$ arcs.

Theorem 2  Let $\mathcal{D}$ be a design such that each product is supplied by exactly two plants, i.e., $|N(u, \mathcal{D})| = 2$ for any $u \in B$. Then the 2-long chain design, $\mathcal{L}C_2$, is more symmetrically robust than $\mathcal{D}$. That is, $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{L}C_2) \geq R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$.

Although Theorem 2 shows the superiority of $\mathcal{L}C_2$, it restricts the optimality of $\mathcal{L}C_2$ over designs in which each product is produced by exactly two plants. In the next theorem, we relax this restriction
for connected designs.

**Theorem 3** Let $\mathcal{D}$ be a connected design such that $|\mathcal{D}| = 2n$, then 2-long chain design, $\mathcal{LC}_2$, is more symmetrically robust than $\mathcal{D}$. That is, $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_2) \geq R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$.

The main idea of the proofs of Theorems 2 and 3 is based on Lemma 3 that provides the exact value of $\min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, \mathcal{LC}_2)$ at any $k$ and $\ell$. In order to establish the superiority of $\mathcal{LC}_2$ over any other class of designs it is sufficient, by Theorem 1, to show that $\min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, \mathcal{LC}_2)\geq \min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, \mathcal{D})$ at any $0 \leq k \leq n$, $0 \leq \ell \leq 2n$ for any $\mathcal{D}$ in that class. Theorems 2 and 3 are generalization of the results in Simchi-Levi and Wei (2015), where the latter results do not take into account supply disruptions. Another key difference of our derivations is the following technical lemma that is employed in the proofs of Theorems 2 and 3, and establishes a special property of the class of flexibility designs in which each product is supplied by two plants.

**Lemma 4** Let $\mathcal{D}$ be a connected design over sets $A$ and $B$ such that for any $u \in B$, $|\mathcal{N}(u, \mathcal{D})| = 2$. Then for any $1 \leq z \leq n$, and any $1 \leq \ell \leq 2n$ there exist some $T \subseteq B$, $|T| = z$ and $E \subseteq \mathcal{D}$, $|E| = \ell$ such that $|\mathcal{N}(T, \mathcal{D} \setminus E)| \leq (z - \lfloor \frac{\ell}{2} \rfloor)$. 

By Theorem 2, we conclude that $\mathcal{LC}_2$ is more symmetrically robust than any $SC_2$ in $\{SC_2\}$. On the other hand, when there is no disruption some studies have shown that the performance of $\mathcal{LC}_2$ is at least as good as any $SC_2$ in the worst-case scenario (Chou et al. 2011) and in the expected performance (Simchi-Levi and Wei 2012). Nonetheless, in the next subsection, we identify conditions under which any $SC_Q$ has the same worst-case performance as $\mathcal{LC}_Q$ under supply and demand uncertainties for any $Q \geq 2$.

### 4.2 Higher Chains

In this subsection, we evaluate the worst-case performance of $Q$-short chains, $SC_Q$, versus $Q$-long chain, $\mathcal{LC}_Q$, for any $Q \geq 2$ when the system is subject to disruptions and demand uncertainty. We show that, in the absence of complete arc disruptions, the performance of $\mathcal{LC}_Q$ is superior to the performance of any $SC_Q$. Additionally, the worst-case performance of any $SC_Q$ is the same as $\mathcal{LC}_Q$ when the number of complete arc disruptions is sufficiently large.

First, we evaluate $\min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, SC_Q)$ for some values of $\ell$, which can be done via the following technical result.

**Lemma 5** For any $SC_Q$ in $\{SC_Q\}$ and $0 \leq k \leq n$ if $\ell = 0$, then

$$\lambda \cdot (n - k - \gamma)^+ \leq \min_{g \in \mathcal{U}_p} \delta^{k,0}(g, SC_Q) \leq \min_{g \in \mathcal{U}_p} \delta^{k,0}(g, \mathcal{LC}_Q).$$

(14)

Additionally, if $(Q - 1)^2 \leq \ell \leq n \cdot Q$, then

$$\min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, SC_Q) = \min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, \mathcal{LC}_Q) = \lambda \cdot (n - k - \lceil \frac{\ell}{Q} \rceil - \gamma)^+.  \quad (15)$$

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Up to this point, we considered a general form of designs wherein arc capacities could be different or limited. Hereafter, throughout the paper, we narrow our attention to designs satisfying the following assumption.

**Assumption 1** Arc capacities are sufficiently large; specifically, \( \beta \cdot C_{ij}^{(a)} > \lambda \cdot c_{i}^{(p)} \) for all \((a_i, b_j) \in D\).

An uncapacitated arc is the extreme case of Assumption 1 which appears in the majority of related studies on process flexibility designs; see, e.g., Chou et al. (2011), Chen et al. (2015), Désir et al. (2016), and Feng and Shen (2018). Note that under Assumption 1 in Equation (10) we have \( \ell^* = \alpha \).

As a direct consequence of Lemma 5 and Theorem 1 under Assumption 1, we obtain the main result of this subsection. In particular, we demonstrate that \( SC_Q \) and \( LC_Q \) have the same performance for sufficiently large number of complete arc disruptions.

**Proposition 3** For any \( SC_Q \) in \( \{SC_Q\} \) and any disruption parameters \( \gamma, \lambda \) and \( \beta \) if \( \alpha = 0 \), then

\[
R(U_d, U_p, U_a, SC_Q) \leq R(U_d, U_p, U_a, LC_Q). \tag{16}
\]

Additionally, if \((Q - 1)^2 \leq \alpha \leq n \cdot Q\), then

\[
R(U_d, U_p, U_a, SC_Q) = R(U_d, U_p, U_a, LC_Q). \tag{17}
\]

Inequality (16) indicates that in the absence of complete arc disruptions – regardless of plant disruptions – the worst-case performance of \( LC_Q \) is at least as good as the worst-case performance of any \( SC_Q \). This result is consistent with the literature that – without disruptions – the performance of \( LC_Q \) is better than that of any \( SC_Q \) in both the worst case (Chou et al. 2011) and in expectation (Jordan and Graves 1995, Simchi-Levi and Wei 2012, Chou et al. 2011).

However, Equation (17) implies that for sufficiently large number of complete arc disruptions any \( SC_Q \) in \( \{SC_Q\} \) has the same worst-case performance as \( LC_Q \). Therefore, if there is at least one complete arc disruption, i.e., \( \alpha \geq 1 \), then from the worst-case performance perspective any \( SC_2 \) is also optimal over the design classes considered in Theorems 2 and 3.

Moreover, note that if the cost of flexibility increases with products dissimilarities, then constructing longer chains is more expensive than multiple shorter ones, e.g., producing two dissimilar products in one plant versus two similar ones (Lim et al. 2011). Therefore, by grouping similar products in short chains we can minimize the cost of flexibility and simultaneously guarantee the optimality of the (worst-case) performance of the resulting short-chain design versus the long-chain design.

The findings of Proposition 3 are also exploited to analyze the sensitivity of chains to disruptions studied in the next section.
5 Fragility

In this section, we analyze the sensitivity of chains to disruptions. Fragility is the notion originally proposed by Lim et al. (2011) to quantify the impact of disruptions on the expected-case performance of flexibility designs, in particular, in the context of analyzing the sensitivity of 2-long and 2-short chains. In the following, we extend the concept of fragility for the worst-case performance. We show that Q-short chain designs are less fragile (sensitive) than Q-long chain if the number of complete arc disruptions are sufficiently large regardless of the other disruption parameters for any $Q \geq 2$. In contrast, Q-long chain design is less fragile than Q-short chains in the case of a complete plant disruption.

Formally, let $R(\mathcal{U}_d, \mathcal{D})$ represent the worst-case performance of design $\mathcal{D}$ without disruptions and only subject to demand uncertainty. Then from Lemma 1, Proposition 1 and Assumption 1 we have:

$$R(\mathcal{U}_d, \mathcal{D}) = R(\mathcal{U}_d, \mathcal{U}_p^{0,1}, \mathcal{U}_a^{0,1}, \mathcal{D}) = \min_{0 \leq k \leq n, \mathcal{d} \in \mathcal{U}_d} \left\{ \delta^{k,0}(\mathcal{e}, \mathcal{D}) + \sum_{j=1}^{k} d_j \right\}. \tag{18}$$

The fragility of design $\mathcal{D}$, denoted by $Fr(\mathcal{D})$, with respect to uncertainty sets $\mathcal{U}_d, \mathcal{U}_p$ and $\mathcal{U}_a$ is the difference in the worst-case performance with and without disruptions, i.e.,

$$Fr(\mathcal{D}) = R(\mathcal{U}_d, \mathcal{D}) - R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D}) = \min_{0 \leq k \leq n, \mathcal{d} \in \mathcal{U}_d} \left\{ \delta^{k,0}(\mathcal{e}, \mathcal{D}) + \sum_{j=1}^{k} d_j \right\} \tag{19}$$

$$- \min_{0 \leq k \leq n, 0 \leq \alpha \leq \mathcal{D}} \left\{ \lambda \cdot (\delta^{k,0}(\mathcal{e}, \mathcal{D}) - \gamma)^+ + \sum_{j=1}^{k} d_j + \beta \cdot \sum_{r=1}^{(\ell-\alpha)^+} \min^r(C^{(a)}) \right\}.$$

When the disruptions occur, the fragility of design $\mathcal{D}$ is denoted by the amount of reduction in $R(\mathcal{U}_d, \mathcal{D})$. In this section, we make, to some extent, a surprising observation that if a system is subject to sufficiently large number of arc disruptions, then the worst-case performance of any SCQ in $\{SCQ\}$ is less sensitive than LCQ, i.e., $Fr(SCQ) \leq Fr(LCQ)$. This result is independent of the other disruption parameters as well as the number of short-chain design components. On the contrary, for a single complete plant disruption we show that $Fr(LCQ) \leq Fr(SCQ)$.

We start with the following result to demonstrate that under complete arc disruptions the fragility of any SCQ is less than that of LCQ.

**Proposition 4** Let designs be subject to sufficiently large number of complete arc disruptions, i.e., $\alpha \geq (Q - 1)^2$, then $Fr(SCQ) \leq Fr(LCQ)$ for any SCQ in $\{SCQ\}$ and any disruption parameters $\beta, \gamma$ and $\lambda$.

**Proof.** First, consider the case without disruptions, i.e., $\alpha = \gamma = 0$ and $\beta = \lambda = 1$. Then by Inequality (16) we have $R(\mathcal{U}_d, \mathcal{U}_p^{0,1}, \mathcal{U}_a^{0,1}, SCQ) \leq R(\mathcal{U}_d, \mathcal{U}_p^{0,1}, \mathcal{U}_a^{0,1}, LCQ)$, or equivalently

$$R(\mathcal{U}_d, SCQ) \leq R(\mathcal{U}_d, LCQ). \tag{i}$$

When designs are subject to $\alpha \geq (Q - 1)^2$ arc disruptions, based on Equation (17) we get
\[ R(U_d, U_p^\alpha, U_a^\alpha, SC_Q) = R(U_d, U_p^\alpha, U_a^\alpha, LC_Q), \]

for any \( \beta, \gamma \) and \( \lambda \). Therefore, by (i), (ii), and the definition of fragility we conclude that Proposition 4 holds.

Although Proposition 4 shows that any \( SC_Q \) in \( \{SC_Q\} \) is less fragile than \( LC_Q \) in the case of sufficiently large number of complete arc disruptions, it does not determine how we can compare two different members of \( \{SC_Q\} \) with respect to fragility. In the following, we address this issue.

Consider two short chains \( SC_Q^{(1)} \) and \( SC_Q^{(2)} \) in \( \{SC_Q\} \), with component sizes \( z_1^{(1)}, \ldots, z_{c(1)}^{(1)} \) and \( z_1^{(2)}, \ldots, z_{c(2)}^{(2)} \), respectively. We say that the components of \( SC_Q^{(1)} \) are decomposition of the components of \( SC_Q^{(2)} \) if for every \( k \in \{0, \ldots, n\} \) such that \( k = \sum_{i \in I_2} z_i^{(2)} \) for some \( I_2 \subseteq \{1, \ldots, c(2)\} \), there exists some \( I_1 \subseteq \{1, \ldots, c(1)\} \) for which \( \sum_{i \in I_1} z_i^{(1)} = k \). Next, we show that if the components of \( SC_Q^{(1)} \) can get decomposed into the components of \( SC_Q^{(2)} \), then \( SC_Q^{(1)} \) is less fragile.

**Proposition 5** Consider two short chains \( SC_Q^{(1)} \) and \( SC_Q^{(2)} \) in \( \{SC_Q\} \) with component sizes \( z_1^{(1)}, \ldots, z_{c(1)}^{(1)} \) and \( z_1^{(2)}, \ldots, z_{c(2)}^{(2)} \), respectively. For any disruption parameters \( \beta, \gamma, \) and \( \lambda \) suppose that \( \alpha \geq (Q-1)^2 \). If the components of \( SC_Q^{(1)} \) are decomposition of the components of \( SC_Q^{(2)} \), then \( Fr( SC_Q^{(1)} ) \leq Fr( SC_Q^{(2)} ) \).

**Proof.** Under Assumption 1, we conclude that \( \ell^* = \alpha \) in Equation (10) for any design \( D \). Moreover, for \( \alpha \geq (Q-1)^2 \) by leveraging Equation (15) in Lemma 5, we get

\[
\min_{g \in U_p} \delta^{k,\alpha}(g, SC_Q^{(1)}) = \min_{g \in U_p} \delta^{k,\alpha}(g, SC_Q^{(2)}) \quad \forall k \in \{1, \ldots, n\};
\]

this leads to \( R(U_d, U_p, U_a, SC_Q^{(1)}) = R(U_d, U_p, U_a, SC_Q^{(2)}) \) by Theorem 1. Next, we ascertain the relationship between the performances without any disruptions, i.e., \( R(U_d, SC_Q^{(1)}) \) and \( R(U_d, SC_Q^{(2)}) \). To this end, if we demonstrate that \( \delta^{k,0}(e, SC_Q^{(1)}) \leq \delta^{k,0}(e, SC_Q^{(2)}) \) holds true for any \( 0 \leq k \leq n \), then we get \( R(U_d, SC_Q^{(1)}) \leq R(U_d, SC_Q^{(2)}) \) according to Lemma 2 and Theorem 1.

Recall that \( \delta^{k,0}(e, D) \) is the minimum number of plants that is required to create a vertex cover along with \( k \) products on design \( D \). Notably, by Inequality (14) we have \( \delta^{k,0}(e, SC_Q^{(1)}) \geq n-k, \ t \in \{1, 2\} \). It is clear that \( \delta^{k,0}(e, SC_Q^{(1)}) = \delta^{k,0}(e, SC_Q^{(2)}) \) for \( k = 0 \) and \( n \). To evaluate \( \delta^{k,0}(e, SC_Q^{(t)}) \), \( t \in \{1, 2\} \) for \( 1 \leq k < n \), we consider the following disjoint cases:

1. There exist some \( I_1 \subseteq \{1, \ldots, c(1)\} \) and \( I_2 \subseteq \{1, \ldots, c(2)\} \), such that \( \sum_{i \in I_1} z_i^{(1)} = \sum_{i \in I_2} z_i^{(2)} = k \). Thus, in \( SC_Q^{(1)} \) (similarly, in \( SC_Q^{(2)} \)), in addition to \( k \) products of components within \( I_1 \) (\( I_2 \)), \( n-k \) plants of components within \( \{1, \ldots, c(1)\} \setminus I_1 \) (for \( SC_Q^{(2)} \) plants in \( \{1, \ldots, c(2)\} \setminus I_2 \)) are required to create a vertex cover. This is the minimum vertex cover for a particular \( k \) since by Inequality (14), we have \( \delta^{k,0}(e, SC_Q^{(t)}) \geq n-k, \ t \in \{1, 2\} \). Therefore, \( \delta^{k,0}(e, SC_Q^{(1)}) = \delta^{k,0}(e, SC_Q^{(2)}) = n-k \).

2. There exists \( I_1 \subseteq \{1, \ldots, c(1)\} \) such that \( k = \sum_{i \in I_1} z_i^{(1)} \); however, there does not exist any \( I_2 \subseteq \{1, \ldots, c(2)\} \) such that \( k = \sum_{i \in I_2} z_i^{(2)} \). Thus, by the argument given in part 1, we have \( \delta^{k,0}(e, SC_Q^{(1)}) = \delta^{k,0}(e, SC_Q^{(2)}) = n-k \).
3. The cases where there exists no $I_1 \subset \{1, \ldots, c(1)\}$ such that $k = \sum_{i \in I_1} z_i^{(1)}$, but there is $I_2 \subset \{1, \ldots, c(2)\}$ such that $k = \sum_{i \in I_2} z_i^{(2)}$, are not considered. These situations are impossible due to the fact that the components of $SC_Q^{(1)}$ are decomposition of the components of $SC_Q^{(2)}$.

4. There does not exist any $I_1 \subset \{1, \ldots, c(1)\}$ such that $k = \sum_{i \in I_1} z_i^{(1)}$. Likewise, there are no $I_2 \subset \{1, \ldots, c(2)\}$ such that $k = \sum_{i \in I_2} z_i^{(2)}$.

Before we proceed, let us consider an arbitrary $SC_Q$ in $\{SC_Q\}$ with the set of components $\{1, \ldots, c\}$ and component sizes $\{z_1, \ldots, z_c\}$. For any given $k$, let $k_i$ represent the number of products of component $i \in \{1, \ldots, c\}$ in a vertex cover. Then we define $J$ as the subset of components that have all of their products within the vertex cover ($k_i = z_i$ for all $i \in J$), and $k_J := \sum_{i \in J} z_i = \sum_{i \in J} k_i$. By the definition of $J$, it is clear that we have $0 \leq k_i < z_i$ for all $i \in \{1, \ldots, c\} \setminus J$. It should also be noted that we must have $k_J + \sum_{i \in \{1,\ldots,c\} \setminus J} k_i = k$. By the assumption there does not exist any $I \subset \{1, \ldots, c\}$ such that $k = \sum_{i \in I} z_i$; it implies that $k_J < k$ and $\sum_{i \in \{1,\ldots,c\} \setminus J} k_i = k - k_J > 0$.

First, we demonstrate that for any given $k$, a minimum vertex cover for $SC_Q$ can be obtained when $k_J$ has the maximum possible value and $\sum_{i \in \{1,\ldots,c\} \setminus J} 1_{\{0 < k_i < z_i\}} = 1$; i.e., exactly one component has at least one ($0 < k_i$), but not all of its products ($k_i < z_i$) within the vertex cover.

It is observed that all arcs of the components in $J$ are covered by $k_J$ products. Since each component $i \in \{1, \ldots, c\} \setminus J$ is a $Q$-long chain, by Equation (12) we require $\min\{z_i, z_i - k_i + Q - 1\}$ of its plants along with its $k_i$ products to cover all its arcs. Thus, the total number of plants that are required to create the vertex cover is

$$(n - k_J) - \sum_{i \in \{1,\ldots,c\} \setminus J} (z_i - \min\{z_i, z_i - k_i + Q - 1\}) = n - k_J - \sum_{i \in \{1,\ldots,c\} \setminus J} \min\{0, -k_i + Q - 1\}. \quad (i)$$

To obtain a minimum vertex cover we need to minimize the right-hand side of (i) that is strictly decreasing in $k_J$ and is non-increasing in $k_i$ for all $i \in \{1, \ldots, c\} \setminus J$. In particular, the unit increase of $k_J$ and $k_i$ decreases the right-hand side of (i) by exactly 1 unit and at the most 1 unit, respectively. Hence, we set $k_J$ at its maximum possible value.

Recall that $\sum_{i \in \{1,\ldots,c\} \setminus J} k_i = k - k_J > 0$. The minimum quantum of the right-hand side of (i) is obtained when along with the largest value of $k_J$, for exactly one component, say $t \in \{1, \ldots, c\} \setminus J$, $0 < k_t < z_t$, i.e., $k_t = k - k_J$ and $k_t = 0$ for all $i \in \{1, \ldots, c\} \setminus (J \cup \{t\})$. To elaborate further, it should be observed that $\min\{0, -k_t + Q - 1\} = -k_t + \min\{k_t, Q - 1\}$. Then we get

$$\sum_{i \in \{1,\ldots,c\} \setminus J} (-k_i + \min\{k_i, Q - 1\}) = -(k - k_J) + \sum_{i \in \{1,\ldots,c\} \setminus J} \min\{k_i, Q - 1\} \geq -k_t + \min\{k_t, Q - 1\},$$
or equivalently \( \sum_{i \in \{1, \ldots, c\}} \min(k_i, Q - 1) \geq \min(k, Q - 1) \) from the last inequality. This is true because if \( k_i \leq Q - 1 \) for all \( i \in \{1, \ldots, c\} \setminus J \), then \( \sum_{i \in \{1, \ldots, c\}} \min(k_i, Q - 1) = \sum_{i \in \{1, \ldots, c\}} k_i = k - k_J \geq \min(k, Q - 1) \). On the other hand, if \( k_i > Q - 1 \) for \( i \in I \) where \( I \subseteq \{1, \ldots, c\} \setminus J \), then \( |I| \cdot (Q - 1) + \sum_{i \in \{1, \ldots, c\} \setminus (J \cup I)} \min(k_i, Q - 1) \geq \min(k, Q - 1) \). It should be noted that there may exist multiple minimum vertex covers; we can obtain one of them in this manner.

Now, by the aforementioned argument, we create a minimum vertex cover in \( \mathcal{SC}_Q^{(1)} \). Let \( J_1 \) be the largest subset of \( \{1, \ldots, c(1)\} \) such that \( \sum_{i \in J_1} z_i^{(1)} \leq k \) and \( k_{J_1} := \sum_{i \in J_1} z_i^{(1)} \). Next, select component \( x \in \{1, \ldots, c(1)\} \setminus J_1 \) and \( k_x := k - k_{J_1} \). Put all products of \( J_1 \) and \( k_x \) products of \( x \) within the vertex cover. By (i), we can see that the minimum number of required plants is \( \delta^{k,0}(e, \mathcal{SC}_Q^{(1)}) = n - k_{J_1} + \min\{0, -k_x + Q - 1\} = \min\{n - k_{J_1}, n - k + Q - 1\} \).

Similarly, in \( \mathcal{SC}_Q^{(2)} \) let \( J_2 \) denote the largest subset of \( \{1, \ldots, c(2)\} \) such that \( \sum_{i \in J_2} z_i^{(2)} \leq k \) and \( k_{J_2} := \sum_{i \in J_2} z_i^{(2)} \). Then we can select component \( y \in \{1, \ldots, c(2)\} \setminus J_2 \) and \( k_y := k - k_{J_2} \). By (i), the minimum number of required plants is \( \delta^{k,0}(e, \mathcal{SC}_Q^{(1)}) = n - k_{J_2} + \min\{0, -k_y + Q - 1\} = \min\{n - k_{J_2}, n - k + Q - 1\} \).

It should be noted that \( k_{J_1} \geq k_{J_2} \) because by assumption, the components of \( \mathcal{SC}_Q^{(1)} \) are decomposition of \( \mathcal{SC}_Q^{(2)} \) components. Therefore, \( \min\{n - k_{J_1}, n - k + Q - 1\} \leq \min\{n - k_{J_2}, n - k + Q - 1\} \).

As a result, \( \delta^{k,0}(e, \mathcal{SC}_Q^{(1)}) \leq \delta^{k,0}(e, \mathcal{SC}_Q^{(2)}) \).

According to parts 1 to 4, we have \( \delta^{k,0}(e, \mathcal{SC}_Q^{(1)}) \leq \delta^{k,0}(e, \mathcal{SC}_Q^{(2)}) \), for all \( 0 \leq k \leq n \). Therefore, by using Lemma 2 and Theorem 1, we get \( R(\mathcal{U}_d, \mathcal{SC}_Q^{(1)}) \leq R(\mathcal{U}_d, \mathcal{SC}_Q^{(2)}) \). Recalling that \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q^{(1)}) = R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q^{(2)}) \), the proof is complete.

The following example illustrates that if the condition of Proposition 5 does not hold, then the fragility of the designs depends on the uncertainty sets considered.

**Example 1** Let \( \mathcal{SC}_3^{(1)} \) and \( \mathcal{SC}_3^{(2)} \) include five and three equal size components as \( \{3, 3, 3, 3, 3\} \) and \( \{5, 5, 5\} \), respectively. By using Equation (15), we can observe that \( \min_{g \in \mathcal{U}_p} \delta^{k,\ell}(e, \mathcal{SC}_3^{(1)}) = \min_{g \in \mathcal{U}_p} \delta^{k,\ell}(e, \mathcal{SC}_3^{(2)}) \) for all \( 0 \leq k \leq n \) and \( 4 \leq \ell \leq 45 \). Thus, Theorem 1 leads to \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_3^{(1)}) = R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_3^{(2)}) \) for \( 4 \leq \alpha \leq 45 \); however, for \( \ell = 0 \), we have \( \delta^{5,0}(e, \mathcal{SC}_3^{(1)}) = 12 > \delta^{5,0}(e, \mathcal{SC}_3^{(2)}) = 10 \) and \( \delta^{6,0}(e, \mathcal{SC}_3^{(1)}) = 9 < \delta^{6,0}(e, \mathcal{SC}_3^{(2)}) = 10 \). Therefore, none of the designs’ fragility dominates the other because by Theorem 1, we can find demand uncertainty sets such that \( R(\mathcal{U}_d, \mathcal{SC}_3^{(1)}) < R(\mathcal{U}_d, \mathcal{SC}_3^{(2)}) \) or \( R(\mathcal{U}_d, \mathcal{SC}_3^{(1)}) > R(\mathcal{U}_d, \mathcal{SC}_3^{(2)}) \).

Let again the components of \( \mathcal{SC}_Q^{(1)} \) be decomposition of the components of \( \mathcal{SC}_Q^{(2)} \). For \( \alpha \geq (Q - 1)^2 \), based on Propositions 4 and 5 we have \( Fr(\mathcal{SC}_Q^{(1)}) \leq Fr(\mathcal{SC}_Q^{(2)}) \leq Fr(\mathcal{LC}_Q) \).

Moreover, by Equation (17) \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q^{(1)}) = R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q^{(2)}) = R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_Q) \).
Hence, we conclude
\[ R(\mathcal{U}_d, SC_Q^{(1)}) \leq R(\mathcal{U}_d, SC_Q^{(2)}) \leq R(\mathcal{U}_d, LC_Q). \]  \hfill (20)

Relation (20) implies that, without disruptions, longer chains (i.e., chains with fewer number of components) have better performance than shorter ones. Moreover, the performance of chains is bounded above by \( R(\mathcal{U}_d, LC_Q) \). On the other hand, recall the discussion in §4.2 that the construction of shorter chains is less costly. Therefore, there is a trade-off between the worst-case performance of a chain without disruptions and the cost of flexibility.

In contrast to Proposition 4, we show in the following result that if there is only one complete plant disruption without any complete arc disruptions, then the fragility of \( LC_Q \) is less than that of any \( SC_Q \).

**Proposition 6** If a design is subject to only one complete plant disruption, i.e., \( \alpha = 0 \) and \( \beta = \gamma = \lambda = 1 \), then \( Fr(LC_Q) \leq Fr(SC_Q) \) for any \( SC_Q \) in \( \{SC_Q\} \).

**Proof.** Equations (10) and (18) represent the worst-case performance of design \( \mathcal{D} \) with and without disruptions, respectively. Let \((k_{SCQ}^*, \ell_{SCQ}^*, d_{SCQ}^*)\) and \((k_{LCQ}^*, \ell_{LCQ}^*, d_{LCQ}^*)\) denote an optimal solution of Equation (10) for \( SC_Q \) and \( LC_Q \). Similarly, let \((k_{SCQ}^*, d_{SCQ}^*)\) and \((k_{LCQ}^*, d_{LCQ}^*)\) represent an optimal solution of Equation (18) for \( SC_Q \) and \( LC_Q \). Since there are no complete arc disruptions (\( \alpha = 0 \)), by Assumption 1, we have \( \ell_{SCQ}^* = \ell_{LCQ}^* = 0 \). Next, we consider the four following disjoint cases on \( k_{SCQ}^* \) and \( k_{LCQ}^* \) in \( \{0, \ldots, n\} \):

1. Let \( k_{SCQ}^* < n \) and \( k_{LCQ}^* < n \). Any vertex cover of general design \( \mathcal{D} \) can be denoted by sets \( S \subseteq B \) and \( N'(B \setminus S, \mathcal{D}) \). In addition, let \( S^* \subseteq B \), \( |S^*| = k_{SCQ}^* < n \) and \( N'(B \setminus S^*, SC_Q) > 0 \) denote a minimum vertex cover of \( SC_Q \) without disruptions. Hence, by Remark 3 part (vi), we have \( \delta^{k_{SCQ}^*, e}(e, SC_Q) = |N(B \setminus S^*, SC_Q)| \) and by Equation (18), we get

\[
R(\mathcal{U}_d, SC_Q) = |N(B \setminus S^*, SC_Q)| + \sum_{j=1}^{k_{SCQ}^*} \min(d_{SCQ}^*) = |N(B \setminus S^*, SC_Q)| + \min_{d \in \mathcal{U}_d} \sum_{j=1}^{k_{SCQ}^*} d_j. \hfill (i)
\]

The second equality in \((i)\) follows by fixing \( k_{SCQ}^* \) in Equation (18). A single complete plant disruption makes the capacity of a plant zero. We claim that in case of a single complete plant disruption, sets \( S^* \) and \( N'(B \setminus S^*, SC_Q) \) indicate a minimum vertex cover of \( SC_Q \) with disruptions, as well. If our claim holds true, then \( |S^*| = k_{SCQ}^* = k_{SCQ}^* \). Moreover, by Equation (10) we will have

\[
R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, SC_Q) = (\delta^{k_{SCQ}^*, e}(e, SC_Q) - 1)^+ + \sum_{j=1}^{k_{SCQ}^*} \min(d_{SCQ}^*) = |N(B \setminus S^*, SC_Q)| - 1 + \min_{d \in \mathcal{U}_d} \sum_{j=1}^{k_{SCQ}^*} d_j. \hfill (ii)
\]
The second equality in (ii) is obtained by fixing $k^*_SC_Q$ in Equation (10), and due the fact that $\delta^{k^*_SC_Q,0}(e, SC_Q) = |N(B \setminus S^*, SC_Q)| > 0$. Thus, by considering (i) and (ii), we get $Fr(SC_Q) = R(U_d, SC_Q) - R(U_d, U_p, U_a, SC_Q) = 1$.

We prove the validity of our claim by contradiction. Suppose $S^* \cup N(B \setminus S^*, SC_Q)$ is the minimum vertex cover of $SC_Q$ without disruptions, but it is not the minimum vertex cover of $SC_Q$ with disruptions, and sets $\tilde{S} \subseteq B$, $|\tilde{S}| = k^*_SC_Q$, and $N(B \setminus \tilde{S}, SC_Q)$ corresponds to the minimum vertex cover of $SC_Q$ with disruptions. In that case,

$$R(U_d, U_p, U_a, SC_Q) = \left(|N(B \setminus \tilde{S}, SC_Q)| - 1\right)^+ + \min_{d \in U_d} \sum_{j=1}^{k^*_SC_Q} d_j < |N(B \setminus S^*, SC_Q)| - 1 + \min_{d \in U_d} \sum_{j=1}^{k^*_SC_Q} d_j.$$

As a result, by (i)

$$|N(B \setminus \tilde{S}, SC_Q)| + \min_{d \in U_d} \sum_{j=1}^{k^*_SC_Q} d_j < R(U_d, SC_Q). \quad (iii)$$

Both $S^* \cup N(B \setminus S^*, SC_Q)$ and $\tilde{S} \cup N(B \setminus \tilde{S}, SC_Q)$ represent vertex covers for $SC_Q$. Thus, Relation (iii) contradicts the minimality of vertex cover $S^* \cup N(B \setminus S^*, SC_Q)$ for $SC_Q$ without disruptions.

For $LC_Q$ by using a similar argument, we prove that $Fr(LC_Q) = 1$. Therefore, $Fr(LC_Q) = Fr(SC_Q) = 1$ when $k^*_SC_Q < n$ and $k^*_LC_Q < n$.

2. Let $k^*_SC_Q < k^*_LC_Q = n$, then by part 1, we have $Fr(SC_Q) = 1$ since $k^*_SC_Q < n$. In addition, by Equation (18), we have $R(U_d, LC_Q) = \sum_{j=1}^{n} d^*_j,LC_Q$. Next, we evaluate $Fr(LC_Q)$ to prove that $Fr(LC_Q) \leq 1$.

Let $Y$ denote the set of products whose demand is larger than 1 in $d^*_j,LC_Q$, i.e., $Y = \{j \in B \mid d^*_j,LC_Q > 1\}$ and $Y' := N(Y, LC_Q)$. Clearly, it can be seen that $|Y| \leq |Y'|$ and set $(B \setminus Y) \cup Y'$ creates a vertex cover on $LC_Q$.

**Fact 1.** Vector $d^*_j,LC_Q$ satisfies

$$|Y'| + \sum_{j=1}^{n-|Y|} \min^j(d^*_j,LC_Q) \geq \sum_{j=1}^{n} d^*_j,LC_Q, \quad (iv)$$

since $(B \setminus Y) \cup Y'$ creates a vertex cover in $LC_Q$ that corresponds to the feasible solution $(k = n - |Y|, d^*_j,LC_Q)$ with the objective value $|Y'| + \sum_{j=1}^{n-|Y|} \min^j(d^*_j,LC_Q)$ for Equation (18). If (iv) does not hold true, then for this feasible solution we have $|Y'| + \sum_{j=1}^{n-|Y|} \min^j(d^*_j,LC_Q) < \sum_{j=1}^{n} d^*_j,LC_Q = R(U_d, LC_Q)$, that contradicts the optimality of $(k^*_LC_Q = n, d^*_j,LC_Q)$.

**Fact 2.** By Equation (10), we get $R(U_d, U_p, U_a, LC_Q) = \left(\delta^{k^*_LC_Q,0}(e, LC_Q) - 1\right)^+ + \sum_{j=1}^{k^*_LC_Q} \min^j(d^*_j,LC_Q)$.

For any $k \in \{0, \ldots, n\}$, we define $G(k) = G_1(k) + G_2(k)$, where $G_1(k) = \left(\delta^{k,0}(e, LC_Q) - 1\right)^+$, and $G_2(k) = \sum_{j=1}^{k} \min^j(d^*_j,LC_Q)$. It is seen that $R(U_d, U_p, U_a, LC_Q) = \min_{0 \leq k \leq n} G(k)$. 

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It is evident that \( G_1(k) \) and \( G_2(k) \) are non-increasing and non-decreasing in \( k \), respectively. Moreover, each value of \( k \) corresponds to a vertex cover that is associated with \( G(k) \) with the total demand \( G_2(k) \) and the total capacity \( G_1(k) \), respectively. Without loss of generality for each \( k \) the vertex cover includes product set \( S = \{1, 2, \ldots, k\} \) and plant set \( \mathcal{N}(B \setminus S, \mathcal{LC}_Q) \) since \( G_1(k) \) is minimized when products with consecutive indices are selected in the vertex cover.

Next, we demonstrate that the value of any local minimum of \( G(k) \) is the same as \( G(\bar{k}) \) at one of the points \( \bar{k} \in \{0, n - |Y|, n\} \). As a result, it is sufficient to compute \( G(k) \) at \( k \in \{0, n - |Y|, n\} \) and consider the minimum one as \( R(\mathcal{U}_2, \mathcal{U}_p, \mathcal{U}_u, \mathcal{LC}_Q) \).

Specifically, define \( k' \) as the largest value of \( k \) such that \( G_1(k) = n - 1 \) for all \( k < k' \). If \( k' \leq n - |Y| - 1 \), then we show that

\[
\begin{align*}
(a) \quad G(n - |Y|) < G(n - |Y| + 1) < \ldots < G(n - 1) \\
(b) \quad G(k') \geq \ldots \geq G(n - |Y| - 2) \geq G(n - |Y| - 1) \\
(c) \quad G(0) \leq G(1) \leq \ldots \leq G(k') \\
(d) \quad G(0) \leq G(1) \leq \ldots \leq G(k') < \ldots < G(n - 1)
\end{align*}
\]

In case \( k' \geq n - |Y| \), we only need to demonstrate that

\[
\begin{align*}
(d) \quad G(0) \leq G(1) \leq \ldots \leq G(k') < \ldots < G(n - 1)
\end{align*}
\]

It should be noted that for \( k = n \), if \( G(n - 1) \geq G(n) \), then \( k = n \) is a local minimum.

- Relation (a) holds true because \( G_1(k - 1) - G_1(k) = 1 \), and \( G_2(k - 1) - G_2(k) = -\min^k(\mathcal{d}_{\mathcal{LC}_Q}) < -1 \); therefore, \( G(k - 1) - G(k) < 0 \) for \( k \in \{n - |Y| + 1, \ldots, n - 1\} \). To elaborate further, utilizing the structure of \( \mathcal{LC}_Q \), product \( k > k' \) has a different neighbor plant from product \( k - 1 \); i.e., \( G_1(k - 1) - G_1(k) = |\mathcal{N}(k, \mathcal{LC}_Q) \setminus \mathcal{N}(k - 1, \mathcal{LC}_Q)| = 1 \). Furthermore, by the definition of set \( Y \) we have \( \min^k(\mathcal{d}_{\mathcal{LC}_Q}) > 1 \) for \( n - |Y| < k \leq n \). As a result, we observe that \( G_2(k - 1) - G_2(k) < -1 \).

If \( G(n - |Y| - 1) \geq G(n - |Y|) \), then \( k = n - |Y| \) is a local minimum. Otherwise, \( k = n - |Y| \) cannot be a local minimum of \( G(k) \).

- Relation (b) holds true because \( G_1(k - 1) - G_1(k) = 1 \), and \( G_2(k - 1) - G_2(k) = -\min^k(\mathcal{d}_{\mathcal{LC}_Q}) \geq -1 \) for \( k \in \{k', \ldots, n - |Y|\} \). By the structure of \( \mathcal{LC}_Q \), product \( k > k' \) has one different neighbor plant from product \( k - 1 \); i.e., \( G_1(k - 1) - G_1(k) = |\mathcal{N}(k, \mathcal{LC}_Q) \setminus \mathcal{N}(k - 1, \mathcal{LC}_Q)| = 1 \). Furthermore, since \( \min^k(\mathcal{d}_{\mathcal{LC}_Q}) \leq 1 \) for \( k \leq n - |Y| \), we have \( G_2(k - 1) - G_2(k) = -\min^k(\mathcal{d}_{\mathcal{LC}_Q}) \geq -1 \).

- Relation (c) holds true because \( G_1(k - 1) - G_1(k) = 0 \) and \( G_2(k - 1) - G_2(k) = -\min^k(\mathcal{d}_{\mathcal{LC}_Q}) \leq 0 \) for \( k \in \{0, \ldots, k'\} \). We get \( G_1(k - 1) - G_1(k) = 0 \) because \( G_1(k) = n - 1 \) for all \( k < k' \). Since there are no negative demands, it follows that \( G_2(k - 1) - G_2(k) \leq 0 \). By Relation (c), we see that \( k = 0 \) is a local minimum for \( G(k) \).
• Relation (d) holds true since Relations (a) and (c) are true. If \( k' \geq n - |Y| \), we have \( G(k - 1) < G(k) \) for \( k, k' \in \{k' + 1, \ldots, n - 1\} \) by Relation (a) and \( G(k - 1) \leq G(k) \) for \( k, k' \in \{0, \ldots, k'\} \) by Relation (b).

It should be noted that \( G(k) \) may have local minima other than \( \{0, n - |Y|, n\} \). However, on the basis of Fact 2, any local minimum takes the value of \( G(k) \) at one of the points \( \{0, n - |Y|, n\} \). Considering Fact 2 and since \( R(U_d, U_p, U_a, \mathcal{L}C_Q) = \min_{0 \leq k \leq n} G(k) \), in the following we evaluate \( Fr(\mathcal{L}C_Q) \) only for \( k_{\mathcal{L}C_Q}^o \in \{0, n - |Y|, n\} \). Recall that it is supposed \( k_{\mathcal{L}C_Q}^* = n \).

2.1. Let \( k_{\mathcal{L}C_Q}^o = n \), then we have \( k_{\mathcal{L}C_Q}^* = k_{\mathcal{L}C_Q}^o = n \). Using Equations (10) and (18), we obtain \( R(U_d, \mathcal{L}C_Q) = \sum_{j=1}^n d_{j, \mathcal{L}C_Q}^* \) and \( R(U_d, U_p, U_a, \mathcal{L}C_Q) = \sum_{j=1}^n d_{j, \mathcal{L}C_Q}^o \), respectively. Thus, \( R(U_d, \mathcal{L}C_Q) = R(U_d, U_p, U_a, \mathcal{L}C_Q) \leq R(U_d, \mathcal{L}C_Q) \leq \sum_{j=1}^n d_{j, \mathcal{L}C_Q}^o \) and since \( (k_{\mathcal{L}C_Q}^o = n, d_{j, \mathcal{L}C_Q}^o) \) is a feasible solution for (18). Hence, \( Fr(\mathcal{L}C_Q) = 0 \).

Based on part 1 we know that \( Fr(\mathcal{S}C_Q) = 1 \) because \( k_{\mathcal{L}C_Q}^* < n \). Therefore, \( Fr(\mathcal{L}C_Q) \leq Fr(\mathcal{S}C_Q) = 1 \).

2.2. Let \( k_{\mathcal{L}C_Q}^o = n - |Y| \), then the corresponding minimum vertex cover for \( k_{\mathcal{L}C_Q}^o = n - |Y| \) is \( Y' \cup (B \setminus Y) \).

By Equation (10) we have \( R(U_d, U_p, U_a, \mathcal{L}C_Q) = |Y'| - 1 + \sum_{j=1}^{|Y|} \min_j (d_{j, \mathcal{L}C_Q}^o) \). According to the definition of fragility coupled with Relation (iv) in Fact 1, we get

\[
Fr(\mathcal{L}C_Q) = R(U_d, \mathcal{L}C_Q) - R(U_d, U_p, U_a, \mathcal{L}C_Q) = \sum_{j=1}^n d_{j, \mathcal{L}C_Q}^* - (|Y'| - 1 + \sum_{j=1}^{|Y|} \min_j (d_{j, \mathcal{L}C_Q}^o)) \leq 1.
\]

Therefore, \( Fr(\mathcal{L}C_Q) \leq Fr(\mathcal{S}C_Q) = 1 \).

2.3. Let \( k_{\mathcal{L}C_Q}^o = 0 \), then \( R(U_d, U_p, U_a, \mathcal{L}C_Q) = n - 1 \) by Equation (10). Importantly, \( R(U_d, \mathcal{L}C_Q) = \sum_{j=1}^n d_{j, \mathcal{L}C_Q}^* \leq n \) because \( k_{\mathcal{L}C_Q}^* = n \); otherwise one can select all plants as the vertex cover and as a result \( R(U_d, \mathcal{L}C_Q) = n \). Hence,

\[
Fr(\mathcal{L}C_Q) = \sum_{j=1}^n d_{j, \mathcal{L}C_Q}^* - (n - 1) \leq 1.
\]

Therefore, \( Fr(\mathcal{L}C_Q) \leq Fr(\mathcal{S}C_Q) = 1 \) when \( k_{\mathcal{S}C_Q}^* < k_{\mathcal{L}C_Q}^* = n \).

3. Let \( k_{\mathcal{S}C_Q} = k_{\mathcal{L}C_Q} = n \), then \( R(U_d, \mathcal{S}C_Q) = R(U_d, \mathcal{L}C_Q) = \sum_{j=1}^n d_{j, \mathcal{S}C_Q}^* = \sum_{j=1}^n d_{j, \mathcal{L}C_Q}^* \). Considering Inequality (14), we have \( \min_{g \in U_p} \delta^{k,0}(g, \mathcal{S}C_Q) \leq \min_{g \in U_p} \delta^{k,0}(g, \mathcal{L}C_Q) \) for all \( k \) and any \( \mathcal{S}C_Q \) in \{\mathcal{S}C_Q\}. Thus, by using Theorem 1, we get \( R(U_d, U_p, U_a, \mathcal{S}C_Q) \leq R(U_d, U_p, U_a, \mathcal{L}C_Q) \). Therefore, \( Fr(\mathcal{L}C_Q) \leq Fr(\mathcal{S}C_Q) \).

4. Let \( k_{\mathcal{L}C_Q}^* < k_{\mathcal{S}C_Q}^* = n \). We demonstrate that this case never occurs. Since \( k_{\mathcal{S}C_Q}^* = n \), using Equation (18), we get \( R(U_d, \mathcal{S}C_Q) = \sum_{j=1}^n d_{j, \mathcal{S}C_Q}^* \). Evidently, \( (k = n, d_{\mathcal{S}C_Q}^*) \) with the objective value

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of $\sum_{j=1}^{n} d_{j,SC_Q}$ is a feasible solution of (18) for $\mathcal{LC}_Q$. Moreover, considering Inequality (14) and Theorem 1, we have $R(U_d, SC_Q) \leq R(U_d, \mathcal{LC}_Q)$. Accordingly, we get $\sum_{j=1}^{n} d_{j,SC_Q} \leq R(U_d, \mathcal{LC}_Q)$.

- If $\sum_{j=1}^{n} d_{j,SC_Q} < R(U_d, \mathcal{LC}_Q)$, then we have a feasible solution $(k = n, d_{SC_Q}^*)$ with the objective value $\sum_{j=1}^{n} d_{j,SC_Q}$ being less than the optimal value $R(U_d, \mathcal{LC}_Q)$; hence, $\sum_{j=1}^{n} d_{j,SC_Q} < R(U_d, \mathcal{LC}_Q)$ cannot occur.

- If $\sum_{j=1}^{n} d_{j,SC_Q} = R(U_d, \mathcal{LC}_Q)$, then $(k = n, d_{SC_Q}^*)$ is an optimal solution for (18). Thus, $k_{\mathcal{LC}_Q}^* = k_{SC_Q}^* = n$ and refer to part 3.

Therefore, on the basis of the discussions in parts 1 to 4, Proposition 6 is proved. □

Proposition 6 implies that $\mathcal{LC}_Q$ is less sensitive than $SC_Q$ under a single plant disruption. The intuition behind this result is that under a single plant disruption without complete arc disruptions – a long chain allows us to better utilize the remaining capacity than localizing the effect of the disruption in short chains.

Finally, from Propositions 4, 5 and 6 we conclude that the impact of a plant disruption differs from that of arc disruptions; since losing a plant leads to a reduction in the plant’s supply capacity whereas losing arcs decreases the flexibility. These results are significant since likelihood of arc and plant disruptions and the cost of flexibility (recall the discussion in §4.2) are different for each industry. Therefore, the preferred flexibility may differ depending on the company needs.

6 Generating Flexibility Designs

Majority of the available algorithms for generating sparse flexibility designs either do not handle well supply uncertainty or are not intended at all for settings where the design is susceptible to disruptions or the flexibility design is unbalanced, e.g., we refer to Chen et al. (2015) and Deng and Shen (2013). In this section, however, we propose a heuristic that exploits the notion of DPCI to take into account possible supply disruptions to generate both balanced and unbalanced designs. We then evaluate the worst-case and the expected performances of these designs, and compare them against designs generated by other heuristics from the literature in settings with and without disruptions.

The idea of our approach is as follows. Consider an initial design $\mathcal{D}$, e.g., the dedicated design (Figure 1b), and complete arc and plant disruption parameters $\alpha$ and $\gamma$. We aim to add $\mathcal{E}$ arcs to this initial design. Our method is based on Theorem 1, which implies that a larger value for $\sum_{0 \leq k \leq n-1} \min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, \mathcal{D})$ may translate into a better performance. Therefore, an increase in the value of DPCI, and subsequently, $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, \mathcal{D})$, may lead to a better worst-case performance. Hence, our algorithm adds arc $(i, j) \notin \mathcal{D}$ to $\mathcal{D}$ at each iteration in order to increase DPCI.

To elaborate further, from Assumption 1 we know that $\ell^* = \alpha$ for $R(U_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{D})$ in Equation (10). Recall Remark 3 part (vi), then for given $k$ and $\alpha$, let $S^*$ and $E^*$ denote an optimal solution for the
following problem:
\[
\min \delta^{k,\alpha}(g, D) = \min_{S \subseteq \mathcal{B} \mid |S| = k, E \subseteq \mathcal{D} \mid |E| = \alpha, a_i \in \mathcal{N}(B \setminus S, D \setminus E)} \sum_{i} c_i^{(p)} g_i.
\]

Sets $S^*$ and $\mathcal{N}(B \setminus S^*, D \setminus E^*)$ create a minimum vertex cover for $D \setminus E$. For a given $k$, we can increase $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, D)$ by adding arc $(i, j) \notin \mathcal{D}$ in such a manner that none of its endpoints are part of the vertex cover, and plant $i$ is not disrupted, i.e., $i \in A \setminus \mathcal{N}(B \setminus S^*, D \setminus E^*)$, $g_i \neq 0$ and $j \in B \setminus S^*$. Thus, we compute $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, D)$ in the algorithm for all $1 \leq k \leq n - 1$. For each value of $k$, we obtain a minimum vertex cover. We then select the arc with endpoints belonging to a fewer number of minimum vertex covers over all $0 \leq k \leq n - 1$. This arc has possibly the greatest effect in increasing $\sum_{0 \leq k \leq n - 1} \min_{g \in \mathcal{U}_p} \delta^{k,\alpha}(g, D)$. By iterating this process for $\mathcal{E}$ times, we can create a highly flexible design under disruptions. The formal pseudo-code is given in Algorithm 1.

Since arc disruptions only impact $\alpha$ arcs of the output design, we consider the arc disruptions only for the last $\alpha$ added arcs. Thus, we set the number of complete arc disruptions as $\alpha' = \max\{0, r - (\mathcal{E} - \alpha)^+\}$ at the $r$-th iteration of the algorithm. Then in STEP 1, we obtain an optimal solution of $\min_{g \in \mathcal{U}_p} \delta^{k,\alpha'}(g, D)$ for all $k \in \{1, \ldots, n - 1\}$. In STEP 2, we define the function $\Psi(p^k_i, q^k_j, g^k_i)$, which yields one if none of the endpoints of arc $(i, j)$ belong to the vertex cover and plant $i$ is not disrupted. Otherwise, it returns zero. Then for each arc, we compute its weight $W(i, j)$ as the sum of $\Psi(p^k_i, q^k_j, g^k_i)$ for different values of $k$. In STEP 3, set $M$ includes all candidate arcs with the maximum weight. For each arc in $M$, we compute $\Omega(i, j)$ as the sum of degrees of its endpoints. Finally, we pick the arc in $M$ with the minimum value of $\Omega(i, j)$.

### Algorithm 1 DPCI-based heuristic algorithm

**Input:** Design $D$, $m, n, \alpha, \mathcal{U}_p^{(1)}$ and the budget of extra arcs $\mathcal{E}$

For $1 \leq r \leq \mathcal{E}$

Set $\alpha' = \max\{0, r - (\mathcal{E} - \alpha)^+\}$.

**STEP 1.**

For $1 \leq k \leq n - 1$

Find $\min_{g \in \mathcal{U}_p^{(1)}} \delta^{k,\alpha'}(g, D)$ as well as its optimal solution $(p^k, q^k, t^k, g^k)$.

**END**

**STEP 2.**

Let $\Psi(p^k_i, q^k_j, g^k_i) = 1$ if $p^k_i = q^k_j = 0$ and $g^k_i = 1$. Otherwise, $\Psi(p^k_i, q^k_j, g^k_i) = 0$.

For $1 \leq i \leq m$, $1 \leq j \leq n$, $(i, j) \notin D$

$W(i, j) = \sum_{k=1}^{n-1} \Psi(p^k_i, q^k_j, g^k_i)$.

**END**

**STEP 3.**

$M = \{(i, j) \mid W(i, j) = \max\{W(i', j') \mid 1 \leq i' \leq m, 1 \leq j' \leq n, (i', j') \notin D\}\}$.

For $(i, j) \in M$

$\Omega(i, j) = \text{deg}_D(i) + \text{deg}_D(j)$.

**END**

Find arc $(i^*, j^*)$ in a manner that $\Omega(i^*, j^*) = \min\{\Omega(i, j) \mid (i, j) \in M\}$ (in case of a tie, we randomly select an arc, albeit uniformly, with the minimum $\Omega(i, j)$).

$D = D \cup (a_{i^*}, b_{j^*})$.

**END**

Both Algorithm 1 and PCI algorithm introduced in Simchi-Levi and Wei (2015) take into account the vertex cover concept to generate a design. However, the key differences of Algorithm 1 are as follows:
• We take into account disruptions by using $\min_{g \in U_p} \delta^{k,\alpha}(g, D)$ instead of PCI in STEP 1 and define function $\Psi(p_i^k, q_i^k, g_i^k)$ in STEP 2.

• If multiple arcs have the maximum weight (i.e., set $M$ includes multiple arcs), then we employ $\Omega(i, j)$ comprising of the information about the degree of vertices in STEP 3 to make the design more symmetric; note that highly connected designs such as Desargues graph (Kutnar and Marušič 2009, Figure 5) or Levi graph (Chou et al. 2011, Figure 1) have this property. In contrast, in PCI algorithm, when multiple arcs have the maximum weight, one arc is selected randomly in a uniform manner.

These features help us to generate designs that are less vulnerable to disruptions than those constructed by earlier methods from the literature.

**Benchmarks.** In order to evaluate the designs generated by Algorithm 1, we select PCI (Simchi-Levi and Wei 2015) and Expander (Chou et al. 2011) algorithms as the benchmarks among available algorithms. As reported by Simchi-Levi and Wei (2015) designs generated by PCI algorithm outperform those generated by existing algorithms in the literature, including the algorithms proposed by Hopp et al. (2004), and Chou et al. (2011). However, under asymmetric demand, designs of Expander algorithm outperform those generated by PCI algorithm.

**Measures.** To evaluate the worst-case and the expected performances of the constructed designs, we take three measures into consideration:

(i) $\Delta_\alpha$ that is the summation of $\min_{g \in U_p} \delta^{k,\alpha}(g, D)$ over all $0 \leq k \leq n$, i.e., $\Delta_\alpha = \sum_{0 \leq k \leq n} \min_{g \in U_p} \delta^{k,\alpha}(g, D)$.

(ii) $\bar{\Delta}$ that is the summation of $\Delta_\alpha$ over all $0 \leq \alpha \leq |D|$, i.e., $\bar{\Delta} = \sum_{0 \leq \alpha \leq |D|} \Delta_\alpha$.

Based on Theorem 1, higher values for $\Delta_\alpha$ and $\bar{\Delta}$ are indicators of better worst-case performances for a particular $\alpha$ and general $\alpha \in \{0, \ldots, |D|\}$, respectively.

(iii) Expected performance, that is the average of $P(d, g, h, D)$ over all demands, plant and arc disruption scenarios.

**Test instances.** To construct test instances, we generate as many as 5,000 demand scenarios. These demands are generated from independent normal distributions, where the demand for the $j$-th product has mean $\mu_j$, and standard deviation 0.5. Furthermore, for fixed $\alpha$ and $\gamma$, we consider 30 arc disruptions and 10 plant disruptions scenarios (binary vectors), generated uniformly and randomly.

We consider three possible disruption settings: no complete disruptions with $\alpha = \gamma = 0$, “medium”-level disruptions with $\alpha = 3$, $\gamma = 1$, and “high”-level disruptions with $\alpha = 5$, $\gamma = 2$. We use three test instances T1, T2 and T3 that are also considered in Simchi-Levi and Wei (2015). For the first instance, the initial design is a dedicated design, $\mathcal{L}C_1$, where $m = n = 10$. For the second and third instances, the initial design is an unbalanced design where $m = 7$, $n = 14$ and $D = \{(1,1), (1,2), (1,3), (2,4), (2,5), \ldots\}$.
In T1 and T2, \( \mu_j = 1 \), for \( j \in B \), but in T3, \( \mu_j \) is chosen uniformly and randomly from \([0.5,1.5]\), for all \( j \in B \). For all test instances \( c_i = \sum_{j \in A(i,D)} \mu_j \), for \( i \in A \).

**Results and discussions.** All computational experiments within this section were performed on a personal computer equipped with an 8-core Intel Xeon processor (CPU 3.6 GHz) and 32 GB of RAM. All algorithms were solved using CPLEX 12.6 IBM (2013) coded in C++ programming environment.

Table 1 reports the performance of designs generated by Expander, PCI and DPCI algorithms with respect to the worst-case and the expected performance measures. Specifically, we make the following observations. The designs generated by DPCI algorithm have the highest values for \( \Delta_{\alpha} \) and \( \bar{\Delta} \) in all of our test instances. Therefore, we expect that the designs generated by DPCI have a better worst-case performance with and without disruptions. Additionally, DPCI has the best expected performance for the balanced design, T1, and when there are no disruptions in T2 and T3. These results are noteworthy since they show that although DPCI algorithm is specifically designed for the worst-case performance under disruptions, its designs outperform designs generated by the other algorithms under no disruption setting and also with respect to the expected performance measure. Nevertheless, for unbalanced designs T2 and T3, the designs generated by Expander algorithm demonstrate a slightly better expected performance under disruptions. Since PCI and DPCI solve MIPs their running times are larger than that of Expander algorithm; nonetheless, all algorithms (on average) generate designs within a minute.

<table>
<thead>
<tr>
<th>Measure</th>
<th>T1 ( \alpha = 0, \beta = 1, \gamma = 0, \lambda = 1 )</th>
<th>T2 ( \alpha = 0, \beta = 1, \gamma = 0, \lambda = 1 )</th>
<th>T3 ( \alpha = 0, \beta = 1, \gamma = 0, \lambda = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_0 )</td>
<td>63</td>
<td>70</td>
<td>70</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>617</td>
<td>623</td>
<td>636</td>
</tr>
<tr>
<td>Exp. Perf.</td>
<td>0.21</td>
<td>14.54</td>
<td>14.98</td>
</tr>
<tr>
<td>Time(sec.)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Measure</th>
<th>T1 ( \alpha = 3, \beta = 1, \gamma = 1, \lambda = 1 )</th>
<th>T2 ( \alpha = 3, \beta = 1, \gamma = 1, \lambda = 1 )</th>
<th>T3 ( \alpha = 3, \beta = 1, \gamma = 1, \lambda = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_3 )</td>
<td>38</td>
<td>37</td>
<td>44</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>452</td>
<td>462</td>
<td>532</td>
</tr>
<tr>
<td>Exp. Perf.</td>
<td>0.28</td>
<td>13.23</td>
<td>20.94</td>
</tr>
<tr>
<td>Time(sec.)</td>
<td></td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Measure</th>
<th>T1 ( \alpha = 5, \beta = 1, \gamma = 2, \lambda = 1 )</th>
<th>T2 ( \alpha = 5, \beta = 1, \gamma = 2, \lambda = 1 )</th>
<th>T3 ( \alpha = 5, \beta = 1, \gamma = 2, \lambda = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_5 )</td>
<td>23</td>
<td>26</td>
<td>27</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>321</td>
<td>362</td>
<td>377</td>
</tr>
<tr>
<td>Exp. Perf.</td>
<td>0.22</td>
<td>14.65</td>
<td>20.87</td>
</tr>
<tr>
<td>Time(sec.)</td>
<td></td>
<td></td>
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<table>
<thead>
<tr>
<th>Measure</th>
<th>Average</th>
<th>T1</th>
<th>T2</th>
<th>T3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_{\alpha} )</td>
<td>41.33</td>
<td>44.33</td>
<td>47</td>
<td>38</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>463.33</td>
<td>482.33</td>
<td>515</td>
<td>429.33</td>
</tr>
<tr>
<td>Exp. Perf.</td>
<td>8.22</td>
<td>8.51</td>
<td>8.54</td>
<td>10.24</td>
</tr>
<tr>
<td>Time(sec.)</td>
<td>0.24</td>
<td>14.14</td>
<td>18.93</td>
<td>0.23</td>
</tr>
</tbody>
</table>
To further examine the efficacy of designs generated by the algorithms, we plot their performances in Figure 2 as a function of the number of added arcs ($E$) for medium-level disruptions with $\alpha = 3$, $\gamma = 1$. This figure supports the earlier observations in Table 1. Moreover, we notice that for sufficiently large values of $E$, there is no significant difference among the expected performances of the constructed designs. On the other hand, the difference in their worst-case performance measures is significant, particularly for unbalanced designs.

![Figure 2: Results for PCI, DPCI and Expander algorithms with respect to $\Delta_3$, $\bar{\Delta}$ and the expected performance measures versus the number of added arcs to the initial design for three test instances T1, T2 and T3. Note that $\alpha = 3$, $\gamma = 3$ and $\beta = \lambda = 1$.](image)

### 7 Conclusion

This paper studies the worst-case performance of process flexibility designs. In addition to the demand uncertainty, we assume that designs are susceptible to complete and/or partial plant and arc disruptions. We define plant cover index under disruptions (DPCI), denoted by $\delta^{k,\ell}(g, D)$, as the minimum required capacity of plants to create a vertex cover on $D$, given that the vertex cover contains exactly $k$ products, exactly $\ell$ arcs are ignored, and plants are disrupted according to $g$. We show that the worst-case performance of any design can be formulated as a function of DPCI and symmetric uncertainty sets.

DPCI also allows us to compare the performance of different designs with no additional information on the demand. In particular, we demonstrate optimality of the well-known 2-long chain design over a broad class of designs in the worst-case performance under disruptions. This result is consistent with earlier studies in the literature that show the superiority of 2-long chain design with respect to the expected and the worst-case performances when no disruptions are present.

Furthermore, we show that, for $Q \geq 2$, any $Q$-short chain has the same performance as $Q$-long
chain if the design is subject to a sufficiently large number of complete arc disruptions; this result holds regardless of the presence of plant disruptions. Therefore, if there exists at least one complete arc disruption, then 2-short chains (which are not often taken into account because of its poor expected performance) are optimal – in the worst case – over all designs for which 2-long chain is optimal.

In the second part of the paper, we consider the notion of fragility that quantifies impacts of disruptions on the worst-case performance. We show that, for \( Q \geq 2 \), \( Q \)-long chain is less fragile (sensitive) than any \( Q \)-short chain design under a single complete plant disruption. In contrast, we demonstrate that \( Q \)-short chain designs are less fragile than \( Q \)-long chain if the number of complete arc disruptions are sufficiently large regardless of the other disruption parameters.

Finally, using the concept of DPCI we develop a heuristic for generating designs that are less vulnerable to supply and demand uncertainties than the designs generated by earlier methods from the literature. Our numerical experiments demonstrate that the designs constructed by the DPCI-based heuristic performs well under supply and demand uncertainties in both the worst and the expected cases.

References


A Proofs

Proof of Lemma 2. Under the assumption $c^{(p)} = e$, the definition of DPCI specifies that $\delta^{k,\ell}(e, D)$ denotes the number of plants required to cover all arcs, given that the vertex cover contains $k$ products and $\ell$ arcs are ignored to be covered in case there are no plant disruptions ($g = e$). Based on Remark 1, in order to get $\min_{g \in U_\ell} \delta^{k,\ell}(g, D)$ in the worst-case scenario, among $\delta^{k,\ell}(e, D)$ plants exactly $\gamma$ plants have zero capacity whereas the others have the capacity of $\lambda$. It can be verified that if $\delta^{k,\ell}(e, D) \leq \gamma$, then the minimum required capacity to form a vertex cover is zero.

Proof of Lemma 3. To prove Relations (11), (12) and (13), separately for each one, we first derive $\delta^{k,\ell}(e, \mathcal{L}_C)$. Then by applying Lemma 2, we obtain the desired result. Recall that the term $\delta^{k,\ell}(e, \mathcal{L}_C)$ is DPCI without any plant disruptions, i.e., the minimum number of plants required to create a vertex cover (under the assumption $c^{(p)} = e$) that includes $k$ products after ignoring $\ell$ arcs.

1. To prove the validity of Relation (11), we evaluate $\delta^{k,\ell}(e, \mathcal{L}_C)$ for any $0 \leq k \leq n$ and $0 \leq \ell \leq n \cdot Q$. We know that $\mathcal{L}_C$ has $n \cdot Q$ arcs. Exactly $k \cdot Q$ arcs are covered by $k$ products. Among the uncovered arcs, $\ell$ arcs are ignored by Equations (3c) and (3d). If $n \cdot Q - k \cdot Q \leq \ell$, then $n \cdot Q - \lfloor \frac{k - \ell}{Q} \rfloor \leq 0$. Thus, $\delta^{k,\ell}(e, \mathcal{L}_C) = (n - k - \lfloor \frac{k - \ell}{Q} \rfloor)^+ = 0$. Otherwise, there remain $Q \cdot (n - k) - \ell$ uncovered arcs.

Since each plant can cover $Q$ uncovered arcs at most, we have $\delta^{k,\ell}(e, \mathcal{L}_C) \geq \frac{Q(n-k)-\ell}{Q}$. Moreover, since $\delta^{k,\ell}(e, \mathcal{L}_C) \in \mathbb{Z}_+$ we have $\delta^{k,\ell}(e, \mathcal{L}_C) \geq \lceil \frac{Q(n-k)-\ell}{Q} \rceil = n - k - \lfloor \frac{k - \ell}{Q} \rfloor$. Therefore, based on Lemma 2, we obtain $\min_{g \in U_\ell} \delta^{k,\ell}(g, \mathcal{L}_C) \geq \lambda \cdot (n - k - \lfloor \frac{k - \ell}{Q} \rfloor - \gamma)^+$ for any $0 \leq k \leq n$ and $0 \leq \ell \leq n \cdot Q$.

2. To prove the validity of Equation (12), we first evaluate $\delta^{k,0}(e, \mathcal{L}_C)$, for given $1 \leq k \leq n - 1$; for $k = 0$ the proof is trivial since by Remark 3 part (i) we know that $\delta^{0,0}(e, \mathcal{L}_C) = n$. Let $S \subset B$ such that $|S| = k$. Put all products belonging to $S$ in the vertex cover. Clearly, on the basis of the definition of $Q$-long chain we have $|N(S, \mathcal{L}_C)| \geq \min\{n, k + Q - 1\}$. For a minimum vertex cover, the smallest value of $|N(S, \mathcal{L}_C)|$ is desired. This is important because it allows the largest number of plants, say $A' \subset N(S, \mathcal{L}_C)$, to be connected with all their $Q$ arcs to set $S$. As a result, the largest number of plants ($A'$) are excluded from the vertex cover by $k$ products, which then leads us to the minimum vertex cover.

Let $S$ be a set of products with consecutive indices, then $|N(S, \mathcal{L}_C)| = \min\{n, k + Q - 1\}$; refer to Chen et al. (2015). Additionally, based on the definition of $Q$-long chain, it is verified that $|A'| = (k - Q + 1)^+$. Set $A \setminus A'$ is required to create a minimum vertex cover along with $S$. Thus, $|A \setminus A'| = n - (k - Q + 1)^+ = \min\{n, n - k + Q - 1\}$ plants are connected to uncovered arcs with at least one endpoint in $B \setminus S$. Therefore, $\delta^{k,0}(e, \mathcal{L}_C) = |A \setminus A'| = \min\{n, n - k + Q - 1\}$, and by applying Lemma 2, it is seen that Equation (12) is valid.

3. To prove the validity of Equation (13), we first evaluate $\delta^{k,\ell}(e, \mathcal{L}_C)$ for all $0 \leq k \leq n$ and $(Q - 1)^2 \leq \ell \leq Q \cdot n$ for two cases $Q = 2$ and $Q \geq 3$ separately:
3.1. Let \( Q = 2 \), then by Inequality (11) we have \( \delta^{k,\ell}(e, \mathcal{L}C_2) \geq (n - k - \lfloor \frac{\ell}{2} \rfloor)^+ \). In the following, for any \( 0 \leq k \leq n \) and \( 1 \leq \ell \leq 2 \cdot n \), we create a vertex cover necessitating \( (n - k - \lfloor \frac{\ell}{2} \rfloor)^+ \) plants. Thus, based on Inequality (11), the created vertex cover is the minimum and we have \( \delta^{k,\ell}(e, \mathcal{L}C_2) = (n - k - \lfloor \frac{\ell}{2} \rfloor)^+ \).

If \( k = n \), then all products are in the vertex cover. Hence, by Remark 3 part (iii), we get \( \delta^{n,\ell}(e, \mathcal{L}C_2) = (n - n - \lfloor \frac{\ell}{2} \rfloor)^+ = 0 \) for any \( 1 \leq \ell \leq 2 \cdot n \). If \( k = 0 \), then we temporarily put all plants in the vertex cover. Next, by ignoring every two arcs connected to a plant in \( \mathcal{L}C_2 \), we can exclude exactly one plant from the vertex cover (in total, \( \lfloor \frac{\ell}{2} \rfloor \) plants are excluded). Thus, \( \delta^{0,\ell}(e, \mathcal{L}C_2) = (n - 0 - \lfloor \frac{\ell}{2} \rfloor)^+ = n - \lfloor \frac{\ell}{2} \rfloor \) for any \( 1 \leq \ell \leq 2 \cdot n \) by Inequality (11).

For \( 0 < k < n \), let \( S \subset \mathcal{B} \), \( |S| = k \) be a set of products with consecutive indices, e.g., \( S = \{1, 2, \ldots, k\} \). After putting \( S \) in the vertex cover, \( 2n - 2k \) uncovered arcs remain. By the selection of \( S \) and the structure of \( \mathcal{L}C_2 \) exactly two uncovered arcs emanate from \( \mathcal{N}(S, \mathcal{L}C_2) \), i.e., arcs with an endpoint in \( B \setminus S \). If \( \ell \geq 2n - 2k \), then all arcs are either covered or ignored. Thus, \( \delta^{k,\ell}(e, \mathcal{L}C_2) = (n - k - \lfloor \frac{\ell}{2} \rfloor)^+ = 0 \). Otherwise, this may be handled by the following.

3.1.1. if \( \ell \geq 1 \) is even, let us first ignore 2 uncovered arcs of \( \mathcal{N}(S, \mathcal{L}C_2) \). Then each plant in \( A \setminus \mathcal{N}(S, \mathcal{L}C_2) \) is connected to two uncovered arcs. If \( \ell - 2 > 0 \), then ignore \( \ell - 2 \) uncovered arcs which are connected to \( \frac{\ell - 2}{2} \) plants in \( A \setminus \mathcal{N}(S, \mathcal{L}C_2) \). Thus, there remain \( 2n - 2k - \ell \) uncovered arcs. The number of plants in \( A \setminus \mathcal{N}(S, \mathcal{L}C_2) \) that are still connected to two uncovered arcs is \( 2n - 2k - \ell = n - k - \frac{\ell}{2} = n - k - \lfloor \frac{\ell}{2} \rfloor \), and we need all of them to create a vertex cover with \( S \). Therefore, by Inequality (11) we get \( \delta^{k,\ell}(e, \mathcal{L}C_2) = (n - k - \lfloor \frac{\ell}{2} \rfloor)^+ \).

3.1.2. if \( \ell \geq 1 \) is odd, let us first ignore uncovered arc(s) of \( \mathcal{N}(S, \mathcal{L}C_2) \). After ignoring \( \ell \) arcs, all plants in \( A \setminus \mathcal{N}(S, \mathcal{L}C_2) \) barring one are connected to two uncovered arcs and only one plant is connected to single uncovered arc. Thus, \( \delta^{k,\ell}(e, \mathcal{L}C_2) \leq \frac{2n - 2k - \ell}{2} + 1 \). Since \( \delta^{k,\ell}(e, \mathcal{L}C_2) \in \mathbb{Z}_+ \) we get \( \delta^{k,\ell}(e, \mathcal{L}C_2) \leq \lfloor \frac{2n - 2k - \ell}{2} \rfloor + 1 \). As a result, by Inequality (11) we get \( \delta^{k,\ell}(e, \mathcal{L}C_2) = (n - k - \lfloor \frac{\ell}{2} \rfloor)^+ \).

Finally, Equation (13) holds true for \( Q = 2 \) using Lemma 2.

3.2. Let \( Q \geq 3 \), then for any \( 0 \leq k \leq n \) and \((Q - 1)^2 \leq \ell \leq Q \cdot n \) suffices to demonstrate that there exist \( S \subset \mathcal{B} \), \( |S| = k \) and \( E \subset \mathcal{L}C_Q \), \( |E| = \ell \) such that \( |\mathcal{N}(B \setminus S, \mathcal{L}C_Q \setminus E)| = (n - k - \lfloor \frac{\ell}{Q} \rfloor)^+ \). It implies that \( \delta^{k,\ell}(e, \mathcal{L}C_Q) = (n - k - \lfloor \frac{\ell}{Q} \rfloor)^+ \) due to Remark 3 part (vi) and Inequality (11). Then by applying Lemma 2, we obtain the desired result, that is \( \min_{g \in U_p} \delta^{k,\ell}(g, \mathcal{L}C_Q) = \lambda \cdot (n - k - \lfloor \frac{\ell}{Q} \rfloor) - \gamma \).

To this end, let \( S = \{1, \ldots, k\} \) and \( Z := B \setminus S = \{k + 1, \ldots, n\} \). Clearly, \(|S| = k \) and \(|Z| = n - k \). Put \( S \) in the vertex cover. Hence, set \( Z \) includes products that are connected to all uncovered arcs by \( S \). We define \( \eta_i \) as the number of uncovered arcs (with an endpoint in \( Z \)) connected to plant \( i \in A \). It should be observed that \( \eta_i > 0 \) for all \( i \in \mathcal{N}(Z, \mathcal{L}C_Q) \) and \( \eta_i = 0 \) for all \( i \in A \setminus \mathcal{N}(Z, \mathcal{L}C_Q) \). Without excluding \( E \) from \( \mathcal{L}C_Q \) the set \( \mathcal{N}(Z, \mathcal{L}C_Q) \) is required to create a vertex cover along with \( S \).
3.2.1. If $|Z| \leq Q - 2$, then $|Z|Q \leq (Q-2)Q < (Q-1)^2 \leq \ell$. For any $\ell \geq (Q-1)^2$ let $E \subseteq \mathcal{LC}_Q$ such that \{(i,j) \in \mathcal{LC}_Q \mid j \in Z\} \subseteq E$, and $|E| = \ell$. Clearly, $|\{(i,j) \in \mathcal{LC}_Q \mid j \in Z\}| = |Z|Q < |E| = \ell$ and $E$ includes all $|Z|Q$ arcs connected to $Z$. Thus, $|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E)| = 0$. It should be observed that $\ell > |Z|Q = (n - k)Q$. Hence, $(n - k - \lfloor \frac{\ell}{Q} \rfloor)^+ = 0$. Therefore, there exist $S \subseteq B$, $|S| = k$ and $E \subseteq \mathcal{LC}_Q$, $|E| = \ell$ in a manner that $|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E)| = (n-k-\lfloor \frac{\ell}{Q} \rfloor)^+ = 0$.

3.2.2. If $|Z| \geq Q - 1$, then define $\tau_t = |\{i \in A \mid \eta_i = t\}|$ and $x = (|Z| + Q - 1 - n)^+$. We observe that either $\eta_i = 0$ or $\eta_i \geq x + 1$. Additionally, $\tau_{x+1} = x + 2$, $\tau_t = 2$ for $x + 2 \leq t \leq Q - 1$, and $\tau_Q = |\mathcal{N}(Z, \mathcal{LC}_Q)| - 2(Q - 1 - (1 + x) - (2 + x) = |\mathcal{N}(Z, \mathcal{LC}_Q)| - 2Q + x + 2$. We first show that, 

**Fact 1.** for any $x + 1 \leq T \leq Q - 1$,

\[
\sum_{t=x+1}^{T} (\tau_t \cdot t) - T = T^2. \tag{i}
\]

We prove Equality (i) by induction on $T$. Let $T = x + 1$, then $\tau_{x+1} = x + 2$. Thus, $\sum_{t=x+1}^{x+1} (\tau_t \cdot t) - (x + 1) = (x + 2) \cdot (x + 1) - (x + 1) = (x + 1)^2$. Next, we need to prove if Equality (i) holds true for $T$, then it also holds true for $T + 1$. Suppose Equality (i) is true for $T$, $x + 1 \leq T < Q - 1$, then by induction hypothesis,

\[
\sum_{t=x+1}^{T} (\tau_t \cdot t) - T = T^2. \tag{ii}
\]

Additionally, since $\tau_{T+1} = 2$, we have $\sum_{t=x+1}^{T+1} (\tau_t \cdot t) = \sum_{t=x+1}^{T} (\tau_t \cdot t) + 2(T + 1)$. Hence, by Equality (ii) we get

\[
\sum_{t=x+1}^{T+1} (\tau_t \cdot t) - (T + 1) = \sum_{t=x+1}^{T} (\tau_t \cdot t) + T + 1 = T^2 + T + T + 1 = (T + 1)^2.
\]

Therefore, Equality (i) is valid.

Recall that $|\mathcal{N}(Z, \mathcal{LC}_Q)|$ plants are required to create a vertex cover along with $S$. By the definition of $Q$-long chain design and since $Z$ includes products with consecutive indices, $|\mathcal{N}(Z, \mathcal{LC}_Q)| = \min\{n, |Z| + Q - 1\}$. Next,

- for $\ell = (Q-1)^2$ let $E_0 = \{(i,j) \in \mathcal{LC}_Q \mid j \in Z, 0 < \eta_i \leq Q-1\} \setminus \{(i,j) \in \mathcal{LC}_Q \mid j \in Z, \eta_i = Q - 1 \text{ for one } i\}$. Set $E_0$ is connected to all plants $i \in \mathcal{N}(Z, \mathcal{LC}_Q)$ except those with $\eta_i = Q$, and one of two plants with $\eta_i = Q - 1$, i.e., it is the set of arcs with an endpoint in $Z$ and connected to $|\mathcal{N}(Z, \mathcal{LC}_Q)| - \tau_Q - 1 = 2Q - x - 3$ plants with the smallest $\eta_i > 0$. Let $T = Q - 1$ in Equality (i). Then, by Fact 1 we have $|E_0| = \sum_{t=x+1}^{Q-1} (\tau_t \cdot t) - (Q-1) = (Q-1)^2 = \ell$. Thus, by excluding $E_0$ from $\mathcal{LC}_Q$ the number of plants required to create a vertex cover, $|\mathcal{N}(Z, \mathcal{LC}_Q)|$, along with $S$ reduces by $2Q - x - 3$, i.e.,

\[
|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = |\mathcal{N}(Z, \mathcal{LC}_Q)| - (2Q - x - 3) = \min\{n, |Z| + Q - 1\} - (2Q - x - 3). \tag{iii}
\]
If $|Z| + Q - 1 \geq n$, then $x = |Z| + Q - 1 - n$ and from Equality (iii),

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = n - 2Q + (|Z| + Q - 1 - n) + 3$$

$$= |Z| - (Q - 2) = |Z| - \left\lfloor \frac{(Q - 1)^2}{Q} \right\rfloor = n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor > 0;$$

else, $x = 0$ and from Equality (iii),

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = |Z| + Q - 1 - 2Q + 0 + 3 = |Z| - (Q - 2)$$

$$= |Z| - \left(\frac{(Q - 1)^2}{Q}\right) = n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor > 0.$$

As a result,

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor > 0. \quad (iv)$$

Note that the value of $\eta_i$ for $i \in \mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)$, the remaining required plants for the vertex cover after ignoring arcs in $E_0$, is $Q - 1, Q, \ldots, Q$, i.e., $\tau_i = 0$ for $t \leq Q - 2$, $\tau_{Q-1} = 1$ and $\tau_Q = |\mathcal{N}(Z, \mathcal{LC}_Q)| - 2Q + x + 2$.

Note that for the remaining required plants for the vertex cover after ignoring arcs in $E_0$, i.e., $i \in \mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)$ we have $\eta_i = Q - 1$ or $Q$. Additionally, $\tau_i = 0$ for $t \leq Q - 2$, $\tau_{Q-1} = 1$ and $\tau_Q = |\mathcal{N}(Z, \mathcal{LC}_Q)| - 2Q + x + 2$.

- for $(Q - 1)^2 < \ell < (Q - 1)^2 + (Q - 1)$ let $E_1 = E_0 \cup \{(i, j) \in \mathcal{LC}_Q \mid j \in Z, (i, j) \notin E_0\}$ such that $|E_1| = \ell$. Excluding $E_1$ from $\mathcal{LC}_Q$ does not remove any more plants from $\mathcal{N}(Z, \mathcal{LC}_Q)$ than by excluding $E_0$. Because $|E_1 \setminus E_0| < Q - 1$, but $\eta_i = Q - 1$ or $Q$ for $i \in \mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)$. Thus, based on $(iv)$, we get

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_1)| = |\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = |Z| - \left(\frac{(Q - 1)^2}{Q}\right) = n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor > 0,$$

for any $(Q - 1)^2 < \ell < (Q - 1)^2 + (Q - 1)$.

- for $\ell = (Q - 1)^2 + (Q - 1) + t \cdot Q = Q^2(Q - 1 + t)$, where $t \in \mathbb{Z}_+ \cup \{0\}$. Let $E_2 = \cup_{i \in A}\{(i, j) \in \mathcal{LC}_Q \mid j \in Z, (i, j) \notin E_0, \eta_i = Q - 1 \text{ or } \eta_i = Q \} \cup E_0$ such that $|E_2| = \ell$. Subsequently, excluding $E_2$ from $\mathcal{LC}_Q$ removes $t + 1$ plant(s) from $\mathcal{N}(Z, \mathcal{LC}_Q)$ more than $E_0$, because $E_2 \setminus E_0$ includes arcs with an endpoint in $Z$ and connected to a plant with $\eta_i = Q - 1$ and $t$ plants with $\eta_i = Q$. Hence, by $(iv)$, we get

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_2)| = |\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| - (t + 1) = |Z| - \left\lfloor \frac{(Q - 1)^2}{Q} \right\rfloor - (t + 1)$$

$$= |Z| - \left(\frac{Q^2(Q - 1 + t)}{Q}\right) = n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor, \quad (v)$$
for $\ell = Q^2(Q - 1 + t)$. It should be noted that $\eta_i = Q$ for $i \in \mathcal{N}(Z, \mathcal{L}C_Q \setminus E_2)$, i.e., for the remaining plants to create the vertex cover after ignoring arcs in $E_2$.

- for $Q^2(Q - 1 + t) < \ell < Q^2(Q - 1 + t) + r$, where $t \in \mathbb{Z}_+ \cup \{0\}$ and $1 \leq r < Q$. Let $E_3 = E_2 \cup \{(i, j) \in \mathcal{L}C_Q | j \in Z, (i, j) \notin E_2\}$ such that $|E_3| = \ell$. It can be clearly seen that excluding $E_3$ from $\mathcal{L}C_Q$ removes no more plants from $\mathcal{N}(Z, \mathcal{L}C_Q)$ than excluding $E_2$, because $|E_3 \setminus E_2| = r < Q$, while $\eta_i = Q$ for $i \in \mathcal{N}(Z, \mathcal{L}C_Q \setminus E_2)$. Thus, by $(v)$

$$|\mathcal{N}(Z, \mathcal{L}C_Q \setminus E_3)| = |\mathcal{N}(Z, \mathcal{L}C_Q \setminus E_2)| = |Z| - \left\lfloor \frac{Q^2(Q - 1 + t)}{Q} \right\rfloor = n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor$$

for any $Q^2(Q - 1 + t) < \ell < Q^2(Q - 1 + t) + r$.

Therefore, there exist $S \subseteq B$, $|S| = k$ and $E \subseteq \mathcal{L}C_Q$, $|E| = \ell$ in a manner that $|\mathcal{N}(Z, \mathcal{L}C_Q \setminus E)| = (n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor)^+$ for $|Z| \geq (Q - 1)$ and any $\ell \geq (Q - 1)^2$.

Finally, according to parts 3.2.1, 3.2.2, Remark 3 part $(vi)$, and Inequality (11) we have $\delta^k,\ell(e, \mathcal{L}C_Q) = |\mathcal{N}(Z, \mathcal{L}C_Q \setminus E)| = (n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor)^+$. Thus, Equation (13) holds true for $Q \geq 3$ using Lemma 2.

Proof of Lemma 4. We prove the statement $|\mathcal{N}(T, \mathcal{D} \setminus E)| \leq (z - \left\lfloor \frac{\ell}{2} \right\rfloor)^+$ for any $1 \leq k \leq n$ and $1 \leq \ell \leq 2n$ using a double induction on $z$ and $\ell$ in the following three steps:

1. **Base case.** Let $z = 1$, $\ell = 1$. For any $u \in B$, consider $T = \{u\}$. Since there exist $a, a' \in A$ such that $|\mathcal{N}(u, \mathcal{D})| = |\{a, a'\}| = 2$, then $|\mathcal{N}(T, \mathcal{D} \setminus \{(u, u)\})| = 1$.

2. **Induction over $z$ for $\ell = 1$.** We need to prove that if the statement holds true for $z < |B|$ and $\ell = 1$, then it also holds true for $z + 1$ and $\ell = 1$ (note that $\left\lfloor \frac{\ell}{2} \right\rfloor = 0$ for $\ell = 1$). Suppose the statement is true for some $z < |B|$ and $\ell = 1$, then by induction hypothesis, there exist sets $T^z \subseteq B$, $|T^z| = z$, and $E^z = \{(a, b)\} \subseteq \mathcal{D}$, $|E^z| = 1$ such that $|\mathcal{N}(T^z, \mathcal{D} \setminus E^z)| \leq z$. We consider the following two cases:

2.1. Let $E^z = \{(a, b)\} \subseteq \mathcal{D} \cap \{\mathcal{N}(T^z, \mathcal{D}) \times T^z\}$. It should be noted that the vertices in $A \cup B$ over $\mathcal{D}$ form a connected graph, and $T^z \subseteq B$. Therefore, there exists some $v \in \mathcal{N}(T^z, \mathcal{D} \setminus E^z)$ and $v' \notin \mathcal{N}(T^z, \mathcal{D} \setminus E^z)$ such that $(v, u)$ and $(v', u)$ are arcs for some $u \notin T^z$. Let $T^{z+1} = T^z \cup \{u\}$ and $E^{z+1} = E^z$, subsequently, we get $|\mathcal{N}(T^{z+1}, \mathcal{D} \setminus E^{z+1})| \leq z + 1$.

2.2. Let $E^z = \{(a, b)\} \notin \mathcal{D} \cap \{\mathcal{N}(T^z, \mathcal{D}) \times T^z\}$. Since $\mathcal{D}$ is connected, there exists $u \in B \setminus T^z$ such that $\mathcal{N}(u, \mathcal{D}) \cap \mathcal{N}(T^z, \mathcal{D}) \neq \emptyset$. Thus, for $T^{z+1} = T^z \cup \{u\}$ and $E^{z+1} = E^z$ we get $|\mathcal{N}(T^{z+1}, \mathcal{D} \setminus E^{z+1})| \leq z + 1$.

3. **Induction over $\ell$ for fixed $z$.** We need to prove that if the statement holds true for some $1 \leq z \leq |B|$ and $1 \leq \ell < 2n$, then it also holds true for $z$ and $\ell + 1$. Suppose the statement is true for some $1 \leq z \leq |B|$ and $1 \leq \ell < 2n$, then by induction hypothesis there exist sets $T^\ell \subseteq B$, $|T^\ell| = z$, and $E^\ell \subseteq \mathcal{D}$, $|E^\ell| = \ell$ such that $|\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| \leq (z - \left\lfloor \frac{\ell}{2} \right\rfloor)^+$. Construct $E^{\ell+1} = E^\ell \cup \{(a, b)\}$ such that $(a, b) \in \mathcal{D} \setminus E^\ell$ and $T^{\ell+1} = T^\ell$, subsequently, we need to consider the following three cases:
3.1. If $|N(T^\ell, D \setminus E^\ell)| < (z - \lceil \ell/2 \rceil)^+$, then $|N(T^{\ell+1}, D \setminus E^{\ell+1})| = |N(T^\ell, D \setminus E^\ell)|$ or $|N(T^{\ell+1}, D \setminus E^{\ell+1})| = |N(T^\ell, D \setminus E^\ell)| - 1$. Similarly, $z - [\ell/2] = z - [\ell/2]$ or $z - [\ell/2] = z - [\ell/2] - 1$. Therefore, we have $|N(T^{\ell+1}, D \setminus E^{\ell+1})| \leq (z - [\ell/2])^+$.

3.2. If $|N(T^\ell, D \setminus E^\ell)| = (z - [\ell/2])^+$ and $\ell$ is even, then $|N(T^{\ell+1}, D \setminus E^{\ell+1})| = |N(T^\ell, D \setminus E^\ell)|$ or $|N(T^{\ell+1}, D \setminus E^{\ell+1})| = |N(T^\ell, D \setminus E^\ell)| - 1$, but $z - [\ell/2] = z - [\ell/2]$. As a result, $|N(T^{\ell+1}, D \setminus E^{\ell+1})| \leq (z - [\ell/2])^+$.

3.3. If $|N(T^\ell, D \setminus E^\ell)| = (z - [\ell/2])^+$ and $\ell$ is odd, i.e., $\ell = 2t - 1$, $t \in \{1, 2, \ldots, n\}$, then $z - [\ell/2] = z - [\ell/2] - 1$. Next, we demonstrate that there exists $a \in N(T^\ell, D \setminus E^\ell)$ which is connected to $T^\ell$ by only one arc:

Set $N(T^\ell, D)$ is connected to $T^\ell$ by $2z$ arcs. Thus, all $a \in N(T^\ell, D \setminus E^\ell)$ cannot be connected to $T^\ell$ by 2 or more than 2 arcs, because in such an instance the number of arcs connecting $T^\ell$ to $N(T^\ell, D)$ would be greater than $2z$ due to:

$$2|N(T^\ell, D \setminus E^\ell)| + \ell = 2 \cdot (z - \frac{2t - 1}{2})^+ + 2t - 1$$
$$= 2 \cdot (z - t + 1)^+ + 2t - 1 \geq 2z + 1 > 2z, \quad \forall t \in \{1, 2, \ldots, n\}.$$

Therefore, by ignoring an odd number of arcs connecting $T^\ell$ to $N(T^\ell, D)$ such that $|N(T^\ell, D \setminus E^\ell)| = (z - [\ell/2])^+$, there exists plant $a \in N(T^\ell, D \setminus E^\ell)$ connected to $T^\ell$ by one arc. Let $(a, b)$ be the arc connecting $b \in T^\ell$ to $a$ and $E^\ell+1 = E^\ell \cup \{(a, b)\}$, $T^{\ell+1} = T^\ell$. Thus, $|N(T^{\ell+1}, D \setminus E^{\ell+1})| = |N(T^\ell, D \setminus E^\ell)| - 1$, and we have $|N(T^{\ell+1}, D \setminus E^{\ell+1})| = (z - [\ell/2])^+.$ 

\[\Box\]

\textbf{Proof of Theorem 2.} If we show that $\delta^{k,\ell}(e, D) \leq \delta^{k,\ell}(e, LC_2)$ for all $0 \leq k \leq n$ and $0 \leq \ell < 2n$, then using Lemma 2 and Theorem 1 it is proved that Theorem 2 holds. If $\ell = 0$, we refer to Simchi-Levi and Wei (2015, Theorem 5). For $0 \leq k \leq n$ and $1 \leq \ell < 2n$, it suffices to show that we can find some sets $S \subseteq B$, $|S| = k$ and $E \subseteq D$, $|E| = \ell$ such that $|N(B \setminus S, D \setminus E)| \leq (n - k - [\ell/2])^+$. Then based on Remark 3 part $(vi)$ and Equation (13),

$$|N(B \setminus S, D \setminus E)| = \delta^{k,\ell}(e, D) \leq \delta^{k,\ell}(e, LC_2) = (n - k - [\ell/2])^+.$$

Observe that if $2n \leq 2k + \ell$, then $\delta^{k,\ell}(-, -) = 0$. Thus, in this proof, we only consider the case wherein $2k + \ell < 2n$, $0 \leq k < n$ and $1 \leq \ell < 2n$.

Assume that design $D$ comprises $c$ connected components named $D_1, \ldots, D_c$ such that $A_w \subseteq A$ and $B_w \subseteq B$ denote the sets of plants and products of the $w$-th component, respectively. Without loss of generality, let us suppose that $|A_w| - |B_w|$ is nondecreasing with $w$. Because $\sum_{w=1}^{c} (|A_w| - |B_w|) = 0$, this assumption implies that $\sum_{w=1}^{t} |A_w| \leq \sum_{w=1}^{t} |B_w|$ for any $t \leq c$.

For any $0 \leq k < n$ and $1 \leq \ell < 2n$ such that $2k + \ell < 2n$, we have $n - k - [\ell/2] > 0$. Let $t_{k\ell}$ denote the largest possible $t$ such that $\sum_{w=1}^{t} |B_w| + [\ell/2] < n - k$. By our choice of $t_{k\ell}$, we get $t_{k\ell} < c$. 

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and \( n - k - \sum_{w=1}^{t_k} |B_w| - \left\lceil \frac{\ell - 1}{2} \right\rceil \leq |B_{t_k+1}| \). Moreover, define \( T_0 \subseteq \bigcup_{w=t_{k+1}}^{c} B_w \) with \( |T_0| = \left\lceil \frac{\ell - 1}{2} \right\rceil \), and \( E_0 = \{(i, j) \in \mathcal{D} \mid j \in T_0\} \); hence \( |E_0| = 2\left\lceil \frac{\ell - 1}{2} \right\rceil \) because \( |N(u, \mathcal{D})| = 2 \) for all \( u \in B \).

Based on Lemma 4, in the connected component \( B_{t_k+1} \), we can find some sets \( T_1 \) and \( E_1 \), where \( T_1 \subseteq B_{t_k+1} \), \( |T_1| = n - k - \sum_{w=1}^{t_k} |B_w| - \left\lceil \frac{\ell - 1}{2} \right\rceil \), and \( E_1 \subseteq N(T_1, \mathcal{D}) \times T_1 \), \( |E_1| = \ell - |E_0| = (\ell - 2\left\lceil \frac{\ell - 1}{2} \right\rceil) \in \{1, 2\} \) such that

\[
|N(T_1, \mathcal{D} \setminus E_1)| \leq n - k - \sum_{w=1}^{t_k} |B_w| - \left\lceil \frac{\ell - 1}{2} \right\rceil - \left\lceil \frac{|E_1|}{2} \right\rceil = n - k - \sum_{w=1}^{t_k} |B_w| - \left\lceil \frac{\ell}{2} \right\rceil. \tag{i}
\]

Next, we select predefined \( T_0 \) such that \( T_0 \cap T_1 = \emptyset \), this is possible because it can be verified that for \( T_0 \subseteq \bigcup_{w=t_{k+1}}^{c} B_w \) and \( T_1 \subseteq B_{t_k+1} \) we have \( |T_0| + |T_1| \leq |\bigcup_{w=t_{k+1}}^{c} B_w| \); by this selection we also have \( E_0 \cap E_1 = \emptyset \). Finally, let \( S := \bigcup_{w=t_{k+1}}^{c} B_w \setminus (T_0 \cup T_1) \), and \( E = E_0 \cup E_1 \); thus, \( |E| = \ell \), and by (i) we get

\[
|N(B \setminus S, \mathcal{D} \setminus E)| \leq |N(T_1, \mathcal{D} \setminus E_1)| + \sum_{w=1}^{t_k} |A_w| \leq n - k - \sum_{w=1}^{t_k} |B_w| - \left\lceil \frac{\ell}{2} \right\rceil + \sum_{w=1}^{t_k} |B_w| = n - k - \left\lceil \frac{\ell}{2} \right\rceil.
\]

We know that \( |T_0| = \left\lceil \frac{\ell - 1}{2} \right\rceil \), \( S \subseteq B \) and

\[
|S| = \left| \bigcup_{w=t_{k+1}}^{c} B_w \setminus (T_0 \cup T_1) \right| = n - \sum_{w=1}^{t_k} |B_w| - |T_0| - |T_1| = n - \sum_{w=1}^{t_k} |B_w| - \left\lceil \frac{\ell - 1}{2} \right\rceil - (n - k - \sum_{w=1}^{t_k} |B_w| - \left\lceil \frac{\ell - 1}{2} \right\rceil) = k.
\]

Since \( |S| = k \), thus \( |B \setminus S| = n - k \); thereby the proof is complete. \( \square \)

**Proof of Theorem 3.** Let \( \hat{n} \) be the size of the smallest system for which there exists a counter example \( \hat{\mathcal{D}} \) to Theorem 3. For \( n = 2 \), \( \mathcal{D} \) is the same as \( \mathcal{L}C_2 \); thus we must have \( \hat{n} \geq 3 \). Moreover, we must have \( 2k + \ell < 2n \); otherwise \( \delta^{k, \ell}(\cdot, \cdot) = 0 \).

For \( \ell = 0 \) refer to Simchi-Levi and Wei (2015, Theorem 6). According to Lemmas 2 and 3, \( \delta^{k, \ell}(e, \mathcal{L}C_2) = n - k - \left\lceil \frac{\ell}{2} \right\rceil \) for \( 2n > 2k + \ell \) and \( \ell \geq 1 \). Since \( \hat{\mathcal{D}} \) is a counterexample, there exists some \( 0 \leq \hat{k} < \hat{n} \) and \( 1 \leq \hat{\ell} < 2\hat{n} \) such that \( \delta^{\hat{k}, \hat{\ell}}(e, \hat{\mathcal{D}}) > \hat{n} - \hat{k} - \left\lceil \frac{\hat{\ell}}{2} \right\rceil > 0 \). We can find \( u \in B \) such that \( |N(u, \hat{\mathcal{D}})| = 1 \); otherwise by the proof of Theorem 2, we have \( \delta^{k, \ell}(e, \mathcal{D}) \leq \delta^{k, \ell}(e, \mathcal{L}C_2) = (n - k - \left\lceil \frac{\ell}{2} \right\rceil)^+ \) for all \( n, k \) and \( \ell \). Let \( \{v\} = N(u, \hat{\mathcal{D}}) \). Since \( \hat{\mathcal{D}} \) is connected, we have \( |N(v, \hat{\mathcal{D}})| \geq 2 \).

Next, we define design \( \mathcal{D}' \) with the set of plants and products \( A \setminus \{v\} \) and \( B \setminus \{u\} \), respectively, such that \( \mathcal{D}' = \{(v', u') \mid (v', u') \in \hat{\mathcal{D}}, u' \neq u, v' \neq v\} \). Design \( \mathcal{D}' \) is not necessarily connected. If \( \mathcal{D}' \) has c
components, then \(|\mathcal{N}(v, \hat{D})| \geq c + 1\). By adding \(c - 1\) arcs to \(\mathcal{D}'\) we can make it connected. Define \(\mathcal{D}''\) as the arc set that contains \(\mathcal{D}'\) and \(c - 1\) added arcs. Hence, \(\mathcal{D}''\) is connected. It should be noted that \(\mathcal{D}''\) is defined on a system with size \(\hat{n} - 1\) and \(|\mathcal{D}''| \leq 2(\hat{n} - 1)\).

Based on the minimality assumption on \(\hat{n}\), \(\delta^{k,\ell}(e, \mathcal{D}'') \leq \hat{n} - \hat{k} - \lfloor \frac{\ell}{Q} \rfloor - 1\). Thus, by Remark 3 part \((vi)\), there exists some \(S \subseteq B \setminus \{u\}, E \subseteq \mathcal{D}'', |E| = \hat{\ell}\), and \(|S| = \hat{n} - \hat{k} - 1\) such that \(|\mathcal{N}(S, \mathcal{D}' \setminus E)| \leq \hat{n} - \hat{k} - \lfloor \frac{\ell}{Q} \rfloor - 1\). This implies that \(S \cup \{u\} \subseteq B\), \(|S \cup \{u\}| = \hat{n} - \hat{k}\) and \(|\mathcal{N}(S \cup \{u\}, \mathcal{D} \setminus E)| \leq \hat{n} - \hat{k} - \lfloor \frac{\ell}{Q} \rfloor\). Hence, by Remark 3 part \((vi)\) we have \(\delta^{k,\ell}(e, \hat{D}) \leq \hat{n} - \hat{k} - \lfloor \frac{\ell}{Q} \rfloor\). This contradicts the assumption that \(\delta^{k,\ell}(e, \hat{D}) > \hat{n} - \hat{k} - \lfloor \frac{\ell}{Q} \rfloor\).

**Proof of Lemma 5.** Let \(\{1, \ldots, c\}\) and \(z_1, \ldots, z_c\) represent the components of \(\mathcal{SC}_Q\) and their sizes, respectively. For each of Relation (14) and Equation (15), we first derive \(\delta^{k,\ell}(e, \mathcal{SC}_Q)\), that is DPCI without plant disruptions or the minimum required number of plants to create a vertex cover along with \(k\) products after ignoring \(\ell\) arcs. Then by applying Lemma 2 we obtain the desired results.

Before we proceed, similar to the proof of Lemma 3 part 1, we can demonstrate that

\[
n - k - \left\lfloor \frac{\ell}{Q} \right\rfloor \leq \delta^{k,\ell}(e, \mathcal{SC}_Q),
\]

(i)

for any \(0 \leq k \leq n\) and \(0 \leq \ell \leq n \cdot Q\).

To prove Relation (14) for \(\ell = 0\):

1. Let \(k = \sum_{i \in I_1} z_i\) for some \(I_1 \subseteq \{1, \ldots, c\}\); then we put all products of components in \(I_1\) in the vertex cover. Clearly, all arcs of components in \(I_1\) are covered. Thus, we need all plants of components in \(\{1, \ldots, c\} \setminus I_1\) to create a vertex cover that is \(n - \sum_{i \in I_1} z_i = n - k\). By (i) we conclude that, for \(\ell = 0\), this constructed vertex cover is the minimum one. Hence, \(\delta^{k,0}(e, \mathcal{SC}_Q) = n - k\). Additionally, by Equation (11) for \(\mathcal{LC}_Q\) we have \(n - k \leq \delta^{k,0}(e, \mathcal{LC}_Q)\). Thus,

\[
n - k = \delta^{k,0}(e, \mathcal{SC}_Q) \leq \delta^{k,0}(e, \mathcal{LC}_Q).
\]

(ii)

2. Let \(k \neq \sum_{i \in I} z_i\) for any \(I \subseteq \{1, \ldots, c\}\), then let \(I_1 \subseteq \{1, \ldots, c\}\) be the largest subset of components such that \(\sum_{i \in I_1} z_i < k\) and \(k_1 := \sum_{i \in I_1} z_i\). Thus, there exists component \(x \in \{1, \ldots, c\} \setminus I_1\) such that \(z_x > k_2\), where \(k_2 := k - k_1\). Now, let us put all products of components of \(I_1\) and \(k_2\) products of component \(x\) into the vertex cover. Since component \(x\) is a \(Q\)-long chain, by Equation (12) we need \(\min\{x, x - k_2 + Q - 1\}\) plants from component \(x\) for the vertex cover. Thus,

\[
\delta^{k,0}(e, \mathcal{SC}_Q) \leq n - k - x + \min\{x, x - k_2 + Q - 1\} = \min\{n - k_1, n - k + Q - 1\}.
\]

(iii)

Moreover, by Equation (12) for \(\mathcal{LC}_Q\) we have

\[
\delta^{k,0}(e, \mathcal{LC}_Q) = \min\{n, n - k + Q - 1\}.
\]

(iv)
By (iii) and (iv), we get
\[ \delta^{k,0}(e, \text{SC}_Q) \leq \delta^{k,0}(e, \text{LC}_Q) = \min\{n, n - k + Q - 1\}. \] (v)

Therefore, by (i), (ii), (v) and Lemma 2 we have
\[ \lambda \cdot (n - k - \gamma)^+ \leq \min_{g \in \mathcal{U}_p} \delta^{k,0}(g, \text{SC}_Q) \leq \min_{g \in \mathcal{U}_p} \delta^{k,0}(g, \text{LC}_Q). \]

To prove the validity of Equation (15) for \((Q - 1)^2 \leq \ell \leq n \cdot Q\) note that by Equation (13), \(\min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, \text{LC}_Q) = \lambda \cdot (n - k - \lfloor \frac{\ell}{Q} \rfloor - \gamma)^+\). Hence, we only need to demonstrate that \(\min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, \text{SC}_Q) = \lambda \cdot (n - k - \lfloor \frac{\ell}{Q} \rfloor - \gamma)^+\), as well. The proof is trivial if \(kQ + \ell \geq nQ\) because in this case, \(\min_{g \in \mathcal{U}_p} \delta^{k,\ell}(g, \mathcal{D}) = 0\) for any design \(\mathcal{D}\); thus, in the following we let \(kQ + \ell < nQ\).

3. Let \(k = \sum_{i \in I_1} z_i\) for some \(I_1 \subseteq \{1, \ldots, c\}\); then we put all products of components in \(I_1\) in the vertex cover. Moreover, let \(I_2\) denote the largest subset in \(\{1, \ldots, c\} \setminus I_1\) such that \(Q \sum_{i \in I_2} z_i < \ell\); then ignore all arcs in \(I_2\). Clearly, all arcs of components in \(I_1 \cup I_2\) are either covered or ignored. Define \(\ell_y = \ell - Q \sum_{i \in I_2} z_i\). Since \(Q \sum_{i \in I_2} z_i < \ell\), we have \(\ell_y > 0\). Hence, there exists component \(y \in \{1, \ldots, c\} \setminus (I_1 \cup I_2)\) such that \(z_y Q \geq \ell_y\). The number of plants required from component \(y\) is \(z_y - \lfloor \frac{\ell_y}{Q} \rfloor\). Because, no products of component \(y\) are included in the vertex cover and by ignoring each batch of \(Q\) arcs from \(y\) we can exclude only one plant of \(y\) from the vertex cover. Therefore, the total number of required plants is
\[
n - \sum_{i \in I_1} z_i - \sum_{i \in I_2} z_i - \lfloor \frac{\ell_y}{Q} \rfloor = n - k - \sum_{i \in I_2} z_i - \left\lfloor \frac{\ell - Q \sum_{i \in I_2} z_i}{Q} \right\rfloor = n - k - \lfloor \frac{\ell}{Q} \rfloor.
\]

By (i), we conclude that the aforementioned constructed vertex cover is the minimum one, and we get \(\delta^{k,\ell}(e, \text{SC}_Q) = n - k - \lfloor \frac{\ell}{Q} \rfloor\).

4. Let \(k \neq \sum_{i \in I} z_i\) for any \(I \subseteq \{1, \ldots, c\}\), then let \(I_1 \subset \{1, \ldots, c\}\) be the largest subset of components such that \(\sum_{i \in I_1} z_i < k\) and define \(k_1 := \sum_{i \in I_1} z_i\). Next, we create a vertex cover in the following manner. Put all products of components in \(I_1\) into the vertex cover (all arcs in \(I_1\) are covered by \(k_1\) products). There exists component \(x \in \{1, \ldots, c\} \setminus I_1\) such that \(z_x > k_2\), \(k_2 := k - k_1\). We also put \(k_2\) products of \(x\) with consecutive indices in the vertex cover. It is seen that, \(Q(z_x - k_2)\) arcs of competent \(x\) remain uncovered.

4.1. If \(\ell \leq Q(z_x - k_2)\), then since component \(x\) is a \(Q\)-long chain and \((Q - 1)^2 \leq \ell\), by Equation (13), the number of plants required from \(x\) to put into the vertex cover is \(z_x - k_2 - \lfloor \frac{\ell}{Q} \rfloor\). In addition, we need all plants in components \(i \in \{1, \ldots, c\} \setminus (I_1 \cup \{x\})\) for the vertex cover. Thus, we need
By Equation (13) and applying Lemma 2 for parts 3 and 4 of this proof, we conclude that
\[
λ = (n - k_1 - z_x) + (z_x - k_2 - \lfloor \frac{ℓ}{Q} \rfloor) = n - k - \lfloor \frac{ℓ}{Q} \rfloor
\] plants along with specified products to create a vertex cover. By (i) we arrive at a minimum vertex cover. Therefore, \( δ^{k,ℓ}(e, SC_Q) = n - k - \lfloor \frac{ℓ}{Q} \rfloor \).

4.2. If \( ℓ > Q(z_x - k_2) \), then ignore all uncovered arcs of \( x \). Let \( I_2 \) denote the largest subset in \( \{1, \ldots, c\} \setminus (I_1 \cup \{x\}) \) such that \( Q \sum_{i \in I_2} z_i < ℓ - Q(z_x - k_2) \). If so, we can ignore all arcs in \( I_2 \). Clearly, all arcs of components in \( I_1 \cup I_2 \cup \{x\} \) are either covered or ignored.

Define \( ℓ_y := ℓ - Q \sum_{i \in I_2} z_i - Q(z_x - k_2) \). We observe that \( ℓ_y > 0 \) since \( Q \sum_{i \in I_2} z_i < ℓ - Q(z_x - k_2) \). Thus, there exists component \( y \in \{1, \ldots, c\} \setminus (I_1 \cup I_2 \cup \{x\}) \) such that \( z_y Q ≥ ℓ_y \). It should be noted that no product of component \( y \) is in the vertex cover. Moreover, by ignoring each batch of \( Q \) arcs connected to a plant only one plant of \( y \) is excluded from the vertex cover. Thus, the required number of plants from component \( y \) in the vertex cover is \( z_y - \lfloor \frac{ℓ_y}{Q} \rfloor \). Therefore, by (i) the minimum number of plants required to create a vertex cover is

\[
δ^{k,ℓ}(e, SC_Q) = n - \sum_{i \in I_1} z_i - \sum_{i \in I_2} z_i - z_x - z_y + (z_y - \lfloor \frac{ℓ_y}{Q} \rfloor) = n - k_1 - \sum_{i \in I_2} z_i - z_x - \lfloor \frac{ℓ - Q \sum_{i \in I_2} z_i - Q(z_x - k_2)}{Q} \rfloor = n - k - \lfloor \frac{ℓ}{Q} \rfloor.
\]

By Equation (13) and applying Lemma 2 for parts 3 and 4 of this proof, we conclude that \( λ · (n - k - \lfloor \frac{ℓ}{Q} \rfloor - γ)^+ = \min_{g \in \mathcal{U}_p} δ^{k,ℓ}(g, SC_Q) = \min_{g \in \mathcal{U}_p} δ^{k,ℓ}(g, LC_Q) \) for any \( 0 < k \leq n \) and \( (Q - 1)^2 \leq ℓ \leq n · Q \). \( \square \)

Proof of Proposition 3. If \( α = 0 \), then by Assumption 1 we have \( ℓ^* = α = 0 \) in Equation (10). Moreover, from Inequality (14) we have \( \min_{g \in \mathcal{U}_p} δ^{k,0}(g, SC_Q) ≤ \min_{g \in \mathcal{U}_p} δ^{k,0}(g, LC_Q) \) at any \( 0 < k \leq n \). Therefore, \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, SC_Q) ≤ R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, LC_Q) \) by Theorem 1.

Similarly, if \( α ≥ (Q - 1)^2 \), then \( ℓ^* = α ≥ (Q - 1)^2 \). Furthermore, \( \min_{g \in \mathcal{U}_p} δ^{k,α}(g, SC_Q) = \min_{g \in \mathcal{U}_p} δ^{k,α}(g, LC_Q) \) by Equality (15). Consequently, from Theorem 1 we obtain \( R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, SC_Q) = R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, LC_Q) \) for any \( (Q - 1)^2 < α ≤ n · Q \). \( \square \)

B Examples for Chains under Disruptions

Let us consider design \( D \) with 10 plants and 10 products with no disruptions such that \( e^{(p)} = 100 · e \) and \( c^{(a)}_{ij} = 100 \) for all \( (i, j) \in D \). The demand for each product is assumed to be from a truncated normal distribution \( N(100, 40^2) \) with support [20, 180].

The second column of Table 2 displays the simulation results for the expected performance of some designs computed by Jordan and Graves (1995). They concluded that (i) the expected performance of \( LC_2 \) is almost the same as \( LC_{10} \), and adding more flexibility is not significantly more beneficial; (ii) fewer and longer chains are preferred to improve the expected performance. In other words, the expected performance of \( LC_2 \) is at least as good as the expected performance of any \( SC_2 \) in \( \{SC_2\} \). These observations primarily explain why the related works often do not take into account either designs with
more than 2n arcs or short-chain designs (Deng and Shen 2013).

To evaluate the worst-case performances and to be consistent with Jordan and Graves (1995), let \( \mathcal{U}_d = \{d \mid 20 \leq d_j \leq 180, \forall j \in \{1, 2, \ldots, n\}\} \). According to Table 2, the worst-case performance of all designs are the same if there are no disruptions (see the third column). In particular, \( \mathcal{SC}_2 \) has the same worst-case performance as \( \mathcal{LC}_2 \) and the full flexibility. However, for even a few complete disruptions the results are different. First, \( \mathcal{LC}_2 \) does not have the same worst-case performance as \( \mathcal{LC}_{10} \). Second, when the design is subject to only arc disruptions (see the fourth column) \( \mathcal{LC}_2 \) and \( \mathcal{SC}_2 \) still have the same performance.

To further illustrate these findings, we compare the performance of \( \mathcal{LC}_Q \) for \( Q \in \{1, 2, 4, 6, 8, 10\} \) and \( \mathcal{SC}_2 \) under disruptions. Figure 3 portrays the (worst-case) performance of chains versus different types and number of disruptions; specifically it demonstrates that as \( Q \) is larger \( Q \)-long chain is less vulnerable to disruptions. Figure 3 also shows that the worst-case performance of designs coincide when there are no disruptions and it is non-increasing with respect to complete arc (\( \alpha \)) and plant (\( \gamma \)) disruptions. Moreover, Figure 3a displays that the performances of \( \mathcal{SC}_2 \) and \( \mathcal{LC}_2 \) are the same for any value of \( \alpha \). However, in the case of plant disruptions, see Figure 3b, the worst-case performance of \( \mathcal{SC}_2 \) is bounded above by the performance of \( \mathcal{LC}_2 \).

Table 2: Expected performances (second column, Jordan and Graves 1995) versus the worst-case performances of sample chains

<table>
<thead>
<tr>
<th>Design</th>
<th>( E(\text{Perf.}) )</th>
<th>without disruptions</th>
<th>with 2 complete arc disruptions,</th>
<th>with 2 complete plant disruptions,</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{LC}_1 )</td>
<td>853</td>
<td>200</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td>( \mathcal{SC}_2 ) (five components)</td>
<td>896</td>
<td>200</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td>( \mathcal{LC}_2 )</td>
<td>950</td>
<td>200</td>
<td>180</td>
<td>180</td>
</tr>
<tr>
<td>( \mathcal{LC}_{10} )</td>
<td>954</td>
<td>200</td>
<td>200</td>
<td>200</td>
</tr>
</tbody>
</table>

Figure 3: Worst-case performances of chains for different types of disruptions