PyMOSO: Software for Multi-Objective Simulation Optimization with R-PεRLE and R-MINRLE

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We present the PyMOSO software package for (1) solving multi-objective simulation optimization (MOSO) problems on integer lattices, and (2) implementing and testing new simulation optimization (SO) algorithms. First, for solving MOSO problems on integer lattices, PyMOSO implements R-PεRLE, a state-of-the-art algorithm for two objectives, and R-MINRLE, a competitive benchmark algorithm for three or more objectives. Both algorithms employ pseudo-gradients, are designed for sampling efficiency, and return solutions that, under appropriate regularity conditions, provably converge to a local efficient set with probability one as the simulation budget increases. PyMOSO can interface with existing simulation software and can obtain simulation replications in parallel. Second, for implementing and testing new SO algorithms, PyMOSO includes pseudo-random number stream management, implements algorithm testing with independent pseudo-random number streams run in parallel, and computes the performance of algorithms with user-defined metrics. For convenience, we also include an implementation of R-SPLINE for problems with one objective. The PyMOSO source code is available under a permissive open source license.

Key words: multi-objective simulation optimization, software

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1. Introduction
We present the PyMOSO software package, written in Python, for using, implementing, and testing multi-objective simulation optimization (MOSO) algorithms. PyMOSO currently implements a state-of-the-art algorithm, R-PεRLE, for solving MOSO problems on integer lattices with two objectives, and a competitive benchmark algorithm, R-MINRLE, for solving MOSO problems on integer lattices with many objectives. Since R-PεRLE and R-MINRLE are both pseudo-gradient-based algorithms that rely on the single-objective SPLINE solver by Wang et al. (2013), for convenience, we also include an implementation of R-SPLINE (Wang et al. 2013) for solving single-objective simulation optimization problems on integer lattices. The initial version of the software, and this paper, serve as companions to Cooper et al. (2018), the paper that introduces and explains the R-PεRLE and R-MINRLE algorithms.
MOSO problems are nonlinear optimization problems with more than one simultaneous objective function, each of which can only be observed with stochastic error. For example, at any feasible point, observations of the objective values may be obtained from a Monte Carlo simulation oracle.

By the nature of the algorithms currently included in PyMOSO, our focus is on MOSO problems in which the feasible set is a subset of an integer lattice. Such problems take the form

\[
\text{Problem } M_d: \min_{x \in \mathcal{X}} \{ g(x) = (g_1(x), \ldots, g_d(x)) := (\mathbb{E}[G_1(x, \xi)], \ldots, \mathbb{E}[G_d(x, \xi)]) \},
\]

where \( \mathcal{X} \subseteq \mathbb{Z}^q \) is the feasible set and \( \xi \) is a random vector. R-P\( \varepsilon \)RLE is designed to solve Problem \( M_2 \) to local optimality, and R-MinRLE is designed to solve Problem \( M_d, \ d \geq 2 \), to local optimality. A local solution to Problem \( M_d \) is called a local efficient set, which we formally define in §1.2.

MOSO problems on integer lattices arise in many application domains including aviation, healthcare, transportation, and manufacturing (Hunter et al. 2019). For example, in aviation, Li et al. (2015) solve a bi-objective aircraft spare parts management problem, and Lee et al. (2008) solve a tri-objective inventory control problem for a network of airports. In healthcare, Chen and Wang (2016) solve a bi-objective capacity allocation problem for a hospital’s emergency department. In transportation, Zhou et al. (2018) solve a bi-objective problem to reduce congestion in a lighterage terminal. Finally, in manufacturing, Andersson et al. (2007) solve a bi-objective problem to increase the throughput and maintain safety stocks for a camshaft production line. In these applications, the decision variables take on integer values, the objective functions can only be observed with stochastic error though a Monte Carlo simulation, and the goal is to retrieve the entire efficient set as input to the multi-criteria decision-making process.

Though MOSO problems arise in many application areas, few software packages exist to solve Problem \( M_d \) for \( d \geq 2 \) objectives. The software packages that do exist, such as OptQuest (OptTek Systems, Inc. 2018, Thengvall et al. 2016) and PaGMO/PyGMO (Biscani and Izzo 2018), tend to implement metaheuristics to solve MOSO problems, which do not provide performance guarantees (Hong et al. 2015). We are not aware of any available MOSO software that provides the sampling efficiency and convergence guarantees of R-P\( \varepsilon \)RLE and R-MinRLE, moreover, no software implementing these two algorithms currently exists.
Finally, we remark here that an implementation of an as-yet-unpublished algorithm, Multi-objective Partitioned Random Search, is available in a source code repository (Weizhi 2017).

1.1. Contributions

As in Schmeiser (2008), we adopt the terms ‘practitioners’ and ‘researchers’ to describe those who seek a solution to Problem $M_d$ to aid in decision-making, and those who intend to create and compare MOSO algorithms, respectively. We discuss our contributions to each group.

PyMOSO provides practitioners with off-the-shelf access to the state-of-the-art, provably convergent, bi-objective solver $R-\varepsilon RLE$, and to the competitive, provably convergent, multi-objective solver $R$-$\text{MinRLE}$. PyMOSO can accommodate any Monte Carlo simulation oracle that can be called from Python. Swain (2017) provides a list of proprietary simulation software packages and indicates whether each package can be invoked by an external program. Most simulation software packages that can be invoked by external programs are, with programming effort, compatible with PyMOSO. After the oracle has been implemented, PyMOSO can obtain simulation replications in parallel using common random numbers (CRN, see Law 2015) by exploiting the stream and substream capabilities of the pseudo-random number generator $\text{mrg32k3a}$ (L’Ecuyer et al. 2002), which may reduce runtime. Finally, PyMOSO provides off-the-shelf access to R-SPLINE in the same software framework. While PyMOSO currently supports only integer decision variables, practitioners may wish to solve problems with continuous decision variables. We urge caution when discretizing continuous problems, as the choice of grid size is non-trivial. We hope that future versions of PyMOSO will include solvers for problems with continuous decision variables.

For researchers who intend to create and compare MOSO algorithms, PyMOSO offers two primary benefits, as follows.

1. Researchers designing new algorithms to solve MOSO problems on integer lattices should compare their algorithms with $R-\varepsilon RLE$ and $R$-$\text{MinRLE}$ on a variety of test problems. PyMOSO enables researchers to compare algorithms by providing an interface for implementing test problems and calculating user-defined metrics. By default, PyMOSO includes all test problems and associated metrics from Cooper et al. (2018). To create additional test problems, we recommend taking inspiration from existing testbeds for deterministic multi-objective optimization (see, e.g., the references in Hunter et al. 2019, p. 25 – 26). Once the desired test problems are implemented, researchers may use PyMOSO to run many independent sample paths of the algorithms in parallel. PyMOSO
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provides pseudo-random numbers using mrg32k3a (L’Ecuyer 1999) and random number
stream management consistent with L’Ecuyer et al. (2002).

2. PyMOSO provides a framework that enables researchers to create and implement new
algorithms. Although any new MOSO algorithm can be implemented in PyMOSO, it
is especially easy to implement algorithms that rely on a version of sample average
approximation called retrospective approximation (RA). (We provide a brief explana-
tion of RA in §1.2; see Pasupathy and Ghosh (2013) for a more thorough explanation.)
In particular, researchers can create new RA algorithms for MOSO by writing “accelera-
tor” functions that provide starting points to the naïve search algorithm Relaxed Local
Enumeration (RLE) in each RA iteration, as we describe in §3.3. Under appropriate
regularity conditions, Cooper et al. (2018) prove that such algorithms converge to a
local efficient set almost surely as the sample size increases.

In what follows, we discuss background concepts in §1.2, including optimality definitions,
RA, and accelerators. Then, we provide introductions to the current version of PyMOSO for
practitioners in §2 and for researchers in §3. Since all information provided for practitioners
is relevant to researchers, we encourage researchers to read both sections. In addition to the
online supplement, PyMOSO installation instructions, source code, and the user manual can
be found at https://github.com/pymoso/.

1.2. Background

Before discussing PyMOSO in detail, for completeness, we provide a sense of the sets returned
by R-PεRLE and R-MinRLE, which we call estimated local efficient sets. Due to space
constraints, we make this section compact. See Cooper et al. (2018) for a complete treatment.

To define a local efficient set (LES), we first define neighborhoods and dominated points.
Let \(d(x, x') := ||x - x'||\) denote the Euclidean distance between two points \(x, x' \in \mathbb{R}^q\).

**Definition 1.** Given \(a \in \mathbb{R}, a \geq 1\), the \(N_a\)-neighborhood of a point \(x \in \mathbb{Z}^q\) is \(N_a(x) := \{x' \in \mathbb{Z}^q : d(x, x') \leq a\}\), and the \(N_a\)-neighborhood of a set \(S \subseteq \mathbb{Z}^q\) is \(N_a(S) := \cup_{x \in S} N_a(x)\).

**Definition 2.** A vector \(g(x^*)\) dominates \(g(x)\), written as \(g(x^*) \leq g(x)\), if \(g_k(x^*) \leq g_k(x)\) for all \(k \in \{1, \ldots, d\}\) and \(g_{k^*}(x^*) < g_{k^*}(x)\) for at least one \(k^* \in \{1, \ldots, d\}\).

**Definition 3** (Cooper et al. 2018). Given \(a \in \mathbb{R}, a \geq 1\), a set \(L_a \subseteq \mathcal{X}, |L_a| \geq 1\) is an
\(N_a\)-local efficient set (\(N_a\)-LES) if (a) for each \(x^* \in L_a\), \(\exists x \in N_a(x^*) \cap \mathcal{X}\) such that \(g(x) \leq g(x^*)\), (b) for each \(x^* \in L_a\), \(\exists x' \in L_a\) such that \(g(x') \leq g(x^*)\), (c) for each \(x \in (N_a(L_a) \setminus L_a) \cap \mathcal{X}\), \(\exists x^* \in L_a\) such that \(g(x^*) \leq g(x)\).
Figure 1 shows two $N_a$-LES’s, $L_1$ and $E$. The set $E$ is also the global efficient set, which we define as a local efficient set with $a = \infty$.

All algorithms currently implemented in PyMOSO solve Problem $M_d$ using an algorithmic framework called RA. RA is a version of sample average approximation that is designed for algorithmic efficiency. Algorithms in an RA framework solve a sequence of sample-path problems at increasing sample sizes, using the solution from the previous RA iteration as a warm start for the current RA iteration. The sample-path problem is defined as

$$
\text{minimize}_{x \in X} \left\{ \bar{G}_{m_\nu}(x) = (G_{1,m_\nu}(x), \ldots, G_{d,m_\nu}(x)) : = \left( \frac{1}{m_\nu} \sum_{i=1}^{m_\nu} G_1(x, \xi_i), \ldots, \frac{1}{m_\nu} \sum_{i=1}^{m_\nu} G_d(x, \xi_i) \right) \right\},
$$

where $G_{m_\nu}(x)$ is an estimator of $g(x)$ constructed in RA iteration $\nu$ with sample size $m_\nu$.

The local solution to the sample-path problem is a sample-path $N_a$-LES, which we define by replacing all quantities in Definition 3 by their respective estimators. Since the estimated LES found in RA iteration $\nu - 1$ is used as a warm start in RA iteration $\nu$ and the sequence of sample sizes $\{m_\nu, \nu = 1, 2, \ldots\}$ is increasing, RA algorithms are efficient because they reserve large sample sizes for points that are "close" to the solution.

In the context of R-PzRLE and R-MinRLE, the algorithms that solve the sample-path problems, also called sample-path solvers, consist of two sub-routines: (1) an accelerator routine which generates a candidate estimated LES, and (2) RLE, which determines whether a set is an estimated LES and, if not, constructs an estimated LES by crawling through the feasible space. Thus, given a starting feasible point, the goal of the accelerator is to find candidate estimated LES members more efficiently than RLE alone would. As $\nu \to \infty$, the sequence of estimated LES’s generated by RLE in an RA framework converge to a LES almost
Cooper and Hunter: PyMOSO: Software for MOSO with R-PεRLE and R-MinRLE surely under appropriate regularity conditions. Loosely speaking, the regularity conditions include that all LES’s are finite, the standard errors of the objective estimators go to zero fast enough as the sample size increases (e.g., the objective estimators are not heavy-tailed), and the objective values of all feasible points are separated from each other. (If this last condition does not hold, the algorithm guarantees a slightly weaker form of convergence.) Under the additional assumptions that the feasible set is finite, all local efficient points are global efficient points, and there is exactly one LES that equals the global efficient set, R-PεRLE converges exponentially fast. Given a distance $a$ and a total simulation budget that limits the number of simulation oracle calls, R-PεRLE and R-MinRLE return an approximate sample-path $N_a$-LES (ALES), which we refer to as an estimated LES. For an exact definition of an ALES, and for a thorough explanation of the regularity conditions required for each type of convergence, see Cooper et al. (2018).

2. Practitioners: Using PyMOSO to Solve a Problem
In this section, we discuss using PyMOSO in the practitioner context. Practitioners use PyMOSO in two steps: first, implement the simulation oracle in PyMOSO, which we discuss in §2.1, and second, use PyMOSO to solve the problem, which we discuss in §2.2.

2.1. Structuring an Oracle for Use in PyMOSO
To structure an oracle for use in PyMOSO, a practitioner should modify the Python source code template provided in Figure 2. Figure 2 implements an oracle named MyProblem in a Python file named myproblem.py. PyMOSO requires that the problem name match the file name, although the capitalization does not need to match.

A practitioner implements a MOSO problem by first modifying the number of objectives and the dimension of the feasible points in Lines 8 and 9 of Figure 2, respectively. Setting

```python
1 # import the Oracle base class
2 from pymoso.chnbase import Oracle
3
class MyProblem(Oracle):
    """Example implementation of a user-defined MOSO problem."
    ""
    def __init__(self, rng):
        """Specify the number of objectives and dimensionality of points."
        self.num_obj = 2
        self.dim = 1
        super().__init__(rng)

    def g(self, x, rng):
        """Check feasibility and simulate objective values."
        #objective_values = (obj1, obj2), is_feasible = True
        return is_feasible, objective_values
```

Figure 2 The Python file myproblem.py is a template PyMOSO oracle. As shown, g(self, x, rng) is incomplete.
the correct values for the practitioner’s Problem $M_d$, and changing nothing else, is sufficient to implement the `__init__` method correctly. Then, the practitioner should replace the comment in Line 14 of Figure 2 with valid Python code that, given a Python tuple $x$ representing a point $x \in \mathbb{R}^q$, generates the following to return in Line 15: (a) a boolean indicator denoting whether $x$ is feasible, and (b) a Python tuple, containing one observation of every objective function at $x$ if $x$ is feasible, and a Python tuple containing `None` for every objective if $x$ is not feasible. In our notation, one observation of every objective function at $x$ is represented by $(G_1(x, \xi), \ldots, G_d(x, \xi))$; alternatively, the practitioner may think of this quantity as one observation of $\bar{G}_n(x)$ where $n = 1$. The function $g(self, x, rng)$ may contain any number of lines and may be a wrapper for an external simulation oracle. Returning the feasibility indicator followed by a Python tuple of the objective values is sufficient to correctly implement the $g(self, x, rng)$ method.

Optionally, a practitioner may use the PyMOSO object $rng$ to generate pseudo-random numbers with `mrg32k3a`, or may use the `mrg32k3a` seed from $rng$ as the seed in an external `mrg32k3a` generator. If `MyProblem` implements either approach, PyMOSO ensures that simulation replications obtained in parallel are independent by exploiting the stream and sub-stream capabilities of `mrg32k3a` (L’Ecuyer et al. 2002). Further, the implemented PyMOSO oracle is compatible with PyMOSO’s common random number (CRN) framework. Practitioners who ignore $rng$ should take care when using PyMOSO’s parallel computing and CRN capabilities. To determine when using CRN is appropriate, we refer the reader to Law (2015). More detailed information about $rng$ is available in the PyMOSO user manual.

In the remainder of the paper, we assume the practitioner has implemented a PyMOSO oracle called `MyProblem`. So the reader can run our examples, we provide a simple example of `MyProblem` in Figure 3, where Problem $M_2$ is $g_1(x) = x^2 + z_0$ and $g_2(x) = (x - 2)^2 + z_1$, $x \in \{-100, -99, \ldots, 100\}$, and $z_0, z_1$ are standard normal random variables. The solution is $\{0, 1, 2\}$. We remark here that Figure 3 contains an example of using $rng$.

### 2.2. Solving a MOSO Problem in PyMOSO

Having implemented a PyMOSO oracle called `MyProblem` in §2.1, we now discuss using PyMOSO to solve `MyProblem`. Practitioners may use PyMOSO in two modes: as a stand-alone solver invoked from the command line, or as a subroutine in a Python program. The former creates a file containing the output of one run of the selected algorithm, which is an estimated LES. The latter returns the estimated LES as a Python set. In either case,
```python
def g(self, x, rng):
    '''Check feasibility and simulate objective values for MyProblem.'''
    feas_range = range(-100, 101)
    obj = []
    is_feas = False
    # check that dimensions of x match self.dim
    if len(x) == self.dim:
        is_feas = True
        for i in x:
            if not i in feas_range:
                is_feas = False
        if is_feas:
            z_vec = [rng.normalvariate(0, 1) for i in [0, 1]]
            obj1 = x[0]*2 + z_vec[0]
            obj2 = (x[0] - 2)**2 + z_vec[1]
    return is_feas, (obj1, obj2)
```

Figure 3  This figure provides an example $g$ function, which we use in MyProblem.

PyMOSO requires the practitioner to specify, using a method we describe, at least the following information: the problem, the algorithm, and an initial feasible point. The method is slightly different depending on the chosen mode. In this section, we only consider the command line mode to solve MyProblem. See the user manual for the subroutine mode.

The simplest viable command to solve a problem in command line mode follows the structure `program command problem solver x0`, where $x0$ denotes the initial feasible point. The solve command takes options, including the total simulation budget and the number of parallel processors. Practitioners may view the full set of available options by entering `pymoso --help` and the full list of PyMOSO solver names by entering `pymoso listitems`. As an example, the command to solve MyProblem using R-PεRLE, starting from the feasible point 97, with a total simulation budget of 10,000 and 4 processors, is below.

```
pymoso solve --budget=10000 --simpar=4 myproblem.py RPERLE 97
```

This invocation requires that `myproblem.py` is in the working directory. Since the feasible points of MyProblem are one-dimensional, $x0$ is a scalar. For feasible points in higher dimensions, separate each component with a space, e.g., a three-dimensional point (97, 23, 18) is written as 97 23 18. After issuing the above command, PyMOSO creates a new subdirectory named `testrun` in the working directory. This subdirectory typically contains two files: one file containing the metadata and one file containing the estimated LES. We exhibit the file containing the estimated LES in Figure 4. If PyMOSO detects an error, it may also write an error file.

In this section, we discuss using PyMOSO in the researcher context. Researchers can use PyMOSO to compare algorithms and to create new algorithms. To compare algorithms, researchers first implement a PyMOSO oracle as in §2.1. Then, they create a PyMOSO tester, which we discuss in §3.1, and run the tester, which we discuss in §3.2. We briefly discuss creating new algorithms in §3.3.

3.1. Structuring a Test Problem for Use in PyMOSO

After implementing a PyMOSO oracle called MyProblem in §2.1, researchers must implement a PyMOSO tester for MyProblem. We provide an example tester called MyTester in Figure 5, where the oracle to be tested in Line 27 is specified as MyProblem.

Technically, a valid PyMOSO tester may consist of only Lines 24–27 in Figure 5. However, Figure 5 illustrates two optional features that researchers may find useful. First, researchers can implement a PyMOSO function that generates feasible starting points by setting self.get_ranx0 to an appropriate function in Line 30. We provide an example function, called get_ranx0, that randomly generates a feasible point for MyProblem in Lines 9–12 of Figure 5. The function must take rng as a parameter and return a Python tuple representing a feasible point. The second feature enables researchers to implement a metric for comparing an estimated solution to the known, true solution.

We provide an example metric that calculates the Hausdorff distance from the expected objective values of the points in the estimated LES, eles, to the image of the known solution, self.answer, in Lines 32–36. To calculate the expected objective values of the points in eles, we implement the function true_g in Lines 15-19. We also specify myanswer in Line 22 and set self.answer and self.true_g as members of MyTester in Lines 28–29. Researchers may replace Lines 34–36 with a metric of their choosing.

3.2. Testing a MOSO Algorithm in PyMOSO

Having implemented both MyProblem, in §2.1, and its tester, MyTester in §3.1, in PyMOSO, we now discuss using PyMOSO to test algorithms on MyProblem. As with practitioners,
solving MOSO problems, researchers can use PyMOSO in two modes for testing algorithms on problems: as a stand-alone solver invoked from the command line, or as a subroutine in a Python program. In this section, we only consider the command line mode to test PyMOSO algorithms. See the user manual for the subroutine mode.

The simplest viable command to test an algorithm in command line mode follows the structure program command tester solver. (Researchers may also specify a feasible starting point if the tester is not programmed to generate them.) The testsolve command takes options, including the number of independent sample paths of the test problem, the number of processors to use, and whether to compute a metric. As an example, the command to test R-PεRLE by running 16 independent sample paths of MyProblem using MyTester, on 4 processors and computing a metric, is below.

```
pymoso --isp=16 --proc=4 --metric testsolve mytester.py RPERLE
```
This invocation requires that `myproblem.py` and `mytester.py` are in the working directory. After issuing the above command, PyMOSO creates a new subdirectory named `testrun` in the working directory. This subdirectory contains (a) one metadata file; (b) 16 data files, each containing a list of estimated LES’s, one for every algorithm iteration; and (c) 16 files containing metric calculations, which are included only when using the `--metric` option. The files containing metric calculations each have data of the form $(\nu, w_\nu, h_\nu)$, where $\nu$ is the RA algorithm iteration number; $w_\nu$ is the cumulative work done, measured as the total number of simulations used at the end of iteration $\nu$; and $h_\nu$ is the metric computed on the estimated LES at iteration $\nu$. We exhibit sample data and metric files in Figures 6 and 7. If PyMOSO detects an error, it may also write an error file.

```
1  {(3 ,)}
2  {(2 ,), (1 ,)}
3  {(0 ,), (1 ,)}
4  {(2 ,), (0 ,), (1 ,)}
5  {(2 ,), (0 ,), (1 ,)}
6  {(2 ,), (1 ,)}
7  {(0 ,), (1 ,)}
8  {(2 ,), (0 ,), (1 ,)}
9  {(2 ,), (0 ,), (1 ,)}
10 { (2 ,), (0 ,), (1 ,)}
```

Figure 6 One of the 16 solution output files from the `testsolve` invocation given in §3.2. Each line contains a set, which is the solution of the iteration corresponding to the line number.

```
1  (0 , 0 , 9.486832980505138)
2  (1 , 15 , 3.1622776601683795)
3  (2 , 33 , 3.1622776601683795)
4  (3 , 51 , 0.0)
5  (4 , 69 , 0.0)
6  (5 , 93 , 3.1622776601683795)
7  (6 , 117 , 3.1622776601683795)
8  (7 , 141 , 0.0)
9  (8 , 171 , 0.0)
10 (9 , 201 , 0.0)
```

Figure 7 One of 16 metric output files from the `testsolve` invocation given in §3.2. Each line contains indicates the iteration number, the number of simulations used at the end of the iteration, and the performance metric of the iteration solution. In this example, the best possible performance is zero.
3.3. Creating New Accelerators in PyMOSO

Though researchers may implement any simulation optimization algorithm in PyMOSO, we discuss how to implement RA algorithms that invoke an “accelerator” followed by RLE in every RA iteration (see Cooper et al. 2018). For example, in R-PεRLE, “Pε” is the accelerator, and in R-MinRLE, “Min” is the accelerator. Users can create new accelerators. We provide an accelerator template in Figure 8. Researchers should replace the comment in Line 9 of Figure 8 with their own code. The function signature must be `accel(self, warm_start)` and the function must return a Python set. After implementing a PyMOSO algorithm, researchers can test it as in §3.2.

```
pymoso --isp=20 --proc=4 --metric testsolve mytester.py myaccel.py
```

For implementing algorithm logic, PyMOSO also provides support for, e.g., obtaining simulation replications from the oracle, computing all points in a set that are non-dominated, and generating neighborhoods of points and sets. For detailed descriptions of the support functions, including working examples, we refer the reader to the user manual.

4. Conclusion

The PyMOSO software package provides open-source, off-the-shelf access to state-of-the-art solvers for simulation optimization on integer lattices in an accessible and popular programming language. PyMOSO also provides a framework and useful tools for researchers who wish to compare and create new algorithms.

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Online Supplement for
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A. PyMOSO User Manual
We provide this user manual to accompany the initial release of PyMOSO.
A.1. Additional Reading

The initial release of PyMOSO contains solvers that implement four total algorithms, in alphabetical order: R-MinRLE, R-P\(\varepsilon\), R-P\(\varepsilon\)RLE, and R-SPLINE. The algorithms R-MinRLE, R-P\(\varepsilon\), and R-P\(\varepsilon\)RLE were introduced in the following paper:


The algorithm R-SPLINE was introduced in the following paper:


We recommend reading these papers to understand the algorithms, what they return, and the algorithm parameter options that we describe in the user manual.

A.2. Installation

Since PyMOSO is programmed in Python, every PyMOSO user must first install Python, which can be downloaded from https://www.python.org/downloads/. PyMOSO is compatible with Python versions 3.6 and higher. In the remainder of this section, we assume an appropriate Python version is installed. We discuss three different methods to install PyMOSO: first, from the Python Packaging Index; second, directly from our source code using git; and third, manually installing PyMOSO from our source code.

A.2.1. Install PyMOSO from the Python Packaging Index using pip

For ease of distribution, we keep stable, recent releases of PyMOSO on the Python Packaging Index (PyPI). Since the program pip is included in Python versions 3.6 and higher, we recommend using pip to install PyMOSO. To do so, open a terminal, type the following command, and press enter.

```
    pip install pymoso
```

Depending on how users configure their Python installation and how many versions of Python they install, they may need to replace pip with pip3, or other variants of pip.
A.2.2. Install PyMOSO from git using pip

Users with git installed can use pip to install the most current version of PyMOSO directly from our source code:

```bash
pip install git+https://github.com/pymoso/PyMOSO.git
```

We consider the latest source to be less stable than the fixed releases we upload to PyPI, and thus we recommend most users install PyMOSO as in §A.2.1.

A.2.3. Install PyMOSO Manually from Source Code

Users may follow the steps below to manually install PyMOSO from any version of the source code.

1. Acquire the PyMOSO source code, for example, by downloading it from the repository
   ```bash
   https://github.com/pymoso/PyMOSO
   ```
2. Install the `wheel` package, e.g. using the `pip install wheel` command.
3. Open a terminal and navigate into the main project directory which contains the file `setup.py`
4. Build the installable PyMOSO package, called a wheel, using the command
   ```bash
   python setup.py bdist_wheel
   ```
   As with `pip`, some users may need to replace `python` with `python3` or something similar. The command should create a directory named `dist` containing the PyMOSO wheel.
5. Install the PyMOSO wheel using `pip install dist/pymoso-x.x.x-py3-none-any.whl`, where users replace `x.x.x` with the appropriate PyMOSO version.

A.3. Command Line Interface (CLI)

PyMOSO users solving MOSO problems and testing MOSO algorithms may do so using the command line interface. First, we show how to access the included help file. Then, we show how to view the lists of solvers, testers, and oracles installed by default with PyMOSO. Finally, we discuss the `solve` and `testsolve` commands.

A.3.1. CLI Help

PyMOSO includes a command line help file. The help file shows syntax templates for every PyMOSO command, the available options, and a selection of example invocations. The `pymoso --help` invocation prints the file to the terminal. The file is also printed when PyMOSO cannot parse an invocation that begins with `pymoso`. We show the current help file in Figure 9.

A.3.2. The listitems Command for Viewing Solvers, Testers, and Oracles Included in PyMOSO

The default installation of PyMOSO includes a selection of solvers, testers, and oracles. Users can view the complete lists of included solvers, testers, and oracles using
Cooper and Hunter: PyMOSO: Software for MOSO with R-PεRLE and R-MinRLE

Usage:

```
pymoso listitems
  <problem> <solver> <x> <y> ...
  [[(--seed <s> <s> <s> <s> <s> <s> <s>))]
  [([--param <param> <val>])...]
```

```
  [[(--seed <s> <s> <s> <s> <s> <s> <s>))]
  [([--param <param> <val>])...]
  <tester> <solver> <x>...
```

```
pymoso -h | --help
pymoso -v | --version
```

Options:

```
--budget=B Set the simulation budget [default: 200]
--odir=D Set the output file directory name. [default: testrun]
--crn Set if common random numbers are desired.
--simpar=P Set number of parallel processes for simulation replications. [default: 1]
--isp=T Set number of algorithm instances to solve. [default: 1]
--proc=Q Set number of parallel processes for the algorithm instances. [default: 1]
--metric Set if metric computation is desired.
--seed Set the random number seed with 6 spaced integers.
--param Specify a solver-specific parameter <param><val>.
-h --help Show this screen.
-v --version Show version.
```

Examples:

```
pymoso listitems
pymoso solve ProbTPA RPERLE 4 14
pymoso solve --budget=100000 --odir=test1 ProbTPB RMINRLE 3 12
pymoso solve --seed 12345 32123 5322 2 9543 666666666 ProbTPC RPERLE 31 21 11
pymoso solve --simpar=4 --param betaeps 0.4 ProbTPA RPERLE 30 30
pymoso solve --param radius 3 ProbTPA RPERLE 45 45
pymoso testsolve --isp=16 --proc=4 TPATester RPERLE
pymoso testsolve --isp=20 --proc=10 --metric --crn TPBTester RMINRLE 9 9
```

Figure 9 PyMOSO displays help when users enter the `pymoso --help` invocation.

the `pymoso listitems` command. We show the current listing in Figure 10. Test problems A, B, and C refer to those in Cooper et al. (2018).

A.3.3. The solve Command The PyMOSO `solve` command is for solving MOSO problems. Users can solve the built-in problems (use the `listitems` command to view the built-in problems), however, PyMOSO `solve` users typically will have their own MOSO problem they wish to solve. Thus, we assume users have implemented a PyMOSO oracle named MyProblem

---

<table>
<thead>
<tr>
<th>Solver</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMINRLE</td>
<td>A solver using R-MinRLE for integer-ordered MOSO.</td>
</tr>
<tr>
<td>RPE</td>
<td>A solver using R-Pε for integer-ordered bi-objective MOSO.</td>
</tr>
<tr>
<td>RPERLE</td>
<td>A solver using R-PERLE for integer-ordered bi-objective MOSO.</td>
</tr>
<tr>
<td>RSPLINE</td>
<td>A solver using R-SPLINE for single objective SO.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problems</th>
<th>Description</th>
<th>Test Name (if available)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ProbSimpleSO</td>
<td>x^2 + noise</td>
<td>SimpleSOTester</td>
</tr>
<tr>
<td>ProbTPA</td>
<td>Test Problem A</td>
<td>TPATester</td>
</tr>
<tr>
<td>ProbTPB</td>
<td>Test Problem B</td>
<td>TPBTester</td>
</tr>
<tr>
<td>ProbTPC</td>
<td>Test Problem C</td>
<td>TPCTester</td>
</tr>
<tr>
<td>BSProb</td>
<td>Bus Scheduling problem</td>
<td>BSTester</td>
</tr>
</tbody>
</table>

Figure 10 The `pymoso listitems` invocation shows the lists of built-in solvers, testers, and oracles.
in myproblem.py. In the examples that follow, we assume the MyProblem implementation in Figure 11, which is a bi-objective oracle with one-dimensional feasible points. See §A.4.1 for instructions on implementing a MOSO problem as a PyMOSO oracle.

The template solve command is pymoso solve oracle solver x0, where oracle is a built-in or user-defined oracle, solver is a built-in or user-defined algorithm, and x0 is a feasible starting point for the solver, with a space between each component. As a first example, we solve the user-defined MyProblem using the built-in R-PεRLE starting at the feasible point 97.

pymoso solve myproblem.py RPERLE 97

Similarly, we can solve built-in problems, such as ProbTPA which has two-dimensional feasible points.

pymoso solve ProbTPA RPERLE 40 40

```python
# import the Oracle base class
from pymoso.chnbase import Oracle

class MyProblem(Oracle):
    '''Example implementation of a user-defined MOSO problem.'''
    def __init__(self, rng):
        '''Specify the number of objectives and dimensionality of points.'''
        self.num_obj = 2
        self.dim = 1
        super().__init__(rng)

    def g(self, x, rng):
        '''Check feasibility and simulate objective values.'''
        # feasible values for x in this example
        feas_range = range(-100, 101)
        # initialize obj to empty and is_feas to False
        obj = []
        is_feas = False
        # check that dimensions of x match self.dim
        if len(x) == self.dim:
            is_feas = True
            # then check that each component of x is in the range above
            for i in x:
                if not i in feas_range:
                    is_feas = False
            # if x is feasible, simulate the objectives
            if is_feas:
                # use rng to generate random numbers
                z0 = rng.normalvariate(0, 1)
                z1 = rng.normalvariate(0, 1)
                obj1 = x[0]**2 + z0
                obj2 = (x[0] - 2)**2 + z1
                obj = (obj1, obj2)
        return is_feas, obj
```

Figure 11 The file myproblem.py implements the example MyProblem.
Henceforth, we present `solve` examples only for solving `MyProblem`. Since `MyProblem` is bi-objective, we recommend using the R-\(\varepsilon\)RLE solver. However, for two or more objectives, PyMOSO has R-MinRLE.

```python
pymoso solve myproblem.py RMINRLE 97
```

For a single objective problem, PyMOSO has R-SPLINE. We remark that if given a multi-objective problem, R-SPLINE will simply minimize the first objective. We do not necessarily prohibit such use, but urge that users take care when using R-SPLINE to minimize one objective of a many-objective problem.

```python
pymoso solve myproblem.py RSPLNE 97
```

Regardless of the chosen solver, PyMOSO creates a new sub-directory of the working directory containing output. There will be a metadata file, indicating the date, time, solver, problem, and any other specified options. In addition, PyMOSO creates a file containing the solver-generated solution. PyMOSO provides additional options for users solving MOSO problems. We present examples of each option below. First, users can specify the name of the output directory.

```python
pymoso solve --odir=OutDirectory myproblem.py RPERLE 45
```

Users can specify the simulation budget, which is currently set to a default of 200.

```python
pymoso solve --budget=100000 myproblem.py RPERLE 12
```

Users may specify to take simulation replications in parallel. We only recommend doing so if the user has thought through appropriate pseudo-random number stream control issues (see §A.4.1). Furthermore, due to the overhead of parallelization, we only recommend using the parallel simulation replications feature if observations are sufficiently “expensive” to compute, e.g. the simulation takes a half second or more to generate a single observation. We remark that the run-time complexity of the simulation oracle may not perfectly indicate when it is appropriate to use parallelization; other factors include, e.g., the total simulation budget.

```python
pymoso solve --simpar=4 myproblem.py RPERLE 44
```

Currently, all PyMOSO solvers support using common random numbers. Users may enable the functionality using the `crn` option.

```python
pymoso solve --crn myproblem.py RMINRLE 62
```

We do not recommend this option unless the oracle is implemented to be compatible, that is, the oracle uses PyMOSO’s pseudo-random number generator to generate pseudo-random numbers or to provide a seed to an external `mrg32k3a` generator (see §A.4.1).
Users may specify an initial seed to PyMOSO’s `mrg32k3a` pseudo-random number generator. Seeds must be 6 positive integers with spaces. The default is 12345 for each of the 6 components.

```
pymoso solve --seed 1111 2222 3333 4444 5555 6666 myproblem.py RPERLE 23
```

Users may specify algorithm-specific parameters (see the papers in which the algorithms were introduced for detailed explanations of the parameters). All parameters are specified in the form `--param name value`. For example, the RLE relaxation parameter can be specified and set as `betadel` to a real number. We refer the reader to Table 1 for the full list of currently available algorithm-specific parameters.

```
pymoso solve --param betadel 0.2 myproblem.py RPERLE 34
```

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Default Value</th>
<th>Affected Solvers</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>mconst</td>
<td>2</td>
<td>R-P(\varepsilon)RLE, R-MinRLE, R-P(\varepsilon), R-SPLINE</td>
<td>Initialize the sample size and subsequent schedule of sample sizes.</td>
</tr>
<tr>
<td>bconst</td>
<td>8</td>
<td>R-P(\varepsilon)RLE, R-MinRLE, R-P(\varepsilon), R-SPLINE</td>
<td>Initialize the search sampling limit and subsequent schedule of limits.</td>
</tr>
<tr>
<td>radius</td>
<td>1</td>
<td>R-P(\varepsilon)RLE, R-MinRLE, R-P(\varepsilon), R-SPLINE</td>
<td>Set the radius (a) that determines a point’s neighborhood, (\mathcal{N}_a) (Wang et al. 2013).</td>
</tr>
<tr>
<td>betadel</td>
<td>0.5</td>
<td>R-P(\varepsilon)RLE, R-MinRLE</td>
<td>An error tolerance parameter for RLE. See Cooper et al. (2018).</td>
</tr>
<tr>
<td>betaeps</td>
<td>0.5</td>
<td>R-P(\varepsilon)RLE, R-P(\varepsilon)</td>
<td>An error tolerance parameter for P(\varepsilon). See Cooper et al. (2018).</td>
</tr>
</tbody>
</table>

Finally, users may specify any number of options in one invocation. However, all options must be specified after the `solve` command and before the `myproblem.py` argument. Furthermore, any `--param` options must be last. (Note that the `\` at the end of the first line continues the command to the second line.)

```
pymoso solve --crn --simpar=4 --budget=10000 --seed 1 2 3 4 5 6 \ 
 --odir=Exp1 --param mconst 4 --param betadel 0.7 myproblem.py RPERLE 97
```

**A.3.4. The testsolve Command** The PyMOSO `testsolve` command tests algorithms on problems using a PyMOSO tester. Users can test built-in or user-defined solvers with built-in or user-defined testers. In the examples that follow, we assume users have implemented `MyProblem` as in Figure 11 and the corresponding tester named `MyTester` in `mytester.py`, shown in Figure 12. See §A.4.2 for instructions on implementing a user-defined tester, including a metric for comparing algorithms, in PyMOSO.
```python
import sys, os
sys.path.insert(0, os.path.dirname(__file__))

# use hausdorff distance (dh) as an example metric
from pymoso.chutils import dh

# import the MyProblem oracle
from myproblem import MyProblem

# optionally, define a function to randomly choose a MyProblem feasible x0
def get_ranx0(rng):
    val = rng.choice(range(-100, 101))
    x0 = (val,)
    return x0

# compute the true values of x, for computing the metric
def true_g(x):
    'Compute the objective values.'
    obj1 = x[0]**2
    obj2 = (x[0] - 2)**2
    return obj1, obj2

# define an answer as appropriate for the metric
myanswer = {(0, 4), (4, 0), (1, 1)}

class MyTester(object):
    'Example tester implementation for MyProblem.'
    def __init__(self):
        self.ranorc = MyProblem
        self.answer = myanswer
        self.true_g = true_g
        self.get_ranx0 = get_ranx0

    def metric(self, eles):
        'Metric to be computed per retrospective iteration.'
        epareto = [self.true_g(point) for point in eles]
        haus = dh(epareto, self.answer)
        return haus
```

Figure 12 The file mytester.py implements the example MyTester.

The template testsolve command is `pymoso testsolve tester solver` where tester is a built-in or user-defined tester, and solver is a built-in or user-defined solver. Users may also specify an x0, as in the solve command, if the tester does not implement the function to generate feasible points. As a first example, we test R-PεRLE on MyProblem using MyTester. Since some options are compatible with both solve and testsolve, we include those options in this example.

```
pymoso testsolve --budget=999 --odir=exp1 \
    --crn --seed 1 2 3 4 5 6 mytester.py RPERLE
```

Users may want to compute some metric on the algorithm-generated solutions. If a metric is defined as part of the tester, such as in MyTester, the testsolve command can compute the metric on every algorithm iteration using the --metric option.

```
pymoso testsolve --metric mytester.py RPERLE
```
The `testsolve` command cannot perform simulation replications in parallel. However, testers can apply the solvers to independent sample paths of the problems. For example, to test R-PεRLE on 100 independent sample paths of `MyProblem`, compute the metrics for each sample path, and use common random numbers in each sample path, use the following command.

```
pymoso testsolve --crn --metric --isp=100 mytester.py RPERLE
```

PyMOSO can perform independent algorithm runs in parallel. Use the `proc` option to specify the number of processes available to PyMOSO.

```
pymoso testsolve --crn --metric --isp=100 --proc=20 mytester.py RPERLE
```

We remark here that, to ensure the algorithm runs remain independent using PyMOSO’s pseudo-random number generator (see §A.4.1), researchers should set the total simulation budget so that the included algorithms do not surpass 200 retrospective approximation (RA) iterations. For reference, using the default settings, the sample size at every point in the 200th RA iteration is almost 380 million.

The `testsolve` command creates a results file for each independent sample path. The file contains the solutions generated at every algorithm iteration, such that the solution of iteration 2 is on line 2, iteration 10 on line 10, and so forth. If `--metric` is specified, PyMOSO generates a second file for each independent sample path containing the collection of triples (iteration number, simulations used at end of iteration, metric).

### A.4. Implementing Oracles, Testers, and Solvers in PyMOSO

To use PyMOSO, users solving MOSO problems must implement a PyMOSO oracle, and users testing MOSO algorithms should implement, at least, a PyMOSO oracle and tester. In this section, we provide template Python code to help users quickly implement oracles, testers, and perhaps solvers in PyMOSO.

#### A.4.1. Implementing PyMOSO Oracles

Usually, implementing a PyMOSO oracle implies implementing a Monte Carlo simulation oracle as a black box function while following the PyMOSO rules put forth in this section. For reference, we discuss the example PyMOSO oracle `MyProblem` in Figure 11. Users may copy the code in Figure 11 and re-implement the function `g` as needed. We now list the basic requirements of every `g` implementation.

1. The function `g` must be an instance method of an `Oracle` sub-class, and thus take `self` as its first parameter.
2. The function $g$ must take an arbitrarily-named second parameter which is a tuple of length `self.dim` and represents a point. Stylistically, PyMOSO consistently names this parameter $x$.

3. The function $g$ must take an arbitrarily-named third parameter which is a modified Python `random.Random` object. Stylistically, PyMOSO consistently names this parameter `rng`.

4. The function $g$ must return a boolean first and a tuple of length `self.num_obj` second.
   - The boolean is `True` if $x$ is feasible, and `False` otherwise.
   - If $x$ is feasible, the tuple contains a single observation of every objective. If $x$ is not feasible, each element in the tuple is `None`.

If users already have an implemented simulation oracle, they may find it convenient to implement $g$ as wrapper which calls that simulation from Python. As an example, suppose a user has implemented a simulation in C which is compiled to a C library called `mysim.so` and placed in the working directory. Suppose further that the simulation function takes the following as parameters: an array of integers representing a point $x \in \mathbb{R}$ and an unsigned integer representing the number of observations to take at $x$. The function output is defined as `struct Simout` with members `feas` set to 0 or 1, `obj` a double array set to the mean of the observed objective values, and `var` a double array set to the sample variance of the observed objective values. Then users can modify the template to wrap the C function `struct Simout c_func(int x, int n)` as in Figure 13.

Figure 13 is a valid PyMOSO oracle which wraps a C function. However, PyMOSO algorithms cannot enable common random numbers on this oracle. Furthermore, PyMOSO cannot guarantee that observations are independent when taken in parallel. To enable these properties, the external simulation must use `mrg32k3a` as the generator and must accept a user-specified seed.

Suppose the library `mysim.so` also implements the function `set_simseed` which accepts a long array representing an `mrg32k3a` seed. We modify the wrapper in Figure 14 for compatibility with common random numbers and to guarantee independence of parallel observations. Figure 14 demonstrates using `rng.get_seed()` to return the current `mrg32k3a` seed.

Alternatively, if the number of required pseudo-random numbers is known, users can use `rng.random()` to generate pseudo-random numbers and then pass them to an external simulation if such functionality is supported.
Figure 13 The g function wraps an external simulation written in C.

The rng object is implemented as a sub-class of Python's random.Random class, thus the official Python documentation for random applies to rng and is found at https://docs.python.org/3/library/random.html. In addition to rng using mrg32k3a as its generator, we also implement rng.normalvariate such that it uses the Beasley-Springer-Moro algorithm (Law 2015, p. 458) to approximate the inverse of the standard normal cumulative distribution function.

When using rng, to ensure independent sampling of observations, PyMOSO “jumps” forward in the pseudo-random number stream after obtaining every simulation replication. Each jump is of fixed size $2^{76}$ pseudo-random numbers. Thus, we require that every simulation replication use fewer than $2^{76}$ pseudo-random numbers. We ensure independence among parallel replications by “giving” each processor a stream (an rng), each of which is $2^{127}$ pseudo-random numbers apart. When using the current PyMOSO algorithms that rely on
RA, each RA iteration begins the next available independent stream \(2^{127}\), where PyMOSO accounts for the possibility of parallel computation within an RA iteration. Thus, in a given RA iteration, a user may simulate 100 million points at a sample size of 1 million, without common random numbers, and easily not reach the limit.

### A.4.2. Implementing PyMOSO Testers

Consider again the example tester in Figure 12. As a minimal valid PyMOSO tester, users may do nothing but assign the `MyTester` member `self.ranorc` to a PyMOSO oracle, such as `MyProblem`, in Line 27. However, we expect most users to leverage PyMOSO features by implementing metrics and feasible point generators.
The function `get_ranx0` allows the tester to generate feasible points to `MyProblem` and `metric` allows the tester to compute a metric on sets returned by a solver. Researchers may implement any number of additional supporting functions, including members and methods of the tester class. The `true_g` function is an example of such a supporting function, which is used to compute the example metric.

First, we list the rules for implementing a feasible point generator.

1. The function is arbitrarily named but must be set to the `self.get_ranx0` member of a tester.
2. The function must take a single parameter, an arbitrarily named `random.Random` object we suggest naming `rng`.
3. The function must return a tuple with length corresponding to the `self.dim` member of the `self.ranorc` member of the tester.

Since a researcher’s desired metric depends on the algorithm capabilities and problem complexity, PyMOSO allows researchers to implement any metric they choose. We provide three example metrics, but first, we list the implementation rules of the `metric` function.

1. The `metric` function must be an instance method of a tester, and thus take `self` as its first parameter.
2. The second parameter of `metric` is arbitrarily named and is a Python set of tuples.
3. PyMOSO does not enforce the return value of `metric`, but we recommend a scalar real number.

The metric implemented in Figure 12 is the Hausdorff distance from (a) the true image of an estimated solution returned by an algorithm, to (b) the true solution hard-coded as `myanswer`.

For an example of a different metric, consider a MOSO problem that has more than one local efficient set (LES) and such that each LES contains no members of another LES. Since an algorithm that converges to a LES is may find only one LES, we may define the metric to compute the Hausdorff distance between the true image of the estimated solution and the “closest” true LES, as follows. Let `self.answer` be implemented as a list of sets, and assume a `self.true_g` implementation. Then Figure 15 implements the described metric.

For single-objective problems with one correct solution $x^*$, a simple metric that takes an estimated solution $X$ is $|g(X) - g(x^*)|$, which we implement in Figure 16 assuming an appropriate implementation of `self.answer` and `self.true_g`. 
A.4.3. Implementing PyMOSO Algorithms

Researchers can implement simulation optimization algorithms in the PyMOSO framework. PyMOSO provides support for algorithms in three categories:

1. PyMOSO provides strong support for implementing new MOSO algorithms that rely on RLE in an RA framework.
2. PyMOSO provides strong support for implementing general RA algorithms.
3. PyMOSO provides basic support, such as pseudo-random number control, for implementing other simulation optimization algorithms.

We provide templates of algorithms implemented in each of these three categories, along with example code snippets.

In the first category, programmers can use PyMOSO to create new RA algorithms that use RLE for convergence. The novel part of these algorithms, created by the user, will be the `accel` function which should collect points to send to RLE for certification. Here, we list the rules for `accel`.

1. The `accel` function must be an instance method of an `RLE Solver` object, and thus its first parameter must be `self`.
2. The second parameter is arbitrarily named and is a set of tuples. We recommend naming the parameter `warm_start`, as it represents the sample-path solution of the previous RA iteration.
3. The return value must be a set of tuples representing feasible points; we do not recommend any particular name.
In every RA iteration, PyMOSO will first call `accel(self, warm_start)` and send the returned set to `rle(self, candidate_les)`. The return value must be a set of tuples. The implementer does not need to implement or call RLE, as in Figure 17.

In the second category, algorithm designers can quickly implement any RA algorithm by sub-classing `RASolver` and implementing the `spsolve` function, as shown in Figure 18. The algorithm can be a single-objective algorithm. PyMOSO cannot guarantee the convergence of such algorithms. Figure 18 is technically valid in PyMOSO but is probably not effective. Though analogous to those of an `RLESolver.accel` method, for completeness, we list the requirements for an `RASolver.spsolve` method.

1. The `spsolve` function must be an instance method of an `RASolver` object, and thus its first parameter must be `self`.
2. The second parameter is arbitrarily named and is a set of tuples. We recommend naming the parameter `warm_start` as it represents the sample-path solution of the previous RA iteration.
3. The return value must be a set of tuples representing feasible points; we do not recommend any particular name.

In the third category, PyMOSO can accommodate any simulation optimization algorithm by implementing the `solve` function of a `MOSOSolver` sub-class as shown in Figure 19. It does not have to be a multi-objective algorithm. PyMOSO will require users to send an
from pymoso.chnbase import MOSOSolver

class MyMOSOAlg(MOSOSolver):
    '''Template implementation of a MOSO solver.'''

    def solve(self, budget):
        while self.numCalls <= budget:
            # implement algorithm logic and return the results
            return results

Figure 19 We provide a template to implement a simulation optimization algorithm.

initial feasible point $x_0$ whether or not the algorithm needs it. The initial feasible point $x_0$ is accessed through `self.x0` which is a tuple. We now list the rules for implementing any `MOSOSolver.solve` function.

1. The `solve` function must be an instance method of `MOSOSolver`, and thus take `self` as its first parameter.
2. The second parameter is the simulation budget, a natural number.
3. The `solve` function must return a dictionary (we name it `results` in our example) with at least 3 keys: 'itersoln', 'simcalls', 'endseed'. Researchers may track additional data and add it to `results` as desired.
   - The 'itersoln' key itself corresponds to a dictionary with a key for each algorithm iteration labeled \{0, 1, \ldots\}. The value at each iteration is a set containing the estimated solution at the end of the iteration.
   - The 'simcalls' key itself corresponds to a dictionary with a key for each algorithm iteration labeled \{0, 1, \ldots\}. The value at each iteration is a natural number containing the cumulative number of simulation replications taken at the end of the iteration.
   - The 'endseed' key corresponds to a tuple of length 6, representing an `mrg32k3a` seed. The algorithm programmer should ensure the stream generated by `results['endseed']` is independent of all streams used by the algorithm.

Researchers may use Figure 19 to implement new simulation optimization algorithms.

For convenience, in the list below, we also provide some example code snippets that we find useful when implementing algorithms in PyMOSO. They work without modification when using the templates above that inherit `RLESolver` or `RASolver`, but some functions may require implementation or modification for use in a `MOSOSolver`. For reference, §B contains a list of most objects accessible to PyMOSO programmers.
- Example code to take simulation replications of a point at some sample size:

```python
# pretend x has not yet been visited in this RA iteration and is feasible
x = (1, 1, 1)

# self.m is the sample size of the current RA iteration
m = self.m

# self.num_calls is the cumulative number of simulations used till now
start_num_calls = self.num_calls

# use estimate to sample x and put results in self.gbar and self.sehat
isfeas, fx, se = self.estimate(x)

calls_used = self.num_calls - start_num_calls
print(m == calls_used)  # True
print(fx == self.gbar[x])  # True
print(se == self.sehat[x])  # True

# estimate will not simulate again in subsequent visits to a point
start_num_calls = self.num_calls
isfeas, fx, se = self.estimate(x)
calls_used = self.num_calls
print(calls_used == 0)  # True
```

- Example code to retrieve a point’s neighbors and take simulation replications:

```python
from pymoso.chnutils import get_nbors

r = self.nbor_rad
nbors = get_nbors(x0, r)
self.upsample(nbors)

for n in nbors:
    print(n in self.gbar)  # True if n feasible else False
# upsample also returns the feasible subset
nbors = self.upsample(nbors)
```

- Example code to sort points by their observed objective values:

```python
# 0 index for first objective
sorted_feas = sorted(nbors | {x}, key=lambda t: self.gbar[t][0])
xmin = sorted_feas[0]
fxmin = self.gbar[x]
```

- Example code to use the built-in SPLINE implementation:

```python
# unconstrained minimize the 2nd objective
x0 = (2, 2, 2)
isfeas, fx, sex = self.estimate(x0)

# the suppressed value is the set visited along SPLINE’s trajectory
xmin, fxmin, sexmin = self.spline(x0, float('inf'), 1, 0)
print(self.gbar[xmin] == fxmin)  # True
```

- Example code to find the non-dominated points in a dictionary:

```python
from pymoso.chnutils import get_nondom

nondom = get_nondom(self.gbar)
```

- Example code to randomly choose points from a set:

```python
solver_rng = self.sprn

# pick 5 points — returns a list, not a set.
ran_pts = solver_rng.sample(list(nondom), 5)
one_in_five = solver_rng.choice(ran_pts)
```
A.5. Using solve and testsolve in Python Programs

Users may invoke the solve and testsolve functions within a Python program.

- Using solve in a Python program is similar to using the CLI solve. We provide the minimal example here.

```python
# import the solve function
from pymoso.chnutils import solve
# import the module containing the RPERLE implementation
import pymoso.solvers.rperle as rp
# import MyProblem - myproblem.py should usually be in the script directory
import myproblem as mp

# specify an x0. In MyProblem, it is a tuple of length 1
x0 = (97,)
soln = solve(mp.MyProblem, rp.RPERLE, x0)
print(soln)
```

- Users can specify options, including algorithm-specific parameters, as shown below.

```python
# example for specifying budget and seed
budget=10000
seed = (111, 222, 333, 444, 555, 666)
soln1 = solve(mp.MyProblem, rp.RPERLE, x0, budget=budget, seed=seed)

# specify crn and simpar
soln2 = solve(mp.MyProblem, rp.RPERLE, x0, crn=True, simpar=4)

# specify algorithm specific parameters
soln3 = solve(mp.MyProblem, rp.RPERLE, x0, radius=2, betaeps=0.3, betadel=0.4)

# mix them
soln4 = solve(mp.MyProblem, rp.RPERLE, x0, crn=True, seed=seed, radius=5)
```

- Using testsolve in a Python program is also similar to using the CLI testsolve. Here, we provide an example with options.

```python
# import the testsolve functions
from pymoso.chnutils import testsolve
# import the module containing RPERLE
import pymoso.solvers.rperle as rp
# import the MyTester class
from mytester import MyTester

# testsolve needs a "dummy" x0 even if MyTester will generate them
x0 = (1,)
run_data = testsolve(MyTester, rp.RPERLE, x0, isp=100, crn=True, radius=2)
```

- When using testsolve in a Python program, users must compute their metric. Here, run_data is a dictionary of the form described in §A.4.3, in the description of Figure 19. In the snippet below, we compute the metric on the 5th algorithm iteration of the 12th independent sample path.

```python
iter5_soln = run_data[11][\{\'itersoln\}\][4]
ispl2_iter5_metric = MyTester.metric(iter5_soln)
```
B. PyMOSO Programming Object List

We describe the object names inside each of the following pymoso modules.

**prng.mrg32k3a** The module exposes the pseudo-random number generator and functions to manipulate it.

  - **MRG32k3a** Sub-class of random.Random, defines all rng objects.
  - **get_next_prnstream(seed)** Return an rng object seeded $2^{127}$ steps from the input seed.
  - **jump_substream(rng)** Seed the input rng object $2^{76}$ steps forward.

**chnbase** The module implements the base classes for programming oracles and solvers.

  - **Oracle** Base class for implementing oracles.
  - **RLESolver** Base class for implementing solvers using RLE.
  - **RASolver** Base class for implementing RA solvers.
  - **MOSOSolver** Base class for all solvers.

**chnutils** The module contains generally useful functions for programming or testing algorithms.

  - **solve(oracle, solver, x0, **kwargs)** See §A.5.
  - **testsolve(tester, solver, x0, **kwargs)** See §A.5.
  - **does_weak_dominates(g, h, relg, relh)** All inputs are tuples of equal length. Returns True if g weakly dominates h with relaxations.
  - **does_dominates(g, h, relg, relh)** Returns True if g dominates h with relaxations.
  - **does_strict_dominates(g, h, relg, relh)** Returns True if g strictly dominates h with relaxations.
  - **get_nondom(obj_dict)** Input: a dictionary with tuples for keys and values. The keys are feasible points; the values are their objective values. Return: a set of tuples representing non-dominated points.
  - **get_nbors(x, r)** Input: a tuple x, a positive real scalar r indicating the neighborhood radius. Return: Set of tuples, the neighbors.
  - **get_setnbors(S, r)** Input: a set of tuples, and the neighborhood radius. Return: $\bigcup_{x \in S} \text{get_nbors}(x, r)$.
  - **dh(A, B)** Returns the Hausdorff distance between set A and set B.
  - **edist(x1, x2)** Returns the Euclidean distance between x1 and x2.
gen_metric(results, tester) Input: results is a dictionary, the output of each sample path of testsolve. tester must implement metric. Returns: The set of triples (iteration, simulation count, metric) for an algorithm run.

Oracle When implementing RAsolver algorithms, programmers may not need to access Oracle objects directly at all. When implementing MOSOSolver algorithms, programmers will use (or wrap) hit and crn_advance().

Oracle.num_obj A positive integer, the number of objectives.
Oracle.dim A positive integer, the dimensionality of feasible points.
Oracle.rng An instance of MRG32k3a internal to the oracle.
Oracle.hit(x, n) Take n observations of x. Return: True, and a tuple containing the mean of the observations for each objective se, and a tuple containing the standard error for each objective if x is feasible. The function handles CRN internally.
Oracle.set_crnflag(bool) Turn CRN on (True) or off.
Oracle.set_crnold(state) Save the rng state as the CRN baseline, e.g. for an algorithm iteration.
Oracle.crn_reset() Back the oracle rng to the CRN baseline.
Oracle.crn_advance() If CRN is on, reset, and then jump to the next independent pseudo-random stream and save the new baseline, e.g. before starting a new algorithm iteration.
Oracle.crn_setobs() Set an intermediate CRN for individual oracle observations.
Oracle.crn_nextobs() Jump the rng forward, e.g. after taking an observation, and set_obs the seed.
Oracle.crn_check() If CRN is on, return to the baseline. Otherwise, use nextobs before taking the next observation.

MOSOSolver The base class provides a basic structure for implementing new MOSO algorithms in PyMOSO.

MOSOSolver.orc The oracle object for the solver to solve.
MOSOSolver.dim Number of dimensions of points in the self.orc’s feasible points.
MOSOSolver.num_obj Similarly, the number of objectives in self.orc.
MOSOSolver.num_calls A running count of the number of observations taken of self.orc.
MOSOSolver.x0 A feasible starting point. This point is additionally supplied to algorithms that don’t need one.

RASolver Implements a common structure for all RA algorithms, including: caching of simulation replications, scheduling and updating of sample sizes and limits, and a wrapper to Oracle.hit.

RASolver.spn An instance of MRG32k3a for the solver to use.

RASolver.nbor_rad The neighborhood radius used by solvers seeking local optimality.

RASolver.gbar A dictionary where every key and value is a tuple. The keys are feasible points, values are their objective values. gbar is “wiped” every retrospective iteration.

RASolver.sehat Exactly like gbar except the values are standard errors.

RASolver.m The sample size of the current iteration.

RASolver.calc_m(nu) Compute the sample size of the current iteration. RA algorithms automatically do this every iteration and assign the value to self.m.

RASolver.b The searching sample limit of the current iteration.

RASolver.calc_b(nu) Exactly as calc_m but for the searching sample limit.

RASolver.estimate(x, c, obj) The estimate function is essentially a smart wrapper for self.orc.hit. Inputs: tuple x to sample, c a feasibility constraint, obj the objective to constrain. Return: same as Oracle.hit. Retrieves or saves the results from/to gbar and sehat as appropriate. Returns not feasible if the otherwise feasible result is not less than the constraint.

RASolver.upsample(mcS) A version of estimate for sets. Returns the feasible subset of mcS.

RASolver.spline(x, c, obmin, obcon) Return a sample path local minimizer. Input: a feasible start, constraint, objective to minimize, objective to constrain. Return: a set of tuples of the trajectory, the minimizer tuple, the minimum tuple, the standard error tuple.

RLESolver Builds on RASolver to add RLE and its relaxation.

RLESolver.betadel Affects the relaxation values computed in RLE.

RLESolver.calc_delta(se) Computes the RLE relaxation given a standard error, using self.m and self.betadel

RLESolver.rle(candidate_les) Input: set of tuples, Returns: set of tuples. Finds the LES at sample size self.m.