New Valid Inequalities for the Fixed-Charge
and Single-Node Flow Polytopes

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Abstract

The most effective software packages for solving mixed 0-1 linear programs use strong valid linear inequalities derived from polyhedral theory. We introduce a new procedure which enables one to take known valid inequalities for the knapsack polytope, and convert them into valid inequalities for fixed-charge and single-node flow polytopes. The resulting inequalities are very different from the previously known inequalities (such as flow cover and flow pack inequalities). In particular, the coefficients tend to be very small integers.

Keywords: polyhedral combinatorics; branch-and-cut; mixed-integer linear programming

1 Introduction

Polyhedral methods have proven to be remarkably useful for solving pure and mixed 0-1 linear programs (see, e.g., [5, 6]). In the case of large, sparse instances without special structure, three families of polytopes have proven to be of particular importance: the knapsack, fixed-charge and single-node flow polytopes. The knapsack polytope is the convex hull of vectors \( y \in \{0,1\}^n \) satisfying

\[
\sum_{j \in N} a_j y_j \leq b,
\]

where \( b \) and the \( a_j \) are positive integers, and \( N \) denotes \( \{1, \ldots, n\} \). The fixed-charge polytope is the convex hull of pairs \( (x,y) \in \mathbb{R}_+^n \times \{0,1\}^n \) satisfying

\[
\sum_{j \in N} x_j \leq d \tag{1}
\]

\[
x_j \leq u_j y_j \quad (j \in N), \tag{2}
\]

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where \( d \) and the \( u_j \) are positive integers \([11]\). Finally, the single-node flow polytope is the convex hull of pairs \((x, y) \in \mathbb{R}^n_+ \times \{0, 1\}^n\) satisfying

\[
\sum_{j \in N^+} x_j - \sum_{j \in N^-} x_j \leq d \\
\ell_j y_j \leq x_j \leq u_j y_j \quad (j \in N^+ \cup N^-),
\]

where \( d \) and the \( u_j \) are again positive integers, the \( \ell_j \) are non-negative integers, \( N^+ \) and \( N^- \) are disjoint sets, and \( n \) now denotes \(|N^+ \cup N^-|\) \([13]\).

Several families of valid linear inequalities are known for the knapsack polytope, including lifted cover inequalities \([3, 16]\), weight inequalities \([15]\) and lifted pack inequalities \([2, 10]\). Inequalities for the fixed-charge polytope include flow cover inequalities \([11]\) and lifted flow cover inequalities \([8, 9]\). Inequalities for the single-node flow polytope include generalized flow cover inequalities \([13, 14]\), reverse flow cover inequalities \([12]\), lifted generalised flow cover inequalities \([8]\) and lifted flow pack inequalities \([1]\).

The purpose of this note is to present a procedure which enables one to take known valid inequalities for the knapsack polytope and convert them into new valid inequalities for the fixed-charge and single-node flow polytopes. We call the resulting inequalities \textit{rotated knapsack inequalities} or RKIs. The RKIs are very different from the inequalities from the literature that we mentioned above. In particular, their coefficients tend to be very small integers. Nevertheless, as we will see, they often define facets.

The paper is structured as follows. Section 2 is a literature review. In Section 3 we present the theory for the new procedure. Section 4 includes some concluding remarks and suggestions for further research.

## 2 Literature Review

We now review the literature. The following subsections are concerned with valid inequalities for the knapsack polytope, the fixed-charge polytope and the the single-node flow polytope.

### 2.1 Knapsack polytope

As mentioned above, many families of inequalities are known for the knapsack polytope. For brevity, we recall here only the \textit{lifted cover inequalities} (LCIs), first defined in \([3, 16]\). A set \( C \subseteq N \) is called a \textit{cover} if \( \sum_{j \in C} a_j > b \). If \( C \) is a cover, then the \textit{cover} inequality \( \sum_{j \in C} x_j \leq |C| - 1 \) is valid. The strongest cover inequalities are obtained when \( C \) is \textit{minimal} (i.e., no proper subset of \( C \) is a cover). Given any minimal cover \( C \), there exists at least one facet-defining LCI of the form

\[
\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leq |C| - 1,
\]
where $\alpha_j \in \mathbb{Z}_+$ for $j \in N \setminus C$. There may also be facet-defining LCIs where the $\alpha_j$ are not integers. Some even more general LCIs are studied in [7, 10, 16]. We omit details for brevity.

### 2.2 Fixed-charge polytope

Padberg et al. [11] presented two families of inequalities for the fixed-charge polytope. The inequalities of the first family are derived as follows. A set $C \subseteq N$ is called a cover if $\sum_{j \in C} u_j > d$. Given a cover $C$ and a (possibly empty) set $L \subseteq N \setminus C$, we let $\lambda$ denote $\sum_{j \in C} u_j - d$ and $u^+$ denote $\max_{j \in C} u_j$. The following flow cover inequality is valid:

$$\sum_{j \in C \cup L} x_j \leq d - \sum_{j \in C} \alpha_j (1 - y_j) + \sum_{j \in L} \alpha_j y_j,$$

where $\alpha_j$ is $\max \{0, u_j - \lambda\}$ for $j \in C$, and $\max \{u^+, u_j\} - \lambda$ for $j \in L$. The flow cover inequalities were slightly strengthened in [8], yielding lifted flow cover inequalities.

The second family is very different. Let $P^\alpha$ be the face of the fixed-charge polytope obtained by setting the inequality (1) to equality. One can check that, if $(x, y) \in P^\alpha$, then $y$ must lie within the knapsack polytope $K = \text{conv} \{y \in \mathbb{Z}_+^n : \sum_{j \in N} u_j y_j \geq d\}$. Let $\alpha^T y \geq \beta$ be any valid inequality for $K$ with $\alpha \in \mathbb{Z}_+^n$ and $\beta \in \mathbb{Z}_+$. It is shown in [11] that there exists a positive rational $\delta$ such that the inequality

$$\sum_{j \in N} x_j \leq d + \delta \left( \sum_{j \in N} \alpha_j y_j - \beta \right)$$

is valid for the fixed-charge polytope. To the best of our knowledge, this procedure has received no attention in the literature.

### 2.3 Single-node flow polytope

Van Roy & Wolsey [13] extended the flow cover inequalities to the single-node flow polytope. Now, a pair $(C^+, C^-)$ is called a generalised cover if $C^+ \subseteq N^+$, $C^- \subseteq N^-$ and $\sum_{j \in C^+} u_j - \sum_{j \in C^-} \ell_j > d$. Given a generalised cover $(C^+, C^-)$ and sets $L^+ \subseteq N^+ \setminus C^+$ and $L^- \subseteq N^- \setminus C^-$, one can construct a valid inequality of the form

$$\sum_{j \in C^+ \cup L^+} x_j - \sum_{j \in N^-} x_j \leq d - \sum_{j \in C^+} \alpha_j (1 - y_j) + \sum_{j \in L^+} \alpha_j y_j$$
\[ + \sum_{j \in C^-} \alpha_j (1 - y_j) - \sum_{j \in L^-} \alpha_j y_j, \]

where \( \alpha_j \in \mathbb{Z}_+ \) for \( j \in C^+ \cup L^+ \cup C^- \cup L^- \). (For brevity, we skip the details on how to compute the \( \alpha_j \).) These are called generalised flow cover (GFC) inequalities. They have been generalised and strengthened in various ways [8, 9, 12, 13]. A related family of inequalities, the lifted flow pack inequalities, were studied by Atamtürk [1].

### 3 The New Procedure

In this section, we present our new procedure. In Subsection 3.1, we show how to extend the second procedure mentioned in Subsection 2.2 to obtain a much wider family of inequalities for the fixed-charge polytope. Then, in Subsection 3.2, we show how to extend the procedure to the more general case of the single-node flow polytope.

#### 3.1 Fixed-charge polytope

Let \( P \) denote the fixed-charge polytope and \( C \subseteq N \) be a cover. The inequality \( \sum_{j \in C} x_j \leq d \) is trivially valid for \( P \). Let \( P^= \) be the face of \( P \) obtained by setting this inequality to equality. That is,

\[
P^= := \text{conv}\left\{ (x,y) \in \mathbb{R}_+^n \times \{0,1\}^n : \sum_{j \in N} x_j \leq d, \sum_{j \in C} x_j = d, x_j \leq u_j y_j (j \in N) \right\}
\]

One can check that the following inequality is valid for \( P^= \):

\[
\sum_{j \in C} u_j y_j \geq d.
\]  

Let \( \bar{y}_j = 1 - y_j \) for all \( j \in C \). Then, we can write (5) as

\[
\sum_{j \in C} u_j \bar{y}_j \leq \sum_{j \in C} u_j - d,
\]

and define the restricted knapsack polytope

\[ K = \text{conv}\left\{ \bar{y} \in \{0,1\}^{|C|} : \sum_{j \in C} u_j \bar{y}_j \leq \sum_{j \in C} u_j - d \right\}. \]
Let $\alpha^T\bar{y} \leq \beta$ be a valid inequality for $K$, with $\alpha \in \mathbb{Z}_+^{|C|}$ and $\beta \in \mathbb{Z}_+$. By substituting $\bar{y}$, we obtain the inequality

$$\sum_{j \in C} \alpha_j y_j \geq \sum_{j \in C} \alpha_j - \beta,$$

which is valid for $P^\neq$. To simplify the notation, let $\beta^* = \sum_{j \in C} \alpha_j - \beta$.

Given that $P^\neq$ is the face of $P$ defined by the equation $\sum_{j \in C} x_j = d$, there exists a rational number $\delta$, such that the inequality

$$\sum_{j \in C} x_j \leq d + \delta \left( \sum_{j \in C} \alpha_j y_j - \beta^* \right)$$

is valid for $P$. This latter inequality is the desired RKI.

In order to determine $\delta$, it is helpful to define the function $\phi : \mathbb{Z}_+ \to \{0,1,\ldots,d\}$ with

$$\phi(t) = \max \left\{ \sum_{j \in C} x_j : (x,y) \in P, \sum_{j \in C} \alpha_j y_j \leq t \right\}.$$

By definition, $\phi(t)$ is an integer-valued and non-decreasing function. Note that (a) $\phi(0) \geq 0$, (b) $\phi(0)$ is strictly positive if some of the $\alpha_j$ are equal to zero, and (c) $\phi(t) = d$ for all $t \geq \beta^*$. One can compute $\phi(t)$, for $t = 0,\ldots,\beta^*$, in $O(|C|\beta^*)$ time, by dynamic programming. Details are given in Algorithm 1.

Once the $\phi(t)$ values have been computed, one can compute $\delta$ as follows:

$$\delta = \min_{0 \leq t < \beta^*} \left\{ \frac{d - \phi(t)}{\beta^* - t} \right\}.$$

We illustrate this theory with an example.

**Example 1:** Let $n = 6$, $d = 11$ and $u = (1,2,3,4,5,6)$. If we let $C = \{2,3,4,5,6\}$, we have

$$K = \left\{ \bar{y} \in \{0,1\}^{|C|} : 2\bar{y}_2 + 3\bar{y}_3 + 4\bar{y}_4 + 5\bar{y}_5 + 6\bar{y}_6 \leq 9 \right\}.$$ 

The inequality

$$\bar{y}_2 + \bar{y}_3 + \bar{y}_4 + 2\bar{y}_5 + 2\bar{y}_6 \leq 3$$

is valid for $K$. (In fact it is a so-called “general LCI”, see [7, 10]). So the inequality $y_2 + y_3 + y_4 + 2y_5 + 2y_6 \geq 4$ is valid for $P^\neq$. One can check that $\phi(0) = 0$, $\phi(1) = u_4 = 4$, $\phi(2) = u_3 + u_4 = 7$, $\phi(3) = u_4 + u_5 = 10$ and $\phi(4) = \phi(5) = d = 11$. The function $\phi(t)$ is illustrated in Figure 1. This yields $\delta = (11 - 10)/(4 - 3) = 1$. Therefore, the RKI

$$x_2 + x_3 + x_4 + x_5 + x_6 \leq 7 + (y_2 + y_3 + y_4 + 2y_5 + 2y_6)$$
Algorithm 1: Computing $\phi(t)$ by dynamic programming

**input**: positive integer $n$, upper bound vector $u \in \mathbb{Z}_n^+$,
flow limit $d \in \mathbb{Z}_+$, cover $C \subseteq \{1, \ldots, n\}$,
left-hand side vector $\alpha \in \mathbb{Z}_{|C|}^+$, right-hand side $\beta^* \in \mathbb{Z}_+$

Let $\text{size} := |C|$
Create an integer $(\beta^* + 1) \times (\text{size} + 1)$ matrix $M$;
for $k = 0$ to $\text{size}$ do
  for $t = 0$ to $\beta^*$ do
    $M[k][t] := 0$;
  end
end
for $k = 1$ to $\text{size}$ do
  Let $j$ be the $k$th item in $C$;
  for $t = 0$ to $\beta^*$ do
    if $\alpha_j \leq t$ then
      $M[k][t] = \max\{M[k - 1][t], u_j + M[k - 1][t - \alpha_j]\}$;
    else
      $M[k][t] = M[k - 1][t]$;
    end
  end
end
Create an integer array $\phi$ of size $\beta^*$;
for $t = 0$ to $\beta^* - 1$ do
  $\phi[t] := M[\text{size}][t]$;
end
for $t = 1$ to $\beta^* - 1$ do
  if $\phi[t - 1] > \phi[t]$ then
    $\phi[t] := \phi[t - 1]$;
  end
end
$\phi[\beta^*] := d$;
**output**: $\phi(0), \ldots, \phi(\beta^*)$. 


is valid for $P$. One can check (either by hand or with a software package such as PORTA [4]) that this inequality defines a facet of $P$. Other RKIs for this instance include, for example, the following:

\[
\begin{align*}
x_2 + x_4 + x_5 + x_6 &\leq 5 + 3(y_4 + y_5 + y_6) & (C = \{2, 4, 5, 6\}) \\
x_3 + x_4 + x_5 + x_6 &\leq 2 + 2(y_4 + y_5 + y_6) & (C = \{3, 4, 5, 6\}) \\
x_2 + x_3 + x_4 + x_5 + x_6 &\leq 9 + 2(y_5 + y_6) & (C = \{2, 3, 4, 5, 6\})
\end{align*}
\]

One can check that these too are facet-defining. □

We remark that each RKI cuts off at least one fractional LP solution satisfying the following three conditions:

1. Constraint (I) holds at equality,
2. $x_j = u_jy_j > 0$ for all $j \in C$,
3. $y_j = x_j = 0$ for all $j \in N \setminus C$.

### 3.2 Single-node flow polytope

We now extend our results to the single-node flow polytope, which we will again denote by $P$. This case turns out to be considerably more complicated.

Let $U^+$ and $L^+$ be disjoint subsets of $N^+$, and let $U^-$ and $L^-$ be disjoint subsets of $N^-$. The sets $U^-$, $L^+$ and $L^-$ are permitted to be empty, but $U^+$ must be non-empty.

Let $P^\geq$ be the convex hull of the feasible solutions that satisfy the inequality

\[
\sum_{j \in U^+} x_j - \sum_{j \in L^-} x_j \geq d + \sum_{j \in U^-} u_j - \sum_{j \in L^+} \ell_j. \tag{6}
\]
(Note that $P^\geq$ is contained in $P$, but is not necessarily a face of $P$.) By definition, all points in $P^\geq$ satisfy the inequality

$$\sum_{j \in U+} x_j - \sum_{j \in L^+} x_j \geq d + 2 \sum_{j \in U^-} u_j - 2 \sum_{j \in L^+} \ell_j.$$ 

Weakening this using (4), we find that all points in $P^\geq$ satisfy

$$\sum_{j \in U+} u_j y_j - \sum_{j \in L^+} \ell_j y_j \geq d + 2 \sum_{j \in U^-} u_j - 2 \sum_{j \in L^+} \ell_j.$$

Now, as before, let $\bar{y}_j$ denote $1 - y_j$. Also let $R^- = N^- \setminus (U^- \cup L^-)$. All points in $P^\geq$ satisfy:

$$\sum_{j \in U+ \cup U^- \cup R^-} \alpha_j \bar{y}_j + \sum_{j \in L^+ \cup L^-} \beta_j y_j \leq \sum_{j \in U+ \cup U^- \cup R^-} u_j - 2 \sum_{j \in L^+ \cup L^-} \ell_j - d. \quad (7)$$

We now define a knapsack polytope, $K$, as the convex hull of pairs $(y, \bar{y})$ that satisfy (7). We remark in passing that a necessary condition for $K$ to be full-dimensional is

$$\sum_{j \in U+ \cup U^- \cup R^-} \alpha_j y_j + \sum_{j \in L^+ \cup L^-} \beta_j y_j \leq \gamma.$$ 

be a valid inequality for $K$ with non-negative coefficients. Complementing yields

$$\sum_{j \in U+ \cup U^- \cup R^-} \alpha_j \bar{y}_j = \sum_{j \in L^+ \cup L^-} \beta_j y_j \leq \gamma.$$ 

To simplify the notation, let $\gamma^* = \sum_{j \in U+ \cup U^- \cup R^-} \alpha_j - \gamma$. Given that $P^\geq$ is defined by the inequality (6), there exists a positive rational $\delta$ such that the inequality

$$\sum_{j \in U^+} x_j - \sum_{j \in L^-} x_j \leq d + \sum_{j \in U^-} u_j - \sum_{j \in L^+} \ell_j$$

$$+ \delta \left( \sum_{j \in U+ \cup U^- \cup R^-} \alpha_j y_j - \sum_{j \in L^+ \cup L^-} \beta_j y_j - \gamma^* \right)$$

is valid for $P$. This latter inequality is the desired RKI.
As before, in order to determine the value of \( \delta \), we need to define an auxiliary function. For \( t \in \mathbb{Z} \), let

\[
\phi(t) = \max \left\{ \sum_{j \in U^+} x_j - \sum_{j \in L^-} x_j : (x, y) \in P, \sum_{j \in U^+ \cup U^- \cup R^-} \alpha_j y_j - \sum_{L^+ \cup L^-} \beta_j y_j \leq t \right\}.
\]

Again, \( \phi(t) \) is an integer-valued, non-decreasing function, but its natural domain is

\[
\mathbb{Z} \cap \left[ -\sum_{L^+ \cup L^-} \beta_j, \sum_{j \in U^+ \cup U^- \cup R^-} \alpha_j \right].
\]

One can compute the values taken by \( \phi(t) \) over this domain efficiently by dynamic programming. (Details omitted for brevity.) Once the \( \phi(t) \) values have been computed, one can compute the value \( \delta \) as follows. Let \( t^* \) be the minimum value of \( t \) such that \( \phi(t) = d + \sum_{j \in U^-} u_j - \sum_{j \in L^+} \ell_j \). (Note that \( t^* \geq \gamma^* \).) Then let

\[
\delta = \min_{-\sum_{j \in U^+ \cup L^-} \beta_j \leq t \leq t^*} \left\{ \frac{d - \phi(t)}{\gamma^* - t} \right\}.
\]

Again, we illustrate this theory with an example.

**Example 2:** Let \( n = 7 \), \( N^+ = \{1, 2, 3\} \), \( N^- = \{4, 5, 6, 7\} \), \( d = 4 \), \( u = (4, 3, 3, 2, 2, 2, 2) \) and \( \ell = (1, 1, 1, 1, 1, 1, 1) \). Suppose we set \( U^+ = \{2, 3\} \), \( L^+ = \emptyset \), \( U^- = \emptyset \) and \( L^- = \{4\} \). The knapsack constraint is \( 3y_2 + 3y_3 - y_4 + 2y_5 + 2y_6 + 2y_7 \geq 4 \). Complementing gives \( 3\bar{y}_2 + 3\bar{y}_3 + y_4 + 2\bar{y}_5 + 2\bar{y}_6 + 2\bar{y}_7 \leq 8 \). The inequality \( 2\bar{y}_2 + 2\bar{y}_3 + y_4 + \bar{y}_5 + \bar{y}_6 + \bar{y}_7 \leq 5 \) is valid for \( K \). (Again, it is a non-simple LCI.) So the inequality \( 2y_2 + 2y_3 - y_4 + y_5 + y_6 + y_7 \geq 2 \) is valid for \( P \). We have \( \phi(-1) = -1 \), \( \phi(0) = 0 \), \( \phi(1) = 2 \), \( \phi(2) = 3 \) and \( \phi(3) = 4 \). So \( t^* = 3 \) and \( \delta = 1 \), and we obtain the RKI \( x_1 + x_3 - x_4 \leq 4 + (2y_2 + 2y_3 - y_4 + y_5 + y_6 + y_7 - 3) \). One can check that this RKI defines a facet of \( P \). Other RKIs for this instance include, for example, the following:

\[
\begin{align*}
x_1 + x_2 - x_4 - x_5 & \leq 4 + (3y_1 + 2y_2 - y_4 - y_5 + y_6 + y_7 - 3) \\
x_2 + x_3 - x_5 & \leq 4 + (2y_2 + 2y_3 + y_4 - y_5 + y_6 + y_7 - 3) \\
x_1 + x_3 - x_7 & \leq 4 + (2y_1 - y_2 + y_3 + y_4 + y_5 + y_6 - y_7 - 1).
\end{align*}
\]

One can check that these too are facet-defining. \( \square \)

We remark that each RKI cuts off at least one fractional LP solution satisfying the following four conditions:

1. Constraint \([3]\) holds at equality,
2. \( x_j = u_j y_j > 0 \) for all \( j \in U^+ \cup U^- \),

\[
\text{prove that each RKI cuts off at least one fractional LP solution satisfying the above four conditions.}
\]

}\]
3. \( y_j = 1 \) and \( x_j = \ell_j \) for all \( j \in L^+ \cup L^- \),

4. \( y_j = x_j = 0 \) otherwise.

4 Concluding Remarks

We have introduced new families of valid inequalities for the fixed-charge and single-node flow polytopes. The inequalities, called rotated knapsack inequalities (RKIs), are completely different to the well-known flow cover inequalities and variants. Note that our procedure can yield a huge number of RKIs, because the number of possible choices for the subsets \( U^+ \), \( U^- \), etc. and the number of facets of the restricted knapsack polytope can both grow exponentially in the size of the problem. Thus, a natural topic for research is to find necessary and/or sufficient conditions for obtaining strong (preferably facet-defining) RKIs. Another pressing question is the development of effective exact or heuristic separation algorithms for the RKIs.

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