Bounds for Probabilistic Constrained Problems

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Abstract

In this paper we develop four upper bounds for single and joint chance constraints with independent matrix vector rows. The deterministic approximations of the probability constraints are based on the one-side Chebyshev inequality, Chernoff inequality, Bernstein inequality and Hoeffding inequality. Various sufficient conditions under which the aforementioned approximations are convex and tractable are derived. Therefore, we reformulate the chance constrained problems as tractable convex optimization problems based on piecewise linear and tangent approximations allowing to reduce further the computational complexity. Finally, numerical results on randomly generated data are discussed allowing to identify the tighter deterministic approximations.

Keywords: stochastic programming, chance-constrained problem, bounds, single chance-constraint, joint chance-constraints, piecewise approximations.

AMS: 90C15, 90C90, 65K05.

1 Introduction

Chance constrained optimization problems is an important class of optimization problems under uncertainty which involve constraints that are required
to hold with specified probabilities [7, 26, 31]. Several applications of chance constraints are considered in economics and finance [3], water reservoir management [2, 28], system optimization [15], electrical industry [36], optimal power flow [37] and many others.

In this paper, we consider the following chance constrained linear program:

$$\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad P\{\Xi x \leq H\} \geq \alpha, \\
& \quad x \in X,
\end{align*}$$

where $H = (h_1, \ldots, h_K) \in \mathbb{R}^K$, $\Xi = [\xi_1, \ldots, \xi_K]^T$ is a $K \times n$ random matrix, $\xi_k$, $k = 1, \ldots, K$ is a random vector in $\mathbb{R}^n$. $P$ is a probability measure, $x$ is a decision vector with feasible set $X \subseteq \mathbb{R}^n_+$, $c \in \mathbb{R}^n$ and $0 < \alpha < 1$ a prespecified confidence parameter.

One of the main goals in this class of problems is to come up with a deterministic equivalent problem such that the feasible set $S(\alpha) = \{x \in X : P\{\Xi x \leq H\} \geq \alpha\}$ of (1) is convex. In order to solve chance constrained problems efficiently, we need indeed both the convexity of the corresponding feasible set and efficient computability of the considered probability. This combination is rare, and very few are the cases in which a chance constraint can be processed efficiently (see [16, 32, 12]). In this context, [31] investigated a wide family of logarithmically concave distributions, showing that under this assumption the set $S(\alpha)$ is convex.

For an individual chance constrained problem given as follows

$$\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad P\{\xi^T x \leq h\} \geq \alpha, \\
& \quad x \in X,
\end{align*}$$

where for instance $\xi$ is a multivariate normally distributed vector with mean $\bar{\xi} = E(\xi)$ and variance $\sigma_\xi^2 = x^T \Sigma x$ with a positive definite variance-covariance matrix $\Sigma$, the following relations hold true:

$$\begin{align*}
P(\xi^T x \leq h) & \geq \alpha, \\
\downarrow & \\
\bar{\xi}^T x + F^{-1}(\alpha) \|\Sigma^{1/2} x\| & \leq h,
\end{align*}$$

where $F^{-1}(\cdot)$ is the inverse of $F$, the standard normal cumulative distribution function, transforming the chance constraint [3] into a Second Order
Cone Programming (SOCP) constraint\(^4\). The same scheme can be applied
to elliptical distributions, e.g., Laplace distribution, t-Student distribution,
Cauchy distribution, Logistic distribution\(^9\)\(^8\).

When the probability distributions are not elliptical or not known in
advance, lower and upper bounds on \(P\{\xi^T x \leq h\} \geq \alpha\), can be very useful.
In the next section we review the related literature.

1.1 Literature Review

For the case of individual chance constraint, the bounds available in the lit-
erature, are mainly based on extensions of Chebyshev inequality, and require
only the first two moments of the distribution or the mean and the support
(see\(^4\)\(^4\)\(^4\)\(^4\)).

For joint chance constraints, deterministic equivalent approximations have
been studied in\(^9\)\(^1\)\(^1\)\(^1\)\(^1\), \(^1\)\(^8\) and for special distributions, such as the
multivariate gamma, in\(^3\)\(^4\)\(^4\). In\(^8\) a new formulation for approximating joint
chance-constrained problems that improves upon the standard approach us-
ing Bonferroni’s inequality is proposed. The approach decomposes the joint
chance constraint into a problem with individual chance constraints, and
then applies safe robust optimization approximation on each one of them.
Connections with bounds on the conditional-value-at-risk (CVaR) measure
are also provided. Besides, in\(^2\)\(^4\) a class of analytical approximations of
single and joint independent chance constraints are developed and referred
to as Bernstein approximations. Unlike\(^3\)\(^0\) and the approach we will adopt
in this paper, they treat the natural scale parameter of Bernstein approxi-
mation as a variable rather than a chosen constant obtaining a bound jointly
convex in the original decision variables and in the scale parameter.

Relaxations and approximations of linear chance constraints in the setting
of a finite distribution of the stochastic parameters has been discussed in\(^1\)
where they review some recent results on improving the relaxation bounds
and constructing approximate solutions for the mixed integer linear program
(MILP) associated with the finite distribution chance constrained problem.
They also discuss a bicriteria approximation algorithm for covering chance
constrained problems.

Relaxations for probabilistically constrained stochastic programming prob-
lems in which the random variables are in the right-hand sides of the stochas-
tic inequalities defining the joint chance constraints are reviewed and pro-
vided in\(^1\)\(^7\).

Another safe approximation of chance constrained problems is given by
the conditional-value-at-risk (CVaR) as discussed in\(^2\)\(^4\)\(^1\). However, the
CVaR approximation can be overly conservative and does not have any optimality guarantee.

An alternative to bounds based on deterministic analytical approximation, is given by scenario approaches, based on Monte Carlo sampling techniques [5], [6], [19], [24], [29], where the probabilistic constraint is replaced by a sampled set of constraints. The sample size is chosen to guarantee that a solution to the sampled problem is feasible to the probabilistic constrained one with a high probability. See [27] for a survey of safe and scenario approximations of chance constraints.

Several bounding techniques have been proposed for two-stage and multistage stochastic programs with expectation (see for instance [4], [25]). This class of problems brings computational complexity which increases exponentially with the size of the scenario tree, representing a discretization of the underlying random process. Even if a large discrete tree model is constructed, the problem might be untractable due to the curse of dimensionality. In this situation, easy-to-compute bounds have been proposed in literature by solving small size sub-problems instead of the big one (see for instance [13], [20], [21], [22], [23]).

In this paper we develop upper bounds for linear single and joint probabilistic constrained problems with independent matrix vector rows. The uncertainty is considered in the left-hand side coefficient matrix. The deterministic approximations of probability inequalities are based on the one-side Chebyshev inequality, Chernoff inequality, Bernstein and Hoeffding inequalities. We derive various sufficient conditions related to the confidence parameter value under which the aforementioned approximations are convex and tractable. The approximations can be computed under different assumptions: more specifically Chebyshev inequality requires the knowledge of the first and second moments of the random variables while Bernstein and Hoeffding ones, their mean and support. On the contrary, Chernoff inequality requires only the moment generating function of the random variables. Approximations based on piecewise linear and tangent are also provided in order to reduce further the computational complexity of the problem. Finally, numerical results on randomly generated data are discussed.

The paper is organized as follows: Section 2 presents Chebyshev and Chernoff bounds, Section 3 discusses Bernstein and Hoeffding bounds. Section 4 gives our numerical results. Conclusions follows.
2 Chebyshev and Chernoff Bounds

In the following, we provide upper bounds based on deterministic approximations of probability inequalities such as the one-side Chebyshev inequality and Chernoff inequality.

2.1 Chebychev bounds

We consider the one-side Chebyshev inequality [30, 35]. We assume that $\xi$ has finite second moments and denote $\sigma^2_\xi = \text{Var}(\xi)$ the variance of $\xi$, and $\bar{\xi} = E(\xi)$ the mean of $\xi$. The one-side Chebyshev inequality is given by

$$P(\xi - \bar{\xi} \geq h) \leq \frac{\sigma^2_\xi}{\sigma^2_\xi + h^2}.$$  \hspace{1cm} (5)

For the individual chance constraint problem, we have the following results:

Theorem 1. Under one-sided Chebyshev inequality (5), Problem (2) can be formulated as follows

$$\min \chi c^T x$$

s.t. $\bar{\xi}^T x + \sqrt{\frac{\alpha}{1 - \alpha}} \| \Sigma^{1/2} x \| \leq h$,

$$x \in X,$$ \hspace{1cm} (6)

Moreover, Problem (6) is a convex problem.

Proof. First, we note that

$$P(\xi^T x \leq h) \geq \alpha,$$ \hspace{1cm} (7)

$$\Downarrow$$

$$P(\xi^T x \geq h) \leq 1 - \alpha,$$ \hspace{1cm} (8)

$$\Downarrow$$

$$P(\xi^T x - \bar{\xi}^T x \geq h - \bar{\xi}^T x) \leq 1 - \alpha.$$ \hspace{1cm} (9)

Then, we apply (5) to (9):

$$P(\xi^T x - \bar{\xi}^T x \geq h - \bar{\xi}^T x) \leq \frac{\sigma^2_\xi}{\sigma^2_\xi + (h - \bar{\xi}^T x)^2},$$ \hspace{1cm} (10)
where $\sigma^2 = x^T \Sigma x$ with variance-covariance matrix $\Sigma$. If \( \frac{\sigma^2}{\sigma^2 + (h - \bar{\xi}^T x)^2} \leq 1 - \alpha \), then $\{7\}$ will be satisfied. Therefore,

\[
\frac{x^T \Sigma x}{x^T \Sigma x + (h - \bar{\xi}^T x)^2} \leq 1 - \alpha, \iff \frac{\alpha}{1 - \alpha} \frac{x^T \Sigma x}{(h - \bar{\xi}^T x)^2} \leq (h - \bar{\xi}^T x)^2,
\]

which is equivalent to

\[
\sqrt{\frac{\alpha}{1 - \alpha}} \| \Sigma^{1/2} x \| \leq h - \bar{\xi}^T x.
\]

Moreover $\{11\}$ is a Second Order Cone Programming (SOCP) constraint and consequently Problem $\{6\}$ is convex.

In the following, we extend our results to the case of independent joint chance constraints.

**Remark 2.** For the sake of clarity, we replace the constraints $x \in X$ by $x \in \mathbb{R}^n$ for the joint constraints case in the remaining of this paper. Notice that additional constraints could be considered if their logarithm transformation preserves the convexity. See \[33\] for examples preserving the convexity.

If we assume that $\xi_k, k = 1, \ldots, K$ are multivariate normally distributed independent row vectors with mean vector $\mu_k = (\mu_{k1}, \ldots, \mu_{kn})$, and covariance matrix $\Sigma_k$, we can derive a deterministic reformulation of problem $\{1\}$.

$P \{ \Xi x \leq H \} \geq \alpha$ is equivalent to

\[
\prod_{k=1}^{K} P \{ \xi_k^T x \leq h_k \} \geq \alpha = \prod_{k=1}^{K} \alpha^{y_k},
\]

with $\sum_{k=1}^{K} y_k = 1, y_k \geq 0, k = 1, \ldots, K$.

We provide now an upper bound to problem $\{1\}$ based on the one-side Chebyshev inequality. We assume that $\xi_k, k = 1, \ldots, K$ has finite second moments. Let $\sigma_{\xi_k} = Var(\xi_k)$ the variance of $\xi_k$ and $\bar{\xi_k} = E(\xi_k)$ its mean.

We have the following result:

**Theorem 3.** Based on one-side Chebyshev inequality, an upper bound for problem $\{1\}$ can be obtained by solving the following deterministic equivalent
\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{s.t.} & \quad \bar{\xi}_k^T x + \sqrt{\frac{\alpha y_k}{1 - \alpha y_k}} \| \Sigma_k^{1/2} x \| \leq h_k, \quad k = 1, \ldots, K, \\
& \quad \sum_{k=1}^{K} y_k = 1, \quad y_k \geq 0, \quad k = 1, \ldots, K, \quad x \in \mathbb{R}_+^n . 
\end{align*}
\]

(13)

Proof.

\[
P(\xi_k^T x \leq h_k) \geq \alpha y_k, \quad k = 1, \ldots, K, 
\]

(14)

\[
P(\xi_k^T x - \bar{\xi}_k^T x \geq h_k - \bar{\xi}_k^T x) \leq 1 - \alpha y_k, \quad k = 1, \ldots, K . 
\]

(15)

We apply Chebyshev inequality to (15):

\[
P(\xi_k^T x - \bar{\xi}_k^T x \geq h_k - \bar{\xi}_k^T x) \leq \frac{\sigma^2_{\xi_k}}{\sigma^2_{\xi_k} + (h_k - \bar{\xi}_k^T x)^2}, \quad k = 1, \ldots, K , 
\]

(16)

where \( \sigma^2_{\xi_k} = x^T \Sigma_k x \) with variance-covariance matrix \( \Sigma_k \). If \( \frac{\sigma^2_{\xi_k}}{\sigma^2_{\xi_k} + (h_k - \bar{\xi}_k^T x)^2} \leq 1 - \alpha y_k \), then (14) will be satisfied. We have

\[
\frac{x^T \Sigma_k x}{x^T \Sigma_k x + (h_k - \bar{\xi}_k^T x)^2} \leq 1 - \alpha y_k \iff \frac{\alpha y_k}{1 - \alpha y_k} x^T \Sigma_k x \leq (h_k - \bar{\xi}_k^T x)^2, \quad k = 1, \ldots, K . 
\]

This is equivalent to

\[
\sqrt{\frac{\alpha y_k}{1 - \alpha y_k}} \| \Sigma_k^{1/2} x \| \leq h_k - \bar{\xi}_k^T x, \quad k = 1, \ldots, K . 
\]

(17)

Problem (13) is not convex but biconvex. To come-up with a tractable convex reformulation, we use the following logarithmic transformation \( z = \ln x \). In this case, Problem (13) can be reformulated as follows

\[
\begin{align*}
\min_{z} & \quad c^T e^z \\
\text{s.t.} & \quad \xi_k^T e^z + \left\| \Sigma_k^{1/2} e^{\ln(\frac{\alpha y_k}{1 - \alpha y_k}) + z} \right\| \leq h_k, \quad k = 1, \ldots, K , \\
& \quad \sum_{k=1}^{K} y_k = 1, \quad y_k \geq 0, \quad k = 1, \ldots, K, \quad z \in \mathbb{R}_+^n . 
\end{align*}
\]

(18)

We now prove that problem (18) is convex for all \( \alpha \in [0, 1] \).
Lemma 4. Let $f : X \rightarrow Y$ be a nonincreasing concave function, $g : Z \rightarrow X$ be a convex function. Then, we have $f(g) : Z \rightarrow Y$ is a concave function.

The proof is given in Appendix A.

Assumption 5. For each $k = 1, \ldots, K$, all the components of $\bar{\xi}_k$ and $\Sigma_k$ are non-negative.

Theorem 6. If Assumption 5 holds, then problem (18) is convex for all $\alpha \in [0, 1]$.

Proof. To show the convexity of problem (18), we firstly need to show the convexity of $\ln \left( \frac{\alpha y_k}{1-\alpha y_k} \right)$. As $\ln \left( \frac{\alpha y_k}{1-\alpha y_k} \right) = \frac{1}{2} \left( y_k \ln \alpha - \ln (1 - \alpha y_k) \right)$, we can deduce the convexity of function $\ln \left( \frac{\alpha y_k}{1-\alpha y_k} \right)$ if $\ln (1 - \alpha y_k)$ is concave.

Since $\log (1 - p)$ is decreasing and concave with respect to $p$ and $\alpha y_k$ is convex with respect to $y_k$, we have that $\ln (1 - \alpha y_k)$ is concave with respect to $y_k$ as shown by Lemma 4.

Since the norm is a convex function and it is also a nondecreasing function when the arguments are nonnegative, the composition function $\left\| \Sigma_k^{1/2} e^{\ln \left( \frac{\alpha y_k}{1-\alpha y_k} \right) + z} \right\|$ is a convex function. The term $\bar{\xi}_k^T e^z$ is a convex function because $\bar{\xi}_k \geq 0$. Hence, the problem (18) is convex for all $\alpha \in [0, 1]$.

2.2 Chernoff bounds

We consider now the Chernoff bound:

$$P(\xi \geq h) \leq \frac{E(e^{t\xi})}{e^{th}},$$

where $E(e^{t\xi})$ is the moment generating function of the random variable $\xi$ and $t > 0$. $\bar{\xi}$ is the mean of $\xi$ and $\sigma^2 = Var(\xi)$ is the variance.

First, we proof the convexity of $E(e^{t\xi^T x})$.

Lemma 7. $E(e^{t\xi^T x})$ is a convex function.

The proof is given in Appendix A.

Theorem 8. If $\xi$ follows a normal distribution with mean vector $\bar{\xi}$ and variance-covariance matrix $\Sigma$, under Chernoff bound, Problem (18) can be formulated as follows
\[
\min_x \, e^T x \\
\xi^T x + \sqrt{2\ln \frac{1}{1 - \alpha}} \| \Sigma^{1/2} x \| \leq h, \\
x \in X.
\]

(20)

Moreover, Problem (20) is a convex problem.

Proof. First, we have from (8)

\[
P(\xi^T x \leq h) \geq \alpha \iff P(\xi^T x \geq h) \leq 1 - \alpha.
\]

This implies

\[
P(\xi^T x \geq h) \leq \mathbb{E}(e^{t\xi^T x}) e^{th}.
\]

(21)

If we choose \( \mathbb{E}(e^{t\xi^T x}) \leq 1 - \alpha \), then we get an upper bound to problem (2) with feasible region

\[
\bar{S}(\alpha) = \left\{ x \in X \subseteq \mathbb{R}_+^n \mid \exists t > 0 : \mathbb{E}(e^{t\xi^T x}) \leq (1 - \alpha)e^{th} \right\},
\]

(22)

which is convex as \( \mathbb{E}(e^{t\xi^T x}) \) is convex as shown by Lemma 7.

As \( \xi \) is normally distributed, we have \( \mathbb{E}(e^{t\xi^T x}) = e^{t\xi^T x} \cdot e^{\frac{1}{2}x^T \Sigma x^2} \). The feasible region \( \bar{S}(\alpha) \) can be written as:

\[
\bar{S}(\alpha) = \left\{ x \in X \subseteq \mathbb{R}_+^n \mid \exists t > 0 : \frac{1}{2}x^T \Sigma x^2 + t\xi^T x - th \leq \ln(1 - \alpha) \right\}.
\]

(23)

The set (23) is equivalent to:

\[
\min_{t > 0} \left\{ \frac{1}{2}x^T \Sigma x^2 + t\xi^T x - th \right\} \leq \ln(1 - \alpha).
\]

(24)

The first derivative in \( t \) is \( \frac{h - \xi^T x}{x^T \Sigma x} \). Let \( h - \xi^T x \geq 0 \). Therefore (23) is equivalent to

\[
\bar{S}(\alpha) = \left\{ x \in X \subseteq \mathbb{R}_+^n \mid (h - \xi^T x)^2 \leq 2\ln(1 - \alpha)x^T \Sigma x \right\},
\]

(25)

which is equivalent to the following convex set:

\[
\bar{S}(\alpha) = \left\{ x \in X \subseteq \mathbb{R}_+^n \mid h - \xi^T x \geq \sqrt{2\ln \frac{1}{1 - \alpha}} \| \Sigma^{1/2} x \| \right\}.
\]

(26)
We extend our results to the case of independent joint chance constraints. If we assume that $\xi_k, k = 1, \ldots, K$ are multivariate normally distributed independent row vectors with mean vector $\bar{\xi}_k = (\bar{\xi}_{k1}, \ldots, \bar{\xi}_{kn})^T$ and covariance matrix $\Sigma_k$, we can derive a deterministic reformulation of problem (1) based on (12).

We consider now an upper bound to problem (1) based on Chernoff bound.

**Theorem 9.** If $\xi_k, k = 1, \ldots, K$, are pairwise independent and normally distributed with mean vector $\bar{\xi}_k$ and covariance matrix $\Sigma_k$, then an upper bound for Problem (1) based on Chernoff bound can be obtained by solving the following problem:

$$
\min_z \; c^T x
$$

$$
\tilde{c}_k^T x + \sqrt{2 \ln \left( \frac{1}{1 - \alpha y_k} \right)} \| \Sigma_k^{1/2} x \| \leq h_k, \; k = 1, \ldots, K,
$$

$$
\sum_{k=1}^K y_k = 1, \; y_k \geq 0, \; k = 1, \ldots, K, \; x \in \mathbb{R}_+^n.
$$

(27)

**Proof.** First, we note that

$$
P(\xi_k^T x \leq h_k) \geq \alpha y_k \iff P(\xi_k^T x \geq h_k) \leq 1 - \alpha y_k.
$$

Chernoff bound leads to

$$
P(\xi_k^T x \geq h_k) \leq \frac{E(e^{t \xi_k^T x})}{e^{th_k}}, \; k = 1, \ldots, K,
$$

(28)

with $t > 0$. An upper bound to problem (1) is then obtained by solving the following problem:

$$
\min_z \; c^T x
$$

$$
E(e^{t \xi_k^T x}) \leq (1 - \alpha y_k)e^{th_k}, \; k = 1, \ldots, K,
$$

$$
\sum_{k=1}^K y_k = 1, \; y_k \geq 0, \; k = 1, \ldots, K, \; x \in \mathbb{R}_+^n.
$$

(29)

However, if the probability distributions of $\xi_k, k = 1, \ldots, K$ are not known, the main difficulty of the model (29) is given by the computation of $E(e^{t \xi_k^T x})$. On the other hand, as $\xi_k, k = 1, \ldots, K$ are normally distributed,
we have that $E(e^{t\xi x}) = e^{t\xi_k x - \frac{1}{2}x^T \Sigma_k x t^2}, k = 1, \ldots, K$. Consequently problem (29) can be written as

\[
\begin{align*}
\min_z & \quad c^T x \\
\text{s.t.} & \quad \frac{1}{2}x^T \Sigma_k x t^2 + t\xi_k x - th_k \leq \ln(1 - \alpha^y), \quad k = 1, \ldots, K, \\
& \quad \sum_{k=1}^K y_k = 1, \quad y_k \geq 0, \quad k = 1, \ldots, K, \quad x \in \mathbb{R}_+^n.
\end{align*}
\] (30)

Similarly, Problem (27) is not a convex optimization problem. Therefore, we apply the transformation $z = \ln x$ and get:

\[
\begin{align*}
\min_z & \quad c^T e^z \\
\text{s.t.} & \quad \xi_k^T e^z + \left|\Sigma_k^{1/2} e^{\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)}\right| \leq h_k, \quad k = 1, \ldots, K, \\
& \quad \sum_{k=1}^K y_k = 1, \quad y_k \geq 0, \quad k = 1, \ldots, K, \quad z \in \mathbb{R}_+^n.
\end{align*}
\] (31)

Moreover, if $\xi_k \geq 0$, $k = 1, 2, \ldots, K$, and the function $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)$ is convex, then Problem ((31)) is convex. The following lemma shows the condition under which the function $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)$ is convex.

**Lemma 10.** If $\alpha \geq 1 - e^{-1} \approx 0.6321$, then $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)$ is convex.

**Proof.** By the convexity theorem of composite function, we only need to prove the convexity of $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)$ with respect to $p$, since the convexity of composite function $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)$ is implied when $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)$ is nondecreasing and convex. As $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)$ is monotone, we need to show the convexity of $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right)$. We can notice that $\ln\left(\sqrt{2 \ln\left(\frac{1}{1-\alpha^y}\right)}\right) = \frac{1}{2} \ln\left(2 \ln\left(\frac{1}{1-\alpha^y}\right)\right)$. Therefore, we only need to focus on the convexity of $\ln\left(2 \ln\left(\frac{1}{1-\alpha^y}\right)\right)$.

The second order derivative of $\ln\left(2 \ln\left(\frac{1}{1-\alpha^y}\right)\right)$ can be written as

\[
\begin{align*}
& \quad - \left[ (1 - p)^{-2} (\ln(1 - p))^{-1} + (1 - p)^{-2} (\ln(1 - p))^{-2} \right], \\
& = \quad - (1 - p)^{-2} (\ln(1 - p))^{-2} (\ln(1 - p) + 1).
\end{align*}
\]
Then, \( \ln \left( 2 \ln \left( \frac{1}{1-p} \right) \right) \) is convex if and only if
\[
\ln(1 - p) + 1 \leq 0.
\]

Therefore, we have \( p \geq 1 - e^{-1} \).

As \( \alpha y_k \) is convex with respect to \( y_k \) and \( \alpha y_k \geq \alpha \) for any \( 0 \leq y_k \leq 1 \), if \( \alpha \geq 1 - e^{-1} \), then the function \( \ln \left( \frac{\sqrt{2} \ln(1 - \alpha y_k)}{1 - \alpha y_k} \right) \) is convex.

Therefore, as \( c \geq 0 \), \( \alpha \geq 1 - e^{-1} \), \( \bar{\xi}_k \geq 0 \), \( k = 1, 2, \ldots, K \), problem (31) is convex.

# 3 Bernstein and Hoeffding Bounds

Bernstein and Hoeffding bounds are considered as exponential type estimates of probabilities. These inequalities are frequently used for investigating the law of large numbers for instance. They are also often used in statistics and probability theory. In this section, we investigate these bounds for the case of individual and joint chance constraints.

## 3.1 Bernstein bounds

In this section, we consider Bernstein bound [30]. We assume that the mean and the range parameters for all independent components \( \xi_i \) of the random vector \( \xi \) are known, i.e. \( l_i \leq \xi_i \leq u_i \), and \( \mathbb{E}(\xi_i) = \bar{\xi}_i \), for \( i = 1, \ldots, n \). Then, the respective values for the random variable \( \xi_i x_i \) are \( l'_i = l_i x_i \), \( u'_i = u_i x_i \), and \( \bar{\xi}'_i = \bar{\xi}_i x_i \). With Bernstein-type exponential estimate, we have
\[
e^{-g^* h} \prod_{i=1}^{n} \left\{ \frac{u_i - \bar{\xi}_i}{u_i - l_i} e^{g^* l_i x_i} + \frac{\bar{\xi}_i - l_i}{u_i - l_i} e^{g^* u_i x_i} \right\} \leq \alpha. \tag{32}
\]

**Theorem 11.** An upper bound for problem (2) can be obtained by solving the following problem
\[
\min_x c^T x \quad \sum_{i=1}^{n} \ln \left\{ \frac{u_i - \bar{\xi}_i}{u_i - l_i} e^{g^* l_i x_i} + \frac{\bar{\xi}_i - l_i}{u_i - l_i} e^{g^* u_i x_i} \right\} \leq \ln(1 - \alpha) + g^* h, \quad x \in X, \tag{33}
\]
for any arbitrary \( g^* > 0 \).
Proof. Follows from Bernstein inequality.

We provide now an upper bound to Problem (4) based on the Bernstein bound. We assume that the mean and the range parameters for all independent components \((\xi_k)_i\) of the random vectors \(\xi_k\) are known, i.e. \((l_k)_i \leq (\xi_k)_i \leq (u_k)_i\), and \(E[(\xi_k)_i] = (\bar{\xi}_k)_i\), for \(k = 1, \ldots, K\) and \(i = 1, \ldots, n\). Then, the respective values for the random variable \((\xi_k)_i x_i\) are \((l'_k)_i = (l_k)_i x_i\), \((u'_k)_i = (u_k)_i x_i\), and \((\bar{\xi}'_k)_i = (\bar{\xi}_k)_i x_i\).

**Theorem 12.** An upper bound to problem (4) can be obtained by solving the following problem

\[
\min_x c^T x \\
\sum_{i=1}^n \ln \frac{(u_k)_i - (\bar{\xi}_k)_i}{(u_k)_i - (l_k)_i} g_k^* (l'_k)_i x_i + \frac{(\bar{\xi}_k)_i - (l_k)_i}{(u_k)_i - (l_k)_i} g_k^* (u'_k)_i x_i \leq g_k^* h_k + \\
\ln(1 - \alpha^{y_k}), \ k = 1, \ldots, K, \\
\sum_{k=1}^K y_k = 1, \ y_k \geq 0, \ k = 1, \ldots, K, \ x \in \mathbb{R}^n_+ ,
\]

for any \(g_k^* > 0\), \(k = 1, \ldots, K\).

**Proof.** According to Bernstein-type exponential estimate we have

\[
e^{-g_k^* h_k} \prod_{i=1}^n \left\{ \frac{(u_k)_i - (\bar{\xi}_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^* (l'_k)_i x_i} + \frac{(\bar{\xi}_k)_i - (l_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^* (u'_k)_i x_i} \right\} \leq \alpha^{y_k}, \ k = 1, \ldots, K ,
\]

with arbitrary \(g_k^* > 0\) which implies \(P(\sum_{i=1}^n (\xi_k)_i x_i \geq h_k) \leq \alpha^{y_k}, \ k = 1, \ldots, K\).

We note that

\[
P \left( \bar{\xi}_k^T x \leq h_k \right) \geq \alpha^{y_k}, \ k = 1, \ldots, K ,
\]

\[
P \left( \sum_{i=1}^n (\xi_k)_i x_i \geq h_k \right) \leq 1 - \alpha^{y_k}, \ k = 1, \ldots, K .
\]

**Problem (37)** can be reformulated as

\[
\sum_{i=1}^n \ln \left\{ \frac{(u_k)_i - (\bar{\xi}_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^* (l'_k)_i x_i} + \frac{(\bar{\xi}_k)_i - (l_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^* (u'_k)_i x_i} \right\} \leq \ln(1 - \alpha^{y_k}) + g_k^* h_k
\]
for any $g_k^* > 0$, $k = 1, \ldots, K$.

From Proposition 4.1 in [30] and the concavity of function $\ln(1 - \alpha^y)$, problem (34) is convex.

### 3.2 Hoeffding bounds

We consider now an approximation based on Hoeffding inequality:

$$P\left(\frac{\xi^T e}{n} - \frac{\bar{\xi}^T e}{n} \geq h\right) \leq e^{-\frac{2h^2}{\sum_{i=1}^n (u_i - l_i)^2}}. \tag{39}$$

**Theorem 13.** An upper bound for Problem (2) can be obtained by solving the following convex problem

$$\min_x c^T x$$

$$\bar{\xi}^T x + \frac{\sqrt{2}}{2} \sqrt{-\ln(1 - \alpha)}\|Mx\| \leq h,$$

$$x \in X,$$ \tag{40}

where $M = \text{diag}(u - l)$, $u = (u_1, \ldots, u_n)^T$, $l = (l_1, \ldots, l_n)^n$.

**Proof.** We note that

$$P\left(\xi^T x \leq h\right) \geq \alpha, \tag{41}$$

$$\Downarrow$$

$$P\left(\xi^T x - \bar{\xi}^T x \geq h - \bar{\xi}^T x\right) \leq 1 - \alpha. \tag{42}$$

Then, we apply (39) to (42) and get:

$$P\left(\xi^T x - \bar{\xi}^T x \geq h - \bar{\xi}^T x\right) \leq e^{-\frac{2(h - \bar{\xi}^T x)^2}{\sum_{i=1}^n (u_i - l_i)^2x_i^2}}. \tag{43}$$

If

$$e^{-\frac{2(h - \bar{\xi}^T x)^2}{\sum_{i=1}^n (u_i - l_i)^2x_i^2}} \leq 1 - \alpha, \tag{44}$$

then (41) will be satisfied. Logarithmic transformation of (44) leads to

$$\frac{-2(h - \bar{\xi}^T x)^2}{\sum_{i=1}^n (u_i - l_i)^2x_i^2} \leq \ln(1 - \alpha), \tag{45}$$

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which can be written as
\[ h - \bar{\xi}^T x \geq \frac{\sqrt{2}}{2} \sqrt{-\ln(1 - \alpha)} \| Mx \|, \tag{46} \]
where \( M = \text{diag}(u - l) \) and then (46) is a linear inequality.

The feasible region \( \tilde{S}(\alpha) \) can be written as:
\[ \tilde{S}(\alpha) = \left\{ x \in X \subseteq \mathbb{R}_+^n \mid h - \xi^T x \geq \frac{\sqrt{2}}{2} \sqrt{-\ln(1 - \alpha)} \| Mx \| \right\}. \tag{47} \]

**Theorem 14.** An approximation of problem (1) based on Hoeffding's inequality can be given by

\[ \min_x \ c^T x \\
\bar{\xi}_k^T x + \frac{\sqrt{2}}{2} \sqrt{\ln \left( \frac{1}{1 - \alpha y_k} \right)} \| M_k x \| \leq h_k, \ k = 1, \ldots, K, \tag{48} \]

where \( M_k = \text{diag}(u_k - l_k), u_k = ((u_k)_1, \ldots, (u_k)_n)^T, \ l_k = ((l_k)_1, \ldots, (l_k)_n)^T, \ k = 1, \ldots, K. \)

**Proof.** Note that
\[ P\left( \xi_k^T x \leq h_k \right) \geq \alpha y_k, \ k = 1, \ldots, K, \] \[ \Downarrow \]
\[ P\left( \xi_k^T x - \bar{\xi}_k^T x \geq h_k - \bar{\xi}_k^T x \right) \leq 1 - \alpha y_k, \ k = 1, \ldots, K. \tag{50} \]

Then, we have
\[ P\left( \xi_k^T x - \bar{\xi}_k^T x \geq h_k - \bar{\xi}_k^T x \right) \leq e^{\sum_{i=1}^n ((u_k)_i - (l_k)_i)^2 x_i^2}, \ k = 1, \ldots, K. \tag{51} \]

If
\[ e^{\sum_{i=1}^n ((u_k)_i - (l_k)_i)^2 x_i^2} \leq 1 - \alpha y_k, \ k = 1, \ldots, K, \tag{52} \]
feasibility holds for problem (49). Problem (52) can be reformulated as
\[ \sum_{i=1}^n ((u_k)_i - (l_k)_i)^2 x_i^2 \leq \ln(1 - \alpha y_k), \ k = 1, \ldots, K, \tag{53} \]
then,
\[ h_k - \xi_k^T x \geq \frac{\sqrt{2}}{2} \sqrt{\ln \left( \frac{1}{1 - \alpha y_k} \right)} \| M_k x \|, \quad k = 1, \ldots, K, \tag{54} \]
where \( M_k = \text{diag}(u_k - l_k) \).

Additionally, an equivalent upper bound for problem (1) based on Hoeffding inequality can be obtained by applying the following transformation \( z = \ln x \):

\[
\begin{align*}
\min_x & \quad c^T e^z \\
\text{subject to} & \quad \xi_k^T e^z + \frac{1}{2} \| M_k e^{\ln \left( \sqrt{2 \ln \left( \frac{1}{1 - \alpha y_k} \right)} \right) + z} \| \leq h_k, \quad k = 1, \ldots, K, \\
& \quad \sum_{k=1}^{K} y_k = 1, \quad y_k \geq 0, \quad k = 1, \ldots, K, \quad z \in \mathbb{R}_+^n. \tag{55}
\end{align*}
\]

From Lemma 10, function \( \ln \left( \sqrt{2 \ln \left( \frac{1}{1 - \alpha y_k} \right)} \right) \) is convex, when \( \alpha \geq 1 - e^{-1} \). Hence, if \( c \geq 0, \alpha \geq 1 - e^{-1} \), Problem (55) is convex.

### 4 Computational Results

Although the problems which give upper bounds obtained by Chebyshev inequality, Chernoff inequality, Bernstein inequality and Hoeffding inequality for problem (1) are convex under some conditions, they are still hard to solve directly by current tools because of the following terms: \( \ln \left( \frac{1}{1 - \alpha y_k} \right) \) and \( \ln \left( \sqrt{2 \ln \left( \frac{1}{1 - \alpha y_k} \right)} \right) \)

In the following we denote theses functions by \( \Upsilon(y_k) \), we propose piecewise linear approximations for \( \Upsilon(y_k) \) based on tangent and segment approximations.

**Tangent approximation**

We choose \( S \) different linear functions:

\[ l_s(y_k) = a_s y_k + b_s, \quad s = 1, \ldots, S, \]

such that

\[ l_s(y_k) \leq \Upsilon(y_k), \quad \forall y_k \in [\rho, 1], \quad k = 1, \ldots, K. \]
Here $\rho \geq 0$ is a constant such that $\Upsilon(y_k)$ is convex on $[\rho, 1)$. Then, $\Upsilon(y_k)$ can be approximated by the following piecewise linear function

$$l(y_k) = \max_{s=1,\ldots,S} l_s(y_k),$$

which provides a lower approximation for $\Upsilon(y_k)$.

In order to achieve the expected precision, we set $l_s(y_k)$ as the tangent line of $\Upsilon(y_k)$ at $S$ points $\tau_1, \ldots, \tau_S$ with $\tau_s \in [\rho, 1)$, $s = 1, \ldots, S$. Then, we have

$$a_s = \frac{d \Upsilon(y_k)}{dy_k} \bigg|_{y_k = \tau_s}, \quad b_s = \Upsilon(\tau_s) - a_s \tau_s.$$

Thanks to these piecewise linear approximations for $\Upsilon(y_k)$, we have the following results:

**Theorem 15.** Under the aforementioned convex conditions, if we replace in problems (18), (31), (55) $\Upsilon(y_k)$ by $l(y_k)$, we obtain their convex approximations. The optimum values of the approximation problems are lower bounds for problems (18), (31), (55), respectively. Moreover, the approximation problems become an equivalent reformulation of problems (18), (31), (55) when $S$ goes to infinity.

**Proof.** As the approximation problems are obtained by relaxing some constraints in problems (18), (31), (55), it is easy to see that the optimal values of the approximation problems are lower bounds for problems (18), (31), (55), respectively.

We know under convex conditions for problems (18), (31), (55), $\Upsilon(y_k)$ is convex for each problem. As the $S$ tangent functions are selected differently, when $S$ goes to infinity, the constraints in the approximation problems are equivalent to the constraints in problems (18), (31), (55), respectively. As the original problems and the corresponding approximation problems are all convex programs, the approximation problems become an equivalent reformulation of problems (18), (31), (55), respectively, when $S$ goes to infinity.

**Segment approximation**

In order to come up with conservative bounds for the optimum values of problems (18), (31), (55), we use the linear segments $\bar{a}_s y_k + \bar{b}_s$, $s = 1, \ldots, S$, between $\tau_1, \tau_2, \ldots, \tau_{S+1} \in [\rho, 1)$ to construct a piecewise linear function

$$\bar{l}(y_k) = \max_{s=1,\ldots,S} \left\{ \bar{a}_s y_k + \bar{b}_s \right\}, \quad (56)$$

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where
\[ \bar{a}_s = \frac{\Upsilon(\tau_{s+1}) - \Upsilon(\tau_s)}{\tau_{s+1} - \tau_s}, \quad \bar{b}_s = \Upsilon(\tau_s) - \bar{a}_s\tau_s, \quad s = 1, \ldots, S. \]

Using the piecewise linear function \( \tilde{l}(y_k) \) to replace \( \Upsilon(y_k) \) in problems (18), (31), (55), gives the corresponding approximation problems.

Similar to Theorem (15), we can derive the following result for the linear approximation:

**Theorem 16.** Under the aforementioned convex conditions, if we replace in problems (18), (31), (55) \( \Upsilon(y_k) \) by \( l(y_k) \), we obtain the convex approximations of these problems.

The optimum values of the approximation problems are an upper bound for problems (18), (31), (55), respectively. Moreover, the approximation problems become an equivalent reformulation of problems (18), (31), (55), respectively, when \( S \) goes to infinity.

The proof of this theorem follows the same pattern as the proof of Theorem (15).

**Numerical experiments**

The bounds have been implemented and compared under Matlab environment using CVX software, a modeling system for constructing and solving convex programs. We run the bounds for 100 instances randomly generated with the following characteristics: in the single chance constraint problem (2) we set \( n = 10, h = 0.5, \alpha = 0.95 \), the constraint \( x \in X \in \mathbb{R}^n_+ \) is given by \( \sum_{i=1}^n x_i = 1 \), \( c \) is a random vector from a uniform distribution in the interval \([0, 1]^n\), \( \bar{\xi}^T \) is uniformly generated in the interval \([0, 10]^n\) and \( \bar{\sigma}^2_\xi \) is uniformly generated in the interval \([0, 1]^{n \times n}\). In the following, we assume that the random variable \( \xi \) is distributed according to a normal distribution with mean \( \bar{\xi}^T \) and variance \( \bar{\sigma}^2_\xi \) generated as described above. This will allow us to make a fair comparison of the bounds with the exact SOCP reformulation. For joint chance constrained problem (1), we set \( n = 10, K = 5 \) and \( h_i = 0.5, i = 1, \ldots, K \). The other parameters are the same as in the individual case.

Numerical results for the single chance constraint case are reported in Figure 1. Figure 1(a) shows a comparison of the objective function values of the four bounds (Chebyshev, Chernoff, Bernstein and Hoeffding) with the exact SOCP reformulation for 100 different randomly generated instances while Figure 1(b) shows the corresponding box-and-whisker plots where the extrema of the box represents the 1/3 and 3/4 quartiles, the band inside the
box is the median and the whiskers the minimum and maximum obtained from 100 randomly generated instances.

![Diagram](image1)

(a)  

![Diagram](image2)

(b)  

Figure 1: (a) Comparison of SOCP versus bounds for individual chance constraint (b) box-and-whisker plots from the same 100 randomly generated instances of (a).

<table>
<thead>
<tr>
<th>Bound</th>
<th>Chebyshev</th>
<th>Chernoff</th>
<th>Bernstein</th>
<th>Hoeffding</th>
<th>SOCP</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU time(s)</td>
<td>0.9815</td>
<td>0.9727</td>
<td>1.2761</td>
<td>1.0850</td>
<td>0.9932</td>
</tr>
<tr>
<td>Gap (%)</td>
<td>7.55</td>
<td>1.58</td>
<td>2.8</td>
<td>13.64</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1: Average results over 100 instances of different bounds and SOCP for individual chance constrained problem.

The average CPU times and gaps for different bounds and SOCP are shown in Table 1 in second and third rows respectively. Results from Figure 1 and Table 1 show that the best upper bound is obtained by Chernoff inequality, followed by Bernstein which in all the instances outperforms one-side Chebyshev and Hoeffding ones. The worst bound is in all the cases obtained by Hoeffding approximation. Notice that the CPU times for Chebyshev, Chernoff, Bernstein, Hoeffding and SOCP are relatively comparable. The reason of the good performance of Chernoff bound compared to the others, could be due to the explicit assumption that the moment generating function of the random variable \( \xi \) follows a normal distribution, information not taken into account in all the other approaches. Very good is the performance of the Bernstein bound, considering that only the mean and the range of the random variable are known.

Numerical results for the joint chance constraints case are reported in Figure 2: Figure 2(a) shows a comparison of the objective function values of the
Figure 2: (a) Comparison of SOCP versus bounds for the joint chance constraints problem. (b) Box-and-whisker plots from the same 100 randomly generated instances of (a).

Table 2 shows the average CPU time (second row) for different bounds and SOCP and the average gap (third row) between different bounds and SOCP for the joint chance constraints case. Results from Figure 2(a)-(b) and Table 2 show that Chebyshev bound always provide the worst bound, while there are not much differences between other bounds. As in the single chance constraint case, the best bound is the Chernoff one followed by the Bernstein’s bound. Notice that the CPU times for Chebyshev, Chernoff, Bernstein, Hoeffding and SOCP are relatively comparable.

<table>
<thead>
<tr>
<th>Bound</th>
<th>Chebyshev</th>
<th>Chernoff</th>
<th>Bernstein</th>
<th>Hoeffding</th>
<th>SOCP</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU time(s)</td>
<td>2.9932</td>
<td>2.8255</td>
<td>3.2030</td>
<td>2.2845</td>
<td>2.2551</td>
</tr>
<tr>
<td>Gap (%)</td>
<td>12.30</td>
<td>3.69</td>
<td>3.91</td>
<td>4.01</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2: Average results of different bounds and SOCP for the joint chance constrained problem.

Notice that in the upper bound problems obtained by Chernoff and Hoeffding inequalities for problem (1), we use tangent and segment approxima-
tions, as described above, to approximate the complex terms: \( \ln \left( \sqrt{2 \ln \left( \frac{1}{1-\alpha k} \right)} \right) \) and \( \ln \left( \sqrt{2 \ln \left( \frac{1}{1-y_k} \right)} \right) \), respectively.

Figure 3: Comparison of SOCP versus bounds for the joint chance constraints problem with different values of \( S \) with segment and tangent approximations.

A sensitivity analysis of the segment and tangent approximations on the number of segments \( S = 3, 10, 20, 50 \) is reported in Figure 3 with corresponding box-and-whisker plots in Figure 4, for 100 instances randomly generated. Results show that when \( S = 50 \), there’s almost no gap between the results obtained by tangent and segment approximations, respectively. And for each \( S \), Chernoff inequality and Hoeffding inequality always provide an upper bound for the result obtained by SOCP.
Figure 4: Box-and-whisker plots corresponding to Figure 3 for different values of $S = 3, 10, 20, 50$ with segment and tangent approximations.

Conclusions

In this paper, we propose deterministic approximations for individual and joint chance constraints with independent matrix vector rows. The bounds are based on classical inequalities from probability theory such as the one-side Chebyshev inequality, Bernstein inequality, Chernoff inequality and Hoeffding inequality and allow to reformulate the problem in in a tractable convex way. Approximations based on piecewise linear and tangent are also provided in case of Chernoff and Hoeffding inequalities allowing to reduce the computational complexity of the problem. Finally numerical results on randomly generated data are provided allowing to identify that the Chernoff bound provides the tighter deterministic approximation while the Chebyshev bound,
requiring the knowledge of the first and second moments, is very loose both for single and joint chance constrained problems. Remarkable is also the performance of Bernstein’s bound, considering that only the mean and the range of the random variables are assumed to be known. In terms of CPU times all the considered bounds are relatively comparable.

Future works will be devoted on the application of the bounds addressed in this paper to more general stochastic optimization problems with chance constraints.

Appendix A

Proof of Lemma (4)

Proof. To show that \( f(g) \) is a concave function, we show that the second order derivative of \( f(g) \) is non-positive. The second order derivative of \( f(g) \) can be written as

\[
  f''(g)(g')^2 + f'(g)g''.
\]

Since \( f \) is nonincreasing and concave and \( g \) is convex, we have \( f''(g) \leq 0 \), \( f'(g) \leq 0 \) and \( g'' \geq 0 \). Therefore, \( f''(g)(g')^2 + f'(g)g'' \leq 0 \). □

Proof of Lemma (7)

Proof. Since \( e^{t\xi^Tx} \) is convex with respect to \( x \in X \), we have that for \( \lambda \in [0,1] \) and \( x_1, x_2 \in X \),

\[
  e^{t\xi^T(\lambda x_1 + (1-\lambda)x_2)} \leq \lambda e^{t\xi^Tx_1} + (1-\lambda) e^{t\xi^Tx_2}.
\]

Therefore,

\[
  \mathbb{E}(e^{t\xi^T(\lambda x_1 + (1-\lambda)x_2)}) \leq \lambda \mathbb{E}(e^{t\xi^Tx_1}) + (1-\lambda) \mathbb{E}(e^{t\xi^Tx_2}).
\]

□

References


