Strong Mixed-Integer Formulations for Power System Islanding and Restoration

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***Abstract*—The Intentional Controlled Islanding (ICI) and the Black Start Allocation (BSA) are two examples of problems in the power systems literature that have been formulated as Mixed Integer Programs (MIPs). A key consideration in both of these problems is that each island must have at least one energized generator. In this paper, we provide three alternative MIP formulations for this restriction, show their equivalence, and prove that two of them are stronger in terms of their linear programming relaxation than the one most commonly used in the power systems literature. Since the time to solve MIPs can vary significantly between equivalent formulations, we also present computational experiments on IEEE test systems for the ICI and BSA problems. We observe that a polynomially separable, exponential in size, strong formulation yields the best performance for the BSA problem and exhibits a comparable performance to a linear in size, weak formulation for the ICI problem.**

***Index Terms*—Black Start Allocation, Power System Restoration, Controlled Islanding, Mixed Integer Programming, Graph Partitioning, Topology Optimization**

I. Introduction

Mixed integer programming (MIP) formulations, i.e. optimization models that involve integer as well as real variables, are becoming ubiquitous in power systems applications (unit commitment, power system restoration, capacity expansion planning, optimal islanding). The main reasons for their popularity is that they offer broad modeling capabilities and that specialized commercial MIP solvers have improved significantly over the past years, making MIPs tractable for many practical applications. In these applications, binary variables are used to represent commitment, scheduling, time dependencies, component energization, as well as to approximate nonlinear curves with piecewise linear functions. Two of the problems that have been formulated as MIPs in power systems are the Intentional Controlled Islanding (ICI) and the Black Start Allocation (BSA).

ICI is a measure employed to prevent cascading power system outage by splitting the grid into smaller, stable and easily controllable islands via switching off lines [1]–[16]. One straightforward approach to model the problem is to represent the switching decisions with binary variables and formulate an optimization problem [1]–[11]. The optimization objective and constraints embody the requirements that the splitting should satisfy, such as minimum power flow disruption, size and capacity of the resulting islands and isolating or grouping generators in coherent sets.

The BSA for Power System Restoration (PSR) problem [17]–[27] aims to allocate Black Start (BS) resources in the grid in an efficient way, in order to ensure a successful restoration of the power system after an outage. The problem is often modeled as a MIP, with binary decisions representing the BSA and the restoration of generators/lines/buses of the system [17]–[24]. The optimization objective is to maximize the energization of the system components over a time horizon, or to minimize the load shedding for critical loads, whereas the various constraints ensure that the allocation and restoration plans are feasible.

Both of the aforementioned problems, and possibly others, allow the switching of lines of the power system. It is therefore often necessary to include a constraint to ensure that at all times each island has at least one generator to set up the voltage. More abstractly, the constraint ensures that a graph is partitioned into connected subgraphs, each of which contains a special type of node. In the power systems context, this node is one with an energized generator or corresponding to a set of coherent generators. We will refer to this constraint as the *island energization* (IE) constraint. An example of a system state that violates this requirement is depicted in Figure 1. Network commodity flow formulations have been used to enforce this requirement [1], [2], [6], [8], [10], [11], [18]–[20] and this is currently the most common approach for power systems applications.

A similar requirement appears in other contexts as well. For instance, in [28], the authors propose three formulations to obtain a connected subgraph that includes a set of terminal nodes in a graph, and provide computational results to test the comparative performance and strength of the different formulations. In their problem, they must select a single connected subgraph that includes certain terminal nodes, whereas we can select multiple connected subgraphs (islands) that must contain at least one generator. While projections of the two problems can be shown to be equivalent using the appropriate network transformations, one of their formulations restricts the connected subgraphs to be trees. For similar reasons, our problem also differs from the Steiner Tree problem and multiple generalizations of it (such as the Prize Collecting Steiner Tree Problem [29]), because these problems restrict their connected subgraphs to be trees, whereas our islands can include cycles. Furthermore, all the previous formulations utilize only edge and/or node binary variables. On the other

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hand, we use binary variables for edges, nodes and terminal nodes, since all of these variables have a physical meaning for power systems applications (buses, branches, generators). As a result, the feasible region we describe in our problem includes the feasible regions for most other problems discussed above.

The motivation for studying equivalent reformulations of the same set of constraints is that, despite the continuous improvements of MIP solvers, the solution times highly depend on the problem formulation employed. More specifically, equivalent reformulations of the same requirement can lead to very different solver performances. For a minimization problem, the lower bounds found by the solvers of a MIP are based on solving successive continuous relaxations of the problem, i.e. optimization problems that relax the integrality requirement of some variables. Therefore, equivalent reformulations of the problem with tighter continuous relaxations can lead to better lower bounds and hence a smaller B&B tree. Unfortunately, tighter formulations usually come at the expense of more variables and/or constraints. As a result, there is a computational trade-off between the use of different formulations that has to be resolved based on theoretical and computational results for every particular problem.

The main contributions of this work are the following:

1) We propose three different formulations of the IE constraint using binary variables for generators, buses and branches: a single commodity flow formulation F_1, a multi-commodity flow formulation F_2, and an exponential in size cutset formulation F_3. We show that they are all equivalent, that the Linear Programming (LP) relaxation of F_2 is stronger than that of F_1, and that the LP relaxations of F_2 and F_3 are equally strong. We proceed to propose a polynomial time separation algorithm to identify a violated constraint of the LP relaxation of the exponential in size formulation F_3.
2) We present a new formulation for the variant of the optimal ICI problem considered in [1]. Our formulation has fewer variables and constraints and exhibits a better computational performance for the instances examined.
3) We demonstrate through computational experiments that: (i) For both problems (ICI and BSA), the size of F_2 makes it impractical to use in realistic applications, despite its strength. (ii) For the optimal BSA problem, formulation F_3 is significantly better (at least one order of magnitude faster) than F_1 in the instances considered. (ii) For the optimal ICI problem, despite its strength, F_3 performs no better than F_1 because of the computational overhead of separating the constraints of F_3.

The rest of this paper is organized as follows: section II describes the requirement that the IE constraint imposes; section III sets up the notation for the paper; section IV gives the equivalent reformulations and proves all the theoretical results of the paper; section V presents the computational experiments; and section VI draws conclusions based on the results. The full formulations of the two power systems problems considered are provided in the Appendix.

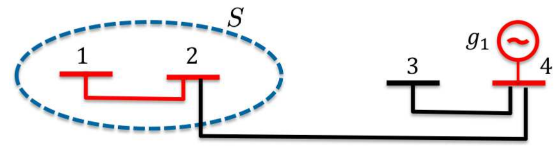

Fig. 1: A small power system with four buses, three branches and one generator. Red color indicates an energized component, whereas black color indicates de-energized components. Note that nodes 1 and 2 and branch $(1, 2)$ form an energized island without any generator, hence this is an infeasible topology that violates the IE constraint.

II. Motivation

The power system restoration (PSR) problem, which is solved for a given allocation of BS units in the system, aims to gradually restore a power system to an operational state after a complete or partial outage. The optimal ICI problem deals with temporarily reconfiguring the grid, by switching lines on and off, as a measure to improve the system security. Both problems involve a series of stepwise actions (usually switchings of lines and generators), while the system moves through a number of different states. Each state can be captured through the status of every bus, line or generator (on or off, i.e. energized or de-energized), as well as through other system characteristics (power flows, generation, etc).

Every step of the process in both problems should respect the island energization (IE) constraint. This requirement may be obvious to the experts that actually perform the switching operations to reconfigure the grid, but when an optimization model is employed to determine a switching plan, the IE constraint has to be imposed. If the ac power flow equations are utilized to model the power system in the optimization model, the IE constraint is implicitly imposed. However, in order to achieve tractability or obtain optimality guarantees, the power flow equations are often relaxed or substituted with approximations and relaxations in the power systems literature which can violate the IE constraint. In such cases, the IE constraint must be explicitly imposed, as in [1], [2], [6], [8], [10], [11], [18]–[20].

III. Notation

Let (N, E) be the directed graph derived from the physical graph of the power system, where buses correspond to nodes (set N) and branches to directed edges (set E). The direction of the edges is defined by arbitrarily picking a "From Node" and "To Node" for every system branch, as is common in power systems literature. Let G be the set of all generators and $G(i)$ the set of generators that are connected to bus $i \in N$. Let u_i denote the energization state of bus $i \in N$ (where $u_i = 1$ indicates an energized bus, whereas $u_i = 0$ indicates a de-energized bus), u_{ij} denote the energization state of branch $(ij) \in E$ ($u_{ij} = 1$ indicates an energized branch) and u_g the energization state of generator $g \in G$ ($u_g = 1$ indicates an energized generator). The energization state of the system is completely described by the binary vector $\mathbf{u} \in \mathbb{B}^{|N| \times |E| \times |G|}$. Finally, if S is a subset of the nodes $S \subseteq N$, the undirected cutset $\delta(S)$ is defined as the set that contains all the edges in E with one node in S and one node not in S, regardless of the direction of the edge.

IV. Formulations

In this section, we describe three different formulations to impose the IE constraint. Constraints (1) and (2) are included in all the formulations.

1) If a generator $g \in G$ connected to node $i \in N$ is energized, then the node is considered energized.

$$u_g \leq u_i, g \in G(i), i \in N \tag{1}$$

2) If a line is energized, both the nodes at its endpoints are considered energized.

$$u_{ij} \leq u_i, u_{ij} \leq u_j, (ij) \in E \tag{2}$$

The first formulation is given by: $F_1 = \{\mathbf{u} \in \mathbb{B}^{|N| \times |E| \times |G|} : \exists f_g \; \forall g \in G, f_{ij} \; \forall (ij) \in E : (1), (2), (3a) - (3c)\}$, where:

$$0 \leq f_g \leq u_g, g \in G \tag{3a}$$

$$-u_{ij} \leq f_{ij} \leq u_{ij}, (ij) \in E \tag{3b}$$

$$\sum_{j:(ji)\in E} f_{ji} - \sum_{j:(ij)\in E} f_{ij} + \sum_{g\in G(i)} f_g = \frac{1}{|N|} u_i, i \in N \tag{3c}$$

This formulation captures exactly the requirement that an energized component should be connected to an energized generator through a path of energized lines. To see that, consider an energized node first. An energized node will act as a sink of $\frac{1}{|N|}$ amount of network flow, due to (3c). Network flow can only be generated from energized generators, due to (3a). Finally, it can only flow through energized lines due to (3b). For example, the topology of Figure 1 is infeasible, since node 1 would act as a source of $1/4$ amount of network flow, but network flow can only be generated at node 4 by the energized generator g_1 and cannot pass through the de-energized line $(2, 4)$. Note that the size of the sinks is $\frac{1}{|N|}$, so that the topology where a single generator (that can generate up to 1 unit of network flow) energizes all the nodes of a (connected) power system belongs in F_1. Now, consider an energized line. Due to (2), both the nodes corresponding to the line would need to be energized, which would mean based on the previous arguments that an energized path leading to them (and hence to the line) exists. This formulation has $|G| + |E|$ flow variables and $|N| + |E| + |G|$ constraints.

An alternative formulation approach, following the same logic, would be to consider a different type of flow corresponding to the energization of each node. This would lead to the following formulation $F_2 = \{\mathbf{u} \in \mathbb{B}^{|N| \times |E| \times |G|} : \exists f_g^k \; \forall k \in N \; \forall g \in G, f_{ij}^k \; \forall k \in N \; \forall (ij) \in E : (1), (2), (4a) - (4d)\}$, where:

$$0 \leq f_g^k \leq u_g, k \in N, g \in G \tag{4a}$$

$$-u_{ij} \leq f_{ij}^k \leq u_{ij}, k \in N, (ij) \in E \tag{4b}$$

$$\sum_{j:(ji)\in E} f_{ji}^k - \sum_{j:(ij)\in E} f_{ij}^k + \sum_{g\in G(i)} f_g^k = u_i, \quad k \in N, i \in N : i = k \tag{4c}$$

$$\sum_{j:(ji)\in E} f_{ji}^k - \sum_{j:(ij)\in E} f_{ij}^k + \sum_{g\in G(i)} f_g^k = 0, \quad k \in N, i \in N : i \neq k \tag{4d}$$

The idea behind this formulation is that each node is treated separately and is associated with its own type of network flow. If node $k \in N$ is energized, then one or more of the energized generators will need to generate the type k network flow, that needs to pass through energized lines. The only sink for that network flow is node k. In this case, the normalization of $\frac{1}{|N|}$ is not necessary in (4c). This formulation has $|N| \times (|G| + |E|)$ flow variables and $|N| \times (|N| + |E| + |G|)$ constraints.

Finally, the third formulation only employs the binary variables, but has an exponential number of constraints. More specifically, $F_3 = \{\mathbf{u} \in \mathbb{B}^{|N| \times |E| \times |G|} : (1), (2), (5)\}$, where:

$$\sum_{(ij)\in\delta(S)} u_{ij} + \sum_{i\in S} \sum_{g\in G(i)} u_g \geq u_n, n \in S, S \subseteq N. \tag{5}$$

The idea behind this formulation is that, given any subset S of the nodes, for any node in that subset to be energized (i.e. for u_n to be equal to 1), either at least one generator in that subset should be energized (i.e. $\sum_{i\in S} \sum_{g\in G(i)} u_g \geq 1$), so that the energizing flow comes from that generator, or at least one edge in the cutset should be energized (i.e. $\sum_{(ij)\in\delta(S)} u_{ij} \geq 1$), so that the energizing flow comes from outside the set S. For example, the topology of Figure 1 is infeasible, since if we pick the set $S = \{1, 2\}$ and the node $n = 1$ with $u_1 = 1$, the left hand side of (5) is zero (since there are no generator nodes in S and the only edge in the cutset has $u_{24} = 0$), while the right hand side is one.

Beyond the above intuitive arguments, we proceed to show that the three formulations are equivalent, i.e. that they represent the exact same binary space.

Proposition 1. *Formulations F_1, F_2 and F_3 are equivalent.*

Proof. We show that $F_1 \subseteq F_3$, $F_3 \subseteq F_2$, and $F_2 \subseteq F_1$.

Part 1. $\mathbf{u} \in F_1 \implies \mathbf{u} \in F_3$

Assume for contradiction that $\mathbf{u} \in F_1$ but $\mathbf{u} \notin F_3$. Based on (5), that means $\exists S_0 \subseteq N, \exists n_0 \in S_0 : \sum_{(ij)\in\delta(S_0)} u_{ij} + \sum_{i\in S_0} \sum_{g\in G(i)} u_g < u_{n_0}$. Since the right hand side is binary, and the left hand side is integer, the only way for strict inequality to hold is if $\sum_{(ij)\in\delta(S_0)} u_{ij} + \sum_{i\in S_0} \sum_{g\in G(i)} u_g = 0$ and $u_{n_0} = 1$. Since the first equality is a sum of non-negative terms equal to zero, each one of them has to equal zero, so we obtain that:

$$u_{ij} = 0, \; (ij) \in \delta(S_0) \tag{6a}$$

$$u_g = 0, \; g \in G(i), i \in S_0 \tag{6b}$$

$$u_{n_0} = 1 \tag{6c}$$

Now, since $\mathbf{u} \in F_1$, by summing over equations (3c) for $i \in S_0$, we obtain:

$$\sum_{i\in S_0} \left(\sum_{j:(ji)\in E} f_{ji} - \sum_{j:(ij)\in E} f_{ij} \right) + \sum_{i\in S_0} \sum_{g\in G(i)} f_g = \frac{1}{|N|} \sum_{i\in S_0} u_i$$

The left hand side (LHS) can be simplified by observing that the sum of the flows inside the set S_0 will cancel each other, while the flows on branches that have only one node in S_0 (i.e. belong in $\delta(S_0)$), are all zero, due to (3b) and (6a).

$$\begin{aligned}\text{LHS} &= \sum_{i\in S_0}\left(\sum_{j:(ji)\in E} f_{ji} - \sum_{j:(ij)\in E} f_{ij}\right) + \sum_{i\in S_0}\sum_{g\in G(i)} f_g \\ &= \sum_{i\in S_0}\sum_{g\in G(i)} f_g \leq \sum_{i\in S_0}\sum_{g\in G(i)} u_g = 0\end{aligned}$$

where the last line uses (3a) and (6b). On the other hand, the right hand side (RHS) yields:

$$\text{RHS} \quad \geq \frac{1}{|N|}u_{n_0} = \frac{1}{|N|}$$

where we used that $n_0 \in S_0$ and the rest of the binary variables in the summation are non negative, together with (6c). Based on the inequalities for the LHS and RHS, we obtain $0 \geq \frac{1}{|N|}$, which is a contradiction.

Part 2. $\mathbf{u} \in F_3 \implies \mathbf{u} \in F_2$

Based on $\mathbf{u}$, construct a directed graph with nodes $N \cup \{t\}$, where t is a dummy node, and edges: for each node $i \in N$ that has at least one generator (i.e. $G(i) \neq \emptyset$), add a directed edge from t to i with capacity $\sum_{g\in G(i)} u_g$, and for each directed edge $(ij) \in E$, add two directed edges, one from i to j and one from j to i, both with capacity u_{ij}. Now pick a node $k \in N$. For $T \subseteq N$ and $S = N \setminus T$, the capacity of any t-k cut is given by (see Figure 2):

$$C(T \cup \{t\}, S) = \sum_{(ij)\in\delta(S)} u_{ij} + \sum_{i\in S}\sum_{g\in G(i)} u_g \tag{7}$$

which is greater than or equal to u_k, since $k \in S$ and $\mathbf{u} \in F_3$. Therefore, the min-cut has capacity $v \geq u_k$, which means the max-flow has capacity v. We can scale all the flows of the max-flow by the positive quantity $\frac{u_k}{v}$, which is less than one, to obtain flows that retain feasibility and inject u_k amount of network flow at node k. Define f_g^k based on the flow on the edge t-i, where $g \in G(i)$ (if more than one generators are connected to bus i, assign to each f_g^k flow proportional to the capacity u_g). For every edge $(ij) \in E$, assign f_{ij}^k equal to the difference of the flows on the arcs in the graph we created, which is guaranteed to be in $[-u_{ij}, u_{ij}]$ due to the feasibility of the max-flows. We can repeat this process for every node $k \in N$, and hence generate feasible flows for formulation F_2, which shows that $\mathbf{u} \in F_2$.

Part 3. $\mathbf{u} \in F_2 \implies \mathbf{u} \in F_1$

Since $\mathbf{u} \in F_2$, based on the multi-commodity flows of formulation F_2, let:

$$\begin{aligned}f_g &= \frac{1}{|N|}\sum_{k\in N} f_g^k, g \in G \\ f_{ij} &= \frac{1}{|N|}\sum_{k\in N} f_{ij}^k, (ij) \in E\end{aligned}$$

It is straightforward to see (by summing over all $k \in N$ the constraints of formulation F_2) that the flows defined above satisfy the constraints of formulation F_1. □

In general, formulations that have a stronger LP relaxation lead to better bounds when solving the underlying MIP using B&B. The relaxation of F_1 is defined as: $\text{LP}(F_1) = \{\mathbf{u} \in [0,1]^{|N|\times|E|\times|G|} : \exists f_g \ \forall g \in G, f_{ij} \ \forall (ij) \in E : (1), (2), (3a) - (3c)\}$, and we have similar definitions for the relaxations of F_2 and F_3. The following propositions give a result for the relative strength of the different formulations.

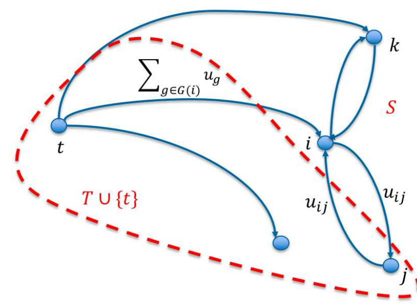

Fig. 2: Graph used in the proofs of Proposition 1, Proposition 3, and Proposition 4.

Proposition 2. *The LP relaxation of F_2 is strictly stronger than the LP relaxation of F_1.*

Proof. We first need to show that $\mathbf{u} \in \text{LP}(F_2) \implies \mathbf{u} \in \text{LP}(F_1)$. To see that, note that the proof employed in Part 3 of Proposition 1 did not use the integrality of the variables in $\mathbf{u}$. Therefore, the same proof can be used to show the inclusion in this case.

To see the strictness of the inclusion, consider a graph with two nodes $N = \{1, 2\}$, one line $E = \{(1, 2)\}$, and one generator $G = \{g_1\}$ connected to node 1. Consider the point $(u_1, u_2, u_{12}, u_{g_1}) = (1, 1, 1/2, 1)$. Picking $f_{g_1} = 1, f_{12} = 1/2$, we can see that $(1, 1, 1/2, 1) \in \text{LP}(F_1)$. However, the point does not belong in $\text{LP}(F_2)$, since for $k = 2$ the line capacity of $1/2$ prevents the 1 unit of type 2 flow to pass from the generator to the sink in node 2. □

Proposition 3. *The LP relaxation of F_2 is the same as the LP relaxation of F_3.*

Proof. To see that $\mathbf{u} \in \text{LP}(F_3) \implies \mathbf{u} \in \text{LP}(F_2)$, notice that the proof in Part 2 of Proposition 1 did not use the integrality of the variables $\mathbf{u}$. Therefore the same proof can be employed to show this result as well.

To see that $\mathbf{u} \in \text{LP}(F_2) \implies \mathbf{u} \in \text{LP}(F_3)$, for each $k \in N$, consider the max-flow problem from t to k in the graph of Figure 2. Due to the constraints in F_2, a feasible flow of at least u_k exists. Therefore, the maximum flow is at least u_k, which means that the minimum cut is at least u_k. That implies that any other cut, whose capacity has the form (7), will be greater than or equal to the minimum cut, so greater than or equal to u_k. Since this holds for all $k \in N$, the constraints of $\text{LP}(F_3)$ are satisfied. □

Finally, the constraints in formulation F_3 are exponentially many. Even though we cannot include all of them in the model that is passed to commercial optimization software, we can actually efficiently identify a violated constraint of $\text{LP}(F_3)$, based on the following proposition. Therefore, we can use solver callbacks to dynamically add the constraints at any point the solver reaches (fractional or not), only if they are violated.

Proposition 4. *Given a point $\mathbf{u} \in [0,1]^{|N|\times|E|\times|G|}$, we can identify a violated constraint from LP(F_3) or verify that none exists (separation problem), in polynomial time.*

Proof. Given a point $\mathbf{u} \in [0,1]^{|N|\times|E|\times|G|}$, for every $k \in N$, construct the graph from Figure 2. Then find the minimum $t-k$ cut in this graph and compare the value to u_k. Given the fact that the capacity of a cut in that graph has the form (7), there are two cases for the capacity of the minimum cut C^k_{min}:

1) If $C^k_{\text{min}} < u_k$, for some $k \in N$, then node k and the minimum cut set S_{min} yield a violated constraint in LP(F_3).
2) If $C^k_{\text{min}} \geq u_k$, for all $k \in N$, then all constraints in LP(F_3) are satisfied.

Since the min-cut problems can be solved in polynomial time, and we only need to solve at most $|N|$ of them, the separation problem can be solved in polynomial time. Note that, if the point is integral, a graph traversal to identify the islands and check if there exists one without a generator, is enough to identify if all constraints are satisfied (if we find an island with no generator, we can then generate a violated constraint with S corresponding to that island). □

V. Simulation Results

All optimization problems were formulated using Gurobi with Python. Each simulation was executed at a single node of the Lawrence Livermore National Laboratory quartz server (128 GB RAM, 2.1 GHz, 18 CPUs). For formulation F_3 all constraints with $|S| = 1$ were *a priori* included in the formulation, and the rest were lazily added using incumbent callbacks, i.e. each candidate incumbent found by the solver is checked for feasibility and a lazy constraint is added if the candidate is infeasible. A $20,000$s time limit was set to the solver and a 1% optimality gap termination was considered (i.e. a solution guaranteed to be within 1% of the optimal is denoted as optimal) for all simulations.

A. Optimal Intentional Controlled Islanding

The most common practice behind ICI is that the generators of the grid can be divided into groups of coherent generators, based on their relative angle response to a disturbance. By isolating unstable generators or grouping together only generators that are coherent to each other, a cascaded outage may be avoided. In [1], an optimal ICI model was devised to propose switching actions. The goal was to create a partition of the grid into islands of coherent generators with minimal power-flow disruption.

The formulation provided in [1], which we denote with F_4, utilizes a mixed integer program to identify the optimal islanding. The authors of [1] deal with the intractability of formulation F_4 by constructing a heuristic based on LP relaxations. In this paper, we present equivalent formulations of F_4 using the IE constraint formulations F_1, F_2 and F_3; see Appendix A for the full formulations.

In order to perform a computational comparison between the different formulations of the problem, we considered two test systems: the IEEE-118 bus system (118 buses, 186 branches, 54 generators) and an instance of the Polish system (3374 buses, 4161 branches, 596 generators). Due to the size of the resulting model and the memory limitations, formulation (F_2) was impractical and was not implemented.

Optimal ICI	B&B Nodes	Gap [%]	UB	LB	Time [s]
IEEE-118	$K=4$				
F_1	1	Optimal	3.6846	3.6846	0.16
F_3	328	Optimal	3.6846	3.6846	1.03
F_4	1	Optimal	3.6846	3.6846	0.94
Polish	$K=3$				
F_1	3	Optimal	17.7717	17.6123	5
F_3	1	Optimal	17.7717	17.7710	29
F_4	1	Optimal	17.7717	17.7717	509
	$K=4$				
F_1	13	Optimal	27.3901	27.3901	12
F_3	1	Optimal	27.4225	27.2373	36
F_4	2635	Optimal	27.3901	27.3901	3098
	$K=5$				
F_1	32	Optimal	44.9159	44.9159	13
F_3	1	Optimal	44.9508	44.8052	34
F_4[1]	5138	1.46	44.9159	44.2587	20000

TABLE I: Optimal ICI for the IEEE-118 and Polish systems, splitting the system into K islands.

The results for the ICI problem are presented in Table I. The IEEE-118 system solves to optimality relatively quickly for all three formulations considered. For the Polish system, we observe that F_4 seems to be two orders of magnitude slower than F_1. The exponential formulation F_3 performs slightly worse than F_1 (with, in fact, more than 80% of its simulation time spent in separating the violated constraints). In this case the trade-off between a tighter formulation and separation of an exponential number of constraints is inconclusive.

B. Optimal Black Start Allocation

The optimal BSA problem aims to allocate the black start resources in the grid in a way that will ensure an efficient restoration of the system in the case of a blackout. The formulation used in this paper is a simplified adaptation of the one explained in [19]. The main simplification employed is that, instead of detailed nodal power balance equations and active and reactive power flow approximations, we only use one aggregate constraint for the active power balance and one for the reactive power capability at every time step of the underlying restoration problem (given the BSA). This is a typical assumption [17], [23], made mostly to ensure tractability of the problem. The complete formulation used in this paper can be found in Appendix B.

We use three test systems: the IEEE-39, the IEEE-118, and the IEEE-300 power systems. The results of the simulations are shown in Table II. The IEEE-39 bus problem solves to optimality relatively quickly for all formulations. Formulation F_2 is slower due to its size (every linear program solved at intermediate steps takes more time), however the number of explored B&B nodes for the solution is smaller (due to the strength of the formulation).

In the larger systems considered, the performance of the formulations is very different. For the IEEE-118 system, F_1 and F_2 are unable to solve the problem to optimality within the

Optimal BSA	B&B Nodes	Gap [%]	UB	LB	Time [s]
IEEE-39					
F_1	1335	Optimal	1550	1536	2
F_2	640	Optimal	1536	1533	22
F_3	108	Optimal	1545	1530	1
IEEE-118					
F_1	474738	4.81	5946	5673	20000
F_2	3177	1.30	5730	5656	20000
F_3	4624	Optimal	5721	5670	99
IEEE-300					
F_1	203933	2.86	12873	12515	20000
F_2[1]	1028	2.15	12840	12570	20000
F_3	5172	Optimal	12668	12593	108

TABLE II: Optimal BSA results for the IEEE-39, IEEE-118 and IEEE-300 systems.

time limit of $20,000$s, whereas F_3 easily achieves a solution with the desired gap within 99 seconds. F_2 performs better than F_1, however due to its size it is unable to explore enough nodes in the B&B tree to reduce the optimality gap to the desired levels. Finally, for the IEEE-300 system F_3 is able to yield an optimal solution in around 100 seconds. F_1 only achieves a 2.86% solution and F_2 only a 2.15% solution in more than 100-fold the time it took for F_3 to optimally solve the problem.

The situation for the optimal BSA problem was different. Formulation F_2 was still impractical, due to its size. However, F_3 significantly outperformed F_1, even though it employed an exponential number of constraints. The reasons for that are twofold: firstly, formulation F_3 employs stronger constraints, as shown in this paper. Secondly, the constraints of F_3 other than those with $|S| = 1$ are rarely violated, so candidates found by solver heuristics early on the tree are often feasible and optimal. The intuition behind this is that for the restoration problem we expect that the direction of the problem is such that edges get energized (rather than de-energized). If lines are mostly getting energized around energized generators, the situation where islands without an energized generator will show up are actually rare, so lazily generating the cuts for F_3 is faster than including the entire formulation F_1 in the optimization solver.

VI. Conclusions

The main message of this work is that, while it might be easy to formulate a power systems problem as a MIP, choosing the right way to formulate the problem can make a significant difference in the solution times. Among others, some issues to consider when selecting the right formulation are the tightness of the formulation (i.e. how tight the relaxation of the feasible region is around the integral points) and the size of the formulation (number of constraints and variables). There is usually a trade-off between the size and the tightness of the formulation, that can be resolved in practice for each problem. Also, even if a problem is intractable in practice with one formulation, a reformulation could yield an acceptable computational performance. We presented empirical proof of all the aforementioned points by examining two problems in power systems.

For the ICI problem, even though formulations F_4 and F_1 are equivalent and are both employing single commodity flow to enforce connectivity, their computational performance was very different, since F_1 is more compact than F_4 (fewer variables and constraints). The problem formulated using F_4 solved orders of magnitude slower compared to F_1. The size of F_2 made it completely intractable. Finally, even though F_3 is imposing tighter constraints in general, the exponential size of the formulation (that forced the implementation to employ lazy constraints added through callbacks) made the computational performance of the formulation slightly worse that the weaker F_1.

[1]These instances caused the B&B tree to run out of memory due to the size of the problem. We present results using settings that limit the solver's memory use by restricting the number of threads to 4 and using the file system as temporary storage.

Appendix A
Optimal Islanding Formulation

The optimal ICI formulation presented here is an equivalent reformulation of the problem formulated in [1], based on a previous work by the same authors in [30]. The goal is to find a minimum cost partitioning of the grid to islands given a partitioning of the generators. More specifically, the generators are divided into $|K|$ subsets of coherent generators $G_h, h \in K$, i.e. of generators that will be in the same island after the reconfiguration of the grid and in different islands from the generators of the other subsets. We note that our single commodity flow formulation approach is similar in nature to the model used in [1], with the exception that we make use of the fact that the generators in each coherent group are forced to belong to the corresponding partition, so one of them can be used as the source of the network flow.

Let $i \in N$ denote a bus of the system, $(ij) \in E$ a branch, and $g \in G$ a generator. We denote the generators connected to node i with $G(i)$. Also, for each set G_h, denote one of the generators (assumed to be the isochronous one, even though the specific choice is not important) with $G_h^{(0)} \in G_h$. We denote the node that generator g is connected by $n(g)$. Let the binary variables $u_i^h/u_{ij}^h/u_g^h$ denote (if equal to 1) that node i/branch (ij)/generator g belongs to partition $h \in K$. Let the binary variable z_{ij} denote that branch (ij) is switched on (i.e. it belongs to some partition). Let the variables f_{ij}^h, f_g^h denote the network flows that will ensure the connectivity of partition $h \in K$. Finally, there is a cost d_{ij} associated with switching off branch (ij) and a minimum size requirement M of every bus set in the partition.

$$\text{minimize} \quad \sum_{(ij)\in E} \frac{1}{2} d_{ij}(1 - z_{ij})$$

s.t.

$$z_{ij} = \sum_{h\in K} u_{ij}^h, (ij) \in E \tag{8a}$$

$$\sum_{h\in K} u_i^h = 1, i \in N \quad (8b)$$

$$\sum_{i\in N} u_i^h \geq M, h \in K \quad (8c)$$

$$u_{ij}^h \leq u_i^h, u_{ij}^h \leq u_j^h, (ij) \in E, h \in K \quad (8d)$$

$$u_g^h \leq u_i^h, g \in G(i) \quad (8e)$$

$$0 \leq f_g^h \leq u_g^h,\ g \in G,\ h \in K, \quad (8f)$$

$$-u_{ij}^h \leq f_{ij}^h \leq u_{ij}^h,\ (ij) \in E,\ h \in K, \quad (8g)$$

$$\sum_{j:(ji)\in E} f_{ji}^h - \sum_{j:(ij)\in E} f_{ij}^h + \sum_{g\in G(i)} f_g^h = \frac{1}{N} u_i^h, \quad i \in N,\ h \in K \quad (8h)$$

$$u_g^h = \begin{cases} 1, & \text{if } g = G_h^{(0)} \\ 0, & \text{otherwise} \end{cases}, h \in K \quad (8i)$$

$$u_{n(g)}^h = 1,\ g \in G_h, h \in K \quad (8j)$$

The objective of the problem minimizes the weighted cost of switching off edges. Constraint (8a) imposes that a bus is switched on if it belongs in one of the partitions. Constraint (8b) ensures that each bus is assigned to exactly one partition, constraint (8c) ensures that each partition has at least M buses, constraints (8d) require that if a branch belongs to a partition, both its endpoints belong to it, constraint (8e) ensures that if a generator belongs to partition h, the node it is connected to will belong to the same partition. Constraints (8f)-(8h), together with (8d) and (8e), are the IE constraints of formulation F_1 imposed for every partition h. These constraints can be equivalently substituted with the constraints of F_2 or F_3, as we have shown in this paper. Finally, (8i) allows only the generator $G_h^{(0)}$ to generate the network flow that ensures the connectivity of each partition h and (8j) forces each node of the coherent generators to belong to the corresponding partition.

Appendix B
Optimal Black Start Allocation Formulation

The optimal BSA problem aims to allocate black start units across the grid so that an efficient system restoration is ensured after a blackout. The formulation used in this paper is a simplified adaptation of the formulation in [19]. Let $t \in T$ denote the set of time instances, starting from $1 \in T$, $g \in G$ the generators, $i \in N$ the buses, and $(ij) \in E$ the branches. We denote the generators connected to node i with $G(i)$. The binary variables u_g^t, u_i^t, u_{ij}^t denote the energization (when set to 1) of generator g/bus i/branch (ij) at time t, and u_g^0, u_i^0 denote the initial state of the generator g and bus i (here assumed zero for all components, i.e. we examine a total blackout). The variables f_g^t, f_{ij}^t denote the network flows of the F_1 formulation, p_g^t is the active generation of generator g, and p_{SH_i} is the load shed at bus i. Finally, the parameters of the problem are the cost C_{BS_g} of allocating generator g to be a BS unit, the total allocation budget B, the generator capability $\overline{P}_g$, cranking time T_{CR_g}, cranking power P_{CR_g}, ramping rate K_{R_g}, and minimum reactive power capability $\underline{Q}_g$, the bus load P_{D_i}, angles ϕ_{D_i} and shunt reactance Q_{SH_i} and the branch susceptance $B_{SH_{ij}}$.

$$\text{maximize} \quad \sum_{i\in N, t\in T} u_i^t + \sum_{(ij)\in E, t\in T} u_{ij}^t + \sum_{g\in G, t\in T} u_g^t$$

s.t.

$$\sum_{g\in G} C_{BS_g} u_{BS_g} \leq B \quad (9a)$$

$$u_g^0 = 0,\ g \in G \quad (9b)$$

$$u_i^0 = 0,\ i \in N \quad (9c)$$

$$u_{ij}^t \leq u_i^{t-1} + u_j^{t-1},\ (ij) \in E,\ t \in T \quad (9d)$$

$$u_i^t \geq u_i^{t-1},\ i \in N,\ t \in T \quad (9e)$$

$$u_g^t \geq u_{BS_g},\ (ij) \in E,\ t \in T \quad (9f)$$

$$u_g^t \leq u_i^t,\ i \in N,\ g \in G(i),\ t \in T \quad (9g)$$

$$u_{ij}^t \leq u_i^t, u_{ij}^t \leq u_j^t,\ (ij) \in E,\ t \in T \quad (9h)$$

$$0 \leq f_g^t \leq u_g^t,\ g \in G,\ t \in T, \quad (9i)$$

$$-u_{ij}^t \leq f_{ij}^t \leq u_{ij}^t,\ (ij) \in E,\ t \in T, \quad (9j)$$

$$\sum_{j:(ji)\in E} f_{ji}^t - \sum_{j:(ij)\in E} f_{ij}^t + \sum_{g\in G(i)} f_g^t = \frac{1}{N} u_i^t, \quad i \in N,\ t \in T \quad (9k)$$

$$0 \leq p_g^\tau \leq \overline{P}_g u_g^t,\ g \in G,\ \tau \in \{t, t+1, \ldots, t + T_{CR_g} + 1\}, \quad t \in T \cup \{0\} \quad (9l)$$

$$p_g^t - p_g^{t-1} \leq K_{R_g},\ g \in G,\ t \in T \quad (9m)$$

$$p_g^{t-1} - p_g^t \leq K_{R_g},\ g \in G,\ t \in T \quad (9n)$$

$$\sum_{g\in G} \left(p_g^t + P_{CR_g}(u_{BS_g} - u_g^t)\right) = \sum_{i\in N} \left(P_{D_i} - p_{SH_i}^t\right), \quad t \in T \quad (9o)$$

$$(1 - u_i^t) P_{D_i} \leq p_{SH_i}^t \leq P_{D_i},\ i \in N,\ t \in T \quad (9p)$$

$$\sum_{i\in N} \sum_{g\in G(i)} \underline{Q}_g u_g^{\max\{0, t - T_{CR_g} - 1\}} + \sum_{(ij)\in E} B_{SH_{ij}} u_{ij}^t + \sum_{i\in N} Q_{SHi} u_i^t - \sum_{i\in N} (P_{D_i} - p_{SH_i}^t) \tan(\phi_{D_i}) \leq 0, t \in T \quad (9q)$$

The objective of the problem maximizes the energization of the system components, (9a) imposes the allocation budget, (9b) and (9c) initialize the outage, (9d) allows the energization of a bus only if at least one of its endpoints was energized at the previous time step, (9e) forces the nodes to stay energized after their initial energization, (9f) forces a black start generator to get energized, (9g) forces the energization of a bus if one of its generators is energized, (9h) impose that if an edge is energized both of its endpoints should be energized. Constraints (9g)-(9k) are the constraints of formulation F_1 for illustration (which, as proven, can be replaced equivalently with F_2 or F_3), repeated for every time step t. Constraints (9l)- (9n) define the generator startup capability curve. Finally, constraints (9o)-(9q) impose aggregate active and reactive power capacity constraints. The underlying assumptions and justification behind this formulation of the problem can be found in [19].

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