Chvátal rank in binary polynomial optimization *

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Abstract

Recently, several classes of cutting planes have been introduced for binary polynomial optimization. In this paper, we present the first results connecting the combinatorial structure of these inequalities with their Chvátal rank. We show that almost all known cutting planes have Chvátal rank 1. All these inequalities have an associated hypergraph that is \( \beta \)-acyclic, thus, in order to derive deeper cutting planes, we consider hypergraphs that contain \( \beta \)-cycles. In particular, we introduce a novel class of valid inequalities arising from odd \( \beta \)-cycles, that generally have Chvátal rank 2. These cuts subsume odd-cycle inequalities for binary quadratic optimization. We then provide an indication of their theoretical power by showing that they yield the multilinear polytope of cycle hypergraphs. To obtain this result, we introduce a novel proof technique that allows us to iteratively obtain multilinear polytopes of increasingly complex hypergraphs.

Key words: Binary polynomial optimization; Chvátal-Gomory cuts; Chvátal rank; Integer nonlinear optimization; Polyhedral relaxations; Multilinear polytope.

1 Introduction

In recent work, Del Pia and Khajavirad introduced the multilinear polytope [7]. In order to define it, let \( V \) be a ground set, let \( E \) be a set of subsets of cardinality at least two of \( V \), and denote by \( G \) the hypergraph \((V, E)\). The multilinear polytope of \( G \), denoted by \( \text{MP}_G \), consists of the convex hull of the binary points that satisfy \( z_e = \prod_{v \in e} z_v \) for every \( e \in E \). The combinatorial structure of the multilinear polytope is highlighted by the fact that its face defined by \( z_e = 0, \forall e \in E \) is an affine transformation of the set covering polytope.

The multilinear polytope plays a fundamental role in integer programming. In fact, the problem of minimizing a multivariate polynomial function over all binary points can be reformulated as a linear program over the multilinear polytope. In this reformulation, each \( v \in V \) corresponds to a variable of the original polynomial problem, and each \( e \in E \) corresponds to a nonlinear monomial in the original objective function.

The strong NP-hardness of binary polynomial programming [13] indicates the high complexity of the multilinear polytope. A much simpler polyhedral relaxation of the multilinear polytope is the so-called standard linearization, denoted by \( \text{MP}_G^{LP} \), and defined by

\[
 z_v \leq 1 \quad \forall v \in V \quad (\alpha_v)
\]

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\[-z_e \leq 0 \quad \forall e \in E \quad (\beta_e)\]
\[\sum_{v \in e} z_v - z_e \leq |e| - 1 \quad \forall e \in E \quad (\varepsilon_e)\]
\[-z_v + z_e \leq 0 \quad \forall v \in e, \forall e \in E. \quad (\delta_{v,e})\]

Since the binary points in $\text{MP}^\text{LP}_G$ coincide with the vertices of $\text{MP}_G$, our binary polynomial program can be now reformulated as an integer linear program over the standard linearization.

In order to obtain tighter polyhedral relaxations of the multilinear polytope, several classes of valid inequalities for $\text{MP}_G$ have been defined (see [7, 6, 9, 8, 11]). These cutting planes have been shown to drastically improve the performance of global solvers [12]. All these inequalities have been derived by directly exploiting the combinatorial nature of the multilinear polytope. In this paper, we take a different approach. Namely, we leverage both the integer programming and combinatorial optimization aspects of the multilinear polytope in order to provide the first links between valid inequalities for $\text{MP}_G$ and Chvátal-Gomory (CG) cuts [17]. CG cuts provide a fundamental class of valid inequalities for general integer programming problems, and have been the subject of extensive research (see, e.g., [17, 4] and references therein). These inequalities are used by all high performance algorithms for solving integer programs, and have been one of the reasons for the great leap in the success of solvers to handle real-world problems in the past 20 years [15].

In Section 2, we show that running intersection inequalities [11] are CG cuts for $\text{MP}^\text{LP}_G$. Since running intersection inequalities subsume 2-link inequalities [6] and flower inequalities [8], this result shows that almost all cutting planes defined so far in the literature are CG cuts for $\text{MP}^\text{LP}_G$. In addition, our result implies that $\text{MP}^\text{LP}_G$ has Chvátal rank 1 when $G$ is $\beta$-acyclic with the simple intersection property, a class introduced in [11] that includes $\gamma$-acyclic hypergraphs. To obtain this result, we heavily exploit the running intersection property, which has been extensively used in the database and machine learning communities [2, 14]. We refer the reader to [8] for an overview of the various types of cycles in a hypergraph.

All running intersection inequalities correspond to $\beta$-acyclic hypergraphs. Thus, in order to derive cutting planes with higher Chvátal rank, in Section 3 we consider hypergraphs that contain $\beta$-cycles. In particular, we introduce the odd $\beta$-cycle inequalities, a novel class of valid inequalities for $\text{MP}_G$, arising from odd $\beta$-cycles, that generally have Chvátal rank 2. These inequalities generalize odd-cycle inequalities for the Boolean quadric polytope [16] as well as their lifting by node addition obtained in [7]. Given the computational impact of odd-cycle inequalities in the quadratic setting [3], we believe that our odd $\beta$-cycle inequalities could lead to a significant speed-up in state-of-the-art solvers.

Section 4 and Section 5 are devoted to attest the theoretical power of odd $\beta$-cycle inequalities. In Section 4, we introduce a technique which allows us to exploit a description of any multilinear polytope $\text{MP}_G$ to obtain a description of $\text{MP}_{G'}$, where $G'$ is a new hypergraph obtained from $G$ by replacing any node with a new edge. This technique provides a general way to iteratively extend convex hull characterizations and decomposability results, and in particular allows us to extend all known decomposability results for the multilinear polytope [9, 8, 11]. In Section 5, we make use of this novel technique to derive an explicit characterization of $\text{MP}_G$ when $G$ is a cycle hypergraph. We show that, for these hypergraphs, $\text{MP}_G$ is given by the standard linearization, flower inequalities, and odd $\beta$-cycle inequalities. Hence, in this case, $\text{MP}^\text{LP}_G$ has Chvátal rank at most 2. We remark that this result provides the first characterization of $\text{MP}_G$ in the case where $\beta$-cycles are present in $G$. 
2 Running intersection inequalities are CG cuts

In this section we analyze the Chvátal rank of running intersection inequalities, a class of valid inequalities for the multilinear polytope introduced in [11]. First, we give the formal definition of running intersection inequalities.

A multiset \( F \) of subsets of a finite set \( V \) has the running intersection property if there exists an ordering \( p_1, p_2, \ldots, p_m \) of the sets in \( F \) such that, for each \( i = 2, \ldots, m \), there exists \( j < i \) such that \( p_i \cap (\bigcup_{k<i} p_k) \subseteq p_j \). An ordering \( p_1, p_2, \ldots, p_m \) satisfying the above condition is called a running intersection ordering of \( F \). Each running intersection ordering \( p_1, p_2, \ldots, p_m \) of \( F \) induces a collection of sets

\[
N(p_1) := \emptyset, \quad N(p_i) := p_i \cap \left( \bigcup_{k<i} p_k \right) \quad \text{for} \; i = 2, \ldots, m.
\]

In the reminder of the paper, for a non-negative integer \( m \), we denote by \([m]\) the set \( \{1, \ldots, m\} \).

**Definition 1.** Consider a hypergraph \( G = (V, E) \). Let \( e_0 \in E \) and let \( e_i, i \in [m] \), be a collection of edges in \( E \), adjacent to \( e_0 \), such that the multiset \( \tilde{E} := \{e_0 \cap e_i : i \in [m]\} \) has the running intersection property. Consider a running intersection ordering of \( \tilde{E} \) with the corresponding sets \( N(e_0 \cap e_i), i \in [m] \). For each \( i \in [m] \) with \( N(e_0 \cap e_i) \neq \emptyset \), let \( u_i \) be a node in \( N(e_0 \cap e_i) \). We define a running intersection inequality as

\[
- \sum_{i \in [m]} z_{u_i} + \sum_{e \in e_0 \setminus \bigcup_{i \in [m]} e_i} z_v + \sum_{i \in [m]} z_{e_i} - z_{e_0} \leq n_0 + \left| \{i \in [m] : N(e_0 \cap e_i) = \emptyset \} \right| - 1, \quad (1)
\]

where \( n_0 \) is the number of nodes in \( e_0 \) not contained in any edge \( e_i, i \in [m] \). We refer to \( e_0 \) as the center and to \( e_i, i \in [m] \), as the neighbors.

**Theorem 1.** Running intersection inequalities are CG cuts for \( M_P^{LP}_G \).

**Proof sketch.** In order to make this proof simpler to describe, we will be using the following class of redundant valid inequalities for \( M_P^{LP}_G \):

\[
z_e \leq 1 \quad \forall e \in E. \quad (\eta_e)
\]

Consider a running intersection inequality \( \text{(1)} \). If \( m = 0 \), then the inequality is in the standard linearization, hence we assume \( m \geq 1 \). To show that it is a CG cut for \( M_P^{LP}_G \), we provide a non-negative combination of the inequalities \( (\alpha_{v, e}), (\beta_{v, e}), (\varepsilon_e), (\delta_{v, e}) \), \( (\eta_e) \), that we denote by \( \pi z \leq \pi_0 \). We prove that \( \pi z \) coincides with the left-hand side of \( \text{(1)} \), and that \( |\pi_0| \) is equal to the right-hand side of \( \text{(1)} \). We denote by \( \alpha_v, \beta_v, \varepsilon_e, \delta_{v, e}, \eta_e \) the multipliers associated with the inequalities \( (\alpha_{v, e}), (\beta_{v, e}), (\varepsilon_e), (\delta_{v, e}), (\eta_e) \), respectively.

Let \( e_0 \) and \( \tilde{E} \) be as in Definition \( \text{(1)} \) and let \( e_0 \cap e_1, e_0 \cap e_2, \ldots, e_0 \cap e_m \) be a running intersection ordering of the multiset \( \tilde{E} \). We partition the nodes of \( e_0 \cap \bigcup_{i \in [m]} e_i \) in two sets \( U, W \), where \( U = \{u_i : i \in [m], N(e_0 \cap e_i) \neq \emptyset\} \) contains the nodes whose variables belong to the first sum of \( \text{(1)} \), and \( W = (e_0 \cap \bigcup_{i \in [m]} e_i) \setminus U \). Define \( \gamma := \left| e_0 \cap \bigcup_{i \in [m]} e_i \right| \geq 1 \). Then, the multipliers are defined as follows. For ease of exposition, all multipliers not explicitly defined are set to zero.

1. Choose \( \varepsilon_{e_0} := \frac{1}{7}, \beta_{e_0} := 1 - \frac{1}{7}, \alpha_v := 1 - \frac{1}{v} \), for every \( v \in e_0 \setminus \bigcup_{i \in [m]} e_i \).

Next, we define the multipliers \( \delta_{v, e}, \eta_e \) of the edges \( e_1, \ldots, e_m \) recursively.
2 Going backwards, consider \( e_i \) with \( i = m, \ldots, 1 \).

2.1 Fix \( \delta_{w,e_i} := \frac{1}{\gamma} \) for every \( w \in W \cap e_i \setminus \bigcup_{j>i} e_j \).

2.2 For each index \( j > i \) such that \( N(e_0 \cap e_j) \neq \emptyset \), \( u_j \in e_i \), and \( u_j \notin e_\ell \) for \( \ell = i+1, \ldots, j-1 \), then set \( \delta_{u_j,e_i} := 1 - (\delta_{u_j,e_j} - \frac{1}{\gamma}) \).

2.3 If \( N(e_0 \cap e_i) \neq \emptyset \), then select \( \delta_{u_i,e_i} := 1 - \sum_{v \in e_i \setminus \{u_i\}} \delta_{v,e_i} \).

Otherwise assign \( \eta_{e_i} := 1 - \sum_{v \in e_i} \delta_{v,e_i} \).

It can be verified that the multipliers are non-negative, that \( \pi z \) is equal to the left-hand side of (1), and that \( \lfloor \pi_0 \rfloor \) coincides with the right-hand side of (1). The proof of these facts requires some involved steps that rely on the combinatorial structure of the support hypergraph of (1). The complete proof can be found in Appendix A. \( \square \)

We remark that Theorem 1 implies an alternative proof of the validity of running intersection inequalities. As running intersection inequalities subsume 2-link inequalities [6] and flower inequalities [5]. Theorem 1 implies that also these cutting planes are CG cuts for MP\(_G^\text{LP} \). In [11], the authors introduce \( \beta \)-acyclic hypergraphs with the simple intersection property, a class that in particular contains all \( \gamma \)-acyclic hypergraphs. The authors further show that, for hypergraphs in this class, the multilinear polytope coincides with the running intersection relaxation. As a consequence, Theorem 1 implies that MP\(_G^\text{LP} \) has Chvátal rank 1 when \( G \) is \( \beta \)-acyclic with the simple intersection property.

3 Odd \( \beta \)-cycle inequalities

In this section, our aim is to introduce valid inequalities for MP\(_G \) that are deeper, in the sense that their Chvátal rank can be larger than 1. Since all the inequalities considered in Section 2 correspond to \( \beta \)-acyclic hypergraphs, we decide to consider here hypergraphs that contain \( \beta \)-cycles. We start by defining our odd \( \beta \)-cycle inequalities. In the remainder of the paper, given a \( \beta \)-cycle \( C = v_1, e_1, v_2, \ldots, v_m, e_m, v_1 \) in a hypergraph \( G = (V,E) \), we denote by \( V(C) = \{v_1, v_2, \ldots, v_m\} \subseteq V \), and by \( E(C) = \{e_1, \ldots, e_m\} \subseteq E \).

**Definition 2.** Consider a hypergraph \( G = (V,E) \), let \( C = v_1, e_1, v_2, \ldots, v_m, e_m, v_1 \) be a \( \beta \)-cycle in \( G \), and let \( E^- \), \( E^+ \) be a partition of \( E(C) \) such that \( k = |E^-| \) is odd and \( e_1 \in E^- \). Let \( D := \{e_{p+1}, e_{p+2}, \ldots, e_m\} \), where \( e_p \) is the last edge in \( C \) that belongs to \( E^- \). We denote by \( f_1, \ldots, f_k \) the subsequence of \( e_1, \ldots, e_m \) of the edges in \( E^- \). Let \( S_1 = (\bigcup_{e \in E^-} e) \setminus \bigcup_{e \in E^+} e \) and \( S_2 = V(C) \setminus \bigcup_{e \in E^-} e \). With this notation in place, we make the following assumptions:

(a) All \( \gamma \)-cycles of \( G \) that can be formed using edges of \( C \) are \( \beta \)-cycles.

(b) For every edge \( e_i \in E^+ \setminus D \), every edge in \( E^- \) adjacent to \( e_i \) (if any) is either \( e_i-1 \) or \( e_i+1 \).

(c) No edge in \( D \) is adjacent to an edge \( f_i \) with \( i \) even.

(d) At least one of the following two conditions holds:

\( \text{(d-1)} \) For every \( v \in S_1 \), either \( v \) is contained in just one edge \( e \in E^- \), or it is contained in two edges \( f_i, f_j \) with \( i \) odd and \( j \) even.

\( \text{(d-2)} \) For every \( e' \in E^- \) and \( e'' \in D \) such that \( e' \cap e'' \neq \emptyset \), then either \( e' = e_1, e'' = e_m \) or \( e' = e_p, e'' = e_{p+1} \).

We then define the odd \( \beta \)-cycle inequality corresponding to \( C \) and \( E^- \) as

\[
\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{k}{2} \right\rfloor. \tag{2}
\]
is satisfied and (d-1) is not. The corresponding odd \( \beta \) inequalities (a)–(d) in Definition 2; in particular, assumption (d-1) is satisfied and (d-2) is not.

If we define edge \( e_1 \) has nonempty intersection only with \( e_{i-1} \) and \( e_{i+1} \) for every \( i \in \{1, \ldots, t\} \), where, for convenience, we define \( e_{m+1} = e_1 \) and \( e_0 = e_m \). In this paper, we refer to a hypergraph of this type as a cycle hypergraph. If \( G \) is a cycle hypergraph, we now explain why assumptions (a)–(d) are trivially satisfied. 

Example 1. Let \( G = (V, E) \) be the hypergraph depicted in Figure 1 and defined by

\[
V = \{v_1, \ldots, v_9\}, \quad E = \{e_{1278}, e_{23}, e_{349}, e_{4578}, e_{569}, e_{16}\},
\]

where the edge \( e_1 \) contains the nodes with indices in \( I \).

Note that a \( \beta \)-cycle in \( G \) is given by

\[
C = v_1, e_{1278}, v_2, e_{23}, v_3, e_{349}, v_4, e_{4578}, v_5, e_{569}, v_6, e_{16}, v_1.
\]

If we define \( E^- = \{e_{1278}, e_{23}, e_{349}\} \), it is simple to check that \( C \) and \( E^- \) satisfy the assumptions (a)–(d) in Definition 2; in particular, assumption (d-1) is satisfied and (d-2) is not. The corresponding odd \( \beta \)-cycle inequality (2) is

\[
z_{v_2} + z_{v_3} - z_{e_{1278}} - z_{e_{23}} - z_{e_{349}} - z_{v_5} - z_{v_6} + z_{e_{4578}} + z_{e_{569}} + z_{e_{16}} \leq 1. \tag{3}
\]

Using the same cycle \( C \), we could define instead \( E^- = \{e_{1278}, e_{23}, e_{4578}\} \). Also in this case, \( C \) and \( E^- \) satisfy the assumptions (a)–(d) in Definition 2. However, in this case, assumption (d-2) is satisfied and (d-1) is not. The corresponding odd \( \beta \)-cycle inequality (2) is

\[
z_{v_2} + z_{v_7} + z_{v_8} - z_{e_{1278}} - z_{e_{23}} - z_{e_{4578}} - z_{v_6} + z_{e_{349}} + z_{e_{569}} + z_{e_{16}} \leq 3. \tag{4}
\]

Later in this section we will show the following result.

**Theorem 2.** Odd \( \beta \)-cycle inequalities (2) are inequalities of Chvátal rank at most 2 for \( MP^L_P \). In particular, they are valid for \( MP_G \).

While the definition of odd \( \beta \)-cycle inequalities is not straightforward, it yields a large class of inequalities that contains odd-cycle inequalities for the Boolean quadric polytope \([10]\), and their lifting by node addition obtained in Corollary 10 in \([7]\). In particular, the inequalities given in \([12]\) have the same form (2) of our odd \( \beta \)-cycle inequalities, but they are defined only in the special case where the hypergraph \( G = (V, E) \), with edges \( e_1, \ldots, e_m \), satisfies \( m \geq 3 \), and every edge \( e_i \) has nonempty intersection only with \( e_{i-1} \) and \( e_{i+1} \) for every \( i \in \{1, \ldots, t\} \), where, for convenience, we define \( e_{m+1} = e_1 \) and \( e_0 = e_m \). In this paper, we refer to a hypergraph of this type as a cycle hypergraph. If \( G \) is a cycle hypergraph, we now explain why assumptions (a)–(d) are trivially satisfied. 

- **Example 1.** Let \( G = (V, E) \) be the hypergraph depicted in Figure 1 and defined by

\[
V = \{v_1, \ldots, v_9\}, \quad E = \{e_{1278}, e_{23}, e_{349}, e_{4578}, e_{569}, e_{16}\},
\]

where the edge \( e_1 \) contains the nodes with indices in \( I \).

Note that a \( \beta \)-cycle in \( G \) is given by

\[
C = v_1, e_{1278}, v_2, e_{23}, v_3, e_{349}, v_4, e_{4578}, v_5, e_{569}, v_6, e_{16}, v_1.
\]

If we define \( E^- = \{e_{1278}, e_{23}, e_{349}\} \), it is simple to check that \( C \) and \( E^- \) satisfy the assumptions (a)–(d) in Definition 2; in particular, assumption (d-1) is satisfied and (d-2) is not. The corresponding odd \( \beta \)-cycle inequality (2) is

\[
z_{v_2} + z_{v_3} - z_{e_{1278}} - z_{e_{23}} - z_{e_{349}} - z_{v_5} - z_{v_6} + z_{e_{4578}} + z_{e_{569}} + z_{e_{16}} \leq 1. \tag{3}
\]

Using the same cycle \( C \), we could define instead \( E^- = \{e_{1278}, e_{23}, e_{4578}\} \). Also in this case, \( C \) and \( E^- \) satisfy the assumptions (a)–(d) in Definition 2. However, in this case, assumption (d-2) is satisfied and (d-1) is not. The corresponding odd \( \beta \)-cycle inequality (2) is

\[
z_{v_2} + z_{v_7} + z_{v_8} - z_{e_{1278}} - z_{e_{23}} - z_{e_{4578}} - z_{v_6} + z_{e_{349}} + z_{e_{569}} + z_{e_{16}} \leq 3. \tag{4}
\]

Figure 1: A hypergraph containing $\beta$-cycles.

Figure 2: A cycle hypergraph.
It can be checked that (3) and (4) are both facet-defining for \( MP_G \). In this example, \( MP_G \) has a total of 156 facet-defining inequalities; 35 of them are in the standard linearization, 2 are flower inequalities, and 70 are odd \( \beta \)-cycle inequalities.

Note that the hypergraph \( G \) given in this example satisfies assumption [(a)] for any choice of \( C \). In this case, all the remaining assumptions are necessary for the validity of (2). Indeed, for any of the three assumptions [(b), (c), (d)] there exists an inequality of the form (2) that does not satisfy that specific assumption but satisfies the remaining two, and is not valid for \( MP_G \). \( \diamond \)

Theorem 2 states that odd \( \beta \)-cycle inequalities have Chvátal rank at most 2 for \( MP_{LP}^G \). This leaves open the possibility that the Chvátal rank of these inequalities could be 1. We show next that this is not the case, by providing an example of an odd \( \beta \)-cycle inequality that is not valid for the Chvátal closure of \( MP_{LP}^G \).

Example 2. Let \( G = (V, E) \) be the cycle hypergraph depicted in Figure 2 and defined by

\[
V = \{v_1, \ldots, v_9\}, \ E = \{e_{123}, e_{345}, e_{4567}, e_{678}, e_{89}, e_{129}\}.
\]

By defining \( E^- = \{e_{345}, e_{678}, e_{129}\} \), we obtain the odd \( \beta \)-cycle inequality

\[
-ze_{345} - ze_{678} - ze_{129} + ze_{123} + ze_{4567} + ze_{89} \leq 1.
\] (5)

In order for (5) to have Chvátal rank 2, we verify that (5) is not valid for the Chvátal closure \( C(MP_{LP}^G) \) of \( MP_{LP}^G \).

Assume, for a contradiction, that (5) is valid for \( C(MP_{LP}^G) \). Since \( G \) is a cycle hypergraph, it follows from Corollary 10 in [7] that (5) is facet-defining for \( MP_G \). As \( MP_G \subseteq C(MP_{LP}^G) \), all the vectors in \( MP_G \) that satisfy at equality (5) belong to \( C(MP_{LP}^G) \) too. Therefore, (5) is facet-defining for \( C(MP_{LP}^G) \) as well. It then follows that (5) is a CG cut for \( MP_{LP}^G \), therefore, if we maximize the left hand side of (5) over \( MP_{LP}^G \), we should get an objective value \( \pi_0 \) such that \( |\pi_0| = 1 \). However, it can be checked that \( \pi_0 = 2 \), therefore \( |\pi_0| = 2 \). We obtained a contradiction, thus we conclude that (5) is not valid for \( C(MP_{LP}^G) \). \( \diamond \)

In order to prove Theorem 2, we show that each odd \( \beta \)-cycle inequality can be obtained as a CG cut for the flower relaxation of \( MP_G \), i.e., the polyhedron obtained from \( MP_{LP}^G \) by adding all flower inequalities. Let us recall here the definition of flower inequalities.

Definition 3. Consider a hypergraph \( G = (V, E) \). Let \( f \in E \) and let \( T \subseteq E \setminus \{f\} \) be a subset of edges adjacent to \( f \) such that \( f \cap e \cap e' = \emptyset \) for all \( e, e' \in T \) with \( e \neq e' \). Then the flower inequality centered at \( f \) with neighborhood \( T \), is given by:

\[
\sum_{e \in f \setminus \cup_{e \in T} e} z_e + \sum_{e \in T} z_e - z_f \leq |f \setminus \cup_{e \in T} e| + |T| - 1. \quad (\theta_f)
\]

Flower inequalities were introduced in [8]. However, our definition is more general than the original one in [8] for three reasons: (i) In Definition 3 the set \( T \) could be empty, while in [8] it must be nonempty. (ii) The condition \( f \cap e \cap e' = \emptyset \) in Definition 3 replaces the stronger assumption \( e \cap e' = \emptyset \) in [8]. (iii) In Definition 3 we require that each edge in \( T \) is adjacent to \( f \), while in [8] it is assumed that \( |f \cap e| \geq 2 \) for every \( e \in T \). Flower inequalities in Definition 3 are still a special case of running intersection inequalities, thus Theorem 1 implies that they are CG cuts for \( MP_{LP}^G \). We are now ready to give the proof of Theorem 2.
Proof of Theorem 2 sketch. Consider an odd $\beta$-cycle inequality (2). We provide a non-negative combination $\pi z \leq \pi_0$ of the inequalities $(\alpha_{v}, \beta_{v}, \epsilon_{v}, \theta_{v,e})$ such that $\pi$ coincides with the left-hand side of (2), and $|\pi_0|$ coincides with the right-hand side of (2). Since inequalities $(\alpha_{v}, \beta_{v}, \epsilon_{v}, \theta_{v,e})$ are present in the standard linearization, and inequalities $(\theta_{v,e})$ are CG cuts for MP-LP by Theorem 1, this implies that (2) has Chvátal rank at most 2. We denote by $\alpha_{v}, \beta_{v}, \epsilon_{v}, \theta_{v,e}$, the multipliers associated with the inequalities $(\alpha_{v}, \beta_{v}, \epsilon_{v}, \theta_{v,e})$, respectively.

For every $f_{i} \in E^-$, we define the set $T_{i} := \{e \in E^+ : e \cap f_{i} \neq \emptyset\}$. We denote by $(\theta_{f_{i}})$ the flower inequality with center $f_{i}$ and neighbors $T_{i}$. If $\sum_{i=1}^{k}|T_{i} \cap D| \geq 1$, we set $T := \sum_{i=1}^{k}|T_{i} \cap D|$. Otherwise, if $\sum_{i=1}^{k}|T_{i} \cap D| = 0$, we set $T := 2$. Note that $\sum_{i=1}^{k}|T_{i} \cap D| \neq 1$, thus we have $T \geq 2$.

For ease of exposition, all multipliers not explicitly defined are set to zero.

1. For $i$ odd, choose $\theta_{f_{i}} := \frac{1}{T}$, and $\beta_{f_{i}} := \frac{T-1}{T}$. For $i$ even, select $\theta_{f_{i}} := \frac{T-1}{T}$, and $\beta_{f_{i}} := \frac{1}{T}$.

2. For every $v$ contained in only one edge $f_{i} \in E^-$ and in no edge of $E^+$, pick $\alpha_{v} := \frac{T}{T-1}$ if $i$ is odd, and set $\alpha_{v} := \frac{T-1}{T}$ if $i$ is even.

3. Consider the multipliers regarding inequalities involving edges in $E^+ \setminus D$. Note that $E^+ \setminus D$ can be partitioned into maximal length substrings of $e_{1}, \ldots, e_{p}$. For each such substring $e_{i}, \ldots, e_{i+h}$:

   3.1 Note that $e_{i-1} = f_{j}$ for some $j$. Then choose $\delta_{v_{i+1},e_{i}} := 1 - \theta_{f_{j}}$.

   3.2 For $l = i + 1, \ldots, i + h - 1$, set $\delta_{v_{l},e_{l}} := \theta_{f_{j}}$, and then $\delta_{v_{i+1},e_{l}} := 1 - \theta_{f_{j}}$.

   3.3 Select $\delta_{v_{i+h},e_{i+h}} := \theta_{f_{j}}$.

4. Let us focus here on the edges in $D$ and their related inequalities. For every edge $e_{i} \in D$ define the number $\Delta_{i} := |\{f \in E^- : e_{i} \cap f \neq \emptyset\}|$.

   4.1 Set $\delta_{v_{p+2},e_{p+1}} := 1 - \frac{\Delta_{p+1}}{T}$.

   4.2 For $i = p + 2, \ldots, m - 1$, choose $\delta_{v_{i+1},e_{i}} := \frac{\sum_{j=p+1}^{i-1} \Delta_{j}}{T}$, and pick $\delta_{v_{i+1},e_{i}} := 1 - \frac{\sum_{j=p+1}^{i} \Delta_{j}}{T}$.

   4.3 Select $\delta_{v_{m},e_{m}} := \frac{\sum_{j=p+1}^{m-1} \Delta_{j}}{T}$.

We remark that, when $\sum_{i=1}^{k}|T_{i} \cap D| = 0$, then $D = \emptyset$, thus we do not need to consider rule 4.

The overall objective of the proof is similar to the one of Theorem 1. In fact, it comprises showing three claims. The first asserts that the multipliers $(\alpha_{v}, \beta_{v}, \epsilon_{v}, \theta_{v,e})$ are non-negative. The second claim states that $\pi z$ is equal to the left-hand side of (2), while the third affirms that $|\pi_0|$ coincides with the right-hand side of (2). To prove these claims we exploit the fact that assumption $(a')$ is equivalent to:

(a') Every node $v \in \bigcup_{i=1}^{m} e_{i}$ is contained in at most two edges among $e_{1}, \ldots, e_{m}$.

This equivalence is proved, together with the three claims, in Appendix A.

We now explain the roles of assumptions $(d-1)$ and $(d-2)$. Assumption $(d-1)$ is used to make sure that the coefficients of the variables related to nodes in $S_{1}$ are ultimately equal to 1 in $\pi z$. Observe that every $v \in S_{1}$ is contained in either one or two edges, because of $(a')$. Therefore, in these two scenarios the multipliers involving $v$ sum to 1 thanks to either rules 1, 2 or to rule 4 respectively, thanks to assumption $(d-1)$. On the other hand, if $v$ is contained in two edges that are both odd the multipliers sum to 2, while, if the two edges are even, the coefficient of the associated variable becomes $\frac{2(T-1)}{T}$. In general these values are not equal to 1. However note that, if assumption $(d-2)$ holds, assumption $(a')$ implies that $T$ is equal to 2 and $\frac{2(T-1)}{T} = 1$, is available in Appendix A.
4 Combining perfect formulations

Here, we present a result of a different flavor that is of independent interest, and that is used in the characterization of MP\(_G\) given in the next section. It deals with extending the multilinear polytope of a hypergraph \(G\) to the multilinear polytope of a new hypergraph \(G'\) obtained from \(G\) by replacing any node with a new edge containing arbitrarily many new nodes. To the best of our knowledge, this is the first result of this type for the multilinear polytope. We first show a very general lemma that allows us to combine two perfect formulations that overlap only in one variable.

**Lemma 1.** Let \(P = \{(x, y) \in \mathbb{R}^{p+1} : A(x, y) \leq b\}, Q = \{(y, z) \in \mathbb{R}^{1+q} : C(y, z) \leq d\}\) be polytopes with binary vertices. Then \(R = \{(x, y, z) \in \mathbb{R}^{p+1+q} : A(x, y) \leq b, C(y, z) \leq d\}\) is a polytope with binary vertices.

**Proof.** Since \(P \subseteq [0,1]^{p+1}\) and \(Q \subseteq [0,1]^{1+q}\), the polyhedron \(R\) is contained in \([0,1]^{p+1+q}\). Let \((\bar{x}, \bar{y}, \bar{z})\) be a vertex of \(R\). We want to show that \((\bar{x}, \bar{y}, \bar{z}) \in [0,1]^{p+1+q}\). Since it is a vertex, there exist \(p + 1 + q\) linearly independent constraints among \(A(x, y) \leq b\), \(C(y, z) \leq d\) that are active at \((\bar{x}, \bar{y}, \bar{z})\). In the inequalities from \(A(x, y) \leq b\), only \(p + 1\) variables can appear with non-zero coefficient, so at most \(p + 1\) of these constraints can be linearly independent. Similarly there can be at most \(1 + q\) linearly independent constraints among the inequalities \(C(y, z) \leq d\). Therefore, there are only two ways the \(p + 1 + q\) inequalities defining \((\bar{x}, \bar{y}, \bar{z})\) can be distributed among the systems \(A(x, y) \leq b\) and \(C(y, z) \leq d\): either \(p + 1\) are in \(A(x, y) \leq b\) and \(q\) are in \(C(y, z) \leq d\), or \(p\) are in \(A(x, y) \leq b\) and \(1 + q\) are in \(C(y, z) \leq d\). By symmetry, we can assume, without loss of generality, that we are in the first case.

The vector \((\bar{x}, \bar{y})\) is in \(P\) since it satisfies \(A(x, y) \leq b\). Moreover, \(p + 1\) linearly independent constraints among \(A(x, y) \leq b\) are active at \((\bar{x}, \bar{y})\). This implies that \((\bar{x}, \bar{y})\) is a vertex of \(P\). Since \(P\) has binary vertices, we obtain \((\bar{x}, \bar{y}) \in [0,1]^{p+1}\). In particular, these \(p + 1\) linearly independent constraints imply the constraint \(y = \bar{y}\).

Consider now the vector \((\bar{y}, \bar{z})\). This vector is in \(Q\), and it satisfies at equality \(q\) linearly independent constraints among \(C(y, z) \leq d\). Moreover, it also satisfies the equation \(y = \bar{y}\). This equation must be linearly independent from the latter \(q\) constraints, since it was obtained from the first \(p + 1\) constraints defining \((\bar{x}, \bar{y}, \bar{z})\). Hence \((\bar{y}, \bar{z})\) must be a vertex of the polytope \(F := \{(y, z) \in Q : y = \bar{y}\}\). Since \(\bar{y} \in \{0,1\}\) and \(Q\) is contained in \([0,1]^{1+q}\), we have that \(F\) is a face of \(Q\). Since \(Q\) has binary vertices, so does \(F\), and we obtain \((\bar{y}, \bar{z}) \in \{0,1\}^{1+q}\). We have shown that \((\bar{x}, \bar{y}, \bar{z}) \in \{0,1\}^{p+1+q}\), thus \(R\) has binary vertices. \(\square\)

**Theorem 3.** Let \(G = (V, E)\) be a hypergraph, let \(w \in V\), and let \(f\) be a set of nodes disjoint from \(V\) with \(|f| \geq 2\). We define the hypergraph \(G' = (V', E')\) as follows:

\[
V' := V \setminus \{w\} \cup f,
E' := \{f\} \cup \{e \in E : w \notin e\} \cup \{e \cup f : e \in E, w \in e\}.
\]

Then, a description of the multilinear polytope MP\(_{G'}\) is obtained from an external description of MP\(_G\) by:

1. Replacing \(z_w\) with \(z_f\) in every inequality of MP\(_G\), and
2. Adding the standard linearization of the edge \(f\).
Proof. Let $R$ be the polyhedron obtained by performing the operations in the statement. We can then apply Lemma 4 where $P = MP_G$, $Q$ is the polytope defined by the standard linearization of the edge $f$, and $y$ is the variable $y_f$. We obtain that $R$ is a polytope with binary vertices. Note that $R$ leaves in a space of dimension $n = |V'| + |E'| = |V| + |E| + |f|$. Therefore, to conclude the proof, we only need to show that $R \cap \{0, 1\}^n = MP_G \cap \{0, 1\}^n$. For ease of notation, in this proof, we denote by $S_G$ and $S_{G'}$ the set of binary point in $MP_G$ and $MP_{G'}$, respectively.

First, we prove the inclusion $R \cap \{0, 1\}^n \supseteq S_{G'}$. Let $z'$ be a vector in $S_{G'}$. As $z'$ is binary, we only need to show that $z'$ is in $R$. Clearly $z'$ satisfies all inequalities of the standard linearization of $f$. Let $\bar{z}$ be the vector in the space of $G$ obtained from $z'$. Formally, $\bar{z}$ is constructed by setting $\bar{z}_v := \bar{z}'_v$ for every $v \in V \setminus \{w\}$, $\bar{z}_w := \bar{z}'_f$, $\bar{z}_e := \bar{z}'_e$ for every $e \in E$ with $w \notin e$, and $\bar{z}_e := \bar{z}'_{e,ef}$ for every $e \in E$ with $w \in e$. Clearly, $\bar{z} \in S_G$, thus it satisfies the constraints in the linear description of $MP_G$. Since $\bar{z}_w = \bar{z}'_f$, we obtain that $\bar{z}'$ satisfies the constraints of $R$ obtained by applying the operation $\square$ in the statement. Therefore we have shown $\bar{z}' \in R$.

Next, we show the reverse inclusion $R \cap \{0, 1\}^n \subseteq S_{G'}$. Let $z'$ be a binary vector not in $S_{G'}$. If $\bar{z}'_f \neq \prod_{e \in f} \bar{z}'_e$, then $\bar{z}'$ does not satisfy some inequality in the standard linearization of $f$ and so $\bar{z}'$ is not in $R$. Assume now $\bar{z}'_f = \prod_{e \in f} \bar{z}'_e$. Since $\bar{z}'$ is not in $S_{G'}$, there exists an edge $g' \in E'$ with $g' \neq f$ such that $\bar{z}'_g' \neq \prod_{v \in g'} \bar{z}'_v$. Let $g$ be the edge of $G$ corresponding to $g'$. As above, we define the vector $\bar{z}$ in the space of $G$ obtained from $\bar{z}'$. Then $\bar{z}_e \neq \prod_{v \in e} \bar{z}_v$, thus $\bar{z}$ is not in $S_G$. This means that there exists an inequality in the linear description of $MP_G$ that is not satisfied by $\bar{z}$. The inequality obtained from it by applying the operation $\square$ is not satisfied by $\bar{z}'$, since $\bar{z}_w = \bar{z}'_f$, which implies that $\bar{z}'$ is not in $R$. □

We remark that Theorem 3 can be used recursively to characterize the multilinear polytope of laminar hypergraphs, providing a simple proof of Theorem 10 in [8]. The new proof is also elementary, as it does not rely on the result by Conforti and Cornuéjols regarding the connection between integral polyhedra and balanced matrices (Theorem 6.13 in [5]).

Theorem 3 can also be used to extend all known decomposition results for the multilinear polytope (see [9,8,10,11]). In fact, let $G$ be a hypergraph, and let $G_1, G_2$ be section hypergraphs of $G$ such that $MP_G$ is decomposable into $MP_{G_1}$ and $MP_{G_2}$. Let $w \in V(G)$, let $G'$ be obtained from $G$ as described in Theorem 3 and let $G'_1, G'_2$ be the section hypergraphs of $G'$ corresponding to $G_1$ and $G_2$. Then, Theorem 3 implies that $MP_{G'}$ is decomposable into $MP_{G'_1}$ and $MP_{G'_2}$. The definition of section hypergraph can be found in any of these cited papers.

5 The multilinear polytope of cycle hypergraphs

In this section, we provide an indication of the theoretical power of odd $\beta$-cycle inequalities. These results provide the first characterizations of $MP_G$ in the case where $\beta$-cycles are present in $G$. First, we show that the multilinear polytope of cycle hypergraphs is fully described by the standard linearization and odd $\beta$-cycle inequalities, if each pair of edges intersects in at most one node.

Proposition 1. Let $G$ be a cycle hypergraph with edges $e_1, \ldots, e_m$ such that $|e_i \cap e_{i+1}| = 1$ for every $i = 1, \ldots, m$, where, for convenience, we define $e_{m+1} = e_1$. Then, $MP_G$ is given by the system comprising of the standard linearization and the odd $\beta$-cycle inequalities.

Proof sketch. We show this result by induction on the number of edges. We first show the base case, which is when $G$ has three edges, by induction on $|V|$. Its own base case is when we are
dealing with a graph. In this situation the claim was proved by Padberg in [16]. In the induction step, we fix a node \( \bar{v} \) at the intersection of two edges and consider the two faces of \( \text{MP}_G \) obtained by fixing \( z_{\bar{v}} \) to zero or one. \( \text{MP}_G \) is a polytope with binary vertices, hence we know that \( \text{MP}_G \) is the convex hull of the union of these two faces. Observe that by the induction hypothesis, we know their perfect formulations. Therefore, we apply Balas’ formulation [1] for the union of polytopes and obtain a perfect formulation for \( \text{MP}_G \) in the extended space. Hence, we reach the thesis for the base case once we apply Fourier elimination on all the additional variables. This projection step involves many calculations, and it can be found in the complete proof of this proposition in Appendix A.

In order to prove the inductive step, we consider a graph \( G \) with \( m \) edges, and we assume that the proposition holds for hypergraphs with the same structure and with at most \( m - 1 \) edges. We create three new hypergraphs: the first is \( G' = (V, E') \) obtained by adding a new edge \( f = \{v', v''\} \) to \( G \), where \( v' = e_1 \cap e_2 \) and \( v'' = e_{m-1} \cap e_m \). Then, we define \( G_1 \) as the section hypergraph of \( G' \) induced by \( e_1 \cup e_m \), and \( G_2 \) as the section hypergraph of \( G' \) induced by \( \bigcup_{i=2}^{m-1} e_i \). By the induction hypothesis, we obtain a characterization of \( \text{MP}_{G_1} \) and \( \text{MP}_{G_2} \), since both \( |E(G_1)|, |E(G_2)| \leq m - 1 \), and subsequently we achieve a description of \( \text{MP}_{G'} \) by applying a decomposition result in [9]. By projecting out \( z_f \) with Fourier elimination, we obtain a perfect formulation for \( \text{MP}_G \), which consists only of inequalities from the standard linearization, and odd \( \beta \)-cycle inequalities.

Next, we present the main result of this section. Specifically, we prove that the multilinear polytope of general cycle hypergraphs is characterized by the standard linearization, all the flower and odd \( \beta \)-cycle inequalities.

**Theorem 4.** Let \( G = (V, E) \) be a cycle hypergraph. Then, \( \text{MP}_G \) is given by the system comprising of the standard linearization, the flower inequalities, and the odd \( \beta \)-cycle inequalities.

**Proof sketch.** We provide an overview of the key steps that comprise this proof. Consider a cycle hypergraph \( G = (V, E) \) with edges \( e_1, \ldots, e_m \). The idea of the proof is to find the description of \( \text{MP}_G \) by building on the multilinear polytope obtained in Proposition 1. We first define the hypergraph \( G' = (V', E') \) obtained from \( G \) by contracting every intersection \( e_i \cap e_{i+1} \) to a new node \( w_i \), for every \( i \in [m] \). The hypergraph \( G' \) satisfies the assumptions of Proposition 1 which then yields a description of \( \text{MP}_{G'} \). At this point, we employ Theorem 3 recursively \( m \) times, starting from the hypergraph \( G' \). Each time, we replace a node \( w_i \), for \( i \in [m] \), with a new edge \( f_i \) containing the original nodes of \( G \) that were contracted to obtain \( w_i \). Hence we achieve \( \text{MP}_{G''} \), where \( G'' = (V, E'') \) is equal to \( G \) with the addition of edges \( f_i = e_i \cap e_{i+1} \), for \( i \in [m] \). In order to derive a description of \( G'' \), we project out these additional variables using the Fourier elimination procedure.

The full proofs of Proposition 1 and Theorem 4 are rather long, mainly due to the computations deriving from the Fourier elimination step in Proposition 1. The complete proofs can be found in Appendix A.

From Theorem 1, Theorem 2, and Theorem 4, we obtain that the polytope \( \text{MP}^\text{LP}_G \) has Chvátal rank at most 2, provided that \( G \) is a cycle hypergraph.
A Appendix

A.1 Proof of Theorem 1

We already defined the multipliers \( \alpha_v, \beta_v, \varepsilon_v, \delta_{v,e}, \eta_v \) in the sketch of the proof of Theorem 1 in Section 2. First, we show that the multipliers are non-negative. Subsequently, we prove that \( \pi \) coincides with the left-hand side of (1). In these arguments, we will be using the notion of connected components of a hypergraph, and we refer to [11] for a formal definition. Let \( \tilde{G} \) be the hypergraph \( (e_0, \tilde{E}) \), and note that \( \tilde{G} \) can contain loops and parallel edges. There is a one-to-one correspondence between edges in \( E \) and in \( \tilde{E} \). Given an edge \( e \in E \), we denote by \( \tilde{e} \) the corresponding edge in \( \tilde{E} \) defined by \( \tilde{e} = e_0 \cap e \); Viceversa, given an edge \( \tilde{e} \in \tilde{E} \), we denote by \( e \) the corresponding edge in \( E \).

For every edge \( e_i, i \in [m], \) let \( J_i \) be the set of indices \( j \in \{i + 1, \ldots, m\} \) for which the condition in rule 2.2 holds. Subsequently, we define recursively the sets

\[
C_i := J_i \cup \bigcup_{j \in J_i} C_j \quad \text{for every } i \in [m],
\]

starting from \( i = m \) and moving to \( i = 1 \). Moreover, for every \( i \in [m], \) let

\[
W^i := \{ w \in W : \delta_{w,e} \neq 0, \text{ for some } h \in \{i\} \cup C_i \},
\]

\[
U^i := \{ u_j \in U : \delta_{u_j,e} \neq 0, \text{ for some } h \in \{i\} \cup C_i, h \neq j \}.
\]

The next claim provides a fundamental tool for the arguments in this proof.

**Claim 1.** For every \( i \in [m], \) we have

\[
\sum_{v \in e_i \setminus \{u_i\}} \delta_{v,e_i} = \frac{|W^i|}{\gamma} + \frac{|U^i|}{\gamma}.
\]

Note that if \( N(\tilde{e}_i) = \emptyset, \) then \( u_i \) is not defined, and the sum ranges over all \( v \in e_i \).

**Proof of claim.** We fix \( i \in [m] \) and prove (6). In order to do this, it will be useful to write the left-hand side of (6) explicitly, in terms of the multipliers defined in rules 2.1 and 2.2. We obtain

\[
\sum_{v \in e_i \setminus \{u_i\}} \delta_{v,e_i} = \sum_{v \in W \cap e_i \setminus \{u_i\}} \frac{1}{\gamma} + \sum_{j \in J_i} \left( 1 - \frac{1}{\gamma} \right).
\]

Notice that, if \( j \in J_i \), then \( u_j \in \tilde{e}_i \cap \tilde{e}_j \), thus \( \tilde{e}_i \) and \( \tilde{e}_j \) belong to the same connected component of \( \tilde{G} \), that we denote by \( C \). Hence, the right-hand side of (7) only depends on the values of the multipliers \( \delta_{v,e} \) for edges \( \tilde{e} \) of \( C \).

Let \( p := |E(C)| \geq 1 \). We can assume, without loss of generality, that the edges of \( C \) correspond to the first \( p \) edges in the running intersection ordering of \( \tilde{E} \). That is, the edges of \( C \) are \( \tilde{e}_1, \ldots, \tilde{e}_p \). Recall that we only need to prove (6) for the fixed index \( i \in [p] \). Instead, we prove (6) for all indices \( i \in [p] \). We do so by induction on \( i \), starting from \( i = p \) and going backwards to 1. Thus, we no longer think of \( i \) as a fixed index.
Consider the base case \( i = p \). Then \( J_p = C_p = \emptyset \). Therefore, the first term of (7) reduces to \( \frac{|W\cap e_p|}{\gamma} \), and the second term is zero. This completes the proof of the base case, since \( U^p = \emptyset \).

Before showing the inductive step, we also provide the argument for the case \( i = p - 1 \) since the idea is the same, but the calculations are significantly simpler. Let \( i = p - 1 \). If \( J_{p-1} = \emptyset \), then also \( C_{p-1} = \emptyset \) and therefore the second term of (7) is zero, and the first term is equal to \( \frac{|W\cap e_{p-1}\setminus e_p|}{\gamma} = \frac{|W_{p-1}|}{\gamma} \). This is true because, in this case, \( W_{p-1} = \{ w \in W : \delta_{v,e_{p-1}} \neq 0 \} \). Note also that \( U^{p-1} \) is the empty set, since \( J_{p-1} = \emptyset \) implies that there is no node \( u \in U \setminus \{ u_{p-1} \} \) for which \( \delta_{u,e_{p-1}} \neq 0 \).

Thus we now assume that \( J_{p-1} = \{ p \} \), which means that \( N(\varepsilon_p) \neq \emptyset \) and \( u_p \in \varepsilon_{p-1} \). Then, \( C_{p-1} = \{ p \} \cup C_p = \{ p \} \). In this case, the first term of (7) reduces to \( \frac{|W \cap e_{p-1}\setminus e_p|}{\gamma} \), and the second term is
\[
1 - \left( \delta_{u_p,e_p} - \frac{1}{\gamma} \right) = 1 - \left( 1 - \sum_{v \in e_p \setminus \{ u_p \}} \delta_{v,e_p} - \frac{1}{\gamma} \right) = \sum_{v \in e_p \setminus \{ u_p \}} \delta_{v,e_p} + \frac{1}{\gamma} = \frac{|W_p|}{\gamma} + \frac{1}{\gamma},
\]
where we have used the definition of \( \delta_{u_p,e_p} \) in rule 2.3 and the fact that \( \sum_{v \in e_p \setminus \{ u_p \}} \delta_{v,e_p} = \frac{|W_p|}{\gamma} \).

We obtain
\[
\sum_{v \in e_p \setminus \{ u_p \}} \delta_{v,e_{p-1}} = \frac{|W \cap e_{p-1}\setminus e_p|}{\gamma} + \frac{|W_p|}{\gamma} + \frac{1}{\gamma} = \frac{|W_{p-1}|}{\gamma} + \frac{|U_{p-1}|}{\gamma}.
\]
This holds because, by definition, \( W_{p-1} = \{ w \in W : \delta_{w,e_h} \neq 0, \text{ for some } h \in \{ p - 1, p \} \} \), and every node \( w \) in \( W \) has exactly one edge \( e \) for which \( \delta_{w,e} \neq 0 \). Hence \( W_{p-1} \) is the disjoint union of \( W \cap e_{p-1}\setminus e_p \) and \( W \setminus e_p \). Similarly, we show \( U^{p-1} = \{ u_p \} \). By definition, \( U^{p-1} = \{ u_j \in U : \delta_{u_j,e_h} \neq 0, \text{ for some } h \in \{ p - 1, p \}, h \neq j \} \). When \( h = p \), then the only non-zero multiplier \( \delta_{u_j,e_p} \) is for \( j = p \), hence \( h = j \), and when \( h = p - 1 \) the multipliers \( \delta_{u_j,e_{p-1}} \) are non-zero only when \( j = p - 1 \). However, \( h \) needs to be different from \( j \), hence the only multiplier that satisfies this condition is \( \delta_{u_j,e_{p-1}} \). This concludes the proof that \( U^{p-1} = \{ u_p \} \).

We now prove the inductive step. Let \( i \in \{ 2, \ldots, p \} \), we suppose that (8) holds for all \( j \geq i \), and we show that (8) still holds for \( i - 1 \). The first term of (7) is then \( \frac{|W \cap e_{i-1}\setminus \bigcup_{j \geq i} e_j|}{\gamma} \), and the second term is
\[
\sum_{j \in \mathcal{J}_{i-1}} \left( 1 - \left( \delta_{u_j,e_j} - \frac{1}{\gamma} \right) \right) = \sum_{j \in \mathcal{J}_{i-1}} \left( 1 - \left( 1 - \sum_{v \in e_j \setminus \{ u_j \}} \delta_{v,e_j} - \frac{1}{\gamma} \right) \right)
\]
\[
= \sum_{j \in \mathcal{J}_{i-1}} \left( \sum_{v \in e_j \setminus \{ u_j \}} \delta_{v,e_j} + \frac{1}{\gamma} \right) = \sum_{j \in \mathcal{J}_{i-1}} \left( \frac{|W_j|}{\gamma} + \frac{|U_j|}{\gamma} + \frac{|\mathcal{J}_{i-1}|}{\gamma} \right),
\]
where we have used the definition of \( \delta_{u_j,e_j} \) in rule 2.3 and the induction hypothesis. We obtain
\[
\sum_{v \in e_{i-1}\setminus \{ u_{i-1} \}} \delta_{v,e_{i-1}} = \frac{|W \cap e_{i-1}\setminus \bigcup_{j \geq i} e_j|}{\gamma} + \sum_{j \in \mathcal{J}_{i-1}} \left( \frac{|W_j|}{\gamma} + \frac{|U_j|}{\gamma} + \frac{|\mathcal{J}_{i-1}|}{\gamma} \right)
\]
\[
= \frac{|W^{i-1}|}{\gamma} + \frac{|U^{i-1}|}{\gamma}.
\]
The last equality follows from two facts that we show next. The first is that \( W^{i-1} \) is the disjoint union of \( W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j \) and \( \bigcup_{j \in I_{i-1}} W^j \), where also all these unions are disjoint.

We prove first that \( W^{i-1} = (W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j) \cup \bigcup_{j \in I_{i-1}} W^j \), by showing the two set inclusion. Recall that \( W^{i-1} = \{ w \in W : \delta_{w,e_h} \neq 0 \text{ for some } h \in \{i-1\} \cup C_{i-1} \} \). We show first the inclusion "\( \subseteq \)". Let \( w \in W^{i-1} \). Then either \( \delta_{w,e_{i-1}} \neq 0 \) or \( \delta_{w,e_h} \neq 0 \) with \( h \in C_{i-1} \). If the first case holds, then \( w \in (W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j) \), by rule 2.1. Assume then that there exists \( h \in C_{i-1} \) such that \( \delta_{w,e_h} \neq 0 \). It implies that \( w \in J_{i-1} \cup \bigcup_{j \in I_{i-1}} C_j \). Then either \( h \in J_{i-1} \) or \( h \in \bigcup_{j \in I_{i-1}} C_j \). If \( h \in J_{i-1} \), there exists \( j' \in J_{i-1} \) such that \( \delta_{w,e_{j'}} \neq 0 \). Hence \( w \in W^{j'} \). On the other hand, if \( h \in \bigcup_{j \in I_{i-1}} C_j \), it means that there exists \( j'' \in J_{i-1} \) such that \( h \in C_{j''} \). Then \( w \in W^{j''} \), by definition of \( W^{j''} \). We prove now the reverse set inclusion "\( \supseteq \)". Let \( w \in (W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j) \cup \bigcup_{j \in I_{i-1}} W^j \). Assume first that \( w \in W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j \). By rule 2.1 we have \( \delta_{w,e_{i-1}} \neq 0 \), thus \( w \in W^{i-1} \). Consider now the other case \( w \in \bigcup_{j \in I_{i-1}} W^j \). This means that there exists \( j \in J_{i-1} \) such that \( w \in W^j \). It follows that there exists \( h \in \{j\} \cup C_j \) such that \( \delta_{w,e_h} \neq 0 \). Note that \( \{j\} \cup C_j \subseteq C_{i-1} \), hence \( h \in C_{i-1} \). Therefore \( w \in W^{i-1} \).

Next we show that \( W^{i-1} \) is the disjoint union of the set \( W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j \), and of the sets \( W_j \), with \( j \in J_{i-1} \). Notice that, by rule 2.1 for every \( w \in W \) there exists an unique edge \( e \) such that \( \delta_{w,e} \neq 0 \). Assume for a contradiction that these unions are not disjoint, i.e. there exists \( w \in W^{i-1} \) such that either \( w \in W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j \) and there exists \( j \in J_{i-1} \) such that \( w \in W^j \), or there exist two distinct indices \( j', j'' \in J_{i-1} \) such that \( w \in W^{j'} \), \( w \in W^{j''} \).

Consider the first case. By assumption, \( w \in W \cap e_{i-1} \setminus \bigcup_{j \geq i} e_j \) which implies that \( w \notin e_j \) for all \( j \geq i \). Observe that, by rule 2.2, all indices \( j \in J_{i-1} \) are greater than or equal to \( i \). Therefore, by definition of \( W^j \), this set contains only nodes \( w' \) for which \( \delta_{w',e_h} \neq 0 \) with \( h \geq i \). Since \( \delta_{w,e_{i-1}} \neq 0 \) by rule 2.1 and since there exists only one edge \( e \) such that \( \delta_{w,e} \neq 0 \), we can conclude that \( w \notin W^j \), which is a contradiction.

Assume the second case holds. Without loss of generality we suppose that \( j' < j'' \). As remarked before there exists an unique index \( h \geq i \) such that \( \delta_{w,e_h} \neq 0 \). By definition of \( W^{j'} \), \( W^{j''} \), it follows that \( h \in (\{j'\} \cup C_{j'}) \cap (\{j''\} \cup C_{j''}) \). We show \( j' \notin C_{j''} \). As \( j' < j'' \) and \( C_{j''} \) only contains indices strictly larger than \( j'' \), then \( j' \notin C_{j''} \). Next, we show \( j'' \notin C_{j'} \). By expanding the definition of \( C_{j'} \), observe that \( C_{j'} \) is a collection of some sets \( J_q \) with \( q \geq j' \). Assume by contradiction \( j'' \in C_{j'} \). Then, there exists an index \( k \geq j' \) such that \( j'' \in J_k \), which means that, by rule 2.2, \( \delta_{w_{j''},e_{k''}} \neq 0 \). By assumption, \( j'' \in J_{i-1} \), hence \( e_{i-1} \) is the only edge \( e \), different from \( e_{j''} \) such that \( \delta_{w_{j''},e_{j''}} \neq 0 \). This is a contradiction because \( k \geq j' > i - 1 \). So we have \( j'' \notin C_{j'} \).

As \( j' \neq j'' \), \( j' \notin C_{j''} \), and \( j'' \notin C_{j'} \), we have \( h \in C_{j'} \cap C_{j''} \). Since \( h \in C_{j'} \), there exists an index \( k' \geq j' \), \( k' \neq h \) such that \( h \in J_{k'} \), therefore \( \delta_{w_{k'},e_{k'}} \neq 0 \). Similarly, since \( h \in C_{j''} \), there exists \( k'' \geq j'' \), \( k'' \neq h \) such that \( \delta_{w_{k''},e_{k''}} \neq 0 \). However, \( \delta_{w_{h},e_{h}} \neq 0 \) with \( e \neq e_h \) is satisfied only by one edge \( e \), thus \( e_{k'} = e_{k''} \). Let us denote this edge by \( e_k \). Because of the recursive nature of the set \( C_{j'} \), there exists a sequence of edges \( e_1, e_2, \ldots, e_p \) such that \( e_1 = e_{j'}, e_{r} = e_k, \) and \( p' \in J_{(p-1)'} \), for every \( p \in \{2, \ldots, t\} \). Similarly, let \( e_{r'}, e_{r''}, \ldots, e_{r'''} \) be a sequence such that \( e_{r'} = e_{r''} = e_{r'''}, \) and \( p'' \in J_{(p-1)''} \), for every \( p \in \{2, \ldots, t'\} \). Since \( k \in J_{(r-1)'} \cap J_{(l-1)''} \) and there exists a unique edge \( e \neq e_k \) for which \( \delta_{w_{e},e} \neq 0 \), it follows that \( e_{(r-1)'} = e_{(l-1)''} \). Recursively, we apply the same uniqueness property until we arrive to the first edge of the shortest sequence. Note that either
Consider the first case. If \( r < t \), it means that \( e_{1'} = e_{q''} \) for some \( q > 1 \). Since \( e_{1'} = e_{j'} \), this implies in turn that \( j' \in C_{j''} \), which is a contradiction as showed earlier. Then suppose \( r > t \). Similarly, we can conclude that \( j'' \in C_{j'} \), which again is a contradiction as proved previously. Hence assume that \( r = t \). In this case, \( e_{1'} = e_{j''} \), which means that \( e_{j'} = e_{j''} \). This is a contradiction because \( e_{j'} \) and \( e_{j''} \) are different edges by assumption. Therefore \( h \notin C_{j'} \cap C_{j''} \) and we have showed that \( W^{j-1} \) is the disjoint union of the set \( W \cap e_{i-1} \setminus \bigcup_{j \geq 1} e_j \), and \( W^j \), with \( j \in J_{i-1} \).

The second fact is that \( U^{i-1} = J_{i-1} \cup \bigcup_{j \in J_{i-1}} U^j \), and in addition \( U^{i-1} \) is the disjoint union of \( J_{i-1} \), \( U^j \), for \( j \in J_{i-1} \). The proof of this is very similar to the one regarding \( W^{j-1} \). This concludes the proof that (4) holds.

\[ \text{Clm 2. The multipliers } \alpha_v, \beta_e, \varepsilon_e, \delta_{v,e}, \eta_e \text{ are non-negative.} \]

\[ \text{Proof of claim. Since } \gamma \geq 1, \text{ we only need to prove that the multipliers } \delta_{u_j,e_i}, \delta_{u_i,e_i}, \text{ and } \eta_{e_i} \text{ are non-negative, for every edge } e_i, \ i \in [m]. \ \text{We first consider multipliers } \delta_{u_i,e_i} \text{ and } \eta_{e_i}, \text{ which are defined in rule (2).} \ \text{Note that, if we are defining } \eta_{e_i} \text{ instead of } \delta_{u_i,e_i}, \text{ it means that } N(e_0 \cap e_i) = \emptyset, \text{ which implies that node } u_i \text{ does not exist. Therefore, to show that multipliers } \delta_{u_i,e_i} \text{ and } \eta_{e_i} \text{ are non-negative, we can equivalently show that } \sum_{v \in e_i \setminus \{u_i\}} \delta_{v,e_i} \leq 1, \text{ for every } i \in [m]. \]

\[ \text{By Claim (4), we obtain} \]

\[ \frac{1}{\gamma} \sum_{v \in e_i \setminus \{u_i\}} \delta_{v,e_i} = \frac{|W^i|}{\gamma} + \frac{|U^i|}{\gamma} \leq \frac{|W|}{\gamma} + \frac{|U|}{\gamma} = \frac{|e_0 \cap \bigcup_{i \in [m]} e_i|}{\gamma} = 1, \]

where the second equality holds since \( W, U \) is a partition of \( e_0 \) and \( \bigcup_{i \in [m]} e_i \), and the last equality is true by definition of \( \gamma \).

\[ \text{Finally, the multipliers } \delta_{u_j,e_i} \text{ are non-negative because } \delta_{u_j,e_i} = 1 + \frac{1}{\gamma} - \delta_{u_j,e_j} \text{ and we just showed that } \delta_{u_j,e_j} \leq 1. \]

\[ \text{Clm 3. The left-hand side of (1) is equal to } \pi z. \]

\[ \text{Proof of claim. We need to check that every entry of } \pi \text{ coincides with the corresponding component of the left-hand side of (1).} \]

Each variable corresponding to an edge different from \( e_0 \) or its neighbors \( e_1, \ldots, e_m \) does not appear in (1), and it does not appear in the inequality \( \pi z \leq |\pi_0| \) either, because their corresponding multipliers are not explicitly defined and therefore are set to zero. An analogous argument holds for the nodes \( v \) that do not belong to \( e_0 \).

Consider the variable \( z_{e_0} \). The only constraints chosen with non-zero multipliers in which it appears are \( \sum_{v \in e_0} z_v - z_{e_0} \leq |e_0| - 1 \) and \( -z_{e_0} \leq 0 \). The first constraint is selected with multiplier equal to \( \frac{1}{\gamma} \), while the second with multiplier \( 1 - \frac{1}{\gamma} \) by rule (1). By summing these two inequalities we obtain that the entry of \( \pi \) related to \( z_{e_0} \) is equal to \( -1 \), as is the coefficient of \( z_{e_0} \) in (1).

Similarly, consider variables \( z_v \), for \( v \in e_0 \setminus \bigcup_{i \in [m]} e_i \). Each of them is involved in just two constraints among the ones picked with non-zero multiplier. These two inequalities are \( \sum_{v \in e_0} z_v - z_{e_0} \leq |e_0| - 1 \) and \( z_v \leq 1 \). Once we sum these two constraints chosen with the multipliers described in rule (1), we obtain that the resulting coefficients of these variables are all equal to 1.
Let \( w \in W \cap \bigcup_{i \in [m]} e_i \). The corresponding variable \( z_w \) is present again in only two constraints among the selected ones: \( \sum_{v \in e_0} z_v - z_{e_0} \leq |e_0| - 1 \) and \( -z_w + z_{e_i} \leq 0 \), where \( i \) is the largest index in the running intersection ordering of \( \hat{E} \) such that \( \hat{e}_i \) contains the node \( w \). By rules [1] and [2.1], the multiplier corresponding to these two inequalities are \( \frac{1}{\gamma} \). Then the component of \( \pi \) corresponding to \( z_w \) is equal to 0, since in one inequality it has coefficient +1 and in the other it has coefficient −1.

Now consider a node \( u \in U \). Let \( e', e'' \) be the two edges in \( \hat{E} \) that contain \( u \), with respectively largest and second largest index in the running intersection ordering of \( \hat{E} \). This time the variable \( z_u \) is present in three different constraints: \( \sum_{v \in e_0} z_v - z_{e_0} \leq |e_0| - 1 \), \( -z_u + z_{e'} \leq 0 \), and \( -z_u + z_{e''} \leq 0 \). The corresponding multipliers are \( \varepsilon_{e_0} = \frac{1}{\gamma} \), \( \delta_{u,e'} \), and \( \delta_{u,e''} = 1 - (\delta_{u,e'} - \frac{1}{\gamma}) \).

Therefore the coefficient of \( z_u \) in \( \pi z \leq \lfloor \pi_0 \rfloor \) is equal to

\[
\frac{1}{\gamma} - \left( \delta_{u,e'} + 1 - \delta_{u,e''} + \frac{1}{\gamma} \right) = \frac{1}{\gamma} - 1 - \frac{1}{\gamma} = -1,
\]

as it is in the left-hand side of [1].

We only need to check the coefficients of the variables \( z_e \) corresponding to the edges \( e_i \) with \( i \in [m] \). The variable \( z_{e_i} \) appears in several \( \lfloor \pi_0 \rfloor \) inequalities. Because of rules [2.1], [2.2] and [2.3], the entry in \( \pi \) corresponding to \( z_{e_i} \) is given either by \( \delta_{u,e_i} + \sum_{v \in e_i \setminus \{u\}} \delta_{v,e_i} \) if \( N(e_0 \cap e_i) \neq \emptyset \), or by \( \eta_{e_i} + \sum_{v \in e_i} \delta_{v,e_i} \). In both cases, by the definition of \( \delta_{u,e_i} \) and \( \eta_{e_i} \) respectively, we can conclude that the coefficient corresponding to \( z_{e_i} \) is equal to 1, for \( i \in [m] \).

**Claim 4.** The right-hand side of [1] coincides with \( \lfloor \pi_0 \rfloor \).

**Proof of claim.** Denote by \( K_i, i = 1, \ldots, l \), the connected components of \( \hat{G} \). For every \( i = 1, \ldots, l \), let \( p_i := |E(K_i)| \). Moreover, let us order the edges of \( E(K_i) \) such that they follow the same order in which they appear in the running intersection ordering of \( \hat{E} \). Then let \( \hat{e}_{ij} \), with \( j = 1, \ldots, p_i \), be the order of the edges of \( E(K_i) \), for every \( i \in \{1, \ldots, l\} \).

Notice that for every connected component \( K_i \), \( i = 1, \ldots, l \), only the first edge \( \hat{e}_{i1} \) satisfies \( N(\hat{e}_{i1}) = \emptyset \), thus it is the only edge that may contribute to \( \lfloor \pi_0 \rfloor \), because it is the only edge for which \( \eta_e \) is possibly non-zero. This contribution is equal to

\[
\eta_{e_{i1}} = 1 - \sum_{v \in e_{i1}} \delta_{v,e_{i1}} = 1 - \left| W^{i1} \right| \left| U^{i1} \right| \gamma,
\]

where the second equality comes from Claim [4].

By rules [1], [2.1], [2.2] and [2.3] we obtain that \( \lfloor \pi_0 \rfloor \) is equal to

\[
\left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \sum_{i=1}^{l} \eta_{e_i} = \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \sum_{i=1}^{l} \left( 1 - \delta_{u,e_i} \right) = \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \sum_{i=1}^{l} \left( \frac{|W^{i1}| - |U^{i1}|}{\gamma} \right).
\]

By [11] we know that we need (8) to be equal to \( |e_0 \setminus \bigcup_{i \in [m]} e_i| + |\{i \in [m] : N(e_0 \cap e_i) = \emptyset\}| - 1 \). Observe that (8) is equal to

\[
\left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + |\{i \in [m] : N(e_0 \cap e_i) = \emptyset\}| + \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| - \sum_{i=1}^{l} \left( |W^{i1}| - |U^{i1}| \right).
\]
Then we simply need to show that

$$\left[ \frac{|e_0|}{\gamma} - \frac{1}{\gamma} \left( 1 + \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \sum_{i=1}^l (|W^{i_1}| + |U^{i_1}|) \right) \right] = -1. \quad (9)$$

Note that $W^{i_1}$ and $U^{i_1}$ contain exactly all the nodes in $V(K_i)$. Moreover, $W^{i_1}$ and $U^{i_1}$ are disjoint and therefore $|W^{i_1}| + |U^{i_1}| = |V(K_i)|$, for every connected component $K_i$. Then, the $2l$ sets $W^{i_1}, U^{i_1}$ for $i = 1, \ldots, l$, form a partition of $e_0 \cap \bigcup_{i \in [m]} e_i$. Thus $\sum_{i=1}^l (|W^{i_1}| + |U^{i_1}|) = \left| e_0 \cap \bigcup_{i \in [m]} e_i \right|$. It follows that the left-hand side of (9) can be written as

$$\left[ \frac{|e_0|}{\gamma} - \frac{1}{\gamma} \left( 1 + \left| e_0 \setminus \bigcup_{i \in [m]} e_i \right| + \left| e_0 \cap \bigcup_{i \in [m]} e_i \right| \right) \right] = \left[ \frac{|e_0|}{\gamma} - \frac{1}{\gamma} (1 + |e_0|) \right] = \left[ -\frac{1}{\gamma} \right] = -1.$$

We can therefore conclude that the claim holds.

\[\square\]

### A.2 Proof of Theorem 2

We first show that assumption (a) in Definition 2 is equivalent to assumption (a’) defined in the proof sketch of Theorem 2. To see that (a’) implies (a), let $C’$ be a $\gamma$-cycle of $G$ with $E(C’) \subseteq \{e_1, \ldots, e_m\}$. We have $V(C’) \subseteq \bigcup_{e \in E(C’)} e \subseteq \bigcup_{i=1}^m e_i$, and (a’) implies that each node in $V(C’)$ is contained in at most two edges in $E(C’)$, thus $C’$ is also a $\beta$-cycle. Conversely, to show that (a) implies (a’), we assume by contradiction that there exists $v \in \bigcup_{i=1}^m e_i$ that is contained in three distinct edges $e_{i_1}, e_{i_2}, e_{i_3} \in \{e_1, \ldots, e_m\}$. Without loss of generality, let us assume that $i_1 < i_2 < i_3$. Let $P$ be the subsequence of $C$ from $e_{i_1}$ to $e_{i_3}$, and define $C’ = v, P, v$. By definition of $\beta$-cycle, the node $v$ is not in $V(C)$, thus $v$ does not appear in the sequence $P$. Since $v$ is contained in three different edges in $E(C’)$, we have shown that $C’$ is a $\gamma$-cycle which is not a $\beta$-cycle.

Next, we prove the three claims mentioned in the sketch of the proof.

**Claim 5.** The multipliers $\alpha_v, \beta_e, \varepsilon_e, \delta_{e,e}, \theta_f$ are non-negative.

*Proof of claim.* If $D = \emptyset$, then $T = 2$. Assume now that $D$ is nonempty. Because there is at least one edge in $E^-$ in the $\beta$-cycle, it follows that $|T_1 \cap D| \neq 0$ and $|T_k \cap D| \neq 0$, hence $T \geq 2$ in any case. Then, all multipliers defined in rule 1 and rule 2 are non-negative. Observe that, the multipliers $\theta_{f_j}$ are defined only in 1. Therefore, all multipliers defined in rule 3 are either $\frac{1}{T}$ or $\frac{T-1}{T}$, thus non-negative. Notice that $\sum_{i=p+1}^m \Delta_j = T$. Therefore, all the sums of $\Delta_j$ considered in rule 4 are between zero and $T$. Hence, also the multipliers defined in rule 4 are non-negative.

\[\square\]

Recall that $\pi z \leq [\pi_0]$ is the inequality provided by the multipliers $\alpha_v, \beta_e, \varepsilon_e, \delta_{e,e}, \theta_f$.

**Claim 6.** $\pi z$ coincides with the left-hand side of (2).

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Proof of claim. The proof of this claim is similar to the analogous one in the proof of Theorem 1. Each variable corresponding to an edge not in the cycle does not appear in (2), and it does not appear in \( \pi z \leq \lceil p_0 \rceil \) either, because its corresponding multipliers are not explicitly defined in rules 1-4 and thus are set to zero. A similar argument is true for the nodes \( v \) that do not belong to any edge of the cycle.

Observe also that the variables representing nodes that are contained in edges of \( E^+ \), but do not belong to \( S_2 \), are not present in any of the inequalities chosen with non-zero multiplier. It is fine, since these variables do not appear in the left-hand side of (2).

It is easy to check that the entries of \( \pi \) corresponding to \( f \in E^- \) and to \( v \in S_1 \cup S_2 \) satisfy the thesis. In fact, each of these variables appears in only two inequalities chosen with non-zero multiplier and these multipliers sum to 1. Indeed, variables related to \( f_i \in E^- \) are present only in the inequalities corresponding to the multipliers \( \theta_{f_2} \) and \( \beta_{f_i} \), which sum to 1 by rule 1. Similarly, each \( z_v \) for \( v \in S_1 \) is present only in one or two different \( \theta_{f_2} \), because of assumption (a'). If it is present only in one flower inequality, then the multiplier \( \alpha_v \) is chosen accordingly such that the two multipliers sum to 1, by rule 2. If, on the other hand, \( z_v \) is contained in two flower inequalities, we must analyze two different cases: whether assumption (d-1) or assumption (d-2) holds. Suppose that assumption (d-1) applies. This implies that the multipliers of the two flower inequalities sum to 1, because of rule 1. Then, assume that assumption (d-2) holds. In this case, we have \( T = 2 \). If \( D \) is non-empty, this follows by assumption (a') while if \( D \) is empty, we have \( \sum_{i=1}^k |T_i \cap D| = 0 \), thus we have set \( T := 2 \). In any case, the two flower inequalities both have multiplier \( \frac{1}{2} \) no matter if the edges in \( E^- \) that contain \( v \) are odd or even. Therefore, the coefficient of \( z_v \) in \( \pi z \) is equal to 1.

The rest of the proof works both for assumption (d-1) and assumption (d-2). Consider \( v_i \in S_2 \). Then the only non-zero multipliers that affect its component in \( \pi \) are \( \delta_{v_i, e_{i-1}} \) and \( \delta_{v_i, e_i} \). By rules 3.2, 4.2, the sum of these two values is equal to 1. Next, we focus on variables \( z_e \), with \( e \in E^+ \). We start analyzing the coefficients of the variables corresponding to \( e \in E^+ \setminus D \). We restrict our attention on one maximal length substring of \( E^+ \setminus D \). Let \( e_1, \ldots, e_{i+h} \) be the substring, and note that \( e_{i-1} = f_j \) for some \( j \). Recall that, by assumption (b) only \( e_i \) and \( e_{i+h} \) are adjacent to edges that belong to \( E^- \). Observe that, if \( e_i = e_{i+h} \), then \( z_e \) appears with non-zero coefficient only in the two flower inequalities \( \theta_{f_i} \) and \( \theta_{f_{i+1}} \). By rule 1 the corresponding multipliers sum to 1. Assume now \( e_i \neq e_{i+h} \). The variable \( z_{e_{i+h}} \) is present in just two inequalities: the flower inequality \( \theta_{f_j} \), and \( \delta_{v_{i+h}, e_i} \). By rule 3.1 the sum of the related multipliers is equal to 1. The coefficients of \( z_{e_i}, \ldots, z_{e_{i+h-1}} \) are correct in \( \pi z \leq \lceil p_0 \rceil \), since these variables are present in only two inequalities with non-zero multipliers and, by rule 3.2, they sum to 1. Consider the entry of \( \pi \) corresponding to the variable \( z_{e_{i+h}} \). This variable appears in two inequalities: one chosen with multiplier equal to \( \delta_{v_{i+h}, e_i} \), and in the flower inequality corresponding to \( e_{i+h} = f_{j+1} \). By rule 1 the multiplier of this flower inequality is equal to \( 1 - \theta_{f_j} \), since there are no edges of \( E^- \) between \( e_i \) and \( e_{i+h} \). Hence, the coefficient of \( z_{e_{i+h}} \) in \( \pi z \leq \lceil p_0 \rceil \) is correctly equal to 1, as it is in the left-hand side of (2).

We check the correctness of the coefficients of \( z_{e_i} \), with \( e_i \in D \), thus we assume \( D \) non-empty. Consider the first edge of \( D \), that is \( e_{p+1} \). Thanks to assumption (c) the variable corresponding to it appears in \( \Delta \) flower inequalities, whose centers are edges \( f_j \) with \( j \) odd, and in \( (\delta_{v_{p+2}, e_{p+1}}) \) if \( |D| \geq 2 \). By rules 1, 4.3, all these multipliers sum to 1. Then consider edges \( e_{p+i} \in D \setminus \{ e_{p+1}, e_m \} \). Because of assumption (c) each of these variables is present in \( \Delta \) flower inequalities, with centers edges \( f_j \in E^- \) with \( j \) odd, and \( \delta_{v_{p+i}, e_{p+1}} \), \( \delta_{v_{p+i+1}, e_{p+i}} \). Then
this sum is equal to
\[
\frac{\Delta_{p+i}}{T} + \delta_{v_{p+i},e_{p+i}} + \delta_{v_{p+i+1},e_{p+i+1}} = \frac{\Delta_{p+i}}{T} + \frac{\sum_{j=p+1}^{p+i-1} \Delta_j}{T} + 1 - \frac{\sum_{j=p+1}^{p+i} \Delta_j}{T} = \\
= \frac{\sum_{j=p+1}^{p+i} \Delta_j}{T} + 1 - \frac{\sum_{j=p+1}^{p+i} \Delta_j}{T} = 1,
\]
where the first equality comes from the definition of the multipliers in rule \([4.2]\). We only need to verify what happens for \(z_{e_m}\). This variable is present in \(\Delta_m\) flower inequalities all with multiplier \(\frac{1}{T}\), and \((\delta_{v_{m},e_m})\). Then, by rules \([11, 13]\) its coefficient is equal to
\[
\frac{\Delta_m}{T} + \frac{\sum_{j=p+1}^{m-1} \Delta_j}{T} = \frac{\sum_{j=p+1}^{m} \Delta_j}{T} = 1,
\]
where last equality follows from \(\sum_{i=p+1}^{m} \Delta_i = T\).

\[\Diamond\]

Claim 7. The right-hand side of \((\ref{eq:claim7})\) is equal to |\(\pi_0\)|.

Proof of claim. Assume first that assumption \([d-1]\) holds. We obtain the following formula for |\(\pi_0\)|, where the first two sums come from inequalities \((\theta_{f_i})\), depending on \(i\) being odd or even, and the last two from \([\alpha]\):

\[
\left|\sum_{i \in \{k\}} \frac{1}{T} \left| f_i \setminus \bigcup_{e \in T_i} e \right| + \left| T_i \right| - 1 \right| + \sum_{i \in \{k\}} \frac{T-1}{T} \left| f_i \setminus \bigcup_{e \in T_i} e \right| + \left| T_i \right| - 1 + \\
+ \sum_{i \in \{k\}} \frac{T-1}{T} \left| f_i \setminus \left( \bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right| + \sum_{i \in \{k\}} \frac{1}{T} \left| f_i \setminus \left( \bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right|.
\]

Remind that \(S_1\) is the set of points contained only in edges of \(E^-\). Then observe that \(|S_1|\) is given exactly by

\[
\left|\sum_{i \in \{k\}} \frac{1}{T} \left| f_i \setminus \bigcup_{e \in T_i} e \right| + \sum_{i \in \{k\}} \frac{T-1}{T} \left| f_i \setminus \bigcup_{e \in T_i} e \right| + \\
+ \sum_{i \in \{k\}} \frac{T-1}{T} \left| f_i \setminus \left( \bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right| + \sum_{i \in \{k\}} \frac{1}{T} \left| f_i \setminus \left( \bigcup_{e \in T_i} e \cup \bigcup_{f \in E^- \setminus \{f_i\}} f \right) \right| \right|. \tag{10}
\]

In fact, each node \(v\) of \(S_1\) is either contained in two edges of \(E^-\), one odd and one even, or only in one edge of \(E^-\), by assumptions \([a']\) and \([d-1]\). In the first case, \(v\) is accounted for in the first two sums one single time, once with multiplier \(\frac{1}{T}\) and once with \(\frac{T-1}{T}\). Summing these two quantities, we obtain that every node \(v \in S_1\) of this type contributes by \(1\) in \((10)\). Consider the second case, i.e. \(v\) is in only one edge of \(E^-\). If \(v\) belongs to an odd edge, it contributes by \(\frac{1}{T}\) in the first sum and by \(\frac{T-1}{T}\) in the third sum. Otherwise, it contributes by \(\frac{T-1}{T}\) in the second sum.
and by \( \frac{1}{T} \) in the last sum. Hence, also when \( v \) is contained in just one edge, by summing these two terms we see that \( v \) contributes by 1 in \([10]\). Then \(|S_1| \leq [10]\). To prove that \([10] \leq |S_1|\), it suffices to see that only nodes in \( S_1 \) are considered in \([10]\) and that no node in considered twice with the same multiplier.

Therefore, we get the following expression for \(|\pi_0|\):

\[
|S_1| + \sum_{i \in [k]} \frac{1}{T}(|T_i \cap D| + |T_i \setminus D| - 1) + \sum_{i \in [k]} \frac{T-1}{T}(|T_i \setminus D| - 1) =
\]

\[
= |S_1| + \left[ 1 + \sum_{i \in [k]} \frac{|T_i \setminus D|}{T} - \frac{1}{T} \left( \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) + \sum_{i \in [k]} \frac{T-1}{T} |T_i \setminus D| - \frac{T-1}{T} \left\lfloor \frac{k}{2} \right\rfloor \right] =
\]

\[
= |S_1| + \left[ 1 + \sum_{i \in [k]} \frac{|T_i \setminus D|}{T} + \sum_{i \in [k]} \frac{T-1}{T} |T_i \setminus D| - \left\lfloor \frac{k}{2} \right\rfloor - \frac{1}{T} \right],
\]

where the first line comes also from the fact that \(|T_i \cap D| = 0\) for all \( i \) even, by assumption \( (c) \) and the first equality follows from the definition of \( T \).

Because of assumption \( (b) \) notice that \( \sum_{i \in [k]} |T_i \setminus D| = \sum_{i \in [k]} |T_i \setminus D| \), since both sums count how many times the sign of the variables \( z_e \) changes from negative to positive in the part of the \( \beta \)-cycle corresponding to the edges of \( E(C) \setminus D \). Let us denote this quantity by \( \sigma \). Then \(|\pi_0|\) is equal to

\[
|S_1| + 1 - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{\sigma}{T} + \frac{T-1}{T} \sigma - \frac{1}{T} \right\rfloor = |S_1| + 1 - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{\sigma - 1}{T} \right\rfloor =
\]

\[
= |S_1| + 1 - \left\lfloor \frac{k}{2} \right\rfloor + \sigma + \left\lfloor \frac{1}{T} \right\rfloor = |S_1| + 1 - \left\lfloor \frac{k}{2} \right\rfloor + \sigma - 1 = |S_1| - \left\lfloor \frac{k}{2} \right\rfloor + \sigma.
\]

In order for it to be equal to the right-hand side of \( (2) \), we need to check that \( -\left\lfloor \frac{k}{2} \right\rfloor + \sigma = -|\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^+\}| + \left\lfloor \frac{k}{2} \right\rfloor \). This is equivalent to \( \sigma + |\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^-\}| = 2 \left\lfloor \frac{k}{2} \right\rfloor = k - 1 \).

The latter equality is true because every edge in \( E^- \), except the last one, is either accounted for in \(|\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^+\}| \) or in \( \sigma \), if its succeeding edge belongs to \( E^+ \). Hence \( \sigma + |\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^-\}| = k - 1 \), since the left-hand side counts all edges of \( E^- \) except for the last one, because its successive edge is in \( D \).

Now assume that \( (d-2) \) holds. In this case \(|\pi_0|\) is given by

\[
\left| \sum_{i \in [k]} \frac{1}{2} \left( f_i \setminus \bigcup_{e \in T_i} e \right) + |T_i| - 1 \right| + \sum_{i \in [k]} \frac{1}{2} \left( f_i \setminus \bigcup_{e \in T_i} e \right) + |T_i| - 1 \right| + \sum_{i \in [k]} \frac{1}{2} \left( f_i \setminus \bigcup_{e \in T_i} e \bigcup_{f \in E^+ \setminus \{f_i\}} f \right) \right|.
\]

and, by similar arguments to the previous case, \( S_1 \) in equal to

\[
\sum_{i \in [k]} \frac{1}{2} \left| f_i \setminus \bigcup_{e \in T_i} e \bigcup_{f \in E^+ \setminus \{f_i\}} f \right| + \sum_{i \in [k]} \frac{1}{2} \left| f_i \setminus \bigcup_{e \in T_i} e \bigcup_{f \in E^- \setminus \{f_i\}} f \right|.
\]

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Thus, at this point we obtain that \(|\pi_0|\) is equal to
\[
|S_1| + \sum_{\substack{i \in [k] \\
i \text{odd}}} \frac{1}{2}(|T_i \cap D| + |T_i \setminus D| - 1) + \sum_{\substack{i \in [k] \\
i \text{even}}} \frac{1}{2}(|T_i \setminus D| - 1).
\]

Note that in this case the only indices \(i\) for which \(|T_i \cap D| \neq 0\) are \(i = p + 1, m\). More precisely they are exactly equal to 1. Because of this, the calculations are analogous to the ones of the first part of the proof. We can conclude that the right-hand side of (2) coincides with \(|\pi_0|\) in every case. 

\[\diamondsuit\]

A.3 Proof of Proposition 1

In the first part of this proof we assume \(G = (V, E)\), with \(E = \{e_1, e_2, e_3\}\) such that \(|e_1 \cap e_2| = |e_2 \cap e_3| = |e_1 \cap e_3| = 1\). We show that the statement holds by induction on \(|V|\). The base case occurs when \(|V| = 3\), that is when \(G\) is a graph. The claim for this case was proved by Padberg in [16]. Thus, we assume the thesis holds when \(|V| \leq k - 1\), for some \(k \in \mathbb{Z}\) with \(k \geq 4\). Now consider \(G\) such that \(|V| = k\). Let \(\bar{v}\) be the only node in \(e_1 \cap e_2\).

We define \(F^1 := \{z \in \text{MP}_G : z_0 = 0\}\), \(F^2 := \{z \in \text{MP}_G : z_0 = 1\}\). By definition, \(\text{MP}_G\) is a binary polytope, hence \(\text{MP}_G\) is the convex hull of the union of \(F^1\) and \(F^2\). Observe that we know the perfect formulations of \(F^1, F^2\). In fact the underlying hypergraph of \(F^1\) is obtained from \(G\) is \((V \setminus \{\bar{v}\}, \{e_3\})\). Note that this hypergraph is Berge-acyclic and by Theorem 7 in [8] we know that its perfect formulation coincides with its standard linearization:

\[
z_{e_1} = z_{e_2} = z_0 = 0 \\
0 \leq z_v \leq 1 \quad \forall v \in e_1 \cup e_2 \setminus \{\bar{v}\} \cup e_3 \\
-z_{e_3} \leq 0 \\
\sum_{v \in e_3} z_v - z_{e_3} \leq |e_3| - 1 \\
z_v \leq 1 \quad \forall v \in e_3 \\
-z_v + z_{e_3} \leq 0 \quad \forall v \in e_3.
\]

Instead, the underlying hypergraph of \(F^2\) is obtained from \(G\) by deleting node \(\bar{v}\), therefore it is Berge-acyclic as well. Then, \(F^2\) is described by the standard linearization:

\[
z_0 = 1 \\
-z_e \leq 0 \quad \forall e \in \{e_1, e_2, e_3\} \\
\sum_{v \in e_1 \setminus \{\bar{v}\}} z_v - z_{e_1} \leq |e_1| - 2 \\
\sum_{v \in e_2 \setminus \{\bar{v}\}} z_v - z_{e_2} \leq |e_2| - 2 \\
\sum_{v \in e_3} z_v - z_{e_3} \leq |e_3| - 1
\]
We apply Balas’ formulation [1] for the union of polytopes and obtain a perfect formulation for MP\(_G\) in the extended space. In order to achieve the thesis we use Fourier elimination on the additional variables \(z^1, z^2, \lambda\). The perfect formulation of MP\(_G\) with the extra variables is

\[
\begin{align*}
z & = z^1 + z^2 \\
0 & \leq \lambda \leq 1 \\
z^1_{e_1} - z^1_{e_2} = & \begin{cases} 
1 & \forall v \in \{\overline{v}\} \\
0 & \forall v \in e_1 \setminus \{\overline{v}\} \\
-1 & \forall v \in e_2 \setminus \{\overline{v}\} \\
0 & \forall v \in e_3 \setminus \{\overline{v}\} \\
\end{cases}
\end{align*}
\]

For every \(v \in e_3\)\( \setminus \{\overline{v}\}\)

\[
\begin{align*}
0 & \leq z^1_v \leq 1 - \lambda \\
0 & \leq z^1_v \leq 0 \\
\sum_{v \in e_3} z^1_v - z^1_{e_3} & \leq (|e_3| - 1)(1 - \lambda) \\
\sum_{v \in e_1 \setminus \{\overline{v}\}} z^2_v - z^2_{e_1} & \leq (|e_1| - 2)\lambda \\
\sum_{v \in e_2 \setminus \{\overline{v}\}} z^2_v - z^2_{e_2} & \leq (|e_2| - 2)\lambda \\
\sum_{v \in e_3} z^2_v - z^2_{e_3} & \leq (|e_3| - 1)\lambda \\
z^2_v & \leq \lambda & \forall v \in V \setminus \{\overline{v}\} \\
-z^2_v + z^2_{e_1} & \leq 0 & \forall v \in e_1 \setminus \{\overline{v}\} \\
-z^2_v + z^2_{e_2} & \leq 0 & \forall v \in e_2 \setminus \{\overline{v}\} \\
-z^2_v + z^2_{e_3} & \leq 0 & \forall v \in e_3, \\
\end{align*}
\]

where the variables \(z^1, z^2\) arise from the systems defining \(F^1, F^2\) respectively.

From (11) we see immediately that \(z_{\overline{v}} = z^1_{\overline{v}} + z^2_{\overline{v}} = \lambda\). Similarly \(z_{e_1} = z^1_{e_1}\) and \(z_{e_2} = z^2_{e_2}\). Hence, projecting out the variables \(z^1_{e_1}, z^2_{e_1}, z^1_{e_2}, z^2_{e_2}, z^1_{\overline{v}}, z^2_{\overline{v}}\) is trivial.

Then, we begin projecting the variables coming from \(F^1\), starting from \(z^1_v\) for every \(v \in e_1 \cup e_2 \setminus \{\overline{v}\} \cup e_3\). These variables only appear in (11), (12). We write these constraints in form of inequality, splitting them in inequalities in which \(z^1_v\) has negative coefficient and inequalities in which its coefficient is positive.

\[
\begin{align*}
-z^1_v & \leq 0 \\
z_v - z^1_v - z^2_v & \leq 0
\end{align*}
\]
\[ z^1_v \leq 1 - \lambda \\
- z_v + z^1_v + z^2_v \leq 0 \]

Therefore from the Fourier elimination for these variables we obtain the following inequalities:

\[ - z_v + z^2_v \leq 0 \quad \forall v \in e_1 \cup e_2 \ \backslash \ \{ \bar{v} \} \cup e_3 \]
\[ z_v - z^2_v \leq 1 - \lambda \quad \forall v \in e_1 \cup e_2 \ \backslash \ \{ \bar{v} \} \cup e_3. \]

Next, we deal with \( z^1_{e_3} \). This variable is present in (11), (13), (14), (16). By splitting the inequalities in two sets we have:

\[ - z^1_{e_3} \leq 0 \]
\[ \sum_{v \in e_3} z^1_v - z^1_{e_3} \leq (|e_3| - 1)(1 - \lambda) \]
\[ z_{e_3} - z^1_{e_3} - z^2_{e_3} \leq 0 \]

\[ - z^1_v + z^1_{e_3} \leq 0 \quad \forall v \in e_3 \]
\[ - z_{e_3} + z^1_{e_3} + z^2_{e_3} \leq 0. \]

After summing every pair of inequalities in which \( z^1_{e_3} \) has different sign, we only keep in the formulation the following inequalities:

\[ \sum_{v \in e_3} z^1_v - z_{e_3} + z^2_{e_3} \leq (|e_3| - 1)(1 - \lambda) \quad (17) \]
\[ - z^1_v + z_{e_3} - z^2_{e_3} \leq 0 \quad \forall v \in e_3 \quad (18) \]
\[ - z_{e_3} + z^1_{e_3} + z^2_{e_3} \leq 0 \quad \forall v \in e_3, \]

as \( - z^1_v \leq 0 \) for every \( v \in e_3 \) is implied by the last two inequalities of the above system, and \( \sum_{v \in e_3 \backslash \{ w \}} z^1_v \leq (|e_3| - 1)(1 - \lambda) \) is implied by (15).

It remains to project out the variables \( z^1_v \), with \( v \in e_3 \), in order to have eliminated all the variables arising from \( F^1 \). These variables appear in (11), (15), (17), (18):

\[ - z^1_v + z_{e_3} - z^2_{e_3} \leq 0 \quad (19) \]
\[ z_v - z^1_v - z^2_v \leq 0 \quad (20) \]

\[ z^1_v \leq 1 - \lambda \quad (21) \]
\[ - z_v + z^1_v + z^2_v \leq 0 \quad (22) \]
\[ \sum_{v \in e_3} z^1_v - z_{e_3} + z^2_{e_3} \leq (|e_3| - 1)(1 - \lambda). \quad (23) \]

Observe that when we projected out the variables \( z^1_v \), with \( v \in e_1 \cup e_2 \ \backslash \ \{ \bar{v} \} \cup e_3 \), we did not impose an elimination order among these variables. This is because no two different variables \( z^1_v, z^1_w \), with \( v \neq w \), appeared in the same inequality. However, it is not the case now because of the presence of (23). Then, let us see how the inequalities change after we eliminate one specific variable, let it be \( z^1_{\hat{v}} \) such that \( \hat{v} \in e_3 \). We leave in the formulation the inequalities:

\[ - z_{\hat{v}} + z^2_{\hat{v}} + z_{e_3} - z^2_{e_3} \leq 0 \quad (24) \]
\[ z_v - z_v^2 \leq 1 - \lambda \]  
\[ z_\hat{v} - z_\hat{v}^2 + \sum_{v \in e_3 \setminus \{\hat{v}\}} z_v^2 - z_{e_3} + z_{e_3}^2 \leq (|e_3| - 1)(1 - \lambda). \]

In fact, the other inequalities are redundant. Indeed, the inequality obtained by (19)+(21) is implied by (24)+(25), and (19)+(23) is implied by (15). Moreover, (20)+(22) provides a trivial inequality, that is \( 0 \leq 0 \).

Note that (26) contains all the remaining \( z_v^1 \) variables, all of them with coefficient +1. Remind that, by Fourier elimination, we can sum (26) with only inequalities in which \( -z_v^1 \) is present, in order to project out the remaining variables. In the system there are only two inequalities of this type: \(-z_v^1 + z_{e_3} - z_{e_3}^2 \leq 0 \) and \( z_v - z_v^1 - z_v^2 \leq 0 \). Similarly to the case in which we projected out \( z_v^1 \), we obtain a redundant inequality when we sum (26) with \(-z_v^1 + z_{e_3} - z_{e_3}^2 \leq 0 \), for the next chosen \( v \) in the elimination order. Hence, the only way in which (26) may lead to a non-redundant inequality is by summing it with \( z_v - z_v^2 - z_v^2 \leq 0 \). This argument holds for any node of \( e_3 \) in the elimination order. Then, we can conclude that after we eliminate all the variables \( z_v^1 \), with \( v \in e_3 \), (17) has become \( \sum_{v \in e_3} (z_v - z_v^2) - z_{e_3} + z_{e_3}^2 \leq (|e_3| - 1)(1 - \lambda) \). Moreover, observe that every time we remove a variable \( z_v^1 \), we obtain the corresponding inequalities (24), (25).

We are done with the variables resulted from \( F^1 \). At this point the system has become:

\[
\begin{align*}
  z_\hat{v} &= \lambda \\
  0 &\leq \lambda \leq 1 \\
  -z_{e_1} &\leq 0 \\
  -z_{e_2} &\leq 0 \\
  -z_{e_3}^2 &\leq 0 \\
  z_{e_3}^2 + z_{e_3} &\leq 0 \\
  z_v^2 &\leq \lambda \quad \forall v \in V \setminus \{\hat{v}\} \\
  -z_v + z_v^2 &\leq 0 \quad \forall v \in e_1 \cup e_2 \setminus (\hat{v} \cup e_3) \\
  z_v - z_v^2 &\leq 1 - \lambda \quad \forall v \in V \setminus \{\hat{v}\} \\
  -z_v^2 + z_{e_1} &\leq 0 \quad \forall v \in e_1 \setminus \{\hat{v}\} \\
  -z_v^2 + z_{e_2} &\leq 0 \quad \forall v \in e_2 \setminus \{\hat{v}\} \\
  -z_v^2 + z_{e_3} &\leq 0 \quad \forall v \in e_3 \\
  \sum_{v \in e_1 \setminus \{\hat{v}\}} z_v^2 - z_{e_1} &\leq (|e_1| - 2)\lambda \\
  \sum_{v \in e_2 \setminus \{\hat{v}\}} z_v^2 - z_{e_2} &\leq (|e_2| - 2)\lambda \\
  \sum_{v \in e_3} z_v^2 - z_{e_3}^2 &\leq (|e_3| - 1)\lambda \\
  \sum_{v \in e_3} (z_v - z_v^2) - z_{e_3} + z_{e_3}^2 &\leq (|e_3| - 1)(1 - \lambda).
\end{align*}
\]

Next, we deal with variables coming from \( F^2 \), starting with \( z_{e_3}^2 \). For simplicity, we display the
inequalities in which \( z_{e_3}^2 \) appears. We divide them in two sets:

\[
\begin{align*}
-z_{e_3}^2 & \leq 0 \\
-z_v + z_{e_3}^2 - z_{e_3}^2 & \leq 0 & \forall v \in e_3 \\
\sum_{v \in e_3} z_v^2 - z_{e_3}^2 & \leq \left| e_3 \right| - 1 \lambda
\end{align*}
\]

After applying Fourier elimination on this variable we keep the following inequalities:

\[
\begin{align*}
-z_{e_3} & \leq 0 \\
-z_v^2 & \leq 0 & \forall v \in e_3 \setminus (e_1 \cup e_2) \\
-z_v + z_{e_3}^2 & \leq 0 & \forall v \in e_3 \\
-z_v + z_{e_3} & \leq 0 & \forall v \in e_3 \\
-z_v + z_{e_3}^2 - z_w^2 + z_{e_3} & \leq 0 & \forall v, w \in e_3, v \neq w \\
\sum_{v \in e_3} (z_v - z_v^2) - z_{e_3} & \leq \left| e_3 \right| - 1 \lambda \\
\sum_{v \in e_3} z_v^2 - z_{e_3} & \leq \left| e_3 \right| - 1 \lambda \\
\sum_{v \in e_3} z_v - z_{e_3} & \leq \left| e_3 \right| - 1
\end{align*}
\]

where (33) comes from (29)+(31) when the nodes \( v \) in the two inequalities are distinct. We discarded (29)+(32), as it is implied by (28). Similarly, we did not write (30)+(31), since it is entailed by (27). Let us remark that we decide to keep some redundant inequalities in the system as long as they contain only variables in the original space, like for example (34), since they will be useful in the next arguments.

Now we continue with the projection of the other variables. We focus first on \( z_v^2 \) with \( v \in e_1 \setminus (\{\bar{v}\} \cup e_3) \). For \( v \in e_2 \setminus (\{\bar{v}\} \cup e_3) \) the calculations are similar, it suffices to swap the roles of \( e_1 \) and \( e_2 \), and therefore we are not going to repeat the computations. As before, we write here the inequalities involving such \( z_v^2 \):

\[
\begin{align*}
z_v - z_v^2 & \leq 1 - \lambda & \forall v \in e_1 \setminus (\{\bar{v}\} \cup e_3) \\
-z_v^2 + z_{e_1} & \leq 0 & \forall v \in e_1 \setminus (\{\bar{v}\} \cup e_3)
\end{align*}
\]

\[
\begin{align*}
z_v^2 & \leq \lambda & \forall v \in e_1 \setminus (\{\bar{v}\} \cup e_3) \\
-z_v + z_v^2 & \leq 0 & \forall v \in e_1 \setminus (\{\bar{v}\} \cup e_3) \\
\sum_{v \in e_1 \setminus \{\bar{v}\}} z_v^2 - z_{e_1} & \leq \left| e_1 \right| - 2) \lambda
\end{align*}
\]
Observe that here, like when we projected out variables \( z_v^1 \) for \( v \in e_3 \), there is one inequality, (38), that contains all the variables \( z_v^2 \). Then, let us fix a node \( \hat{v} \in e_1 \setminus (\{\hat{v}\} \cup e_3) \). After performing Fourier elimination on this variable, we keep the following set of inequalities:

\[
\begin{align*}
    z_{\hat{v}} &\leq 1 \\
    z_{e_1} &\leq \lambda \\
    -z_{\hat{v}} + z_{e_1} &\leq 0 \\
    z_{\hat{v}} + \sum_{v \in e_1 \setminus \{\hat{v}\}} z_v^2 - z_{e_1} &\leq 1 + (|e_1| - 3)\lambda.
\end{align*}
\]

(39)

In fact, (35) + (37) provides the trivial inequality \( 0 \leq 1 - \lambda \), and (36) + (38) produces \( \sum_{v \in e_1 \setminus \{\hat{v}\}} z_v^2 \leq (|e_1| - 2)\lambda \), which is implied by (27).

Observe that (39) contains all the \( z_v^2 \) variables that still need to be projected out, with \( v \in e_1 \setminus (\{\hat{v}\} \cup e_3) \). When we sum (39) with (36), chosen for the variable \( z_{\hat{v}} \) that we are currently eliminating, we obtain a redundant inequality. In fact, it is dominated by the bounds \( z_{\hat{v}} \leq 1 \), for all \( w \in e_1 \setminus (\{\hat{v}\} \cup e_3) \) for which \( z_w^2 \) has already been projected out, and \( z_{\hat{v}}^2 \leq \lambda \), for the variables that are still to be eliminated. Hence, (39) can lead to a non-redundant inequality only when it is summed with (35). This holds for every \( v \in e_1 \setminus (\{\hat{v}\} \cup e_3) \), and therefore we can conclude that (38) leads to

\[
\sum_{v \in e_1 \setminus \{\hat{v}\}} z_v + \sum_{v \in e_1 \cap e_3} z_v^2 - z_{e_1} \leq |e_1| - 2.
\]

Observe that the second sum actually involves only one variable, since \( |e_1 \cap e_3| = 1 \) by assumption.

At this point, the variables left to project out are \( z_v^2 \), for \( v \in e_3 \), and \( \lambda \). We focus on \( z_v^2 \), with \( v \in e_3 \setminus (e_1 \cup e_2) \). Consider a specific node \( \hat{v} \) in this set. Then \( z_{\hat{v}}^2 \) is present in the following system of inequalities:

\[
\begin{align*}
    -z_{\hat{v}}^2 &\leq 0 \\
    z_{\hat{v}} - z_{\hat{v}}^2 &\leq 1 - \lambda \\
    -z_{w} + z_w^2 - z_{\hat{v}}^2 + z_{e_3} &\leq 0 \quad \forall w \in e_3, \ w \neq \hat{v} \\
    \sum_{v \in e_3} (z_v - z_{\hat{v}}) - z_{e_3} &\leq (|e_3| - 1)(1 - \lambda) \\
    -z_{\hat{v}} + z_{\hat{v}}^2 &\leq 0 \\
    z_{\hat{v}}^2 &\leq \lambda \\
    -z_{w} + z_w^2 - z_{\hat{v}}^2 + z_{e_3} &\leq 0 \quad \forall w \in e_3, \ w \neq \hat{v} \\
    \sum_{v \in e_3} z_v^2 - z_{e_3} &\leq (|e_3| - 1)\lambda.
\end{align*}
\]

We first provide the inequalities that we do not discard after this iteration of Fourier elimination and next we explain why we removed the other inequalities. We keep:

\[
\begin{align*}
    z_{\hat{v}} &\leq 1 \\
    z_{\hat{v}} + \sum_{e_1 \setminus \{\hat{v}\}} (z_v - z_{\hat{v}}^2) - z_{e_3} &\leq (|e_3| - 1)(1 - \lambda) + \lambda
\end{align*}
\]

(48)
\begin{align*}
z_0 + \sum_{v \in 3 \backslash \{\emptyset\}} z_v^2 - z_{e_3} & \leq 1 + (|e_3| - 2)\lambda. \\
\end{align*}

(49)

All inequalities coming from (10) are redundant. Inequality (10)+(14) is equivalent to the sum of \(-z_0 + z_{e_3} \leq 0\) and \(-z_{e_3} \leq 0\), however these two inequalities are already in the system and therefore (10)+(14) is implied by \(-z_0 \leq 0\), which is always true, and (10)+(15) is implied by \(-z_0 + z_{e_3} \leq 0\) and \(-z_0^2 \leq 0\) for all \(w \in 3, w \neq \emptyset\). Observe that if \(w \in 3 \backslash (e_1 \cup e_2)\), then \(-z_0^2 \leq 0\) is in the system, otherwise \(-z_0^2 \leq 0\) is obtained by the sum of \(-z_0^2 + z_{e_3} \leq 0\) and \(-z_{e_3} \leq 0\), depending on \(w = 3 \cap e_1\) or \(w = 3 \cap e_2\). Furthermore, (10)+(17) is implied by (27) and \(-z_{e_3} \leq 0\).

Consider inequality (11). When summed with (14), it provides the trivial inequality \(0 \leq 1 - \lambda\). Additionally, (11)+(16) is redundant as it is implied by \(-z_0 + z_0^2 - z_0^2 + z_{e_3} \leq 0\) and \(z_0 - z_0^2 \leq 1 - \lambda\), for \(v \neq w, v \in 3\). Such a \(v\) exists, since we have not projected out the nodes in the intersections \(e_1 \cap 3\) and \(e_2 \cap 3\) yet.

Also all inequalities deriving from (12) are redundant. The calculations regarding (12)+(14), (12)+(15), (12)+(16) are similar to the previous cases in which we considered (10). Inequality (12)+(17) is implied instead by \(-z_0^2 + z_0^2 \leq 0\) and (27), for all \(w \neq \emptyset\).

Then, let us analyze what happens to (13). The sum (13)+(14) can be obtained by summing (28) and \(-z_{e_3} \leq 0\). Similarly, (13)+(16) is implied by \(-z_0^2 \leq 0\), for all \(w \neq \emptyset\) and (28). Lastly, observe that (13)+(17) coincides with (31), which is already in the system.

Now, observe that both (18) and (19) contain all the variables \(z_{e_3}^2\) corresponding to the other nodes in \(3 \backslash (e_1 \cup e_2)\). Like for inequalities (26), (39), notice that (18) and (19) might become non-redundant only when they are summed with respectively (15), (11), where these inequalities are chosen for the node \(v\) whose variable is currently being eliminated. Therefore, after projecting out all variables corresponding to nodes in \(3 \backslash (e_1 \cup e_2)\), (18) and (19) become

\begin{align*}
z_v + \sum_{v \in 3 \backslash (e_1 \cup e_2)} \sum_{e_3 \cap (e_1 \cup e_2)} (z_v - z_0^2) - z_{e_3} & \leq |3 \backslash (e_1 \cup e_2)| + 1 - \lambda \\
& \sum_{v \in 3 \backslash (e_1 \cup e_2)} z_v + \sum_{e_3 \cap (e_1 \cup e_2)} z_v^2 - z_{e_3} & \leq |3 \backslash (e_1 \cup e_2)| + \lambda. \\
\end{align*}

For ease of notation, let \(w_1 = e_1 \cap 3, w_2 = e_2 \cap 3\). There are only three variables left to eliminate: \(z_{w_1}, z_{w_2}, \lambda\). Consider \(z_{w_1}\). It is present in:

\begin{align*}
z_{w_1} - z_{w_1}^2 & \leq 1 - \lambda \tag{50} \\
-z_{w_1}^2 + z_{e_1} & \leq 0 \tag{51} \\
-z_{w_2}^2 + z_{w_2} - z_{w_1}^2 + z_{e_3} & \leq 0 \tag{52} \\
\sum_{v \in 3 \backslash \{w_1, w_2\}} z_v + z_{w_1} - z_{w_1}^2 + z_{w_2} - z_{w_2}^2 - z_{e_3} & \leq |3 \backslash (e_1 \cup e_2)| + 1 - \lambda \tag{53}
\end{align*}

\begin{align*}
z_{w_1}^2 & \leq \lambda \tag{54} \\
-z_{w_1}^2 + z_{w_1}^2 & \leq 0 \tag{55} \\
-z_{w_1}^2 + z_{w_1}^2 - z_{w_2}^2 + z_{e_3} & \leq 0 \tag{56} \\
\sum_{v \in e_1 \backslash \{\emptyset, w_1\}} z_v + z_{w_1} - z_{e_1} & \leq |e_1| - 2 \tag{57}
\end{align*}
\[
\sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v + z_{w_1}^2 + z_{w_2}^2 - z_{e_3} \leq |e_3 \setminus (e_1 \cup e_2)| + \lambda. \tag{58}
\]

We keep these inequalities in the formulation:

\[
\begin{align*}
\sum_{v \in e_1 \setminus \{v\}} z_v & \leq 1 \\
-z_{w_1}^2 + z_{e_3} & \leq 1 - \lambda \\
\sum_{v \in e_3 \setminus \{w_2\}} z_v - z_{e_1} & \leq |e_1| - 1 - \lambda \tag{59} \\
\sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v + z_{w_1}^2 + z_{w_2}^2 - z_{e_3} & \leq |e_3 \setminus e_2| \\
-z_{w_1} + z_{e_1} & \leq 0 \tag{60} \\
-z_{w_1} + z_{e_1} & \leq 0 \tag{61} \\
\sum_{v \in e_3 \setminus \{w_1\}} z_v - z_{w_2} + z_{w_2}^2 - z_{e_1} + z_{e_3} & \leq |e_1| - 2 \\
\sum_{v \in e_3 \setminus \{w_2\}} z_v + z_{w_2}^2 + z_{e_1} - z_{e_3} & \leq |e_3 \setminus (e_1 \cup e_2)| + \lambda \tag{62} \\
\sum_{v \in e_3 \setminus \{w_2\}} z_v + z_{w_2}^2 - z_{w_2}^2 - z_{e_3} & \leq |e_3 \setminus e_2| \tag{63} \\
\sum_{v \in e_1 \setminus \{v, w_1\}} z_v + z_{w_2}^2 - z_{w_2}^2 - z_{e_1} - z_{e_3} & \leq |e_3 \setminus (e_1 \cup e_2)| + |e_1| - 1 - \lambda. \tag{64}
\end{align*}
\]

All the other inequalities are redundant, we check them one by one. Inequality (50)+ (55) is equal to \(0 \leq 1 - \lambda\), which is always true since \(0 \leq \lambda \leq 1\). This is the only inequality certainly redundant arising from (50). Then, let us move on to (51). The result of (51)+ (54) is equal to \(0 \leq 1\), and \(z_{v_2} \leq 1\). Finally, inequality (52)+ (55) is implied by (61). The result of (51)+ (54) is equal to \(z_{e_1} \leq \lambda\), which is already in the system. We can discard also (51)+ (54), since \(z_v \leq 1\) holds for every variable corresponding to nodes \(v\) for which we have already projected out \(z_{v^2}\).

Now consider inequalities coming from (52). Inequality (52)+ (54) is redundant because it can be obtained equivalently by summing \(-z_{w_2} + z_{e_3} \leq 0\) and \(z_{w_2}^2 \leq \lambda\). Similarly, (52)+ (54) is implied by \(-z_{w_1} + z_{e_3} \leq 0\) and \(-z_{w_2} + z_{w_2} \leq 0\). Along the same lines, we can obtain (52)+ (50) as the sum of \(-z_{w_1} + z_{e_3} \leq 0\) and \(-z_{w_2} + z_{e_3} \leq 0\). Finally inequality (52)+ (55) can be discarded, as the combination of \(z_v \leq 1, v \in e_3 \setminus \{w_1, w_2\}\), \(-z_{w_1} + z_{w_1}^2 \leq 0\), and \(z_{w_2}^2 \leq \lambda\) provides the same inequalities.

It remains to check the inequalities originating from (53). The first redundant inequality is (53)+ (55), since \(z_{w_2} - z_{w_2}^2 \leq 1 - \lambda, -z_{e_3} \leq 0,\) and \(z_v \leq 1\) for all \(v \in e_3 \setminus \{w_1, w_2\}\). Similarly, we can remove from the formulation (53)+ (56), because it can be obtained by summing \(z_{w_2} - z_{w_2}^2 \leq 1 - \lambda, -z_{w_2}^2 \leq 0,\) and \(z_v \leq 1\), if \(v \in e_3 \setminus \{w_1, w_2\}\). Ultimately, (53)+ (55) is implied by (54) and the usual bound on the variables \(z_v\). We finished with projecting out the variable \(z_{w_1}^2\).

Next, we apply Fourier elimination on \(z_{w_2}^2\). This variable is present in the following inequalities:

\[
\begin{align*}
z_{w_2} - z_{w_2}^2 & \leq 1 - \lambda \tag{60} \\
-z_{w_2}^2 + z_{e_3} & \leq 0 \tag{61} \\
-z_{w_2}^2 + z_{e_3} & \leq 1 - \lambda \tag{62}
\end{align*}
\]
\[-z_{w_1} + z_{e_1} - z_{w_2}^2 + z_{e_3} \leq 0 \quad (63)\]
\[\sum_{v \in e_3 \setminus \{w_2\}} z_v + z_{w_2} - z_{w_2}^2 - z_{e_1} - z_{e_3} \leq |(e_3 \cup e_1) \setminus e_2| - \lambda \quad (65)\]
\[\sum_{v \in e_2 \setminus \{\bar{v}, w_2\}} z_v + z_{w_2} - z_{w_2}^2 - z_{e_1} - z_{e_3} \leq (e_3 \setminus e_1) - 1 \quad (69)\]
\[\sum_{v \in e_2 \setminus \{\bar{v}, w_2\}} z_v - z_{w_2} + z_{w_2}^2 - z_{e_1} + z_{e_3} \leq |e_1| - 2 \quad (70)\]
\[\sum_{v \in e_2 \setminus \{\bar{v}, w_2\}} z_v + z_{w_2}^2 + z_{e_1} - z_{e_3} \leq |e_3 \setminus (e_1 \cup e_2)| + \lambda \quad (71)\]

We keep the following inequalities and discard the rest:
\[z_{w_2} \leq 1\]
\[-z_{w_2} + z_{e_2} \leq 0\]
\[\sum_{v \in e_2 \setminus \{\bar{v}\}} z_v - z_{e_2} \leq |e_2| - 1 - \lambda\]
\[\sum_{v \in e_1 \setminus \{\bar{v}, w_1\}} z_v - z_{w_1} - z_{e_1} + z_{e_2} + z_{e_3} \leq |e_1| - 2\]
\[\sum_{v \in e_2 \setminus \{\bar{v}, w_2\}} z_v - z_{w_1} + z_{e_1} - z_{e_2} + z_{e_3} \leq |e_2| - 2\]
\[\sum_{v \in e_3 \setminus \{w_1, w_2\}} z_v + z_{e_1} + z_{e_2} - z_{e_3} \leq |e_3| - 2 + \lambda\]
\[\sum_{v \in V \setminus \{\bar{v}\}} z_v - z_{e_1} - z_{e_2} - z_{e_3} \leq |V| - 2 - \lambda.\]

Here we show why we decided to discard all other inequalities. Let us start from analyzing the ones coming from (60). Inequality (60)+(64) is equal to $0 \leq 1 - \lambda$, hence it is always true since $\lambda$ is bounded to lie in $[0, 1]$. Also (60)+(69) is redundant, as it is implied by (64). Observe that (60)+(70) can be obtained as sum of (59) and $-z_{w_1} + z_{e_3} \leq 0$, therefore we are allowed to ignore it. Similarly, (60)+(71) is equal to the sum of (59) and $-z_{w_1} + z_{e_1} \leq 0$.

Next, consider the inequalities coming from (61). Note that (61)+(66) was already found when projecting out nodes in $e_2 \setminus (e_1 \cup e_3)$. Inequalities (61)+(68) and (61)+(69) are implied by the fact that $z_v \leq 1$, for all $v \neq w_2$.

All inequalities resulting from (62) are redundant. In fact, (62)+(66) can be obtained by summing inequalities already in the system, which are $-z_v + z_{e_3} \leq 0, z_v \leq 1$ for any $v \in e_3, v \neq w_2$. Then, (62)+(67) is equal to $(\delta_{w_2, e_1})$, which was found earlier. Inequality (62)+(68) is
redundant for similar reasons to (60)+(70). Instead, (62)+(69) is implied by the fact that \( z_v \leq 1 \). Additionally, (62)+(70) can be achieved as sum of (59), \(-z_{w_1} + z_{e_3} \leq 0\), and \(-z_{w_2} + z_{e_3} \leq 0\). Moreover, (62)+(71) can be discarded, as it is implied by \( z_v \leq 1 \) and \( z_{e_1} \leq \lambda \), for all involved nodes \( v \).

We move on to consider (63). Notice that (63)+(66) is implied by \(-z_{w_1} + z_{e_3} \leq 0\) and \( z_{e_1} \leq \lambda \). Furthermore, (63)+(67) is equal to the sum of \(-z_{w_2} + z_{e_3} \leq 0\) and \(-z_{w_1} + z_{e_1} \leq 0\). Also, (63)+(69) can be obtained by \( z_v \leq 1 \), for all \( v \in e_3 \setminus \{ w_2 \} \), and \(-z_{w_1} + z_{e_1} \leq 0\). Similarly, inequality (63)+(70) is implied by the sum of \( z_v \leq 1 \), for suitable nodes \( v \), \(-z_{w_1} + z_{e_3} \leq 0\), and \(-z_{w_2} + z_{e_3} \leq 0\). Lastly, (63)+(71) is redundant, since it is the sum of \( z_v \leq 1 \), \(-z_{w_1} + z_{e_1} \leq 0\), and \( z_{e_1} \leq \lambda \), all inequalities already in the system.

Here we focus on the next to last inequality in which \( z_{w_2}^2 \) has negative coefficient. (64)+(66) is implied by (64). Inequality (64)+(67) instead is redundant because of \( z_v \leq 1 \), and \(-z_{e_3} \leq 0\). Then, (64)+(68) is implied by (64), \( z_v \leq 1 \), \(-z_{e_2} \leq 0\). Furthermore, (64)+(69) is similar to (64)+(68). Also, (64)+(70) is redundant because of \( z_v \leq 1 \), \(-z_{e_1} \leq 0\). Observe that (64)+(71) is obtained by summing (64), \( z_v \leq 1 \) for the remaining \( z_v \), and \( z_{e_1} \leq \lambda \), \( z_{e_3} \leq 0\).

We focus now on (65). Similarly to before, (65)+(66) is equal to the sum of (64), \( z_v \leq 1 \) for the variables not included in (64), and \(-z_{e_1} \leq 0\). Analogously, (65)+(67) is implied by (59), \( z_v \leq 1 \), and \(-z_{e_3} \leq 0\). Inequality (65)+(69) can be obtained by (59), (64), \( z_v \leq 1 \), and \(-z_{e_3} \leq 0\). Along the same lines, (65)+(70) coincides with summing (59), \( z_v \leq 1 \), \(-z_{e_1} \leq 0\). Finally, (65)+(71) is redundant because implied by (64), \( z_v \leq 1 \), and \(-z_{e_3} \leq 0\).

At this point the only variables appearing in the formulation are \( z \) and \( \lambda \). The system is:

\[
\begin{align*}
z_0 & = \lambda \\
0 & \leq \lambda \leq 1 \\
-z_{e_1} & \leq 0 \\
-z_{e_2} & \leq 0 \\
-z_{e_3} & \leq 0 \\
-\lambda & + z_{e_1} \leq 0 \\
-\lambda & + z_{e_2} \leq 0 \\
-\lambda & + z_{e_3} \leq 0 \\
z_v & \leq 1 \quad \forall v \in V \setminus \{ \bar{v} \} \\
-z_v + z_{e_1} & \leq 0 \quad \forall v \in e_1 \setminus \{ \bar{v} \} \\
-z_v + z_{e_2} & \leq 0 \quad \forall v \in e_2 \setminus \{ \bar{v} \} \\
-z_v + z_{e_3} & \leq 0 \quad \forall v \in e_3 \\
\sum_{v \in e_1 \setminus \{ \bar{v} \}} z_v + \lambda - z_{e_1} & \leq |e_1| - 1 \\
\sum_{v \in e_2 \setminus \{ \bar{v} \}} z_v + \lambda - z_{e_2} & \leq |e_2| - 1 \\
\sum_{v \in e_3} z_v - z_{e_3} & \leq |e_3| - 1 \\
\sum_{v \in e_1 \setminus \{ \bar{v}, w_1 \}} z_v - z_{w_2} - z_{e_1} + z_{e_2} + z_{e_3} & \leq |e_1| - 2 \\
\sum_{v \in e_2 \setminus \{ \bar{v}, w_2 \}} z_v - z_{w_1} + z_{e_1} - z_{e_2} + z_{e_3} & \leq |e_2| - 2 
\end{align*}
\]
\[ \sum_{v \in E \setminus \{w_1, w_2\}} z_v - \lambda + z_{e_1} + z_{e_2} - z_{e_3} \leq |e_3| - 2 \]

\[ \sum_{v \in V \setminus \{v\}} z_v + \lambda - z_{e_1} - z_{e_2} - z_{e_3} \leq |V| - 2. \]

Note that by (72), it suffice to substitute \( \lambda \) with \( z_v \) in the above system in order to project out the variable \( \lambda \). Since the system consists of all the standard linearization inequalities and the four odd \( \beta \)-cycle inequalities that arise in this case, we obtain that the claim holds when the hypergraph \( G \) contains only three edges.

This concludes the base case of the overall proof of this proposition, which is by induction on the number of edges of \( G \). Let us move on to the inductive step. Consider \( G \) a cycle hypergraph with \( m \) edges, \( e_1, \ldots, e_m \), such that \( |e_i \cap e_{i+1}| = 1 \) for every \( i = 1, \ldots, m \), where \( e_{m+1} = e_1 \). Let \( v' = e_1 \cap e_2 \), and \( v'' = e_{m-1} \cap e_m \). We set \( f = \{v', v''\} \), and define \( G' = (V, E') \), where \( E' = E \cup \{f\} \). We also consider two hypergraphs \( G_1, G_2 \) that are the section hypergraphs induced by \( e_1 \cup e_m \), and \( \bigcup_{i=2}^{m-1} e_i \) respectively. Note that \( G_1 \) and \( G_2 \) satisfy the assumptions of Proposition 1 and both have less than \( m \) edges. Hence, we apply the inductive hypothesis in order to obtain the perfect formulation of \( \text{MP}_{G_1}, \text{MP}_{G_2} \). Observe that \( G_1 \) and \( G_2 \) satisfy the hypothesis of Theorem 1 in [9], because \( G_1 \cup G_2 = G' \) and \( G_1 \cap G_2 = \{(v', v''), \{f\}\} \) is a complete hypergraph. This implies that a perfect formulation for \( \text{MP}_{G'} \) is obtained by juxtaposing the perfect formulations of \( \text{MP}_{G_1} \) and \( \text{MP}_{G_2} \).

In order to achieve a description of \( \text{MP}_{G} \), we need to project the variable \( z_f \) out of the system defining \( \text{MP}_{G'} \). We do so by applying the Fourier elimination procedure. We separate here in two sets the inequalities of the formulation for \( \text{MP}_{G'} \) in which \( z_f \) appears with coefficient \(-1\) and \(+1\) respectively.

\[ -z_f \leq 0 \]  
\[ z_{v'} + z_{v''} - z_f \leq 1 \]  
\[ -z_v + z_{e_1} + z_{e_m} - z_f \leq 0 \]  
where \( v = e_1 \cap e_m \)  
\[ \sum_{v \in e \cup e_m} z_v - z_{e_1} - z_{e_m} - z_f \leq |e_1 \cup e_m| - 2 \]  
\[ \sum_{v \in S_1(G_2)} z_v - \sum_{e \in E^-(G_2)} z_e = \sum_{v \in S_2(G_2)} z_v + \sum_{e \in E^+(G_2)} z_e \leq |S_1(G_2)| - |I(G_2)| + \left[ \frac{|E^-(G_2)|}{2} \right] \] (77)

\[ -z_{v'} + z_f \leq 0 \]  
\[ -z_{v''} + z_f \leq 0 \]  
\[ \sum_{v \in e_1 \setminus \{v''\} \cup e_m} z_v - z_{v'} - z_{e_1} + z_{e_m} + z_f \leq |e_1| - 2 \]  
\[ \sum_{v \in e_m \setminus \{v'\} \cup e_1} z_v - z_{v'} + z_{e_1} - z_{e_m} + z_f \leq |e_m| - 2 \]  
\[ \sum_{v \in S_1(G_2)} z_v - \sum_{e \in E^-(G_2)} z_e = \sum_{v \in S_2(G_2)} z_v + \sum_{e \in E^+(G_2)} z_e \leq |S_1(G_2)| - |I(G_2)| + \left[ \frac{|E^-(G_2)|}{2} \right] \] (82)

In the above system, inequality (77) holds for every odd subset \( E^-(G_2) \) of \( E(G_2) \) containing \( f \), while inequality (82) holds for each odd subset \( E^-(G_2) \) of \( E(G_2) \) such that \( f \notin E^-(G_2) \). In these inequalities, \( I(G_2) \) denotes the set of edges in \( E^-(G_2) \) such that also the next edge in the \( \beta \)-cycle belongs to \( E^-(G_2) \) as well. We refer the reader to the definition of odd \( \beta \)-cycle inequalities for the meaning of the sets \( E^+(G_2), S_1(G_2), \) and \( S_2(G_2) \).
Note that the sums \((73)+(78), (73)+(79), (73)+(80), (73)+(81), (73)+(82), (74)+(78), (74)+(79), (74)+(80), (74)+(81), (75)+(78), (75)+(79), (75)+(80), (76)+(78), (76)+(79), (76)+(80), (77)+(78), (77)+(79), (77)+(80), (77)+(81), (77)+(82)\) have a support hypergraph which is Berge-acyclic. This is true, by the assumptions on \(G\) and since none of these sums contains all edges of \(E\). By [3], we know that the only non-redundant inequalities for these hypergraphs are those of the standard linearization and therefore are redundant for the multilinear polytopes deriving from their support hypergraphs. Moreover, by Proposition 6 in [7], it follows that these inequalities are redundant also for \(MP_G\), since the support hypergraphs of any of these sums is a partial hypergraph of \(G\). We refer the reader to [7] for the definition of partial hypergraph.

Then, it remains to check \((75)+(82), (76)+(82), (77)+(80), (77)+(81)\). Consider \((77)+(80)\). We define \(E^{-}(G) = \{e_1\} \cup E^{-}(G_2) \setminus \{f\}\), and denote by \(S_1(G), S_2(G)\) the corresponding sets in Definition 2. Notice immediately that \(|E^{-}(G)| = 1 + |E^{-}(G_2) \setminus f|\), therefore \(|E^{-}(G)|\) is an odd number. Moreover, \(\left\lfloor \frac{|E^{-}(G)|}{2} \right\rfloor = \left\lfloor \frac{|E^{-}(G_2)|}{2} \right\rfloor\), and \(\left\lfloor \frac{|e_1|}{2} \right\rfloor = 0\). We need to examine the different cases where each of \(e_2, e_{m-1}\) belongs to \(E^{-}(G_2)\) or \(E^+(G_2)\).

Then, let us start from the case \(e_2 \in E^{-}(G_2)\). With this assumption, the node \(v'\) is in \(S_1(G)\), since \(e_1, e_2 \in E^{-}(G)\). Indeed, \(v' \in S_1(G_2)\) and the corresponding variable does not appear in \((80)\), therefore \(z_{v'}\) appears with coefficient \(+1\) in \((77)+(80)\). Next, we further assume that \(e_{m-1} \in E^+(G_2)\), hence \(v'' \in S_2(G)\). The variable \(z_{v''}\) does not appear in \((77)\), while it appears with coefficient \(-1\) in \((80)\). Observe that this is exactly what we expect from the definition of odd \(\beta\)-cycle. We need to check the correctness of the right-hand side. It follows that \(|S_1(G)| = |e_1| - 2 + |S_1(G_2)|\), as \(S_1(G)\) is the disjoint union of \(e_1 \setminus \{\{v'\} \cup e_m\}\) and \(S_1(G_2)\). Secondly, \(|I(G_2)| = \{|i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^{-}(G)\}|\), since \(e_1\) replaces \(f\) as predecessor of \(e_2\). Therefore we obtain the desired right-hand side in this case.

Then, consider \(e_2, e_{m-1} \in E^{-}(G_2)\). As a consequence, we have that \(e_{m-1} \in E^{-}(G), e_m \in E^+(G)\), therefore \(z_{v''}\) must not be present in the resulting inequality. This is true, since there is \(-z_{v''}\) in the left-hand side of \((80)\) and \(+z_{v''}\) in \((77)\) since \(e_{m-1}, f \in E^{-}(G_2)\). Once we sum the two inequalities, \(z_{v''}\) vanishes. However, we need to be more careful in the analysis of the correctness of right-hand side. In fact, in this case \(|S_1(G)| = |e_1| - 2 + |S_1(G_2)\setminus \{v''\}| = |e_1| - 2 + |S_1(G_2)| - 1\). Moreover, we now have that \(|\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^{-}(G)\}|\) is equal to \(|I(G_2)| - 1\), since the edge after \(e_{m-1}\) does not belong to \(E^{-}(G)\). Then, the right hand side of the odd \(\beta\)-cycle of \(G\) corresponding to \(E^{-}(G)\) in this case is equal to

\[
|S_1(G)| - |\{i \in \{1, \ldots, m\} : e_i, e_{i+1} \in E^{-}(G)\}| + \left\lfloor \frac{|E^{-}(G)|}{2} \right\rfloor = |e_1| - 2 + |S_1(G_2)| - 1 - |I(G_2)| + 1 + \left\lfloor \frac{|E^{-}(G_2)|}{2} \right\rfloor = |e_1| - 2 + |S_1(G_2)| - |I(G_2)| + \left\lfloor \frac{|E^{-}(G_2)|}{2} \right\rfloor,
\]

which coincides with the sum of the right-hand sides of \((80)\) and \((77)\). This concludes the case \(e_2 \in E^{-}(G_2)\).

Now assume instead that \(e_2 \in E^+(G_2)\). Therefore, \(z_{v'}\) does not appear in \((80)\) nor \((77)\), hence this variable is not present in their sum. Let \(e_{m-1} \in E^+(G_2)\). This case is similar to the
previous case \( e_2 \in E^-(G_2), e_{m-1} \in E^+(G_2) \). Moreover, the case \( e_{m-1} \in E^-(G_2) \) has similar calculations to the case \( e_2, e_{m-1} \in E^-(G_2) \) above.

We can conclude that by summing (77) with (80) we obtain the odd \( \beta \)-cycle inequalities arising from \( G \). Similar arguments hold in the other sums (75)+(82), (76)+(82), (77)+(81).

### A.4 Proof of Theorem 4

We prove this theorem by using Proposition 1, Theorem 3, and subsequently by projecting out the additional variables introduced by the application of Theorem 3.

Let \( G = (V, E) \) be a cycle hypergraph with edges \( e_1, \ldots, e_m \). We create first a new hypergraph \( G' = (V', E') \) obtained by contracting every intersection \( e_i \cap e_{i+1} \) to a new node \( w_i \), for every \( i \in [m] \), where we define \( e_{m+1} = e_1 \). Observe that \( G' \) satisfies the hypothesis of Proposition 1 since it is still a cycle hypergraph and every non-empty intersection of two edges has cardinality equal to one. Hence \( MP_{G'} \) is described by the system consisting of the standard linearization and the odd \( \beta \)-cycle inequalities. Moreover, note that in this case the odd \( \beta \)-cycle inequalities coincide with the inequalities first introduced in [7].

Then, starting from \( G' \), we construct a second hypergraph \( G'' = (V, E'') \) by applying Theorem 3 to every node \( w_i \), for \( i \in [m] \), in order to obtain the same node set \( V \) of the original hypergraph \( G \). In this way we obtain \( MP_{G''} \). By the application of Theorem 3 it follows that \( E'' \neq E \). Indeed, the application of Theorem 3 yields a new edge for every node \( w_i \), and therefore \( E'' = E \cup \{ f_1, \ldots, f_m \} \), where \( f_i = e_i \cap e_{i+1} \).

Thus, in order to achieve a description of \( MP_G \), it remains to eliminate the variables corresponding to the new edges \( f_1, \ldots, f_m \). We do so by using the Fourier elimination on these variables one by one. We start from projecting out the variable \( z_{f_1} \). We write here the inequalities in which this variable is present and we divide the inequalities in two sets: one in which \( z_{f_1} \) has coefficient \(-1\), while in the second set its coefficient is equal to \(+1\).

\[
\sum_{v \in f_1} z_v - z_{f_1} \leq |f_1| - 1
\]  

(83)

\[
- z_{f_1} + z_{e_1} \leq 0
\]  

(84)

\[
- z_{f_1} + z_{e_2} \leq 0
\]  

(85)

\[
\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{f \in S_2} z_f + \sum_{e \in E^+} z_e \leq |S_1| - |\{ i : e_i, e_{i+1} \in E^- \}| + \left\lceil \frac{|E^-|}{2} \right\rceil
\]  

(86)

\[
- z_v + z_{f_1} \leq 0 \quad \forall v \in f_1
\]  

(87)

\[
\sum_{v \in e_1 \setminus (f_1 \cup f_m)} z_v + z_{f_1} + z_{f_m} - z_{e_1} \leq |e_1 \setminus (f_1 \cup f_m)| + 1
\]  

(88)

\[
\sum_{v \in e_2 \setminus (f_1 \cup f_2)} z_v + z_{f_1} + z_{f_2} - z_{e_2} \leq |e_2 \setminus (f_1 \cup f_2)| + 1
\]  

(89)

\[
\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{f \in S_2} z_f + \sum_{e \in E^+} z_e \leq |S_1| - |\{ i : e_i, e_{i+1} \in E^- \}| + \left\lceil \frac{|E^-|}{2} \right\rceil
\]  

(90)

In the above system, inequality (86) holds for each odd subset \( E^- \) of \( \{ e_1, \ldots, e_m \} \) with \( e_1, e_2 \notin E^- \), while inequality (90) holds for each odd subset \( E^- \) of \( \{ e_1, \ldots, e_m \} \) with \( e_1, e_2 \in E^- \). In these
inequalities, $E^+ = \{e_1, \ldots, e_m\} \setminus E^-$. Moreover, the set $S_1$ contains the edges $f_i = e_i \cap e_{i+1}$ such that $e_i, e_{i+1} \in E^-$ as well as the nodes in $V$ contained only in one edge among $e_1, \ldots, e_m$, and this edge belongs to $E^-$. The set $S_2$ contains the edges $f_i$ such that $e_i, e_{i+1} \in E^+$. Therefore, in inequality (86) we have $f_1 \in S_2$, and in inequality (90) we have $f_1 \in S_1$. Note that we did not write the inequalities $-z_{f_1} \leq 0$ and $z_{f_1} \leq 1$ in the above system, the first coming from the addition of the standard linearization of $f_1$ and the second derives from the replacement of $z_{w_1}$ with $z_{f_1}$ in $z_{w_1} \leq 1$. This is because these two inequalities are redundant and we discard them.

In fact, the first inequality can be obtained by summing $-z_{f_1} + z_{e_1} \leq 0$ and $-z_{e_1} \leq 0$. The second inequality is redundant because it is implied by $-z_v + z_{f_1} \leq 0$ and $z_v \leq 1$ for any $v \in f_1$.

Observe that, without doing any calculations, we know a priori that the inequalities obtained from the sums $(84)+(90)$, $(85)+(90)$, $(86)+(88)$, $(86)+(89)$, $(86)+(90)$ are redundant also for the new system of inequalities, since the support hypergraphs of each of these inequalities is a partial hypergraph of $(V, E'' \setminus \{f_1\})$.

By Proposition 6 in [7], it follows that inequalities $(84)+(90)$, $(85)+(90)$, $(86)+(88)$, $(86)+(89)$, $(86)+(90)$ are redundant for the new system, as they can be achieved equivalently by summing, for each variable appearing in the left-hand side, the bound that every variable is less than or equal to 1.

Note that also the inequalities obtained from $(83)+(87)$, $(84)+(88)$, $(85)+(89)$ are redundant for the new system, as they can be achieved equivalently by summing, for each variable appearing in the left-hand side, the bound that every variable is less than or equal to 1.

On the other hand, all the remaining inequalities obtained by the Fourier elimination might be facet-defining for the resulting polytope and will not be discarded. Inequalities $(83)+(88)$, $(83)+(89)$ are the flower inequalities with center $e_1$, $e_2$, and neighbor $f_m$, $f_2$ respectively. Inequality $(83)+(90)$ has the same form of inequality (80) if we redefine $S_1$ by replacing $f_1$ with all its nodes. Notice that the obtained inequality is the first step in achieving the odd $\beta$-cycle inequalities in which the nodes in $e_1 \cap e_2$ belong to the resulting set $S_1$.

The inequalities obtained from $(85)$ are analogous to $(84)$, as the role of $z_{e_1}$ in $(84)$ is the same of $z_{e_2}$ in $(85)$. Hence, we only consider the inequalities arising from $(84)$. These are achieved by summing $(84)$ with $(87)$ and $(89)$. From the first we get $(\delta_{v,e_1})$ for all $v \in e_1 \cap e_2$, while from the second we gain the flower inequality with center $e_2$ and neighbors $\{e_1, f_2\}$.

It remains to check what happens for $(86)+(87)$. In this case we obtain $|e_1 \cap e_2|$ inequalities. Each of these inequalities has the same form of (86), where the set $S_2$ is redefined by replacing the edge $f_1$ with one of the nodes in $f_1$. This inequality will lead to the odd $\beta$-cycles in which there is one node $v \in e_1 \cap e_2$ in $S_2$.

We are done with eliminating the variable $z_{f_1}$ and now we move on to $z_{f_2}$. Next, we consider the system obtained from the Fourier elimination of variable $z_{f_1}$. To project out the variable $z_{f_2}$, we focus on the inequalities with a non-zero coefficient for $z_{f_2}$.

\begin{align}
\sum_{v \in f_2} z_v - z_{f_2} &\leq |f_2| - 1 \\
- z_{f_2} + z_{e_2} &\leq 0 \\
- z_{f_2} + z_{e_3} &\leq 0 \\
\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{p \in S_2} z_p + \sum_{e \in E^+} z_e &\leq |S_1| - |\{i : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor
\end{align}
Like in the previous system, inequality (94) holds for each odd subset $S$ of $e_2, e_3 \not\in E^-$, while inequality (98) holds for each odd subset $E^{-}$ of $e_1, \ldots, e_m$ with $e_2, e_3 \in E^-$. In these inequalities, $E^+ = \{e_1, \ldots, e_m\} \setminus E^-$. The set $S_1$ contains the edges $f_i$, for $i \in \{2, \ldots, m\}$, such that $e_i, e_{i+1} \in E^-$, all the nodes in $e_1 \cap e_2$ if $e_1, e_2 \in E^-$, and all the nodes in $V$ contained only in one edge among $e_1, \ldots, e_m$, and this edge belongs to $E^-$. The set $S_2$ contains the edges $f_i$, for $i \in \{2, \ldots, m\}$, such that $e_i, e_{i+1} \in E^+$, and one node in $e_1 \cap e_2$ if $e_1, e_2 \in E^+$. In particular, in inequality (94) we have $f_2 \in S_2$, and in inequality (98) we have $f_2 \in S_1$.

We remark that the structure of inequalities (91)-(98) is almost identical to that of the system (83)-(90). In particular, $f_1, e_1, e_2$ are replaced by $f_2, e_2, e_3$, respectively. The only difference is between inequality (88) and (96), since the variable $z_{f_1}$ has been already projected out. The same analysis as before holds in this case, except for the inequality obtained by (91)+(96). In fact, in this case we obtain inequality $(e_{e_2})$ of the standard linearization instead of a flower inequality.

Recursively, we project out all the additional variables until we are left with only $z_{f_m}$. This variable is present in the following inequalities:

$$
\sum_{v \in f_m} z_v - z_{f_m} \leq |f_m| - 1
$$

$$
- z_{f_m} + z_{e_1} \leq 0
$$

$$
- z_{f_m} + z_{e_m} \leq 0
$$

$$
\sum_{v \in S_1} z_v - \sum_{e \in E^-} z_e - \sum_{p \in S_2} z_p + \sum_{e \in E^+} z_e \leq |S_1| - |\{i : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor
$$

$$
- z_v + z_{f_m} \leq 0 \quad \forall v \in f_n
$$

$$
\sum_{v \in e_1 \setminus f_m} z_v + z_{f_m} - z_{e_1} \leq |e_1 \setminus f_m|
$$

$$
\sum_{v \in e_m \setminus f_m} z_v + z_{f_m} - z_{e_m} \leq |e_m \setminus f_m|
$$

$$
\sum_{p \in S_1} z_p - \sum_{e \in E^-} z_e - \sum_{v \in S_2} z_v + \sum_{e \in E^+} z_e \leq |S_1| - |\{i : e_i, e_{i+1} \in E^-\}| + \left\lfloor \frac{|E^-|}{2} \right\rfloor
$$

Inequality (102) holds for each odd subset $E^-$ of $e_1, \ldots, e_m$ with $e_m, e_1 \not\in E^-$, and inequality (106) holds for each odd subset $E^-$ of $e_1, \ldots, e_m$ with $e_m, e_1 \in E^-$. The set $S_1$ contains the edge $f_m$ if $e_m, e_1 \in E^-$, all the nodes in $e_i \cap e_{i+1}$, for $i \in \{1, \ldots, m-1\}$, if $e_i, e_{i+1} \in E^-$, and all the nodes in $V$ contained only in one edge among $e_1, \ldots, e_m$, and this edge belongs to $E^-$. The set $S_2$ contains the edge $f_m$ if $e_m, e_1 \in E^+$, and one node in $e_i \cap e_{i+1}$, for $i \in \{1, \ldots, m-1\}$,
if \( e_i, e_{i+1} \in E^+ \). In particular, in inequality \( (102) \) we have \( f_m \in S_2 \), and in inequality \( (106) \) we have \( f_m \in S_1 \).

The variable \( z_{f_m} \) can be projected out by following the same arguments of the previous steps, since the inequalities \( (99)-(106) \) have the same structure. However, observe that the inequalities obtained by \( (99)+(106) \), \( (102)+(103) \) are exactly the odd \( \beta \)-cycle inequalities for which \( e_m, e_1 \) both belong to either \( E^- \) or \( E^+ \).

Once we have eliminated all variables \( z_{f_i} \) from the description of \( MP_{G''} \), the obtained system contains only inequalities of the standard linearization, flower inequalities, and odd \( \beta \)-cycle inequalities. This concludes the proof of the theorem. \( \square \)

References


