A class of spectral bounds for Max $k$-cut

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Abstract

Let $G$ be an undirected and edge-weighted simple graph. In this paper we introduce a class of bounds for the maximum $k$-cut problem in $G$. Their expression notably involves eigenvalues of the weight matrix together with some other geometrical parameters (distances between a discrete point set and a linear subspace). This extends a bound recently introduced by Nikiforov. We also show cases when the provided bounds strictly improve over other eigenvalue bounds from the literature.

Keywords: Max $k$-cut, Adjacency matrix eigenvalues, Adjacency matrix eigenvectors

1 Introduction

Let $G = (V, E)$ be an undirected simple graph having node set $V = \{1, 2, \ldots, n\}$, edge set $E$, and let $w \in \mathbb{R}^E$ denote a weight function on the edges. Let $k$ denote a positive integer. Given any partition $(V_1, V_2, \ldots, V_k)$ of $V$ into $k$ subsets $V_1, V_2, \ldots, V_k$ (some of which may be empty), the $k$-cut defined by this partition is the set $\delta(V_1, V_2, \ldots, V_k)$ of edges in $E$ having their endpoints in different sets of the partition. And the weight of a $k$-cut is the sum of the weights of the edges it contains. Given this, the maximum $k$-cut problem consists in finding $\max \{mc_k(G, W): k \in \mathbb{N} \}$: the maximum weight of a $k$-cut in $G$.

In what follows, let $W \in \mathbb{R}^{n \times n}$ denote the weighted adjacency matrix whose entries are defined by $W_{ij} = w_{ij}$ if $ij \in E$ and $W_{ij} = 0$ otherwise. So in particular, it is a symmetric matrix with a zero diagonal. Given two disjoint node subsets $A, B$, let $w[A, B]$ denote the sum of the weights of the edges having one endpoint in $A$ and the other in $B$: $w[A, B] = \sum_{(i,j)\in A \times B: ij \in E} w_{ij}$. Similarly, $w[A]$ denotes the sum of the weights of the edges with both endpoints in $A$: $w[A] = \sum_{(i,j)\in A^2: ij \in E, i < j} w_{ij}$. Now, let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ denote

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the eigenvalues of $W$ and let $\nu_1, \nu_2, \ldots, \nu_n$ be the corresponding unit and pairwise orthogonal eigenvectors. For any positive integer $q$, let $\mathbf{1}_q$ stand for the $q$-dimensional all ones vector. Given any vector $x \in \mathbb{R}^n$, $\text{Diag}(x)$ stands for the square diagonal matrix of order $n$, having $x$ for diagonal. The Laplacian matrix is $L = \text{Diag}(W\mathbf{1}_n) - W$. Its maximum eigenvalue is denoted by $\lambda_n(L)$.

In this paper we are interested in bounds for the maximum $k$-cut problem that involve eigenvalues of the Laplacian $L$ or of the weight matrix $W$. For the particular case when $k = 2$, Mohar and Poljak \cite{MoharPoljak} proved the inequality $mc_2(G,W) \leq \frac{n}{4}\lambda_n(L)$. More recently, van Dam and Sotirov \cite{vanDamSotirov} proved the following upper bound on $mc_k(G,W)$, still making use of the largest eigenvalue of the Laplacian and providing in the same reference several graphs for which this bound is tight together with some comparisons with other bounds stemming from semidefinite formulations.

**Theorem 1.1.** \cite{vanDamSotirov}

$$mc_k(G,W) \leq \frac{n(k - 1)}{2k} \lambda_n(L). \quad (1)$$

Also recently, Nikiforov \cite{Nikiforov} introduced an upper bound for the maximum cardinality of a $k$-cut in $G$ (i.e. the maximum $k$-cut problem with $w_e = 1, \forall e \in E$), which may be easily extended to the weighted case and can be formulated as follows.

**Theorem 1.2.** \cite{Nikiforov}

$$mc_k(G,W) \leq \frac{k - 1}{k} \left(w[V] - \frac{\lambda_1 n}{2}\right) \quad (2)$$

As he notes, the bounds from Theorems 1.1 and 1.2 are equivalent for regular graphs but they are incomparable in general.

In this paper we show the bound from Theorem 1.2 can be still further improved by making use of the whole spectrum (i.e. all eigenvalues and eigenvectors) of the matrix $W$ in lieu of its smallest eigenvalue only. This is achieved by introducing a natural extension of an earlier work done for the maxcut problem (i.e. the maximum $k$-cut problem for the particular case when $k = 2$) \cite{BencsikSotirov}.

We mention some additional notation to be used. Given a positive integer $q$, $[q]$ stands for the set of integers $\{1, 2, \ldots, q\}$. The inner scalar product is denoted by $\langle \cdot, \cdot \rangle$, and the Euclidean norm by $\| \cdot \|$.

## 2 Spectral bounds

With no loss of generality, we assume the graph $G$ is complete (setting zero weights on non existing edges). Given $r \in \mathbb{R} \setminus \{0, 1\}$, let $d_{j,r}$ denote the distance
between the set of vectors \( \{r, 1\}^n \) and the subspace \( \text{Vect}(\nu_1, \nu_2, \ldots, \nu_j) \) that is generated by the first \( j \) eigenvectors of \( W \):

\[
d_{j,r} = \min \{ \|z - y\| : z \in \{r, 1\}^n, y \in \text{Vect}(\nu_1, \nu_2, \ldots, \nu_j) \}. \tag{3}\]

**Theorem 2.1.** For any \( r \in \mathbb{R} \setminus \{0, 1\} \),

\[
mc_k(G, W) \leq \frac{1}{2(r - 1)^2} \left( (r^2 + k - 1)(2w[V] - \lambda_1 n) - k \sum_{i \in [n-1]} (\lambda_{i+1} - \lambda_i) d_{i,r}^2 \right) \tag{4}
\]

**Proof.** Let \((V_1, V_2, \ldots, V_k)\) denote a partition of \( V \) corresponding to an optimal solution of the maximum \( k \)-cut problem.

For all \( i \in [k] \), let the vector \( y^i \in \{r, 1\}^n \) be defined as follows: \( y^i_l = r \) if \( l \in V_i \) and \( 1 \) otherwise. We have:

\[
\langle y^i, W y^i \rangle = 2r^2 w[V_i] + 2 \sum_{j \in [k]\setminus\{i\}} w[V_j] + 2r \sum_{j \in [k]\setminus\{i\}} w[V_i, V_j] + 2 \sum_{(j,l) \in ([k]\setminus\{i\})^2} w[V_j, V_l], \tag{5}
\]

Let us now compute the sum of each term occurring in the right-hand-side of (5) over all \( i \in [k] \).

\[
\begin{align*}
\sum_{i \in [k]} 2r^2 w[V_i] & = 2r^2 \left( w[V] - mc_k(G, W) \right), \\
\sum_{i \in [k]} 2 \sum_{j \in [k]\setminus\{i\}} w[V_j] & = 2(k - 1) \left( w[V] - mc_k(G, W) \right), \\
\sum_{i \in [k]} 2r \sum_{j \in [k]\setminus\{i\}} w[V_i, V_j] & = 4r mc_k(G, W), \\
\sum_{i \in [k]} 2 \sum_{(j,l) \in ([k]\setminus\{i\})^2} w[V_j, V_l] & = 2(k - 2)mc_k(G, W).
\end{align*}
\]

Thus, we deduce

\[
\sum_{i \in [k]} \langle y^i, W y^i \rangle = 2mc_k(G, W)(-r^2 + 2r - 1) + 2w[V](r^2 + k - 1). \tag{6}
\]

We now derive a lower bound on \( \langle y^i, W y^i \rangle \) making use of the spectrum of \( W \). For, we mention some preliminary properties. Note that since \( W \) is symmetric we may assume \((\nu_1, \nu_2, \ldots, \nu_n)\) forms an orthonormal basis, and consider the expression of \( y^i \) in this basis: \( y^i = \sum_{\alpha \in [n]} \alpha \nu_i \) with \( \alpha \in \mathbb{R}^n \). Then, we have \( \|y^i\|^2 = \sum_{\alpha \in [n]} \alpha_i^2 = n + |V_i|(r^2 - 1) \). From the definition of the distance defined above we deduce

\[
d_{j,r}^2 \leq \sum_{l = j+1}^{n} \alpha_l^2, \forall j \in [n - 1].
\]

Thus, we have

\[
\begin{align*}
\langle y^i, W y^i \rangle & = \sum_{\alpha \in [n]} \lambda_i \alpha_i^2 \\
& = \lambda_1 \left( n + |V_i|(r^2 - 1) \right) - \sum_{l = 2}^{n} \lambda_l \alpha_l^2 \\
& = \lambda_1 \left( n + |V_i|(r^2 - 1) \right) + \sum_{l = 2}^{n} (\lambda_l - \lambda_1) \alpha_l^2
\end{align*}
\]

Then, iteratively making use of the inequality \( \alpha_j^2 \geq d_{j-1,r}^2 - \sum_{l = j+1}^{n} \alpha_l^2 \) for \( j = 2, \ldots, n \), we deduce

\[
\langle y^i, W y^i \rangle \geq \lambda_1 \left( n + |V_i|(r^2 - 1) \right) + \sum_{l = 2}^{n} (\lambda_{l+1} - \lambda_l) d_{l,r}^2.
\]
And summing these inequalities for all \( i \in [k] \) we obtain
\[
\sum_{i \in [k]} \langle y^i, Wy^i \rangle \geq \lambda_1 n (k + r^2 - 1) + k \left( \sum_{l \in [n-1]} (\lambda_{l+1} - \lambda_l) d_{l,r}^2 \right) \tag{7}
\]
Finally combining (6) and (7), the result follows.

Note that all the terms occurring in the last sum of the inequality (4) are nonnegative, so that removing from the right-hand side some or all of the terms involved in this sum, the expression obtained still provides an upper bound on \( mck(G, W) \). In particular, inequality (2) follows as a corollary of Theorem 2.1 taking \( r = 1 - k \) and removing the last sum from the right-hand side of inequality (4).

Remark Enforcing the value ‘1’ among the two possible values for the components of the vectors used in the definition of the distances (3) is done just to slightly simplify the presentation. We are basically interested in the distance between \( \text{Vect}(\nu_1, \nu_2, \ldots, \nu_j) \) and a set of vectors whose components are restricted to take any of two nonzero values. If we denote by \( d_{j,r_1,r_2} \) the distance between \( \text{Vect}(\nu_1, \nu_2, \ldots, \nu_j) \) and the set of vectors \( \{r_1, r_2\}^n \) with \((r_1, r_2) \in (\mathbb{R} \setminus \{0\})^2\), then
\[
d_{j,r_1,r_2} = |r_1| d_{j,r_1,r_2}^r, \forall j \in [n], \text{ and the results we get by using such vectors are equivalent to the ones presented.}
\]

Remark In view of the bound (4) on \( mck(G, W) \), one may ask for the best choice for the parameter \( r \). If we consider the “truncated” bound obtained from (4) by removing the last sum, we can show the ratio \( \frac{r^2 + k - 1}{r^2} \) is minimized for \( r = 1 - k \), which is the value used by Nikiforov [6] and leads to formula (2).

One may ask for the best such choice by considering the whole expression of the bound in (4). Preliminary computational experiments show that other values of \( r \) may lead to strictly better bounds, depending on the instance.

The approach undertaken to prove Theorem 2.1 can also be used to obtain lower bounds on the weight of any \( k \)-cut. Let \( lck(G, W) \) denote the minimum weight of a \( k \)-cut in \( G \) and let \( \overline{d}_{j,r} \) denote the distance between the set of vectors \( \{r, 1\}^n \) and the subspace \( \text{Vect}(\nu_j, \nu_{j+1}, \ldots, \nu_n) \) that is generated by the last \( n - j + 1 \) eigenvectors of \( W \):
\[
\overline{d}_{j,r} = \min \{ \|z - y\| : z \in \{r, 1\}^n, y \in \text{Vect}(\nu_j, \nu_{j+1}, \ldots, \nu_n) \}. \tag{8}
\]

Theorem 2.2.
\[
lck(G, W) \geq \frac{1}{2(r - 1)^2} \left( (r^2 + k - 1)(2w[V] - \lambda_n n) + k \sum_{l \in [n - 1]} (\lambda_{l+1} - \lambda_l) \overline{d}_{l+1,r}^2 \right) \tag{9}
\]
Proof. Similar to that of Theorem 2.1. Or we can also use Theorem 2.1 with the weight matrix \(-W\) instead of \( W \), which gives an upper bound on \(-lck(G, W)\).
Theorems 2.1 and 2.2 lead to the definition of the spectral bound gap, which is the difference between the upper and lower spectral bounds:

$$\frac{1}{2(r-1)^2} \left( (r^2 + k - 1)n(\lambda_n - \lambda_1) - k \sum_{l \in [n-1]} (\lambda_{l+1} - \lambda_l) \left( d_{l+1,r}^2 + d_{l,r}^2 \right) \right).$$

3 On some particular cases

Generally, computing the distances \((d_{j,r})_{j=1}^{n-1}\) involved in the expression of the bound (4) is \(NP\)-hard (see Proposition 4.4 in [1]). In this section we provide an upper bound on \(m_{c_k}(G,W)\) for the particular case when \(\vec{1}_n\) is an eigenvector of \(W\). (This is notably the case when considering the Max \(k\)-cut problem in regular graphs with unit edge weights). Its expression does not involve distances and leads to an upper bound on \(m_{c_k}(G,W)\) that is lower than or equal to the bounds of Theorems 1.1-1.2.

We start with an auxiliary result on the minimum squared distance between any vector in \(\{1,r\}^n\) and the subspace in \(\mathbb{R}^n\) that is orthogonal to \(\text{Vect}(\vec{1}_n)\), denoted by \(\text{Vect}(\vec{1}_n)^\perp\).

**Proposition 3.1.**

$$\min \left\{ \|y - z\|^2 : y \in \{1,r\}, z \in \text{Vect}(\vec{1}_n)^\perp \right\} = \begin{cases} n & \text{if } r \geq 1, \\ \min \left( \frac{(s+r-1)^2}{n}, \frac{s^2}{n} \right) & \text{otherwise}, \end{cases}$$

with \(n \equiv s \mod (1-r)\), \(0 \leq s < 1 - r\), for the case when \(r < 1\).

**Proof.** Let \(p \in \{0,1,\ldots,n\}\) and \(\hat{y} \in \{r,1\}^n\) such that \(\hat{y}\) has exactly \(p\) entries with value \(r\). Let \(\hat{d}^2\) denote the squared distance between \(\hat{y}\) and \(\text{Vect}(\vec{1}_n)^\perp\), that is, the quantity

$$\hat{d}^2 = \frac{\langle \hat{y}, \vec{1}_n \rangle^2}{n} = \frac{(p(r-1) + n)^2}{n}.$$

For \(p \in \{0,1,\ldots,n\}\), the minimum of \(\hat{d}^2\) is obtained for \(p = 0\) if \(r \geq 1\) and for \(p = \left[ \frac{n}{1-r} \right]\) or \(p = \left[ \frac{n}{1-r} \right] - 1\), otherwise.

Using Proposition 3.1 together with the fact that \(d_{j,r} \geq d_{j+1,r}, \forall j \in [n-1]\) the next result follows.

**Corollary 3.2.** If \(\vec{1}_n\) is an eigenvector of \(W\) associated with the eigenvalue \(\lambda_q\), then

$$m_{c_k}(G,W) \leq \frac{1}{2(r-1)^2} \left( (r^2 + k - 1)(2w[V] - \lambda_1 n) - \frac{k}{2} \min((s+r-1)^2, s^2) \sum_{l \in [n-1]} (\lambda_{l+1} - \lambda_l) \right)$$

with \(r < 1\) and \(n \equiv s \mod (1-r)\), \(0 \leq s < 1 - r\). \(\square\)

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Taking $r = 1 - k$ (as is done in Nikiforov’s proof [6] of Theorem 1.2) leads to the following simpler expression.

**Corollary 3.3.** If $G$ is a complete graph and $W$ is its adjacency matrix, then

$$mc_k(G, W) \leq \frac{1}{2k} \left((k-1)n^2 - \min((s-k)^2, s^2)\right),$$

with $n \equiv s \mod k$, $0 \leq s < k$.

Proof. The eigenvalues of the adjacency matrix of the complete graph $K_n$ are $-1$ with multiplicity $n-1$ and $n-1$ with multiplicity 1. The vector $\vec{1}$ is an eigenvector associated with the eigenvalue $\lambda_{n-1} = n-1$. The result follows from (10) with $q = n$ and $r = 1 - k$.

Corollary 3.3 gives an infinite class of graphs (complete graphs such that $\min((s-k)^2, s^2) > 0$) where our new bound (4) strictly improves over Nikiforov’s bound (2).

The bound (11) has also the feature of coinciding with the optimal objective value of Max $k$-cut for some cases. Indeed, by Turán’s Theorem, the maximum cardinality of a $k$-cut in the complete graph $K_n$ is \( \left\lfloor \frac{n^2(k-1)}{2k} \right\rfloor \), and this corresponds to the bound (11) if $s^2 = n^2(k-1) \mod 2k$, where $n \equiv s \mod k$, $0 \leq s < k$. For $k = 2$, it follows that the bound (11) coincides with the optimal objective value of $mc_k(G, W)$ for all complete graphs (see also Proposition 4.4 in [2]), whereas this fails for the bounds of Theorems 1.1 and 1.2 for complete graphs having an odd number of vertices.

**References**


