Intersection disjunctions for reverse convex sets

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Abstract

We present a framework to obtain valid inequalities for optimization problems constrained by a reverse convex set, which is defined as the set of points in a polyhedron that lie outside a given open convex set. We are particularly interested in cases where the closure of the convex set is either non-polyhedral, or is defined by too many inequalities to directly apply disjunctive programming. Reverse convex sets arise in many models, including bilevel optimization and polynomial optimization. Intersection cuts are a well-known method for generating valid inequalities for a reverse convex set. Intersection cuts are generated from a basic solution that lies within the convex set. Our contribution is a framework for deriving valid inequalities for the reverse convex set from basic solutions that lie outside the convex set. We begin by proposing an extension to intersection cuts that defines a two-term disjunction for a reverse convex set. Next, we generalize this analysis to a multi-term disjunction by considering the convex set’s recession directions. These disjunctions can be used in a cut-generating linear program to obtain disjunctive cuts for the reverse convex set.

Keywords: Mixed-integer nonlinear programming; valid inequalities; reverse convex sets; disjunctive programming; intersection cuts

1 Introduction

A structure that appears or can be derived in many nonconvex optimization problems is a reverse convex set. A reverse convex set is a set of the form \( P \setminus C \), where \( P \subseteq \mathbb{R}^n \) is a polyhedron and \( C \subseteq \mathbb{R}^n \) is an open convex set. This is a general set structure arising in the context of mixed-integer nonlinear programming (MINLP). In this setting, \( P \) is a linear programming relaxation of the MINLP feasible region, and \( C \) contains no solutions feasible to the problem. We are motivated by cases where \( \text{cl}(C) \) is either non-polyhedral or is defined by a large number of linear inequalities. If \( \text{cl}(C) \) is a polyhedron defined by a small number of inequalities, we can optimize and separate over \( \text{cl} \text{conv}(P \setminus C) \) efficiently with a disjunctive program. We study valid inequalities for reverse convex sets. These inequalities can be used to strengthen the convex relaxation of any problem for which an open convex set containing no feasible points can be identified; such sets are known as convex S-free sets [9]. The reverse convex constraint \( x \notin C \) defines the reverse convex set.

Intersection cuts are valid inequalities for \( P \setminus C \) [2, 34]. These inequalities are generated from basic solutions of \( P \) that lie within \( C \). A basic solution \( \bar{x} \) of \( P \) corresponding to basis \( B \) forms the apex of a simplicial cone \( P^B \) defining a relaxation of \( P \). For each extreme ray of this cone, a point on \( \text{bd}(C) \) that intersects the extreme ray is found. A hyperplane \( c^\top x = d \) is formed, such that the hyperplanes passes through all of these points and satisfies \( c^\top \bar{x} > d \). For an extreme ray \( r \) of \( P^B \) that recedes into \( C \), it must instead hold that \( c^\top r = 0 \). The intersection cut \( c^\top x \leq d \) is valid for \( P \setminus C \). For a detailed review of intersection cuts, see Section 2.2.

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Our main contribution in this paper is to show how information about the extreme rays of $P^B$ intersect $C$ can be used to construct valid inequalities for $P \setminus C$ in the case where $\bar{x} \notin \text{cl}(C)$. Because $\bar{x} \notin C$, intersection cuts generated using the cone $P^B$ are not valid in general. However, under the assumption that each extreme ray of $P^B$ intersects $C$, we present two linear inequalities that form a two-term disjunction (union of sets) containing $P \setminus C$. If $P$ intersected with one of these inequalities is empty, the inequality defining the other disjunctive term is valid for $P \setminus C$. We call inequalities obtained in this manner external intersection cuts. If both disjunctive terms are nonempty, we can generate disjunctive cuts using the standard cut-generating linear program (CGLP) for disjunctive programming [3, 4]. We refer to these disjunctions as intersection disjunctions.

We extend this analysis by presenting a relaxation of $P^B \setminus C$ that considers $\text{recc}(C)$, the recession cone of $C$. We provide a class of valid inequalities for the relaxation which grows exponentially with the problem size. We derive a polynomial-size extended formulation that captures the full strength of this exponential family of inequalities. We then prove that the proposed relaxation of $P^B \setminus C$ is equivalent to the union of at most $n$ possibly nonconvex sets, thereby forming a disjunction for the reverse convex set. Under some assumptions, we propose a polyhedral relaxation of each disjunctive term individually. Given these polyhedral relaxations, we can use a CGLP to generate disjunctive cuts for $P \setminus C$.

This paper is organized as follows. In Section 3, we present a two-term disjunction for $P \setminus C$ generated by basic solutions of $P$ that lie outside of $C$. In Section 4, we extend this analysis by presenting a multi-term disjunction for $P \setminus C$ by considering $\text{recc}(C)$. We propose extended formulations that can be used to define polyhedral relaxations of the disjunctive terms. Throughout, we consider a fixed basis $B$ and corresponding basic solution $\bar{x}$.

## 2. Related literature

The problem of optimizing a linear function over a reverse convex set is known as linear reverse convex programming (LRCP). Tuy shows that any convex program with multiple reverse convex constraints can be reduced to one with a single reverse convex constraint with the introduction of an additional variable and an additional convex constraint [36]. By reduction from a concave minimization problem, optimizing a linear function over a reverse convex set is NP-hard, even in special cases restricting the structure of the linear constraints or the convex set $C$ [15, 25]. Hillestad and Jacobsen define the concept of a basic solution for LRCP. They show the convex hull of the feasible region of LRCP is a polytope if the linear constraints form a polytope and the functions defining the reverse convex constraints are differentiable [22]. Sen and Sherali extend this result, showing that the closure of the convex hull of any polyhedron intersected with a finite number of reverse convex constraints is a polyhedron [33]. Hillestad and Jacobsen also propose a cutting plane algorithm that uses intersection cuts to generate a sequence of relaxations for $\text{conv}(P \setminus C)$, although these relaxations do not necessarily converge to $\text{conv}(P \setminus C)$ [22]. Additional algorithms for reverse convex optimization have subsequently been developed and analyzed [14, 20, 23, 27, 30, 37].

### 2.1 Examples of reverse convex sets

There are many problems where reverse convex sets can be used to supplement a convex formulation of the problem. One example is difference of convex (DC) functions [35]. A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a DC function if there exist convex functions $g, h: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f(x) = g(x) - h(x)$ for all $x \in \mathbb{R}^n$. A DC set can be written as

$$\{x \in \mathbb{R}^n : g(x) - h(x) \leq 0\}. \quad (2.1)$$

Equivalently, we can write (2.1) as $\text{proj}_x(Z)$, where $Z := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : g(x) - t \leq 0, h(x) - t \geq 0\}$. The convex set $C = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : h(x) - t < 0\}$ contains no points feasible to $Z$. The class of DC functions is broad, subsuming all twice continuously differentiable functions [21].

Reverse convex sets also appear in the context of polynomial optimization. Bienstock et al. consider the set of symmetric matrices representable as outer-products: $\{xx^T : x \in \mathbb{R}^n\}$ [8]. Polynomial optimization problems can be reformulated to include the constraint that a square matrix of variables is outer-product representable. Bienstock et al. construct non-polyhedral outer-product-free sets $C$ that do not contain any
matrices representable as an outer-product of some vector, and as such are not feasible to the problem. Accordingly, they present families of cuts for \( P \setminus C \), where \( P \) is formed by the linear constraints of the problem reformulation. They characterize sets that are \textit{maximal} outer-product-free, that is, not contained in any other outer-product-free sets. For the specific case of quadratically constrained programs (QCPs), Saxena et al. use disjunctive programming techniques to derive valid inequalities for a reverse convex set in an extended variable space [31]. In a companion paper, they suggest an \textit{eigen-reformulation} of the quadratic constraint \( x^T A x + a^T x + b \leq 0 \) [32]:

\[
\sum_{j: \lambda_j > 0} \lambda_j (v_j^T x)^2 + \sum_{j: \lambda_j < 0} \lambda_j s_j + a^T x + b \leq 0
\]

\[
s_j = (v_j^T x)^2 \quad \forall j: \lambda_j < 0,
\]

where \( \lambda_1, \ldots, \lambda_n \) denote the eigenvalues of \( A \), and \( v_1, \ldots, v_n \) denote the corresponding eigenvectors. The convex set \( \{(x, s) \in \mathbb{R}^n \times \mathbb{R}^n : s_j > (v_j^T x)^2 \forall j \text{ s.t. } \lambda_j < 0\} \) does not contain any points feasible to QCP.

Reverse convex sets can also be used to define relaxations of bilevel optimization problems. Bilevel programs include constraints of the form

\[
d \in \mathbb{R}^m \quad \text{where} \quad \lambda \in \mathbb{R}^n,
\]

\[
\mathbb{R}^n \times \mathbb{R}^n
\]

where \( \lambda_1, \ldots, \lambda_n \) denote the eigenvalues of \( A \), and \( v_1, \ldots, v_n \) denote the corresponding eigenvectors. The convex set \( \{(x, s) \in \mathbb{R}^n \times \mathbb{R}^n : s_j > (v_j^T x)^2 \forall j \text{ s.t. } \lambda_j < 0\} \) does not contain any points feasible to QCP.

Reverse convex sets can also be used to define relaxations of bilevel optimization problems. Bilevel programs include constraints of the form \( d^T y \leq \Phi(x) \), where \( \Phi(x) \) is the “value function” for a fixed \( x \):

\[
\Phi(x) := \min_y \{d^T y : Ax + By \leq b\}.
\]

The function \( \Phi(\cdot) \) is convex. The set \( \{(x, y) : d^T y > \Phi(x)\} \) is defined by a reverse convex inequality and does not contain any points feasible to the bilevel program. The closure of this set is polyhedral, but may be defined by a large number of linear inequalities. Fischetti et al. propose intersection cuts for a specific class of bilevel integer programming problems [13].

### 2.2 Intersection cut review

We briefly review intersection cuts, following the presentation of Conforti et al. [10]. Let \( A \in \mathbb{R}^{m \times n} \) be a matrix with full row rank and let \( b \in \mathbb{R}^m \). Let \( P = \{x \in \mathbb{R}_+^n : Ax = b\} \) be a polyhedron. Let \( C \) be an open convex set. We are interested in valid inequalities for the reverse convex set \( P \setminus C \).

For a basis \( B \) of \( P \), let \( N = \{1, \ldots, n\} \setminus B \) be the nonbasic variables. For some \( \bar{a} \in \mathbb{R}^{|B| \times |N|} \) and \( \bar{b} \in \mathbb{R}^{|B|} \), we can rewrite \( P \) as

\[
P = \left\{ x \in \mathbb{R}^n : x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j, i \in B, x_j \geq 0, j = 1, \ldots, n \right\}.
\]

The basic solution corresponding to basis \( B \) is \( \bar{x} \), where \( \bar{x}_i = \bar{b}_i \) if \( i \in B \), and 0 if \( i \in N \). By removing the nonnegativity constraints on variables \( x_i \), \( i \in B \), we obtain \( P^B \), the cone admitted by the basis \( B \). The basic solution \( \bar{x} \) forms the apex of \( P^B \supseteq P \). There is an extreme ray \( \bar{r}^j \) of \( P^B \) for each \( j \in N \):

\[
\bar{r}^j_k = \begin{cases} 
-\bar{a}_{kj} & \text{if } k \in B \\
1 & \text{if } k = j \\
0 & \text{if } k \in N \setminus \{j\}
\end{cases}
\]

The conic hull of the extreme rays \( \{\bar{r}^j : j \in N\} \) forms the recession cone of \( P^B \). Together, the basic solution \( \bar{x} \) and these extreme rays provide a complete internal representation of \( P^B \), namely, \( P^B = \{\bar{x} + \sum_{j \in N} x_j \bar{r}^j : x \in \mathbb{R}_+^{|N|}\} \).

Intersection cuts are valid inequalities for \( P^B \setminus C \) constructed from basic solutions of \( P \) that lie within \( C \). These cuts are transitively valid for \( P \setminus C \subseteq P^B \setminus C \). Let \( \bar{\beta}_j \) be defined as

\[
\bar{\beta}_j := \sup \{\beta \geq 0 : \bar{x} + \beta \bar{r}^j \in C\} \quad \forall j \in N.
\]

The set \( \{\bar{x} + \bar{\beta}_j \bar{r}^j : j \in N\} \) is the points where the extreme rays of \( P^B \) emanating from \( \bar{x} \) leave the set \( C \). Because \( C \) is open, \( \bar{\beta}_j > 0 \) for all \( j \in N \). If \( \bar{\beta}_j = +\infty \), \( \bar{r}^j \) lies in the recession cone of \( C \). We use the convention \( 1/\pm \infty = 0 \).
The following inequality is valid for $P \setminus C$ [2]:

$$\sum_{j \in N} \frac{x_j}{\beta_j} \geq 1. \quad (2.2)$$

We refer to (2.2) as the standard intersection cut.

In cutting plane algorithms leveraging intersection cuts, inequalities of the form (2.2) may be constructed from the basis corresponding to the current optimal solution. Infeasible basic solutions of $P$ within $C$ are also candidates for generating intersection cuts for $P \setminus C$ and may yield intersection cuts that are not dominated by those generated from feasible basic solutions. Gomory mixed-integer (GMI) cuts for mixed-integer linear programming (MILP) behave similarly, and are strengthened versions of intersection cuts. In particular, the intersection of GMI cuts from all basic solutions is equivalent to the split closure [28]. This is not true when considering only GMI cuts from basic feasible solutions [11]. Every GMI cut is an intersection cut obtained from a particular split. Intersection cuts are sufficient to define the split closure of a MILP [1, 6].

Intersection cuts can be generated from any convex set that does not contain feasible points in its interior. In integer programming, these are maximal lattice-free convex sets. More generally, these types of sets are from a particular split. Intersection cuts are sufficient to define the split closure of a MILP [1, 6]. Considering only GMI cuts from basic feasible solutions [11]. Every GMI cut is an intersection cut obtained from a particular split. Intersection cuts are sufficient to define the split closure of a MILP [1, 6].

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In Sections 3 and 4, we assume the basic solution $\mathbf{\bar{x}}$ lies outside of $\text{cl}(C)$. We refer to (2.2) as the standard intersection cut.

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Let $\mathbb{R} := \mathbb{R} \cup \{-\infty, +\infty\}$ be the extended real numbers. For a nonzero vector $r \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, we define the line segment $(\alpha, \beta)r := \{\lambda r : \lambda \in (\alpha, \beta)\}$. Closed brackets (e.g., $[\alpha, \beta)r$) denote the inclusion of one or both endpoints of the line segment. The set $(\alpha, \beta)r$ is unbounded if and only if $\alpha = -\infty$ or $\beta = +\infty$.

Recall $P^B$ is a simplicial cone with apex $\mathbf{\bar{x}}$ and linearly independent extreme rays $\{\mathbf{r}^j : j \in N\}$. Let

$$\alpha_j := \inf\{\alpha \geq 0 : \mathbf{\bar{x}} + \alpha \mathbf{r}^j \in C\} \quad \forall j \in N$$

$$\beta_j := \sup\{\beta \geq 0 : \mathbf{\bar{x}} + \beta \mathbf{r}^j \in C\} \quad \forall j \in N.$$

For $j \in N$, we use the convention $\alpha_j = +\infty$ and $\beta_j = -\infty$ if the set $\{\mathbf{\bar{x}}\} + [0, +\infty)\mathbf{r}^j$ does not intersect $C$. If $\text{cl}(C)$ is polyhedral and $\{\mathbf{\bar{x}}\} + [0, +\infty)\mathbf{r}^j$ intersects $\text{bd}(C)$ finitely many times, $\alpha_j$ and $\beta_j$ can be obtained by solving a linear program. If $\text{cl}(C)$ is non-polyhedral, a convex program may be required to obtain these
parameters. An exception is the case where \( C \) is bounded and a point in \( C \cap (\{\bar{x}\} + [0, +\infty)\bar{v}^j) \) is known a priori, in which case a binary search can be performed to find the values of \( \alpha_j \) and \( \beta_j \).

We partition \( N \) into the following three sets:

\[
N_0 := \{ j \in N : \alpha_j = +\infty \} \\
N_1 := \{ j \in N : \alpha_j < +\infty, \beta_j = +\infty \} \\
N_2 := \{ j \in N : \alpha_j < +\infty, \beta_j < +\infty \}.
\]

For \( j \in N_0 \), the halfline \( \{\bar{x}\} + [0, +\infty)\bar{v}^j \) does not intersect \( C \). Observe \( \bar{v}^j \in \text{recc}(C) \) for \( j \in N_1 \).

Throughout Section 3, we make the following assumption.

**Assumption 1.** It holds that \( \bar{v}^j \in \text{recc}(C) \) for all \( j \in N_0 \).

Theorem 1 proposes a disjunction for \( P^B \setminus C \supseteq P \setminus C \). This two-term disjunction can be used in a disjunctive framework to generate valid inequalities for \( P \setminus C \).

**Theorem 1.** For every \( x \in P^B \setminus C \), either

\[
\sum_{j \in N} x_j \alpha_j \leq 0, \text{ or } \sum_{j \in N} x_j \beta_j \geq 1. \tag{3.1}
\]

**Proof.** If \( N = N_0 \), then \( \alpha_j = +\infty \) for all \( j \in N \), and all \( x \in P^B \setminus C \) trivially satisfy \( \sum_{j \in N} x_j / \alpha_j \leq 1 \). We prove the result for \( N \neq N_0 \). Assume \( \hat{x} \in P^B \) satisfies \( \sum_{j \in N} \hat{x}_j / \alpha_j > 1 \) and \( \sum_{j \in N} \hat{x}_j / \beta_j < 1 \). We show \( \hat{x} \in C \), and hence \( \hat{x} \in P^B \setminus C \). Define \( f : \mathbb{R}_+ \to \mathbb{R} \) as

\[
f(\gamma) = \sum_{j \in N_1} \frac{x_j}{\alpha_j + \gamma} + \sum_{j \in N_2} \frac{(\gamma + 1)x_j}{\alpha_j + \gamma \beta_j}.
\]

The function \( f \) is continuous on its domain. Observe \( f(0) = \sum_{j \in N} \hat{x}_j / \alpha_j > 1 \), and \( \lim_{\gamma \to \infty} f(\gamma) = \sum_{j \in N} \hat{x}_j / \beta_j < 1 \). By the Intermediate Value Theorem, there exists \( \gamma > 0 \) such that \( f(\gamma) = 1 \). For \( j \in N \setminus N_0 \), define \( \theta_j := \hat{x}_j / (\alpha_j + \gamma) \) if \( j \in N_1 \), and \( \theta_j := [(\gamma + 1)\hat{x}_j / (\alpha_j + \gamma \beta_j) \) if \( j \in N_2 \). By construction, \( \sum_{j \in N \setminus N_0} \theta_j = 1 \) and \( \theta_j \geq 0 \) for all \( j \in N \setminus N_0 \). For \( j \in N \setminus N_0 \), let

\[
z^j := \begin{cases} (\alpha_j + \gamma)\bar{v}^j + \sum_{i \in N_0} \bar{x}_i \bar{v}^i & \text{if } j \in N_1 \\ \left( \frac{1}{\gamma + 1} \alpha_j + \frac{\gamma}{\gamma + 1} \beta_j \right) \bar{v}^j + \sum_{i \in N_0} \bar{x}_i \bar{v}^i & \text{if } j \in N_2. \end{cases}
\]

Let \( \hat{v} := \sum_{j \in N_0} \hat{x}_j \bar{v}^j \). Observe \( \hat{x} = \sum_{j \in N \setminus N_0} \theta_j (\hat{x} + z^j) \):

\[
\hat{x} = \bar{x} + \sum_{j \in N} \hat{x}_j \bar{v}^j \\
= \bar{x} + \sum_{j \in N_1} \frac{\hat{x}_j}{\alpha_j + \gamma}(\alpha_j + \gamma)\bar{v}^j + \sum_{j \in N_2} \frac{(\gamma + 1)\hat{x}_j}{\alpha_j + \gamma \beta_j} \left( \frac{1}{\gamma + 1} \alpha_j + \frac{\gamma}{\gamma + 1} \beta_j \right) \bar{v}^j + \hat{v} \\
= \sum_{j \in N \setminus N_0} \theta_j (\hat{x} + z^j).
\]

For all \( j \in N \setminus N_0 \), we have \( \hat{x} + z^j \in (\{\bar{x}\} + (\alpha_j, \beta_j)\bar{v}^j) + \text{recc}(C) \subseteq C \). Then \( \hat{x} \) is a convex combination of points in \( C \).

**Remark 1.** We consider the relationship between the two-term disjunction (3.1) and standard intersection cuts. The two-term disjunction (3.1) assumes that the basic solution \( \bar{x} \) does not lie within \( \text{cl}(C) \). If \( \bar{x} \in C \), then \( N_0 = \emptyset \) (trivially, every extreme ray of \( P^B \) emanating from \( \bar{x} \in C \) intersects \( C \)) and \( \alpha_j = 0 \) for all \( j \in N \). Because \( \alpha_j = 0 \) for all \( j \in N \), the inequality \( \sum_{j \in N} x_j / \alpha_j \leq 1 \) of (3.1) is ill-defined. Instead, we can show that all points in \( P^B \setminus C \) lie in either \( \bar{x} \) or \( \{x \in \mathbb{R}^n : \sum_{j \in N} x_j / \beta_j \geq 1 \} \). However, because \( \{\bar{x}\} \subseteq C \), we can conclude that the inequality \( \sum_{j \in N} x_j / \beta_j \geq 1 \) is valid for \( P^B \setminus C \). This is precisely the standard intersection cut of Balas [2].
Example 1. Let $P = \mathbb{R}^2_+$ and $C = \{(x_1, x_2) \in \mathbb{R}^2: (x_1 - 1)^2 - x_2 < 1/2\}$. Consider $P^B$ generated by the (only) basic solution of $P$, $\bar{x} = (0, 0) \notin C$. In this case, $P^B = P$. The feasible region $P \setminus C$ is the disconnected set shaded in Figure 1a. The inequalities (3.1) form a disjunction for $P \setminus C$, shown in Figure 1b.

Proposition 1 states that if $C$ is bounded, then the inequality defining each term of (3.1) is sufficient to define the convex hull of the points in $P^B \setminus C$ satisfying that inequality. This is not true in general.

**Proposition 1.** If $C$ is bounded, then

\[
\text{conv}\{\{x \in P^B \setminus C: \sum_{j \in N} x_j/\alpha_j \leq 1\}\} = \{x \in P^B: \sum_{j \in N} x_j/\alpha_j \leq 1\}, \quad \text{and}
\]

\[
\text{conv}\{\{x \in P^B \setminus C: \sum_{j \in N} x_j/\beta_j \geq 1\}\} = \{x \in P^B: \sum_{j \in N} x_j/\beta_j \geq 1\}.
\]

**Proof.** We show only that $\text{conv}\{\{x \in P^B \setminus C: \sum_{j \in N} x_j/\alpha_j \leq 1\}\} = \{x \in P^B: \sum_{j \in N} x_j/\alpha_j \leq 1\}$, as the second statement can be shown using similar techniques. Under Assumption 1, $C$ bounded implies $N = N_i$.

Because $P^B \setminus C \subseteq P^B$ and $\{x \in P^B: \sum_{j \in N} x_j/\alpha_j \leq 1\}$ is convex, $\text{conv}\{\{x \in P^B \setminus C: \sum_{j \in N} x_j/\alpha_j \leq 1\}\} \subseteq \{x \in P^B: \sum_{j \in N} x_j/\alpha_j \leq 1\}$. Next, let $\bar{x} \in P^B$ satisfy $\sum_{j \in N} \bar{x}_j/\alpha_j \leq 1$. Then

\[
\bar{x} = \bar{x} + \sum_{j \in N} \bar{x}_j/\alpha_j = \sum_{j \in N} \bar{x}_j/\alpha_j + \left(1 - \sum_{j \in N} \bar{x}_j/\alpha_j \right) \bar{x}
\]

\[
\in \text{conv}\{\{x \in P^B \setminus C: \sum_{j \in N} x_j/\alpha_j \leq 1\}\} \subseteq \text{conv}(P \setminus C).
\]

By the definition of $\bar{x}$ in Section 2.2, $\sum_{j \in N} \bar{x}_j/\alpha_j = 0$. Furthermore, for any $i, j \in N$, $[\bar{x} + \alpha_j \bar{r}_j]$, equals 1 if $i = j$, and 0 otherwise. Continuing from (3.2), we have $\bar{x} \in \text{conv}\{\{x \in P^B \setminus C: \sum_{j \in N} x_j/\alpha_j \leq 1\}\}$.

The disjunction presented in Theorem 1 can be particularly useful if $P$ is empty when intersected with one of the inequalities (3.1). In this case, the inequality defining the other disjunctive term is valid for $P^B \setminus C$.

**Definition 1.** If $\{x \in P: \sum_{j \in N} x_j/\beta_j \geq 1\} = \emptyset$, we refer to the inequality $\sum_{j \in N} x_j/\alpha_j \leq 1$ as an **external intersection cut**. We say the same for the inequality $\sum_{j \in N} x_j/\beta_j \geq 1$ if $\{x \in P: \sum_{j \in N} x_j/\alpha_j \leq 1\} = \emptyset$.

External intersection cuts are valid for $P \setminus C$.

**Example 2.** Let

\[
P = \{(x_1, x_2) \in \mathbb{R}^2_+: -x_1 + 3x_2 \leq 3/2\}
\]

\[
C = \{x \in \mathbb{R}^2: ||x||_2 < 1\}.
\]

As can be seen in Figure 2a, no standard intersection cut is able to generate the inequality that is facet-defining for $\text{conv}(P \setminus C)$. However the basic solution $\bar{x} = (-3/2, 0) \notin \text{cl}(C)$ corresponding to the constraints $x_1 \geq 0$ and $-x_1 + 3x_2 \leq 3/2$ generates this inequality as an external intersection cut. For this basic solution, the set $P \cap \{x \in \mathbb{R}^n: x_j/\alpha_j \leq 1\}$ is empty, implying the inequality $\sum_{j \in N} x_j/\beta_j \geq 1$ is valid for $P \setminus C$. Figure 2b shows the inequalities (3.1) for this example.
The facet-defining inequality for the reverse convex set in Example 2 is not obtainable from a standard intersection cut.

Figure 2: Example of an external intersection cut generating a cut unobtainable by standard intersection cuts.

The external intersection cut \( \sum_{j \in N} x_j / \alpha_j \leq 1 \) defines the facet-defining inequality for \( \text{conv}(P \setminus C) \) for Example 2.

Figure 3: External intersection cut \( \sum_{j \in N} x_j / \alpha_j \leq 1 \) when \( \{ x \in P : \sum_{j \in N} x_j / \beta_j \geq 1 \} = \emptyset \).

Our final example of this section motivates considering how the recession cone can be used to derive more general valid disjunctions for \( P_B \setminus C \).

Example 3. Let \( P = \mathbb{R}^2_+ \), and

\[
C = \left\{ (x_1, x_2) : \left( x_1 - \frac{3}{4} \right)^2 + \left( x_2 - \frac{1}{4} \right)^2 < \frac{1}{4} \right\} + \text{cone} \left( \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right).
\]

Figure 5 provides a graphical representation of \( P_B \setminus C \), where \( P_B \) is generated from the basic solution \( \bar{x} = (0, 0) \). Assumption 1 does not hold; namely, \( 2 \in N_0 \), but \( \bar{r}^2 \notin \text{recc}(C) \). However, there exists a two-term disjunction for \( P_B \setminus C \) that cannot be obtained with the theory of this section.

4 Valid inequalities and intersection disjunctions using \( \text{recc}(C) \)

In this section, we generalize the results of Section 3 by considering the full recession cone of \( C \). In Section 4.1, we construct an inner approximation of \( C \) and analyze its relationship to \( P_B \setminus C \). We derive inequalities to...
(a) Theorem 1’s disjunction fails if there exists $j \in N_0$ such that $\bar{r}^j \notin \text{recc}(C)$

Figure 4: Assumption 1 is necessary for the disjunction of Theorem 1

(b) The disjunction of Theorem 1 requires Assumption 1

Figure 5: Although a two-term disjunction for Example 3 exists, our inequalities are insufficient to obtain it, because Assumption 1 is not satisfied

define a polyhedral relaxation of $P_B \setminus C$ in Section 4.2. In Section 4.3, we generalize the two-term disjunction of Theorem 1 to a multi-term disjunction that uses the recession cone of $C$. We propose polyhedral relaxations of each of these disjunctive terms in Sections 4.4 and 4.5.

4.1 An inner approximation of $C$

Let $N_0^* := N_1 \cup N_2$. We define $T$ and $T^C$ as follows:

$$T := \{\bar{x}\} + \text{conv}\left(\bigcup_{j \in N_0^*} (\alpha_j, \beta_j)\bar{r}^j\right), \quad T^C := T + \text{recc}(C). \quad (4.1)$$

Both $T$ and $T^C$ are subsets of $C$. We derive inequalities valid for $P_B \setminus T^C \supseteq P_B \setminus C$. We illustrate the set $T^C$ graphically in the example that follows.

Example 4. Let $P = \mathbb{R}_+^2$ and $C = \{(x_1, x_2) : x_2 > \sqrt{(x_1 - 1)^2 + 1} - 1.1\}$. Let $\bar{x} = (0, 0)$ be the basic solution of $P$, corresponding to basis $B$. The sets $P_B$ and $C$ are shown in Figure 6a. Figures 6b and 6c show the sets $T^C$ and $P_B \setminus T^C$, respectively.

We motivate the study of $P_B \setminus T^C$ by showing $P_B \setminus T^C$ retains the strength of $P_B \setminus C$ under the convex hull operator. We define $R$ and $R^C$ as follows:

$$R := \{\bar{x}\} + \text{conv}\left(\bigcup_{j \in N_1} (\alpha_j, \beta_j)\bar{r}^j\right), \quad R^C := R + \text{recc}(C).$$

Theorem 2. It holds that

$$\text{clconv}(P_B \setminus C) = \text{clconv}(P_B \setminus T^C) = \text{clconv}(P_B \setminus R^C).$$

Proof. Observe $\{\bar{x}\} + \bigcup_{j \in N_1} (\alpha_j, \beta_j)\bar{r}^j \subseteq \{\bar{x}\} + \bigcup_{j \in N_0^*} (\alpha_j, \beta_j)\bar{r}^j \subseteq C$. By definition, $R^C \subseteq T^C \subseteq C$, which implies

$$\text{conv}(P_B \setminus C) \subseteq \text{conv}(P_B \setminus T^C) \subseteq \text{conv}(P_B \setminus R^C).$$

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To complete the proof, we show \( P^B \setminus R^C \subseteq \text{clconv}(P^B \setminus C) \). Let \( y \in P^B \setminus R^C \). Assume \( y \in C \), or we have nothing to prove. Because \( y \in P^B \), we have \( y = \bar{x} + \sum_{j \in N} y_j \bar{r}^j \), where \( y_j \geq 0 \) for all \( j \in N \). Let \( \eta := \sum_{j \in N_1} y_j / \alpha_j \).

To begin, assume \( \eta < 1 \). Assume also that \( \sum_{j \in N_0 \cup N_2} y_j > 0 \), otherwise \( y \) is a convex combination of the points \( \{ \bar{x} \} \cup \{ \bar{x} + \alpha_j \bar{r}^j : j \in N_1 \} \subseteq P^B \setminus C \):

\[
y = (1 - \eta)\bar{x} + \sum_{j \in N_1} \frac{y_j}{\alpha_j} (\bar{x} + \alpha_j \bar{r}^j).
\]

Let \( \lambda := \sum_{j \in N_0 \cup N_2} y_j / (1 - \eta) \). We rewrite \( y \) as

\[
y = \sum_{j \in N_1} \frac{y_j}{\alpha_j} (\bar{x} + \alpha_j \bar{r}^j) + \sum_{j \in N_0 \cup N_2} \frac{y_j}{\lambda} (\bar{x} + \lambda \bar{r}^j).
\]

We have \( \bar{x} + \alpha_j \bar{r}^j \in P^B \setminus C \) for all \( j \in N_1 \). Additionally, \( \bar{x} + \lambda \bar{r}^j \in P^B \setminus C \) for all \( j \in N_0 \). For \( j \in N_2 \), \( \bar{x} + \delta \bar{r}^j \in \text{conv}(P^B \setminus C) \), because \( \bar{x} \in P^B \setminus C \) and \( \bar{x} + \delta \bar{r}^j \in P^B \setminus C \) for a sufficiently large \( \delta > \lambda \). The coefficients on the vectors \( \{ \bar{x} + \alpha_j \bar{r}^j : j \in N_1 \} \cup \{ \bar{x} + \lambda \bar{r}^j : j \in N_0 \cup N_2 \} \) are nonnegative and sum to one. Then \( y \in \text{conv}(P^B \setminus C) \).

Next, assume \( \eta = 1 \). It follows from the above analysis that \( y \in \text{clconv}(P^B \setminus C) \).

Finally, assume \( \eta > 1 \). For \( \epsilon \in [0, 1) \), let \( z^\epsilon \) be the following:

\[
z^\epsilon := \bar{x} + \sum_{j \in N_1} \frac{y_j}{\eta}(1 - \epsilon)\eta + \epsilon \bar{r}^j.
\]

Observe \( z^\epsilon \in R \) for all \( \epsilon \in [0, 1) \), because \( z^\epsilon = \bar{x} + \sum_{j \in N_1} (y_j/\eta\alpha_j)((1 - \epsilon)\eta + \epsilon)|\alpha_j| \bar{r}^j \). For all \( \epsilon \in [0, 1) \), it must be the case that \( y - z^\epsilon \notin \text{recc}(C) \). If not, we have \( y = z^\epsilon + q \) for some \( q \in \text{recc}(C) \), implying \( y \in R^C \) and contradicting \( y \in \text{clconv}(P^B \setminus C) \).

It holds that \( y - z^\epsilon \in \text{recc}(P^B) \):

\[
y - z^\epsilon = \sum_{j \in N_0 \cup N_2} y_j \bar{r}^j + \sum_{j \in N_1} \frac{y_j}{\eta}(\eta - 1)\epsilon \bar{r}^j.
\]

Then we have \( y - z^\epsilon \in \text{recc}(P^B) \setminus \text{recc}(C) \). For a sufficiently large \( \gamma_\epsilon > 0 \), \( y + \gamma_\epsilon(y - z^\epsilon) \notin C \), because \( y - z^\epsilon \notin \text{recc}(C) \). Because \( y - z^\epsilon \in \text{recc}(P^B) \), \( y + \gamma_\epsilon(y - z^\epsilon) \in P^B \setminus C \). Let \( \hat{z} \in \text{conv}(P^B \setminus C) \) be defined as follows:

\[
\hat{z} := \bar{x} + \sum_{j \in N_1} \frac{y_j}{\eta} \bar{r}^j = \sum_{j \in N_1} \frac{y_j}{\eta\alpha_j}(\bar{x} + \alpha_j \bar{r}^j) \in \text{conv}(\{ \bar{x} + \alpha_j \bar{r}^j : j \in N_1 \})
\]

For any \( \epsilon \in [0, 1) \), let \( v^\epsilon \) be the following convex combination of \( \hat{z} \) and \( y + \gamma_\epsilon(y - z^\epsilon) \):

\[
v^\epsilon := \frac{\gamma_\epsilon}{\gamma_\epsilon + 1} \hat{z} + \frac{1}{\gamma_\epsilon + 1} (y + \gamma_\epsilon(y - z^\epsilon)) \in \text{conv}(P^B \setminus C).
\]
Observe that \( \lim_{\epsilon \to 1} z^\epsilon = \hat{z} \), and \( \gamma_\epsilon/(\gamma_\epsilon + 1) \in [0, 1) \) for all \( \gamma_\epsilon > 0 \). Then
\[
\lim_{\epsilon \to 1} \left| \frac{\gamma_\epsilon}{\gamma_\epsilon + 1} (\hat{z} - z^\epsilon) \right| \leq \lim_{\epsilon \to 1} \left| \frac{\gamma_\epsilon}{\gamma_\epsilon + 1} ||\hat{z} - z^\epsilon|| \right| = 0.
\]
Rearranging the definition of \( v^\epsilon \) from (4.2), we have \( y = v^\epsilon + [\gamma_\epsilon/(\gamma_\epsilon + 1)](z^\epsilon - \hat{z}) \). Thus,
\[
y = \lim_{\epsilon \to 1} v^\epsilon \in \text{clconv}(P^B \setminus C).
\]

Theorem 2 supports our selection of \( P^B \setminus T^C \) as a relaxation of \( P^B \setminus C \), as we do not lose anything when considering \( \text{clconv}(P^B \setminus T^C) \). For the remainder of Section 4, we make the following assumption.

**Assumption 2.** The recession cone of \( C \) is contained in the recession cone of \( P^B \).

If Assumption 2 does not hold, we can consider the convex set \( P^B \cap C \) instead of \( C \). Indeed, \( \text{recc}(P^B \cap C) \subseteq \text{recc}(P^B) \). Our analysis only requires the set \( C \) to be relatively open in \( P^B \), not necessarily open. By Corollary 1, replacing \( C \) with \( P^B \cap C \) does not change the strength of our relaxation of \( P^B \setminus C \) with respect to the convex hull operator.

**Corollary 1.** It holds that \( \text{clconv}(P^B \setminus C) = \text{clconv}(P^B \setminus T^{P^B \cap C}) \).

**Proof.** The statement follows directly from Theorem 2:
\[
\text{clconv}(P^B \setminus C) = \text{clconv}(P^B \setminus (P^B \cap C)) = \text{clconv}(P^B \setminus T^{P^B \cap C}).
\]

If \( P^B \) in Corollary 1 is replaced with \( P \), the statement is no longer true in general. This is relevant because we present a disjunction for \( P^B \setminus T^C \) later in this section. When we add the constraints of \( P \) to the disjunction formulation of \( P^B \setminus T^C \), the cuts obtained from a CGLP could be stronger than those obtained from a CGLP built from a disjunction of \( P^B \setminus T^{P^B \cap C} \). Thus, substituting \( C \) with \( P^B \cap C \) has the potential to weaken the generated disjunctive cuts.

### 4.2 Polyhedral relaxation of \( P^B \setminus T^C \)

We present valid inequalities for \( P^B \setminus Q^D_\emptyset \), where \( D \subseteq N^*_0 \) is fixed and
\[
Q_D := \{ \bar{x} \} + \text{conv} \left( \bigcup_{j \in D} (\alpha_j, +\infty) \bar{r}^j \right), \quad Q^C_D := Q_D + \text{recc}(C).
\]
The set \( T^C \) is equivalent to \( Q^C_D \) when \( D = N_1 \). We consider this more general set structure \( Q^C_D \) to be able to apply this analysis in Section 4.4. Observe that \( \text{recc}(Q^C_D) = \text{cone} \{ \bar{r}^j : j \in D \} + \text{recc}(C) \). Furthermore, \( Q^C_D \subseteq C \) if and only if \( D \subseteq N_1 \).

Let \( F_{ij} := \text{cone} \{ \bar{r}^i, \bar{r}^j \} \) be the cone formed by extreme rays \( \bar{r}^i \) and \( \bar{r}^j \) \( (i, j \in N) \). For \( (i, j) \in D \times (N \setminus D) \), let \( \gamma_{ij} \) be the following:
\[
\gamma_{ij} := \sup \{ \gamma \geq 0 : \alpha_i \bar{r}^i + \gamma \bar{r}^j \in \text{recc}(Q^C_D) \}.
\]
It holds that \( \gamma_{ij} = +\infty \) if \( j \in N \setminus D \) and \( \bar{r}^j \in \text{recc}(Q^C_D) \). In all other cases, \( \gamma_{ij} \) is finite and its supremum is attained, because \( \text{recc}(Q^C_D) \) is a closed convex cone. The parameter \( \gamma_{ij} \) depends on \( D \), but we suppress this dependence for notational simplicity.

Let \( D^* \) be defined as follows:
\[
D^* := \{ i \in D : \gamma_{ij} > 0 \ \forall j \in N \setminus D \}.
\]
The set \( D^* \) is composed of indices \( i \in D \) corresponding to extreme rays of \( P^B \) that exhibit the following property: for every \( j \in N \setminus D \), the cone \( F_{ij} \) contains a nontrivial element of \( \text{recc}(Q^C_D) \) (i.e., anything outside of \([0, +\infty)\bar{r}^i \)).
Proposition 2. Let \((i, j) \in D^* \times (N \setminus D)\). For any \(\gamma \in [0, \gamma_{ij})\), \(\alpha_i \bar{r}^i + \gamma \bar{r}^j \in \text{recc}(Q_D^C)\).

Proof. If \(\gamma_{ij} = +\infty\), then \(\bar{r}^j \in \text{recc}(Q_D^C)\), and the point \(\alpha_i \bar{r}^i + \gamma \bar{r}^j\) lies in \(F_{ij} \subseteq \text{recc}(Q_D^C)\). Assume \(\gamma_{ij} < +\infty\). The point \(\alpha_i \bar{r}^i + \gamma \bar{r}^j\) is a convex combination of \(\alpha_i \bar{r}^i + \gamma_{ij} \bar{r}^j\) and \(\alpha_i \bar{r}^i\), both of which lie in \(\text{recc}(Q_D^C)\):

\[
\alpha_i \bar{r}^i + \gamma \bar{r}^j = \frac{\gamma}{\gamma_{ij}} (\alpha_i \bar{r}^i + \gamma_{ij} \bar{r}^j) + \left(1 - \frac{\gamma}{\gamma_{ij}}\right) \alpha_i \bar{r}^i \in \text{recc}(Q_D^C).
\]

For \(j \in N \setminus D\) and \(U \subseteq D^*\), we define \(\gamma^*_j(U)\) to be

\[
\gamma^*_j(U) = \begin{cases} 
\min_{i \in U} \gamma_{ij} & \text{if } U \neq \emptyset \\
+\infty & \text{otherwise.}
\end{cases}
\]

The parameter \(\gamma^*_j(U)\) also depends on \(D\). We again omit this dependence for notational simplicity.

Theorem 3 presents a family of valid inequalities for \(P^B \setminus Q_D^C\).

Theorem 3. Let \(U \subseteq D^*\). The inequality

\[
\sum_{j \in U} \frac{x_j}{\alpha_j} - \sum_{j \in N \setminus D} \gamma^*_j(U) \leq 1 \tag{4.3}
\]

is valid for \(P^B \setminus Q_D^C\).

Example 4 (continued). Consider \(D = N_1\). Figures 7a and 7b provide a graphical representation of Theorem 3 applied to Example 4, using the relation \(T^C = Q_D^C\). The selection of \(\gamma_1^*(U)\) is shown in Figure 7a, where \(U = \{2\}\). The vector \(\bar{r}^1(U) \bar{r}^i + \alpha_2 \bar{r}^2\) lies on the boundary of \(\text{recc}(Q_D^C) \cap F_{12}\). We note \(\text{recc}(Q_D^C) = \text{recc}(C)\). Figure 7b shows the valid inequality of Theorem 3 using this selection of \(U\).

Prior to proving Theorem 3, we use Farkas Lemma to derive a result on the existence of solutions to a particular family of linear systems.

Lemma 1. Let \(M_1, M_2\) be two finite index sets. Let \(a \in \mathbb{R}_{+}^{\mid M_1\mid}\) and \(c \in \mathbb{R}_{+}^{\mid M_2\mid}\) satisfy \(\sum_{i \in M_1} a_i - \sum_{j \in M_2} c_j > 0\). Then there exists \(\theta \in \mathbb{R}_{+}^{\mid M_1\mid \times \mid M_2\mid}\) such that

\[
\sum_{i \in M_1} \theta_{ij} = 1 \quad \forall j \in M_2
\]

\[
\sum_{j \in M_2} c_j \theta_{ij} \leq a_i \quad \forall i \in M_1. \tag{4.4}
\]

Proof. Let \(\bar{c} := \sum_{j \in M_2} c_j\). If \(\bar{c} = 0\), then any \(\theta \in \mathbb{R}_{+}^{\mid M_1\mid \times \mid M_2\mid}\) satisfying \(\sum_{i \in M_1} \theta_{ij} = 1\) for all \(j \in M_2\) is a solution to system (4.4). Therefore, assume \(\bar{c} > 0\).
Assume for contradiction (4.4) does not have a solution. By Farkas Lemma, there exist \( y \in \mathbb{R}^{\lvert M_2 \rvert} \) and \( z \in \mathbb{R}^{\lvert M_1 \rvert} \) such that
\[
y_j + c_j z_i \geq 0 \quad \forall i \in M_1, j \in M_2
\]
\[
\sum_{j \in M_2} y_j + \sum_{i \in M_1} a_i z_i < 0.
\]
(4.5a) (4.5b)

We multiply (4.5a) by \( a_i / \bar{c} \) to obtain
\[
a_i \bar{c} y_j + c_j / \bar{c} a_i z_i \geq 0 \quad \forall i \in M_1, j \in M_2.
\]

Summing this expression over \( i \in M_1 \) and \( j \in M_2 \) produces the inequality
\[
\sum_{i \in M_1} a_i \bar{c} \sum_{j \in M_2} y_j + \sum_{i \in M_1} a_i z_i \geq 0.
\]
(4.6)

By assumption, \( \sum_{i \in M_1} a_i / \bar{c} > 0 \iff \sum_{i \in M_1} a_i \bar{c} > 1 \). Because \( \sum_{i \in M_1} a_i z_i \geq 0 \), we conclude from (4.5b) that \( \sum_{j \in M_2} y_j < 0 \). Thus, (4.6) implies
\[
\sum_{j \in M_2} y_j + \sum_{i \in M_1} a_i z_i \geq 0
\]
(4.7)

Inequality (4.7) contradicts (4.5b). Therefore, (4.4) has a solution.

\[\square\]

Proof of Theorem 3. The statement is trivially true if \( U = \emptyset \). Therefore, assume \( U \neq \emptyset \). For ease of notation, let \( \gamma_j^* := \gamma_j^*(U) \) for \( j \in N \setminus D \). Let \( E := \{ j \in N \setminus D: \gamma_j^* < +\infty \} \) and \( E^* := \{ j \in N \setminus D: \gamma_j^* = +\infty \} \). Note \( j \in E^* \) if and only if \( r^j \in \text{recce}(Q_D^*) \). Let \( \hat{x} \in P^B \) satisfy \( \sum_{j \in U} \hat{x}_j / \alpha_j - \sum_{j \in E} \hat{x}_j / \gamma_j^* \geq 1 \). We show \( \hat{x} \in Q_D^* \).

We apply Lemma 1 with \( M_1 := U, M_2 := E, a_i := \hat{x}_i / \alpha_i \) for \( i \in U \), and \( c_j := \hat{x}_j / \gamma_j^* \) for \( j \in E \). Thus, there exists \( \bar{\theta} \in \mathbb{R}_+^{\lvert U \rvert \times \lvert E \rvert} \) satisfying
\[
\sum_{i \in U} \theta_{ij} = 1 \quad \forall j \in E
\]
(4.8a)
\[
\sum_{j \in E} \frac{\hat{x}_j}{\gamma_j^*} \theta_{ij} \leq \frac{\hat{x}_i}{\alpha_i} \quad \forall i \in U.
\]
(4.8b)

By Proposition 2, \( q^{ij} := \alpha_i \hat{r}^i + \gamma_j^* \hat{r}^j \in \text{recce}(Q_D^*) \) for all \( i \in U \) and \( j \in E \), because \( 0 < \gamma_j^* \leq \gamma_{ij} \). Consequently,
\[
r^j = \frac{1}{\gamma_j^*} q^{ij} - \frac{1}{\gamma_j^*} \alpha_i \hat{r}^i \quad \forall i \in U, j \in E.
\]
(4.9)

We use \( \theta \) and (4.9) to rewrite \( r^j \), \( j \in E \):
\[
r^j = \sum_{i \in U} \theta_{ij} \left( \frac{1}{\gamma_j^*} q^{ij} - \frac{1}{\gamma_j^*} \alpha_i \hat{r}^i \right).
\]
(4.10)

Substituting (4.10) into the definition of \( \hat{x} \), we have:
\[
\hat{x}_i = \hat{x}_i + \sum_{i \in U} \hat{x}_i \hat{r}^i + \sum_{i \in D \setminus U} \hat{x}_i \hat{r}^i + \sum_{j \in E} \hat{x}_j \hat{r}^j + \sum_{j \in E^*} \hat{x}_j \hat{r}^j
\]
\[
= \hat{x}_i + \sum_{i \in U} \hat{x}_i \hat{r}^i + \sum_{j \in E} \sum_{i \in U} \hat{x}_j \theta_{ij} \left( \frac{1}{\gamma_j^*} q^{ij} - \frac{1}{\gamma_j^*} \alpha_i \hat{r}^i \right) + \sum_{j \in E^*} \hat{x}_j \hat{r}^j
\]
\[
= \hat{x}_i + \sum_{i \in U} \left( \frac{\hat{x}_i}{\alpha_i} - \sum_{j \in E} \theta_{ij} \frac{\hat{x}_j}{\gamma_j^*} \right) \alpha_i \hat{r}^i + \sum_{i \in U} \sum_{j \in E} \theta_{ij} \frac{\hat{x}_j}{\gamma_j^*} q^{ij} + \sum_{j \in E^* \cup (D \setminus U)} \hat{x}_j \hat{r}^j
\]
(4.11)
By (4.8a), the weights on the terms \( \alpha_i r^i, i \in U \) in (4.11) are greater than 1:

\[
\sum_{i \in U} \left( \frac{\hat{x}_i}{\alpha_i} - \sum_{j \in E} \theta_{ij} \frac{\hat{x}_j}{\gamma^j} \right) = \sum_{i \in U} \frac{\hat{x}_i}{\alpha_i} - \sum_{j \in E} \frac{\hat{x}_j}{\gamma^j} > 1. 
\] (4.12)

By (4.8b), each individual coefficient on \( \alpha_i r^i, i \in U \) in (4.11) is nonnegative. Together with (4.12), we have

\[
\bar{x} + \sum_{i \in U} \left( \frac{x_i}{\alpha_i} - \sum_{j \in E} \theta_{ij} \frac{x_j}{\gamma^j} \right) \alpha_i r^i \in \{ \bar{x} \} + \text{conv} \left( \bigcup_{i \in U} (\alpha_i, +\infty) r^i \right). 
\] (4.13)

Continuing from (4.13), \( \{ \bar{x} \} + \text{conv}(\cup_{i \in U}(\alpha_i, +\infty) r^i) \subseteq Q_D \), because \( U \subseteq D \). Furthermore, recession cone membership is preserved under addition:

\[
\sum_{i \in U} \sum_{j \in E} \theta_{ij} \frac{x_j}{\gamma^j} + \sum_{j \in D \setminus U} \hat{x}_j f^j + \sum_{j \in E^c} \hat{x}_j r^j \in \text{recc}(Q_D^C). 
\] (4.14)

By (4.13) and (4.14), it holds that \( \bar{x} \in Q_D + \text{recc}(C) = Q_D^C \).

As a result of Theorem 3 and Proposition 2, for any \( U \subseteq D^* \) and \( \gamma \in [|N|,|D|] \) satisfying \( \gamma_j \in (0, \gamma^*_j(U)) \) for all \( j \in N \setminus D \), the inequality

\[
\sum_{j \in U} \frac{x_j}{\alpha_j} - \sum_{j \in N \setminus D} \frac{x_j}{\gamma_j} \leq 1
\]

is valid for \( P^B \setminus Q_D^C \). However, our choice of \( \gamma^*_j \) yields a stronger inequality than inequalities corresponding to smaller choices of \( \gamma \).

We next consider the problem of selecting a subset of \( D^* \) that yields the most violated inequality of the form (4.3) to cut off a candidate solution \( \bar{x} \in P^B \). That is, we are interested in the separation problem

\[
\max_{U \subseteq D^*} \sum_{j \in U} \frac{\hat{x}_j}{\alpha_j} - \sum_{j \in N \setminus D} \frac{\hat{x}_j}{\gamma^*_j(U)}. 
\] (4.15)

For each \( j \in N \setminus D \), we define the function \( f^*_j : 2^{D^*} \rightarrow \mathbb{R} \) to be \( f^*_j(U) = -\hat{x}_j / \min_{i \in U} \gamma_{ij} \) if \( U \neq \emptyset \), and 0 otherwise. The value \( f^*_j(U) \) is the contribution of index \( j \in N \setminus D \) to the objective function (4.15) for a given \( U \).

**Proposition 3.** The maximization problem (4.15) is a supermodular maximization problem.

**Proof.** The separation problem (4.15) can be equivalently written as

\[
\max_{U \subseteq D^*} \sum_{j \in U} \frac{\hat{x}_j}{\alpha_j} + \sum_{j \in N \setminus D} f^*_j(U). 
\] (4.16)

From standard properties of the min operator, the objective function of (4.16) is the sum of supermodular (and modular) functions. \( \square \)

By Proposition 3, the separation problem (4.15) can be solved in strongly polynomial time [17, 18, 29].

We next propose an extended formulation for the relaxation of \( P^B \setminus Q_D^C \) defined by inequality (4.3) for all \( U \subseteq D^* \):

\[
H_D := \left\{ x \in \mathbb{R}^{|N|}: \sum_{j \in U} \frac{x_j}{\alpha_j} - \sum_{j \in N \setminus D} \frac{x_j}{\gamma^*_j(U)} \leq 1 \ \forall U \subseteq D^* \right\}. 
\]

Let \( p_1 := |D^*| \) and \( p_2 := |N \setminus D| \). Let \( D^* = \{1,\ldots,p_1\} \). For each \( j \in N \setminus D \), let \( d_1^j, d_2^j, \ldots, d_{p_2}^j \) be ordered such that \( \gamma_{d_1^j} \leq \gamma_{d_2^j} \leq \ldots \leq \gamma_{d_{p_2}^j} \). Similarly, for any \( i \in D^* \), let \( \ell_j(i) \) satisfy \( d_{\ell_j(i)}^j = i \). For all \( j \in N \setminus D \),
let \( \gamma_{0j} := +\infty, \theta_{0j} := 0, v_{0j} := 0, v_{p1+1,j} := 0, d_{p1}^j := 0, \) and \( d_{p1+1}^j := 0 \). We define \( G_D \) to be the set of 
\( (x, \theta, v, \lambda) \in \mathbb{R}^{p_1 \times N} \times \mathbb{R}^{p_1 \times p_2} \times \mathbb{R}^{p_1 \times p_2} \times \mathbb{R}^{p_2} \) such that 
\[
\sum_{i \in D^*} \sum_{j \in N \setminus D} \theta_{ij} + \sum_{j \in N \setminus D} \lambda_j \leq 1
\]
\[
\theta_{ij} + v_{ij} - v_{i+1,j} + \left( -\frac{1}{\gamma_{d_{i+1,j}}^j} - \frac{1}{\gamma_{d_{ij}}^j} \right) x_j \geq 0 \quad \forall i = 0, \ldots, p_1, j \in N \setminus D
\]
\[
\sum_{j \in N \setminus D} \theta_{ij(i),j} - \frac{1}{\alpha_i} x_i \geq 0 \quad \forall i = 1, \ldots, p_1.
\]

Theorem 4 establishes the relationship between \( G_D \) and the relaxation \( H_D \).

**Theorem 4.** The polyhedron \( G_D \) is an extended formulation of \( H_D : \text{proj}_x(G_D) = H_D \).

**Proof.** We first argue that the following linear program solves the separation problem (4.15) for a fixed \( \hat{x} \in P^B \):

\[
\max_{y,z} \sum_{i \in D^*} \frac{x_i}{\alpha_i} z_i - \sum_{j \in N \setminus D} \sum_{i=1}^{p_1} \frac{\hat{x}_j}{\theta_{ij}^*} (y_{i-1,j} - y_{ij}) \tag{4.17a}
\]

s.t.
\[
y_{0j} = 1 \quad \forall j \in N \setminus D \tag{4.17b}
\]
\[
y_{ij} + z_{d_{ij}^j} \leq 1 \quad \forall i = 1, \ldots, p_1, j \in N \setminus D \tag{4.17c}
\]
\[
y_{ij} - y_{i-1,j} \leq 0 \quad \forall i = 1, \ldots, p_1, j \in N \setminus D \tag{4.17d}
\]
\[
y_{ij} \geq 0 \quad \forall i = 0, \ldots, p_1, j \in N \setminus D \tag{4.17e}
\]
\[
z_i \geq 0 \quad \forall i \in D^* \tag{4.17f}
\]

The constraint matrix of (4.17) is totally unimodular. To see this, we complement the \( z_i \) variables with \( 1 - z_i \) for all \( i \in D^* \) to obtain an equivalent problem. The resulting constraint matrix has 0, ±1 entries, and each row contains no more than one 1 and one -1.

We show (4.17) correctly models the separation problem (4.15). First, let \( U^* \) be the optimal solution of (4.15). We construct \((y^*, z^*)\) feasible to (4.17) with objective function value equal to \( \sum_{i \in U^*} \frac{x_i}{\alpha_i} - \sum_{j \in N \setminus D} \frac{\hat{x}_j}{\gamma_{d_{ij}}^j} (y_{i-1,j} - y_{ij}) \). If \( U^* = \emptyset \), then the optimal objective value of the separation problem is 0. In this case, set \( y_{i,j}^* = 1 \) for all \( i = 0, 1, \ldots, p_1 \) and \( j \in N \setminus D \), and set \( z_{d_{ij}^j}^* = 0 \) for all \( i \in D^* \). Then \((y^*, z^*)\) is feasible to (4.17) with objective value 0. If \( U^* \neq \emptyset \), set \( z_{d_{ij}^j}^* = 1 \) if \( i \in U^* \), and 0 otherwise. For all \( j \in N \setminus D \), let \( j' \in U^* \) be the smallest index satisfying \( d_{ij'}^j \in \arg \min_{i \in U^*} \gamma_{ij} \); that is, \( \gamma_{ij'}^j(U^*) = \gamma_{d_{ij'}^j} \). For each \( j \in N \setminus D \), set \( y_{i,j}^* = 1 \) for all \( i = 0, 1, \ldots, j'-1 \), and set \( y_{i,j}^* = 0 \) for all \( i = j', \ldots, p_1 \). By construction, \((y^*, z^*)\) satisfies (4.17b)–(4.17f). For a fixed \( j \in N \setminus D \), we have

\[
\sum_{i=1}^{p_1} \frac{\hat{x}_j}{\gamma_{d_{ij}}^j} (y_{i-1,j}^* - y_{ij}^*) = \frac{\hat{x}_j}{\gamma_{d_{ij}}^j(U^*)}
\]

and the objective function (4.17a) evaluates to the desired value of

\[
\sum_{i \in D^*} \frac{x_i}{\alpha_i} z_i - \sum_{j \in N \setminus D} \sum_{i=1}^{p_1} \frac{\hat{x}_j}{\gamma_{d_{ij}}^j} (y_{i-1,j}^* - y_{ij}^*) = \sum_{i \in U^*} \frac{x_i}{\alpha_i} - \sum_{j \in N \setminus D} \frac{\hat{x}_j}{\gamma_{d_{ij}}^j(U^*)}.
\]

Now, let \((y^*, z^*)\) be an optimal solution to (4.17). Set \( U^* = \{ i \in D^* : z_i^* = 1 \} \). It remains to show \( \sum_{i \in U^*} \frac{x_i}{\alpha_i} - \sum_{j \in N \setminus D} \frac{\hat{x}_j}{\gamma_{ij}} (y_{i-1,j}^* - y_{ij}^*) \) is not less than the optimal objective value of (4.17). Recall the constraint matrix of (4.17) is totally unimodular, so \((y^*, z^*)\) is 0–1 valued. If \( z_i^* = 0 \) for all \( i \in D^* \), then the separation problem objective evaluated at \( U^* = \emptyset \) is 0 and the optimal objective value of (4.17) is nonpositive, as desired. Next, assume there exists \( i \in D^* \) such that \( z_i^* = 1 \). By constraints (4.17c), for each \( j \in N \setminus D \), \( y_{ij}^* \)}
is not equal to 1 for all $i \in D^*$. By constraints (4.17b) and (4.17d), for each $j \in N \setminus D$, there exists $j'$ such that $y_{ij} = 1$ for $i = 0, \ldots, j' - 1$ and $y_{ij} = 0$ for $i = j', \ldots, p_1$. Then the optimal objective value of (4.17) is

$$
\sum_{i \in D^*} \hat{x}_i - \sum_{j \in N \setminus D} \sum_{i=1}^{p_1} \gamma_{d_{ij}} - \sum_{j \in N \setminus D} \gamma_{d_{ij}^*} = \sum_{i \in D^*} \hat{x}_i - \sum_{j \in N \setminus D} \gamma_{d_{ij}^*} - \sum_{j \in N \setminus D} \gamma_{d_{ij}^*}.
$$

Consider a fixed $j \in N \setminus D$. By constraints (4.17c), $z_{d_{ij}} = 0$ for $i = 1, \ldots, j' - 1$. Then $\arg \min \{i \in D^* : z_{ij} = 1\} \geq j'$. Due to the ordering $\gamma_{d_{ij}^*} \leq \cdots \leq \gamma_{d_{ij}^*}$, we have $\gamma_{j}^*(U^*) = \min_{i \in U^*} \gamma_{ij} \geq \gamma_{d_{ij}^*}$. Therefore, the optimal objective value of the separation problem evaluated at $U^*$ is at least as large as (4.18):

$$
\sum_{i \in U^*} \hat{x}_i - \sum_{j \in N \setminus D} \gamma_{j}^*(U^*) \geq \sum_{i \in D^*} \hat{x}_i - \sum_{j \in N \setminus D} \gamma_{d_{ij}^*}.
$$

Hence, (4.17) models the separation problem (4.15) for a fixed $\hat{x}_j \in P_B$.

The point $\hat{x}$ lies in $H_D$ if and only if the primal objective (4.17a) does not exceed 1. The linear program (4.17) is feasible and bounded, so strong duality applies. Thus, $\hat{x} \in H_D$ if and only if the dual of (4.17) has objective value less than or equal to 1. Because the dual of (4.17) is a minimization problem, we enforce this condition with the constraint $\sum_{i \in D^*} \sum_{j \in N \setminus D} \theta_0 + \sum_{j \in N \setminus D} \lambda_j \leq 1$. We also replace the fixed $\hat{x}$ in the dual of (4.17) with the nonnegative variable $x \in \mathbb{R}^{|N|}$. Thus, $x \in H_D$ if and only if there exists $(\theta, v, \lambda)$ satisfying the dual constraints of (4.17) and the aforementioned dual objective cut. These constraints define $G_D$.

For a fixed $j \in N \setminus D$, the constraints (4.17b), (4.17d), and (4.17e) form an instance of the mixing set [19]. The proof's derivation of the extended formulation $G_D$ follows results from Luethke and Ahmed [24] and Miller and Wolsey [26].

Proposition 4 states that if no cuts of the form (4.3) exist, then there exist no valid inequalities for $\text{cconv}(P_B \setminus C)$ other than those defined in $P_B$.

**Proposition 4.** If $D^* = \emptyset$, then $\text{cconv}(P_B \setminus Q_D^C) = P_B$.

**Proof.** It suffices to show $\{\hat{x}\} + [0, +\infty)^N \subseteq \text{cconv}(P_B \setminus Q_D^C)$ for $i \in N$. Because $D \subseteq N_0$, this condition holds for $i \in N_0$. For $i \in N_2 \setminus D$, we have $\hat{x} \in P_B \setminus Q_D^C$ and $\hat{x} + M\gamma^i \in P_B \setminus C$ for a sufficiently large $M > 0$. Consider $i \in D$, $\lambda > 0$, and $\gamma > 0$. Because $D^* = \emptyset$, there exists $j \in N \setminus D$ such that $\lambda \gamma^i + \gamma \gamma^j \notin \text{recc}(Q_D^C)$. Then for a sufficiently large $M > 0$, $\hat{x} + \lambda \gamma^i + \gamma \gamma^j$ is a convex combination of $\hat{x} \notin Q_D^C$ and $\hat{x} + M(\lambda \gamma^i + \gamma \gamma^j) \notin Q_D^C$. Thus, $\hat{x} + \lambda \gamma^i + \gamma \gamma^j \in \text{cconv}(P_B \setminus Q_D^C)$ for all $\gamma > 0$, so $\hat{x} + \lambda \gamma^i \in \text{cconv}(P_B \setminus Q_D^C)$.

In the case where $D = N_1$ ($Q_D^C = T_C$) and $D^* = \emptyset$, Theorem 2 and Proposition 4 together give us $D^* = \emptyset \implies \text{cconv}(P_B \setminus C) = P_B$.

### 4.3 A multi-term disjunction valid for $P_B \setminus T_C$

We are interested in a disjunction valid for $P_B \setminus T_C \supseteq P_B \setminus C$. The set $P_B \setminus T_C$ has the potential to be a stronger relaxation of $P_B \setminus C$ than the two-term disjunction of Theorem 1, because it considers the full structure of $\text{recc}(C)$. In this section, we derive a disjunction for $P_B \setminus T_C$ that contains $|N_2| + 1$ disjunctive terms. These terms are defined by nonconvex sets, but we derive polyhedral relaxations of each term. The disjunctive programming approach of Balas can be applied to construct a CGLP to find a valid inequality that separates a candidate solution from $\text{cconv}(P \setminus T_C)$.

Let $S_0^C$ be defined as follows:

$$
S_0^C := \{\hat{x}\} + \text{conv} \left( \bigcup_{j \in N_0} (\alpha_j, +\infty)^j \right) + \text{recc}(C).
$$

We define the following sets for $k \in N_2$:

$$
S_k^C := \{\hat{x}\} + \text{conv} \left( \bigcup_{j \in N_2} (0, \beta_j)^j \right) + (-\infty, 0]^k + \text{recc}(C).
$$

The sets $S_k^C$ and $S_k^C$ ($k \in N_2$) are the foundation of our multi-term disjunction for $P_B \setminus T_C$, presented later in this section.

Observation 1 is a consequence of the relation $\text{conv}(A_1) + \text{conv}(A_2) = \text{conv}(A_1 + A_2)$ for nonempty sets $A_1$ and $A_2$. 

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Because \( k \in N_2 \), \( S^C_k \) can be written as
\[
S^C_k = \{ \bar{x} \} + \text{conv} \left( \bigcup_{j \in N_2} (\alpha_j, \beta_j) \bar{r}_j \right) + (-\infty, 0] \bar{r}_k + \text{recc}(C).
\] (4.21)

Theorem 5 presents a disjunctive representation of \( PB \setminus TC \). Each of these disjunctive terms is itself a reverse convex set. In Sections 4.4 and 4.5, we describe polyhedral relaxations of each term of this disjunction. Throughout, let \( N_2^0 := N_2 \cup \{0\} \).

**Theorem 5.** It holds that
\[
PB \setminus TC = \bigcup_{k \in N_2^0} (PB \setminus S^C_k).
\] (4.22)

Before proving Theorem 5, we prove a consequence of Farkas Lemma.

**Lemma 2.** Let \( a, c \in \mathbb{R}^n_+ \), where \( \sum_{i=1}^n a_i > 0 \). There exists \( \theta \in \mathbb{R}^{n+1}_+ \) such that
\[
\sum_{i=0}^s \theta_i = 1
\]
\[
a_i \theta_0 - c_i \theta_i = 0 \quad i = 1, \ldots, s.
\]

**Proof.** By Farkas Lemma, either system (4.23) has a solution, or there exists \( y \in \mathbb{R}^{n+1} \) such that
\[
y_0 + \sum_{i=1}^s a_i y_i \geq 0
\]
\[
y_0 - c_i y_i \geq 0 \quad i = 1, \ldots, s
\]
\[
y_0 < 0.
\]
Assume for contradiction there exists a \( y \) satisfying (4.24). The nonnegativity of \( c \), (4.24b), and (4.24c) imply \( y_i < 0 \) for all \( i = 1, \ldots, s \). The vector \( a \) is nonnegative and by assumption sums to a strictly positive value. We conclude \( y_0 + \sum_{i=1}^s a_i y_i < 0 \), contradicting (4.24a).

**Proof of Theorem 5.** It suffices to show \( TC = \bigcap_{k \in N_2} S^C_k \). If \( N_2 = \emptyset \), we have \( TC = S^C_0 \) by (4.1) and (4.19). Therefore, assume \( N_2 \neq \emptyset \). By construction, \( TC \subseteq S^C_k \) for all \( k \in N_2^0 \), implying \( TC \subseteq \bigcap_{k \in N_2} S^C_k \).

Let \( \hat{x} \in \cap_{k \in N_2} S^C_k \). By \( \hat{x} \)’s membership in \( S^C_0 \), there exist \( \lambda^0 \in \mathbb{R}^{\left| N_2^0 \right|}_+, \mu \in \mathbb{R}^{\left| N_2^0 \right|}_+, \delta^0 \in \mathbb{R}^{\left| N_2^0 \right|}_+ \), and \( q^0 \in \text{recc}(C) \) such that \( \lambda^0_j \in (\alpha_j, \beta_j) \) for all \( j \in N_0^0 \), \( \sum_{j \in N_0^0} \delta^0_j = 1 \), and
\[
\hat{x} = \bar{x} + \sum_{j \in N_1} \delta^0_j \lambda^0_j \bar{r}_j + \sum_{j \in N_2} \delta^0_j (\lambda^0_j + \mu_j) \bar{r}_j + q^0.
\] (4.25)

If \( \delta^0_j = 0 \) for all \( j \in N_2 \), then \( \hat{x} \in \{ \bar{x} \} + \text{conv}(\bigcup_{j \in N_1} (\alpha_j, \beta_j) \bar{r}_j) + \text{recc}(C) \subseteq TC \) and we have nothing left to prove. We therefore assume \( \sum_{j \in N_2} \delta^0_j > 0 \). From definition (4.21) of \( S^C_k \), \( \hat{x} \in \cap_{k \in N_2} S^C_k \) implies that for all \( k \in N_2 \), there exist \( \lambda^k \in \mathbb{R}^{\left| N_2 \right|}_+, \eta_k \in \mathbb{R}_+, \delta^k \in \mathbb{R}^{\left| N_2 \right|}_+ \), and \( q^k \in \text{recc}(C) \) such that \( \lambda^k_j \in (\alpha_j, \beta_j) \) for all \( j \in N_0^k \), \( \sum_{j \in N_0^k} \delta^k_j = 1 \), and
\[
\hat{x} = \bar{x} + \sum_{j \in N_0} \delta^k_j \lambda^k_j \bar{r}_j - \eta_k \bar{r}_k^k + q^k.
\] (4.26)

Because \( \mu \in \mathbb{R}^{\left| N_2 \right|}_+ \) and \( \sum_{j \in N_2} \delta^0_j > 0 \), it holds that \( \sum_{j \in N_2} \delta^0_j \mu_j > 0 \). We apply Lemma 2 with \( s := \left| N_2 \right| \), \( a_j := \delta^0_j \mu_j \) for \( j \in N_2 \), and \( c_j := \eta_j \) for \( j \in N_2 \). Then there exists \( \theta \in \mathbb{R}^{\left| N_2^0 \right|}_+ \) such that \( \sum_{j \in N_2} \theta_j = 1 \).
and \( \theta_0 \delta^2_{mk} = \theta_k \eta_k \) for all \( k \in N_2 \). We use this \( \theta \) as convex combination multipliers on (4.25) and (4.26) to rewrite \( \bar{x} \) as

\[
\bar{x} = \bar{x} + \sum_{k \in N^0_2} \sum_{j \in N^0_0} \theta_k \delta^k_{j} \bar{r}^j + \sum_{k \in N^0_2} \theta_k q^k. 
\] (4.27)

For every \( j \in N^0_2 \) and \( k \in N^2_0 \), \( \lambda^k_j \bar{r}^j \in (\alpha_j, \beta_j) \bar{r}^j \). The coefficients on the terms \( \lambda^k_j \bar{r}^j \) \((j \in N^0_2, k \in N^2_0)\) in (4.27) are nonnegative and sum to one:

\[
\sum_{k \in N^0_2} \sum_{j \in N^0_0} \theta_k \delta^k_{j} = \sum_{k \in N^0_2} \theta_k \sum_{j \in N^0_0} \delta^k_{j} = 1.
\]

It follows that \( \bar{x} + \sum_{k \in N^0_2} \sum_{j \in N^0_0} \theta_k \delta^k_{j} \lambda^k_j \bar{r}^j \in T \). Lastly, we have \( \sum_{k \in N^0_2} \theta_k q^k \in \text{recc}(C) \). Thus, \( \bar{x} \in T^C \). \( \square \)

The multi-term disjunction (4.22) is a generalization of the two-term disjunction of Theorem 1. Recall this two-term disjunction does not account for the recession structure of \( C \) beyond the property that \( \bar{r}^j \in \text{recc}(C) \) for all \( j \in N_1 \) and the assumption \( \bar{r}^j \in N_0 \) for all \( j \in N_0 \). If \( C \) is bounded and Assumption 1 holds, it can be shown that the multi-term disjunction decomposes the simple two-term disjunction of Theorem 1. In particular, we have

\[
P^B \setminus S^C_0 = \{ x \in P^B : \sum_{j \in N} x_j/\alpha_j \leq 1 \}
\]

\[
P^B \setminus S^C_k = \{ x \in P^B : \sum_{j \in N} x_j/\beta_j \geq 1 \} \quad \forall k \in N_2.
\]

Remark 2. The multi-term disjunction (4.22) for \( P \setminus C \) can be extended to the case \( \bar{x} \in C \). Specifically, if \( \alpha_j = 0 \) for all \( j \in N \), the set \( S^C_0 \) defined in (4.19) contains every point in \( P^B \) except for \( \bar{x} \). Because \( \bar{x} \in C \), we know that \( P^B \setminus S^C_0 \) (one of the terms of the disjunction (4.22)) is empty.

Example 3 (continued). Using the two-term disjunction from Section 3, we were unable to derive meaningful cuts for \( P \setminus C \) from Example 3. In contrast, Theorem 5 provides a disjunction for \( P \setminus C \). A graphical representation of the relationship \( T^C = \bigcap_{k \in N^0_2} S^C_k \) for this example is shown in Figures 8a–8c. In this example, \( |N_0| = |N_2| = 1 \). The disjunction of Theorem 5 can be seen in Figure 9.

Based on the disjunction (4.22), the inequalities (4.3), which are valid for \( P^B \setminus T^C \), are also valid for \( P^B \setminus S^C_k \) for all \( k \in N^0_2 \).

The sets \( P^B \setminus S^C_k \), \( k \in N^0_2 \), are nonconvex in general. In Sections 4.4 and 4.5, we derive polyhedral relaxations of these sets. Together, these relaxations form \( |N_2| + 1 \) polyhedra whose union contains the feasible region \( P \setminus C \).

### 4.4 Polyhedral relaxation of \( P^B \setminus S^C_0 \)

The set \( S^C_0 \) is equivalent to \( Q^C_0 \) from Section 4.2 when \( D = N^0_2 \). As such, the theory of Section 4.2 can be applied to the specific case \( D = N^0_2 \) to obtain an exponential family of inequalities for \( P^B \setminus S^C_0 \) and a polynomial-size extended formulation of these inequalities. This extended formulation can be used to obtain a polyhedral relaxation of \( P^B \setminus S^C_0 \) used in the disjunctive formulation of \( P^B \setminus C \) detailed in Section 4.3.
Example 3 (continued). Let $D$ from Section 4.2 equal $N_0^c$. Consider $P$ and $C$ defined in Example 3. Figure 10a shows the selection of $\gamma^*_2(U)$ for $U = \{1\}$. This $\gamma^*_2(U)$ is then used to construct the inequality of Theorem 3 in Figure 10b.

4.5 Polyhedral relaxation of $P^B \setminus S^C_k$, $k \in N_2$

Let $k \in N_2$ be fixed. In this section, we describe a polyhedral relaxation of the set $P^B \setminus S^C_k$. These results apply to the remaining $|N_2|$ terms of the disjunction (4.22).

Let $J_k$ be defined as follows:

$$J_k := \{i \in N : \bar{r}^i \in \text{recc}(S^C_k)\}.$$ 

Because recc$(C) \subseteq$ recc$(S^C_k)$, we have $N_1 \subseteq J_k$.

Observation 2. It holds that recc$(S^C_k) = \text{recc}(C) + (-\infty,0]\bar{r}^k$.

Proposition 5. The index $k$ is not in $J_k$.

Proof. Assume for contradiction $k \in J_k$. By Observation 2, there exists $q \in \text{recc}(C)$ and $\lambda \geq 0$ such that $\bar{r}^k = q - \lambda \bar{r}^k$, which implies $\bar{r}^k \in \text{recc}(C)$. This is a contradiction; $k \in N_2$, so the halfline $[0,\infty)\bar{r}^k$ extending from $\bar{x}$ intersects $C$ on a finite interval. \hfill \Box

Proposition 6 characterizes the points where $S^C_k$ intersects each edge of $P^B$.

Proposition 6. Let $j \in N$. Then

$$\beta^*_j := \sup\{\lambda \geq 0 : \bar{x} + \lambda \bar{r}^j \in S^C_k\} = \begin{cases} 0 & \text{if } j \in N_0 \setminus J_k \\ \beta_j & \text{if } j \in N_2 \setminus J_k \\ +\infty & \text{if } j \in J_k. \end{cases}$$
Proof. Let $j \in J_k$. By Observation 2, there exists $\lambda \geq 0$ such that $\bar{r}^j + \lambda \bar{r}^k \in \text{recc}(C)$. Then for all $\gamma > 0$, $\bar{x} + \gamma (\bar{r}^j + \lambda \bar{r}^k) \in S^C_{k}$. Because $-\bar{r}^k \in \text{recc}(S^C_{k})$, we have $\bar{x} + \gamma \bar{r}^j \in S^C_{k}$. Thus, $\beta^*_j = +\infty$.

Next, let $j \in N_2 \setminus J_k$. By the construction of $S^C_{k}$ in (4.20), $\beta^*_j \geq \beta_j$. Assume for contradiction $\beta^*_j > \beta_j$. There exists $\theta \in \mathbb{R}_{++}^{\lvert N_2 \rvert}$, $\delta \in \mathbb{R}_{++}^{\lvert N_2 \rvert}$, $\gamma \geq 0$, and $q \in \text{recc}(C)$ such that $\sum_{i \in N_2} \theta_i = 1$, $\delta_i \in [0, \beta_i)$ for $i \in N_2$, and

$$\bar{x} + \beta^*_j \bar{r}^j = \bar{x} + \sum_{i \in N_2} \theta_i \delta_i \bar{r}^i - \gamma \bar{r}^k + q \Rightarrow q = \beta^*_j \bar{r}^j - \sum_{i \in N_2} \theta_i \delta_i \bar{r}^i + \gamma \bar{r}^k. \quad (4.28)$$

Observe $\theta_i \delta_i = 0$ for all $i \in N_2 \setminus \{j, k\}$ and $\gamma \geq \theta_k \delta_k$; if not, $q \notin \text{recc}(P^B)$ from (4.28), contradicting Assumption 2. Therefore,

$$q = (\beta^*_j - \theta_j \delta_j) \bar{r}^j + (\gamma - \theta_k \delta_k) \bar{r}^k.$$

Because $\bar{r}^k \in \text{recc}(S^C_{k})$, $q - (\gamma - \theta_k \delta_k) \bar{r}^k = (\beta^*_j - \theta_j \delta_j) \bar{r}^j \in \text{recc}(S^C_{k})$. This contradicts $q \notin J_k$.

Finally, let $j \in N_0 \setminus J_k$. Assume for contradiction $\beta^*_j > 0$. We follow the definitions in the previous case ($j \in N_2 \setminus J_k$) to obtain

$$q = \beta^*_j \bar{r}^j + (\gamma - \theta_k \delta_k) \bar{r}^k.$$

Again, we obtain $\beta^*_j \bar{r}^j \in \text{recc}(S^C_{k})$, contradicting $q \notin J_k$. \qed

The proof of Proposition 6 shows that without Assumption 2, it may be the case that $\beta^*_j > \beta_j$ for some $j \in N$. This is due to the addition of $(-\infty, 0) \bar{r}^k$ to $\text{recc}(C)$.

Corollary 2 follows from Proposition 6.

Corollary 2. If $N_0 \subseteq J_k$, then there exists $\epsilon > 0$ such that $\bar{x} + \epsilon \bar{r}^j \in S^C_{k}$ for all $j \in N$.

By Corollary 2, if $N_0 \subseteq J_k$, $\bar{x}$ lies in the relative interior of $S^C_{k}$. We can construct a polyhedral relaxation of $P^B \setminus S^C_{k}$ by using intersection cuts generated by the cone $P^B$. Methods for strengthening intersection cuts (e.g., Glover [16]) can be used to obtain a strengthened polyhedral relaxation. For this reason, we present inequalities only for the case $N_0 \subseteq J_k$.

Assumption 3. There exists $j \in N_0$ such that $\bar{r}^j \notin \text{recc}(S^C_{k})$, i.e., $N_0 \not\subseteq J_k$.

For $i \in J_k$ and $j \in N \setminus J_k$, let

$$\omega_{ij} := \sup\{\omega \geq 0 : \bar{r}^i + \omega \bar{r}^j \in \text{recc}(S^C_{k})\}.$$

We define $D^*_k$ to be the indices of $J_k$ that satisfy the following property:

$$D^*_k := \{i \in J_k : \omega_{ij} > 0 \forall j \in N \setminus J_k\}.$$

For any $i \in D^*_k$ and $j \in N \setminus J_k$, $\text{recc}(S^C_{k})$ intersected with the cone $F_{ij}$ contains something other than the trivial directions $[0, +\infty) \bar{r}^i \subseteq \text{recc}(S^C_{k})$.

The proof of Proposition 7 is similar to that of Proposition 2.

Proposition 7. Let $(i, j) \in D^*_k \times (N \setminus J_k)$. For any $\omega \in [0, \omega_{ij})$, $\bar{r}^i + \omega \bar{r}^j \in \text{recc}(S^C_{k})$.

For $U \subseteq D^*_k$ and $j \in N \setminus J_k$, we define $\omega^*_j(U)$ to be

$$\omega^*_j(U) = \begin{cases} \min_{i \in U} \omega_{ij} & \text{if } U \neq \emptyset \\ +\infty & \text{otherwise.} \end{cases}$$

By Proposition 7, if $U \neq \emptyset$, $\bar{r}^i + \omega^*_j(U) \bar{r}^j \in \text{recc}(S^C_{k})$ for all pairs $(i, j) \in U \times (N \setminus J_k)$.
Theorem 6. Let $U \subseteq D_k^*$. The inequality
\[ \sum_{i \in U} x_i - \sum_{j \in N \setminus J_k} \frac{x_j}{\omega_j^*(U)} \leq 0 \] (4.29)
is valid for $P^B \setminus S^C_k$.

Proof. Assume $U \neq \emptyset$, or the result trivially holds. By construction, $\omega_j^*(U) > 0$. Let $\hat{x} \in P^B$ satisfy $\sum_{i \in U} \hat{x}_i - \sum_{j \in N \setminus J_k} \hat{x}_j/\omega_j^*(U) > 0$. We show $\hat{x} \in S^C$. For ease of notation, let $\omega_j := \omega_j^*(U)$.

By Proposition 7, for $(i, j) \in U \times (N \setminus J_k)$, there exists $q_{ij} \in \text{recc}(S^C)$ such that $q_{ij} = \hat{r}_i + \omega_j^* \hat{r}_j$. Then
\[ \hat{r}_i = \frac{1}{\omega_j} q_{ij} - \frac{1}{\omega_j^*} \hat{r}_i \quad \forall i \in U, j \in N \setminus J_k. \]

By Lemma 1, there exists $\theta \in \mathbb{R}^{U \times |N \setminus J_k|}_+$ such that
\[ \sum_{i \in U} \theta_{ij} = 1 \quad \forall j \in N \setminus J_k \] (4.30a)
and
\[ \sum_{j \in N \setminus J_k} \theta_{ij} \frac{\hat{x}_j}{\omega_j} \leq \hat{x}_i \quad \forall i \in U. \] (4.30b)
This result is obtained with $M_1 := U$, $M_2 := N \setminus J_k$, $a_i := \hat{x}_i$ for all $i \in U$, and $c_j := \hat{x}_j/\omega_j^*$ for all $j \in N \setminus J_k$.

With the $\theta$ satisfying (4.30), we have
\[ \hat{r}_i = \sum_{i \in U} \theta_{ij} \left( \frac{1}{\omega_j} q_{ij} - \frac{1}{\omega_j^*} \hat{r}_i \right) \quad \forall j \in N \setminus J_k. \] (4.31)
Using (4.31), $\hat{x}$ is equivalent to
\[ \hat{x} = \bar{x} + \sum_{i \in U} \hat{x}_i \hat{r}_i + \sum_{i \in J_k \setminus U} \hat{x}_i \hat{r}_i + \sum_{j \in N \setminus J_k} \hat{x}_j \hat{r}_j \]
\[ = \bar{x} + \sum_{i \in U} \hat{x}_i \hat{r}_i + \sum_{j \in N \setminus J_k} \hat{x}_j \sum_{i \in U} \theta_{ij} \left( \frac{1}{\omega_j} q_{ij} - \frac{1}{\omega_j^*} \hat{r}_i \right) + \sum_{j \in J_k \setminus U} \hat{x}_j \hat{r}_i \]
\[ = \bar{x} + \sum_{i \in U} \left( \hat{x}_i - \sum_{j \in N \setminus J_k} \theta_{ij} \frac{\hat{x}_j}{\omega_j} \right) \hat{r}_i + \sum_{j \in N \setminus J_k} \theta_{ij} \frac{\hat{x}_j}{\omega_j} q_{ij} + \sum_{j \in J_k \setminus U} \hat{x}_j \hat{r}_i. \]

By (4.30b), the coefficients on the terms $\hat{r}_i$, $i \in U$ are nonnegative. Observe that
\[ \left( \hat{x}_i - \sum_{j \in N \setminus J_k} \theta_{ij} \frac{\hat{x}_j}{\omega_j} \right) \hat{r}_i \in \text{recc}(S^C_k) \quad \forall i \in U \]
\[ \theta_{ij} \frac{\hat{x}_j}{\omega_j} q_{ij} \in \text{recc}(S^C_k) \quad \forall i \in U, j \in N \setminus J_k \]
\[ \hat{x}_i \hat{r}_i \in \text{recc}(S^C_k) \quad \forall i \in J_k \setminus U. \]
It follows that $\hat{x} \in \{\bar{x}\} + \text{recc}(S^C_k) \subseteq S^C_k$. \hfill \Box

We next consider the separation problem for $P^B \setminus S^C_k$. In particular, given some $\hat{x} \in P^B$, we are interested in finding a subset of $D_k^*$ that maximizes the violation of an inequality of the form (4.29):
\[ \max_{U \subseteq D_k^*} \sum_{i \in U} \hat{x}_i - \sum_{j \in N \setminus J_k} \frac{\hat{x}_j}{\omega_j^*(U)}. \] (4.32)

Proposition 8. The separation problem (4.32) is a supermodular maximization problem.
Similar to the derivation of $G_D$ in Section 4.2, we derive an extended formulation for the relaxation of $P^B \setminus S^C_k$ defined by inequality (4.29) for all $U \subseteq D_k^*$. Let $D_k^* = \{1, \ldots, q_1\}$, where $q_1 := |D_k^*|$. Let $q_2 := |N \setminus J_k|$. For all $j \in N \setminus J_k$, let $t_{i_1, j}, t_{i_2, j}, \ldots, t_{i_q, j}$ be ordered to satisfy $\omega_{t_{i_1, j}} \leq \omega_{t_{i_2, j}} \leq \ldots \leq \omega_{t_{i_q, j}}$. For $i \in D_k^*$, let $m_j(i)$ be the unique integer satisfying $t_{m_j(i), j} = i$. For all $j \in N \setminus J_k$, let $\omega_{t_{i_0, j}} := +\infty$, $\theta_{t_{i_0, j}} := 0$, $v_{t_{i_0, j}} := 0$, $v_{t_{i_1,j},j} := 0$, $t_{i_0} := 0$, and $t_{i_q+1} := 0$. We define $G_k$ to be the set of $(x, \theta, v, \lambda) \in \mathbb{R}^{\lceil N \rceil} \times \mathbb{R}^{N \times q_2} \times \mathbb{R}^{N \times q_2} \times \mathbb{R}^{q_2}$ such that

$$\sum_{i \in D_k^*} \sum_{j \in N \setminus J_k} \theta_{ij} + \sum_{j \in N \setminus J_k} \lambda_j \leq 0$$

$$\theta_{ij} + v_{ij} - v_{i+1,j} + \left(\frac{1}{\omega_{t_{i+1,j}}} - \frac{1}{\omega_{t_{ij}}}\right) x_j \geq 0 \quad \forall i = 0, \ldots, q_1, \ j \in N \setminus J_k$$

$$\sum_{j \in N \setminus J_k} \theta_{m_j(i), j} - x_i \geq 0 \quad \forall i = 1, \ldots, q_1$$

Let $H^k_D$ equal the following:

$$H^k_D := \left\{ x \in \mathbb{R}^{\lceil N \rceil} : \sum_{i \in U} x_i - \sum_{j \in N \setminus J_k} x_j \omega_j^U \leq 0 \quad \forall U \subseteq D_k^* \right\} .$$

**Theorem 7.** It holds that $\text{proj}_x(G_k) = H^k_D$.

The proof of Theorem 7 mirrors that of Theorem 4. We can use the extended formulation $\text{proj}_x(G_k)$ to construct a polyhedral relaxation of $P^B \setminus S^C_k$ from the multi-term disjunction (4.22). We can then build a CGLP from polyhedral relaxations of this disjunction to generate cuts for $P \setminus C$.

The nontrivial inequalities of Theorem 6 rely on the existence of a nonempty $U \subseteq D_k^*$. We end this section by stating that if no such subset exists (i.e., $D_k^* = \emptyset$), then no nontrivial inequalities exist for $P^B \setminus S^C_k$.

**Proposition 9.** If $D_k^* = \emptyset$, then $\text{clconv}(P^B \setminus S^C_k) = P^B$.

**Proof.** By Assumption 3, $N_0 \setminus J_k \neq \emptyset$. By Proposition 6, $\bar{x} \in \text{cl}(P^B \setminus S^C_k)$. We show $\{\bar{x}\} + [0, +\infty)^{2q} \subseteq \text{clconv}(P^B \setminus S^C_k)$ for all $i \in N$. This is true for all $N_0 \setminus J_k$ by Proposition 6. Consider $i \in N_0^s$, $\lambda > 0$, and $\omega > 0$. Because $D_k^* = \emptyset$, there exists $j \in N_0$ such that $\lambda \bar{x}^i + \omega \bar{r}^j \notin \text{recc}(S^C_k)$. Then there exists $M > 0$ such that $\bar{x} + M(\lambda \bar{x}^i + \omega \bar{r}^j)$ lies outside of $S^C_k$. Therefore, $\bar{x} + \lambda \bar{x}^i + \omega \bar{r}^j$ is a convex combination of $\bar{x} \in \text{cl}(P^B \setminus S^C_k)$ and $\bar{x} + M(\lambda \bar{x}^i + \omega \bar{r}^j) \in P^B \setminus S^C_k$. This holds for an arbitrary $\omega > 0$, so $\bar{x} + \lambda \bar{x}^i \in \text{clconv}(P^B \setminus S^C_k)$. \qed

## 5 Discussion and future work

Our analysis requires the basic solution $\bar{x}$ to lie outside $\text{cl}(C)$. We showed in Section 3 that if $\bar{x} \in C$, we obtain the standard intersection cut of Balas. It remains to discuss how we can derive valid inequalities for $P \setminus C$ when $\bar{x} \in \text{bd}(C)$.

Under Assumption 1, our analysis still applies if $\bar{x} \in \text{bd}(C)$. To demonstrate this, assume for simplification that $N_0 = \emptyset$ (this is a more restrictive version of Assumption 1). It follows that $\alpha_j = 0$ for all $j \in N$. Similar to the observation made in Remark 1 for the case $\bar{x} \in C$, we can show that every point in $P^B \setminus C$ lies in $\{\bar{x}\}$ or $\{x \in P^B : \sum_{j \in N} x_j / \beta_j \geq 1\}$. We can generate inequalities for $P \setminus C$ in a disjunctive CGLP using the two polyhedra defined by the constraints of $P$ added to each of these two sets. Similarly, if $\bar{x} \in \text{bd}(C)$ and Assumption 1 holds, the term $P^B \setminus S^C_k$ of the multi-term disjunction (4.22) is equal to $\{\bar{x}\}$. We can again use disjunctive programming to generate cuts for $P \setminus C$ with the knowledge that $P^B \setminus S^C_k = \{\bar{x}\}$. Polyhedral relaxations for the remaining disjunctive terms can still be generated using the methods discussed in Section 4.5.

We conclude with some ideas for future work. One direction is to study the computational strength of cuts obtained using these ideas. Another possibility is to generalize this disjunctive framework to allow for cuts to be generated by bases of rank less than $m$ (i.e., bases that do not admit a basic solution). A final direction is to determine if the polyhedral relaxations derived in Section 4 define the convex hulls of their respective disjunctive terms.
References


