An Optimal Control Theory for Accelerated Optimization

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Abstract

Accelerated optimization algorithms can be generated using a double-integrator model for the search dynamics imbedded in an optimal control problem.

Keywords: conjugate gradient, Polyak’s heavy ball method, Transversality Mapping Principle, control Lyapunov function, Lie derivative

1. Introduction

In [1], we proposed an optimal control theory for solving a constrained optimization problem,

\[(N) \begin{cases} \text{Minimize} & E(x_f) \\ x_f \in C \subseteq \mathbb{R}^{N_x} \end{cases} \tag{1} \]

where \( C \) is a constraint set in \( \mathbb{R}^{N_x} \), \( N_x \in \mathbb{N}^+ \) and \( E : x_f \supseteq \mathbb{R}^{N_x} \to \mathbb{R} \) is an objective function. A key concept in this framework was to view an algorithmic map

\[ x_0, x_1, \ldots, x_k, x_{k+1}, \ldots \]

in terms of a discretization of a controllable, continuous-time trajectory, \( t \mapsto x \in \mathbb{R}^{N_x} \), whose dynamics is given by the single integrator model,

\[ \dot{x} = u \tag{2} \]

where \( t \mapsto u \in \mathbb{R}^{N_x} \) is a control trajectory that must be designed such that at some time \( t = t_f \), \( x(t_f) = x_f \) is a solution to the given optimization problem. Starting with this simple idea, it is possible to generate a wide variety
of well-known algorithms such as Newton’s method and the steepest descent method. Because continuous versions of “momentum” optimization methods involve second derivatives [2, 3], we explore the ramifications of replacing (2) by a double-integrator model,

\[ \ddot{x} = u \]  

(3)

In essence, we show that the application of the theory presented in [1] with (2) replaced by (3) generates accelerated optimization techniques.

**Remark 1.** Rewriting (3) in state-space form,

\[ \dot{x} = v, \quad \dot{v} = u \]  

(4)

it follows from (2) that a momentum method is essentially adding “inertia” to the “inertia-less” control of the single-integrator model.

**Remark 2.** A conjugate gradient (CG) method may also be viewed as an accelerated optimization technique in the context of (3). This observation follows by considering a generic CG method,

\[ x_{k+1} = x_k + \alpha_k v_k \]  

(5a)

\[ v_k = -g_k + \beta_{CG}^k v_{k-1} \]  

(5b)

where \( \alpha_k \geq 0 \) is the step length, \( v_k \) is the search direction, \( g_k \) is the gradient of the objective function, and \( \beta_{CG}^k \geq 0 \) is the CG update parameter. Rewriting (5) as single equation,

\[ x_{k+1} = x_k - \alpha_k g_k + \beta_k (x_k - x_{k-1}), \quad \beta_k := \left( \frac{\alpha_k \beta_{CG}^k}{\alpha_{k-1}} \right) \]  

(6)

it follows that (6) is a discretization of

\[ \dot{x} = u, \quad u = ag + bv, \quad a \in \mathbb{R}, b \in \mathbb{R} \]  

(7)

in much the same way as Polyak’s heavy ball method [2]. Note that the control in (7) is in a feedback form.
2. Background: Optimal Control Theory for Optimization

Consider some optimal control problem \((M)\) whose cost functional is given by a “Mayer” cost function \(E : x_f \mapsto \mathbb{R}\), where, \(x_f = x(t_f)\) is constrained to lie in a target set \(C\). A transversality condition for Problem \((M)\) is given by,

\[
\lambda_x(t_f) \in \nu_0 \partial E(x_f) + N_C(x_f) \quad (8)
\]

where, \(\lambda_x(t_f)\) is the final value of an adjoint arc \(t \mapsto \lambda_x\) associated with \(t \mapsto x\), \(\nu_0 \geq 0\) is a cost multiplier and \(N_C(x_f)\) is the limiting normal cone\([4]\) to the set \(C\) at \(x_f\). If Problem \((M)\) is designed so that \(\lambda_x(t_f)\) vanishes, then the transversality condition \((8)\) reduces to the necessary condition for Problem \((N)\),

\[
0 \in \nu_0 \partial E(x_f) + N_C(x_f) \quad (9)
\]

In \([1]\), we showed the existence of Problem \((M)\) by direct construction for the case when \(C\) is given by functional constraints,

\[
C = \{ x \in \mathbb{R}^{N_x} : e^L \leq e(x) \leq e^U \} \quad (10)
\]

where, \(e : x \mapsto \mathbb{R}^{N_e}\) is a given function, and \(e^L\) and \(e^U\) are the specified lower and upper bounds on the values of \(e\). In this paper, we briefly review and revise the results obtained in \([1]\) in the context \((4)\). Furthermore, for the purposes of brevity and clarity, we limit the discussions to the unconstrained optimization problem given by,

\[
(S) \left\{ \begin{array}{l}
\text{Minimize} \quad E(x_f) \\
\quad x_f \in \mathbb{R}^{N_x}
\end{array} \right. \quad (11)
\]

In following \([1]\), we create a vector field by “sweeping” the function \(E\) backwards in time according to,

\[
y(t) := E(x(t)) \quad (12)
\]

Differentiating \((12)\) with respect to time we get,

\[
y = [\partial_x E(x)]^T \dot{x} = [\partial_x E(x)]^T v, \quad \dot{v} := u \quad (13)
\]
Collecting all relevant equations, we define the following candidate optimal control problem \((R)\) that purportedly solves the optimization problem \((S)\):

\[
\begin{align*}
\text{Minimize} \quad & J[y(\cdot), x(\cdot), v(\cdot), u(\cdot), t_f] := y_f \\
\text{Subject to} \quad & \dot{x} = v \\
& \dot{v} = u \\
& \dot{y} = [\partial_x E(x)]^T v \\
& (x(t_0), t_0) = (x^0, t^0) \\
& y(t_0) = E(x^0) \\
& v(t_f) = 0
\end{align*}
\] (14)

where, \(x^0\) is an initial “guess” of the solution (to Problem \((S)\)). The variables that are free are \(t_f, x(t_f)\) and \(v(t_0)\).

**Remark 3.** The main difference between the optimal control problem for unconstrained optimization considered in [1] and (14) is the acceleration equation \(\dot{v} = u\) and its associated endpoint condition \(v(t_f) = 0\).

**Lemma 1.** Problem \((R)\) has no abnormal extremals.

**Proof.** The Pontryagin Hamiltonian[5] for this problem is given by,

\[
H(\lambda_x, \lambda_v, \lambda_y, x, v, y, u) := \lambda_x^T v + \lambda_v^T u + \lambda_y [\partial_x E(x)]^T v
\] (15)

where, \(\lambda_x, \lambda_v\) and \(\lambda_y\) are costates that satisfy the adjoint equations,

\[
\begin{align*}
\dot{\lambda}_x &= -\partial_x H = -\lambda_v \partial_x^2 E(x) v \\
\dot{\lambda}_v &= -\partial_v H = -\lambda_x - \lambda_y \partial_x E(x) \\
\dot{\lambda}_y &= -\partial_y H = 0
\end{align*}
\] (16)

The transversality conditions for Problem \((R)\) are given by,

\[
\begin{align*}
\lambda_x(t_f) &= 0 \\
\lambda_v(t_0) &= 0 \\
\lambda_y(t_f) &= \nu_0 \geq 0
\end{align*}
\] (17)
where, \( \nu_0 \) is the cost multiplier. From (16c) and (17c) we have,

\[
\lambda_y(t) = \nu_0 \quad (18)
\]

If \( \nu_0 = 0 \), then \( \lambda_y(t) \equiv 0 \). This implies from (16a) and (17a) that \( \lambda_x(t) \equiv 0 \). Similarly, \( \lambda_v(t) \equiv 0 \) from (16b) and (17b). The vanishing of all multipliers violates the nontriviality condition. Hence \( \nu_0 > 0 \). 

**Theorem 1.** All extremals of Problem \((R)\) are singular. Furthermore, the singular arc is of infinite order. 

**Proof.** The Hamiltonian is linear in the control variable and the control space is unbounded; hence, if \( u \) is optimal, it must be singular. Furthermore, from the Hamiltonian minimization condition we have the first-order condition,

\[
\partial_u H = \lambda_v(t) = 0 \quad \forall t \in [t_0, t_f] \quad (19)
\]

Differentiating (19) with respect to time, we get,

\[
\frac{d}{dt} \partial_u H = \dot{\lambda}_v(t) = -\lambda_x - \nu_0 \partial_x E(x) = 0 \quad (20)
\]

Equation (20) does not generate an expression for the control function; hence, taking the second time derivative of \( \partial_u H \) we get,

\[
\frac{d^2}{dt^2} \partial_u H = -\ddot{\lambda}_x - \nu_0 \partial_x^2 E(x) \dot{x} \\
= -\ddot{\lambda}_x - \nu_0 \partial_x^2 E(x) \nu \\
= 0 \quad (21)
\]

where, the last equality follows from (16a) and Lemma 1. Hence, we have,

\[
\frac{d^k}{dt^k} \partial_u H = 0 \quad \text{for } k = 0, 1 \ldots
\]

and no \( k \) yields an expression for \( u \). 

**Theorem 2** (A Transversality Mapping Theorem). The necessary condition for Problem \((S)\) is part of the transversality condition for Problem \((R)\).
Proof. From (20), we have

\[ \lambda_x(t) = -\nu_0 \partial_x E(x(t)) \]  

From (17a) and Lemma 1, it follows that \( \partial_x E(x_f) = 0 \). \( \square \)

3. Minimum Principles for Accelerated Optimization

From the results of the previous section, it follows that the primal-dual control dynamical system generated by Problem \( R \) is given by,

\[
\begin{align*}
\dot{x} &= v \\
\dot{\lambda}_x &= -\lambda_y \partial_x^2 E(x) v \\
\dot{v} &= u \\
\dot{\lambda}_v &= -\lambda_x - \lambda_y \partial_x E(x) \\
\dot{y} &= [\partial_x E(x)]^T v \\
\dot{\lambda}_y &= 0
\end{align*}
\]  

The boundary conditions for (23) are given by,

\[
\begin{align*}
x(t_0) &= x^0 \\
v(t_f) &= 0 \\
y(t_0) &= E(x^0) \\
\lambda_x(t_0) &= 0 \\
\lambda_y(t_0) &= \lambda_y(t_f) = \nu_0 > 0
\end{align*}
\]  

Because the optimal control is singular of infinite order, any control trajectory that satisfies (23) and (24) is optimal. Along a singular arc, \( \lambda_x(t) \equiv 0 \); hence, the auxiliary controllable dynamical system of interest [1] resulting from (23) is given by,

\[
\begin{align*}
(A) \left\{ \begin{array}{l}
\dot{\lambda}_x = -\partial_x^2 E(x) v \\
\dot{v} = u
\end{array} \right.
\end{align*}
\]  

where, we have scaled the adjoint equation by \( \nu_0 > 0 \). The target final-time condition for \( A \) is given by,

\[
(T) \left\{ \begin{array}{l}
\lambda_x(t_f) = 0 \\
v(t_f) = 0
\end{array} \right.
\]

That is, any singular control that satisfies (25) and (26) generates a candidate "optimal" continuous-time algorithm for Problem \( S \).
Remark 4. The auxiliary controllable dynamical system is equivalent to the time-derivative of the swept-back gradient function.

3.1. Application of a Minimum Principle Presented in [1]

Let $\beta$ be a control vector field defined according to,

$$
\beta(x, v, u) := \begin{bmatrix} -\partial_x^2 E(x) v \\ u \end{bmatrix}
$$

(27)

Let $V : (\lambda_x, v) \rightarrow \mathbb{R}$ be a control Lyapunov function for the $(A, T)$ pair. Let $\mathcal{L}_\beta V$ be the Lie derivative of $V$ along the vector field $\beta$. Then, a sufficient condition for producing a globally convergent algorithm[1] is to design a singular control function such that $V$ is dissipative (when $x \neq x_f$),

$$
\mathcal{L}_\beta V = \left[ \partial V(\lambda_x, v) \right]^T \beta(x, v, u) < 0
$$

(28)

In [1], it is proposed that this objective can be achieved via the Minimum Principle,

$$
\begin{array}{ll}
\text{(P)} \\
\text{Minimize} & \mathcal{L}_\beta V := \left[ \partial V(\lambda_x, v) \right]^T \beta(x, v, u) \\
\text{Subject to} & u \in U(x, \lambda_x, v, t)
\end{array}
$$

(29)

where, $U(x, \lambda_x, v, t)$ is an appropriate compact set that may vary with respect to the tuple $(x, \lambda_x, v, t)$. In an “unaccelerated” method, a solution to Problem (P) ensures the satisfaction of (28) when $U$ is chosen to metricize the control space[1]. Because of the presence of a drift vector field in the auxiliary dynamical system $(A)$, the Minimum Principle $(P)$ cannot guarantee $\mathcal{L}_\beta V < 0$; this follows by simply inspecting the expression for $\mathcal{L}_\beta V$,

$$
\mathcal{L}_\beta V = -\left[ \partial_{\lambda_x} V(\lambda_x, v) \right]^T \partial_x^2 E(x) v + \left[ \partial_v V(\lambda_x, v) \right]^T u
$$

(30)

To ensure $\mathcal{L}_\beta V < 0$, we impose the following requirement on $V$,

$$
\partial_{\lambda_x} V(\lambda_x, v)^T \partial_x^2 E(x) v > 0 \quad \text{if} \quad \partial_v V(\lambda_x, v) = 0 \text{ and } (\lambda_x, v) \neq 0
$$

(31)

All of these results — in their general form — are well-known in nonlinear feedback control theory[6]; hence, they are, technically, not new. What is new is their application to static optimization.
3.2. An Alternative Minimum Principle

We can formulate an alternative Minimum Principle that essentially exchanges the cost and constraint functions in (29). Let \( \rho : (\lambda_x, v, x) \mapsto \mathbb{R}_+ \) be an appropriate design function such that \( -\rho \) specifies a rate of descent for \( \dot{V} \).

We propose to select a singular control \( u \) such that,

\[
\mathcal{L}_\beta V = \left[ \partial V(\lambda_x, v) \right]^T \beta(x, v, u) \leq -\rho(\lambda_x, v, x)
\]  

That is, in contrast to (28), we seek a singular control that merely achieves a specified rate of descent given in terms of \( \rho \). Let \( D : (u, x, \lambda_x, v, t) \mapsto \mathbb{R} \) be an appropriate objective function. Then, a singular control \( u \) that solves the optimization problem,

\[
(P^*) \begin{cases} 
\text{Minimize} & D(u, x, \lambda_x, v, t) \\
\text{Subject to} & \mathcal{L}_\beta V + \rho(\lambda_x, v, x) \leq 0
\end{cases}
\]  

is a candidate (continuous-time) solution to the accelerated optimization problem.

Remark 5. Problem \( (P^*) \) has been widely used in control theory for generating feedback controls\[6, 7\]. Note also that condition on \( V \) specified by (31) is implicit in (33).

Remark 6. LaSalle’s Invariance Principle\[6\] may be used to relax the positive definite condition on \( V \) and the negative definite condition on \( \mathcal{L}_\beta V \).

4. Accelerated Optimization via Minimum Principles

Following [1], we consider

\[
\mathbb{U}(x, \lambda_x, v, t) := \{ u : u^T W(x, \lambda_x, v, t) u \leq \Delta \}
\]  

where, \( \Delta \neq 0 \) is a positive real number and \( W(x, \lambda_x, v, t) \) is some appropriate positive definite matrix function that metricizes the space \( \mathbb{U} \). Applying the Minimum Principle given by (29), it is straightforward to show that if \( \partial_v V(\lambda_x, v) \neq 0 \), then \( u \) is given explicitly by,

\[
u = -\sigma W^{-1} \partial_v V(\lambda_x, v), \quad \sigma > 0
\]  

where,
\[ \sigma^2 = \frac{\Delta}{[\partial_v V(\lambda_x, v)]^T W^{-1} [\partial_v V(\lambda_x, v)]} \]  
(36)

and \( W \equiv W(x, \lambda_x, v, t) \). Switching to Minimum Principle (\( P^* \)) and using

\[ D(u, x, \lambda_x, v, t) = \frac{1}{2} (u^T W(x, \lambda_x, v, t) u) \]

we get,
\[ u = -\sigma^* W^{-1} \partial_v V(\lambda_x, v), \quad \sigma^* > 0 \]  
(37)

where,
\[ \sigma^* = \frac{\rho(\lambda_x, v, x) - [\partial_{\lambda_x} V(\lambda_x, v)]^T \partial^2_x E(x) v}{[\partial_v V(\lambda_x, v)]^T W^{-1} [\partial_v V(\lambda_x, v)]} \]  
(38)

In other words, both minimum principles (\( P \) and \( P^* \)) generate the same functional form for \( u \) but with different interpretations for the control “multipliers” given by \( \sigma \) and \( \sigma^* \).

**Proposition 1.** Suppose we choose a quadratic positive definite Lyapunov function,
\[ V(\lambda_x, v) = (a/2)\lambda_x^T \lambda_x + (b/2)v^T v + c\lambda_x^T v \]  
(39)

where,
\[ a > 0, \quad b > 0, \quad c \neq 0, \quad ab - c^2 > 0 \]  
(40)

are constants. Then, condition (31) requires that \( c < 0 \) for a positive definite Hessian. Furthermore, the singular control resulting from either minimum principle (\( P \) or \( P^* \)) generates the continuous version of Polyak’s heavy ball method.

**Proof.** Applying (31) we get,
\[ \partial_v V = c \lambda_x + bv = 0 \Rightarrow \lambda_x = -(b/c)v \]  
(41)

Hence,
\[ \partial_{\lambda_x} V(\lambda_x, v)]^T \partial^2_x E(x) v = [a \lambda_x + cv]^T \partial^2_x E(x) v \]  
(42)
\[ = \left( \frac{-ab + c^2}{c} \right)v^T \partial^2_x E(x) v \]  
(43)
Thus, it follows that $c < 0$ if $\partial_x^2 E(x) > 0$.

The control solution resulting from the Minimum Principle $P$ or $P^*$ can be written as,

$$u = -\sigma_q W^{-1} [c\lambda_x + bv]$$ \hspace{1cm} (44)

where $\sigma_q$ is given by $\sigma$ or $\sigma^*$ depending upon the choice of the $P$ or $P^*$ respectively. Substituting (44) in the double integrator model $\ddot{x} = u$ and using the fact that $\lambda_x = -\partial_x E(x)$ we have,

$$\ddot{x} = -(\sigma_q k_x) W^{-1} \partial_x E(x) - (\sigma_q b) W^{-1} \dot{x}$$ \hspace{1cm} (45)

where $k_x := -c > 0$. Setting $W$ to the identity matrix in (45) generates the continuous version of Polyak’s heavy ball method[2].

References


