Abstract. Complementarity problems are often used to compute equilibria made up of specifically coordinated solutions of different optimization problems; e.g., in game-theoretic settings like the bimatrix game or in energy market models like for electricity or natural gas. In all of these situations, rational choice represented by formal optimization principles is almost always due to incomplete information about the optimization problem’s data in practice. However, while optimization under such uncertainties is rather well-developed, the field of equilibrium models represented by complementarity problems under uncertainty is still in its infancy. Being more specific, although there is a reasonable amount of literature on the stochastic treatment of uncertain complementarity problems, only very little is known about robust techniques for these problems. In this paper, we extend the theory of strictly robust linear complementarity problems (LCPs) to $\Gamma$-robust settings in the sense of Bertsimas and Sim. As in the strictly robust case, there is almost no hope for existence of worst-case-hedged equilibria. Thus, we study the minimization of the worst-case gap function of $\Gamma$-robust counterparts of LCPs. For box and $\ell_1$-norm uncertainty sets we derive tractable convex counterparts for monotone LCPs and study their feasibility as well as the existence and uniqueness of their solutions. To this end, we consider both the situations of uncertainty in the LCP vector $q$ and in the LCP matrix $M$. We additionally study so-called $\rho$-robust solutions, i.e., solutions of relaxed uncertain LCPs. Finally, we illustrate the $\Gamma$-robust concept applied to LCPs in the light of the above mentioned classical examples of bimatrix games and market equilibrium modeling.

1. Introduction

Optimization under uncertainty has become a very active field of research in the last decades. While often deterministic assumptions are used regarding the parameters of optimization models, optimization under uncertainty explicitly takes into account that many parameters of practical models are not known. In mathematical optimization, there are two main approaches towards uncertainty—namely stochastic [6] and robust optimization [1, 3, 30]. Both methodologies have been proven to be useful in various fields. However, in the area of equilibrium models much less research has been done although this setting seems to be a rather canonical field of application of optimization under uncertainty.

In this paper, we consider a special type of equilibrium models—namely the linear complementarity problem (LCP); see the book [10] for an overview and many details. In most cases, if at all, uncertain LCPs are treated in a stochastic setting. That means, one assumes knowledge about the distribution of the uncertain parameters of the LCP and then minimizes the expected value of the so-called gap function of the LCP; see, e.g., [7–9, 22] and the references therein. One important drawback of stochastic optimization and thus also of the stochastic consideration of LCPs is that one needs knowledge about the distributions of the uncertain parameters. This,
however, is often not the case in practice, which is why robust optimization has become important. In robust optimization, one does not impose distributions but consider given uncertainty sets in which the parameters may vary. Hence, one does not minimize expected values but hedges against the worst-case realization within these uncertainty sets. The major criticism of this approach is that it tends to yield highly conservative solutions, which lead to the development of less conservative notions of robustness; see, e.g., [5] for \( \Gamma \)-robustness, [21] for recoverable robustness, [12] for light robustness, or [2] for adjustable robustness.

Regarding the application of robust techniques to complementarity problems there are only a few publications. The earliest one—at least to the best of our knowledge—is the paper [31]. The authors consider the concept of strict robustness and develop the notions of \( \rho \)-robust counterparts and \( \rho \)-robust solutions. The main contribution is the development of sufficient and necessary conditions for \( \rho \)-robust solutions for the case of different uncertainty sets. More recently, tractable strictly robust counterparts of uncertain LCPs have been analyzed in detail in [32, 33]. The authors consider different uncertainty sets and distinguish between monotone and non-monotone LCPs. However, existence and uniqueness of robust solutions are not considered in the mentioned articles. There are also only a few papers that apply robust techniques to complementarity problems. We are only aware of [19, 25] that both consider robust LCPs in the context of energy market models.

In this paper we adopt the concept of \( \Gamma \)-robustness, which has been developed in [4, 5], to consider uncertain LCPs. This means that the parameters of the LCP may vary in given uncertainty sets and that only \( \Gamma \) many of them may realize in a worst-case sense. For \( \Gamma \) being the total number of parameters, this corresponds to the classical concept of strict robustness. However, adjusting \( \Gamma \) enables the modeler to control on how conservative the solutions will be.

The contribution of this paper is the following. First, we state \( \Gamma \)-robust LCPs and derive the equivalence to a properly chosen gap minimization problem. This equivalence reveals that existence of robust LCP solutions cannot be expected in the classical sense of robust optimization. This has also been discussed in [32] for the case of strict robustness. As a consequence, we afterward analyze global minimizers of the worst-case gap function and analyze the settings of uncertain LCP vector and LCP matrix separately. We are able to derive finite-dimensional robust counterparts, which are typically convex if the LCP matrix is positive semidefinite. Thus, we can explicitly characterize the tractable cases, which is a classical topic in robust optimization; see, e.g., [3] for a survey on tractability of robust counterparts for linear programming, quadratic programming, and even more general but still convex problems. The tractable convex cases then allow to consider the worst-case gap minimization problem again as an LCP for which we then consider the classical questions of existence and uniqueness of solutions. Finally, we also study the concept of \( \rho \)-robust solutions as it is introduced in [31]. Due to its relation to the worst-case gap function minimization, we readily yield results in analogy to those in [31] also for the \( \Gamma \)-robust case. Finally, we briefly study the impact of the degree of uncertainty on the robust outcomes for a market equilibrium model. This case study again reveals that robust LCP solutions cannot be expected—even for rather simple models. Instead, the global minimizers of the LCP’s gap function yield points that are “almost equilibria”. In this context we also want to mention LCPs with additional integer restriction where the results are rather similar; see [14–16, 18].

The remainder of this paper is structured as follows. We start in Section 2 by discussing two very classical applications of LCPs, namely market equilibrium modeling and the bimatrix game, and illustrate the meaning of robustness for these settings. In Section 3, we then review LCPs, their gap function formulation, and
afterward state the uncertain and robustified LCP that we then study. Uncertainty in the LCP vector is analyzed in Section 4 whereas uncertainty in the LCP matrix is the topic of Section 5. We then return to the example of robust market equilibria in Section 6 and briefly study the dependence of the global LCP gap function minimizers on the uncertain data. Finally, the paper closes with some concluding remarks and possible future directions of research in Section 7.

2. Motivating Examples

In this section, we provide two classical examples of linear complementarity problems and illustrate the meaning of $\Gamma$-robustifications. We start with an easy model of a market equilibrium and afterward discuss the LCP model of a bimatrix game. The nominal problems are both taken from the seminal textbook [10].

2.1. Market Equilibrium Modeling. We consider a stylized example of an economy in which production of goods satisfies the respective demands at equilibrium prices. To this end, we model the production side using the linear optimization problem

$$\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^\top x \\
\text{s.t.} & \quad Ax \geq b, \\
& \quad Bx \geq r^*, \\
& \quad x \geq 0.
\end{align*}$$

Here, $x$ represents the vector of production activity levels and the optimization goal is to minimize production costs given by the vector $c$. Constraint (1b) models technological production constraints and (1c) ensures that production meets the demand $r^*$. The latter depends on the market prices $p^*$ and is given by the market demand function $Q$ that is chosen to be affine-linear here, i.e.,

$$r^* = Q(p^*) = Dp^* + d.$$

Finally, we need the equilibrating condition $p^* = \pi^*$, where $\pi^*$ is the vector of dual variables of the primal demand constraints in (1c). If we now state the Karush–Kuhn–Tucker (KKT) conditions of the production problem (1) and use both the equilibrating condition as well as the market demand function, we obtain the system

\begin{align*}
0 \leq x \perp c - A^\top \lambda - B^\top p & \geq 0, \\
0 \leq \lambda \perp -b + Ax & \geq 0, \\
0 \leq p \perp -Dp - d + Bx & \geq 0.
\end{align*}

That is, with

$$z = \begin{pmatrix} x \\ \lambda \\ p \end{pmatrix}, \quad M = \begin{bmatrix} 0 & -A^\top & -B^\top \\ A & 0 & 0 \\ B & 0 & -D \end{bmatrix}, \quad q = \begin{pmatrix} c \\ -b \\ -d \end{pmatrix},$$

we obtain the LCP

$$z \geq 0, \quad Mz + q \geq 0, \quad z^\top(Mz + q) = 0.$$
sets whereas all other demands are certain. In this setting, a robust LCP asks for a market equilibrium for every possible demand realization.

The same can be done with, e.g., the production matrix $B$. Thus, a certain number of technological data, for instance production capacities, may vary in an uncertainty set and one wants to hedge against the worst-case of $\Gamma$ many of such uncertainties. In the former case, the uncertainty appears in the LCP vector $q$, whereas in the latter case the matrix $M$ is affected. We will later return to this example in the case study in Section 6, where we numerically analyze the dependence of robust solutions on the uncertain data.

2.2. Bimatrix Games. Another classical example is the bimatrix game. Consider two players 1 and 2 with $m$ resp. $n$ pure strategies. The cost incurred for player 1 if she plays strategy $i \in \{1, \ldots, m\}$ and if player 2 plays strategy $j \in \{1, \ldots, n\}$ is given as the entry $a_{ij}$ of the non-negative matrix $A \in \mathbb{R}^{m \times n}$. The analogous costs for player 2 are given in the non-negative matrix $B \in \mathbb{R}^{m \times n}$. A mixed strategy for player 1 is a non-negative vector $x \in \mathbb{R}^{m}$ with $\sum_{i=1}^{m} x_i = 1$. A mixed strategy for the other player is defined in the same way. The expected costs of the players thus are $x^\top Ay$ and $x^\top By$, respectively, and a pair $(x^*, y^*)$ of mixed strategies is called a Nash equilibrium \[26, 27\] if

\[
(x^*)^\top Ay^* \leq x^\top Ay \quad \text{for all } x \geq 0 \text{ with } \sum_{i=1}^{m} x_i = 1,
\]

\[
(x^*)^\top Ay^* \leq (x^*)^\top Ay \quad \text{for all } y \geq 0 \text{ with } \sum_{j=1}^{n} y_j = 1.
\]

It can be shown that computing a Nash equilibrium of a bimatrix game is equivalent to solving the LCP\((q, M)\) with data

\[
q = \begin{pmatrix} -e_m \\ -e_n \end{pmatrix}, \quad M = \begin{bmatrix} 0 & A \\ B^\top & 0 \end{bmatrix};
\]

see \[20\] for an early study of this relation. This LCP is of rather special type since $q$ does not depend on the problem’s data but only contains ones and also $M$ has a special structure. This example shows that for some LCPs, the consideration of uncertain $q$ is not reasonable. Here, perturbations in $q$ would yield an LCP that has no connection anymore to the original bimatrix game. As a consequence, only $M$ can be reasonably considered uncertain, which corresponds to uncertain payoffs of the players; cf., e.g., \[17\].

Of course, there are many other important classical LCP models in which the robust treatment of the problem’s data is reasonable. Let us finally mention the modeling of traffic equilibria \[11\]. Here, e.g., traveling times are usually uncertain. See also \[32\] for a brief discussion of robust LCPs in the context of traffic equilibria.

3. Problem Statement

We consider the LCP\((q, M)\), which is the problem to find a point $x \in \mathbb{R}^n$ satisfying

\[
0 \leq x \perp Mx + q \geq 0,
\]

where $M \in \mathbb{R}^{n \times n}$ is a given matrix and $q \in \mathbb{R}^n$ is a given vector, or to show that no such point exists. Here and in what follows, we use the standard $\perp$-notation, which abbreviates

\[
0 \leq a \perp b \geq 0 \iff 0 \leq a, b \geq 0, \quad ab = 0.
\]

In the following, let $\mathcal{X} := \{ x \in \mathbb{R}^n : x \geq 0, Mx + q \geq 0 \}$ be the set of feasible points of \(2\). The gap function formulation of the LCP \(2\) is the quadratic optimization
problem (QP)
\[
\min_{x \in \mathbb{R}^n} \quad g(x) := x^\top (Mx + q) \\
\text{s.t.} \quad x \geq 0, \\
Mx + q \geq 0.
\] (3)

Here, the objective function \( g \) is the so-called gap function of the LCP (2). Obviously, the objective function \( g \) of Problem (3) is bounded below by zero on its polyhedral feasible set \( X \), which ensures the existence of a minimizer by the theorem of Frank-Wolfe [13].

In this paper, we consider the situation in which the entries in the problem’s data \( M \) and \( q \) are uncertain, i.e., we have \( M(u_i) \) and \( q(u_2) \) with \( u_i \in U_i \), \( i = 1, 2 \), and \( U_i \) are given uncertainty sets. For example, we define \( q(u_2) := \tilde{q} + u_2 \) in the following, where \( \tilde{q} \) is the vector containing the so-called nominal values of the LCP vector \( q \). The definition of \( M(u_1) \) will be detailed out later. Taking these uncertainty sets into account means that we have an infinite family of complementarity problems

\[
\{ 0 \leq x \perp M(u_1)x + q(u_2) \geq 0 \}_{(u_1, u_2) \in U_1 \times U_2} 
\] (4)

instead of the single nominal LCP (2). We call Problem (4) an uncertain linear complementarity problem (ULCP). In this uncertain setting, there exists a lot of work, e.g., [7–9], focusing on the minimization of the expected gap function. For this stochastic approach, one needs access to distributional information about the uncertainty and then considers the problem

\[
\min_{x \in X} \mathbb{E}_{(u_1, u_2)} [g(x; u_1, u_2)], \quad g(x; u_1, u_2) := x^\top (M(u_1)x + q(u_2)),
\]

instead of (3). In this paper, our focus however is not on the minimization of the expected gap function. Instead, we consider worst-case minima, i.e., the robust case. Thus, we study the problem

\[
\min_{x \in X(u_1, u_2)} \sup_{(u_1, u_2) \in U_1 \times U_2} g(x; u_1, u_2)
\] (5)

with the robust feasible set

\[
X(u_1, u_2) := \{ x \in \mathbb{R}^n : x \geq 0, M(u_1)x + q(u_2) \geq 0 \text{ for all } u_1 \in U_1, u_2 \in U_2 \}.
\]

Note that this can be seen as the feasible set of a semi-infinite optimization problem; see, e.g., [28]. To be more specific, we consider the \( \Gamma \)-robust setting. This means, there are at most \( \Gamma_M \in \{ 1, \ldots, n^2 \} \) many values in \( M(u_1) \) and \( \Gamma_q \in \{ 1, \ldots, n \} \) many values in \( q(u_2) \), which are allowed to realize in a worst-case way.

In the following sections, we consider Problem (5) for uncertainty realizing in \( q \) or \( M \) and for different types of uncertainty sets \( U_i \), \( i = 1, 2 \).

4. \( \Gamma \)-Uncertainty in \( q \)

In this section, we consider uncertainty in the vector \( q \) of the ULCP (4). The entries in \( M \) are considered to be certain. That is, the problem reads

\[
\{ 0 \leq x \perp Mx + q(u) \geq 0 \}_{u \in U}
\]

for a given uncertainty set \( U \subset \mathbb{R}^n \). In this setting, the concept of strict robustness is discussed in detail in [32]. We instead focus on \( \Gamma \)-robustifications, i.e., we consider the uncertainty set

\[
U_\Gamma := \{ u \in U : | \{ i \in [n] : u_i \neq 0 \} | \leq \Gamma \}.
\]
Here and in what follows, \( \Gamma \in \{1, \ldots, n\} \) describes the number of deviations in \( q \) we hedge against and we use the abbreviation \([n] := \{1, \ldots, n\}\). We now consider the \( \Gamma \)-robust LCP gap minimization problem
\[
\min_{x \in \mathcal{X}(u)} \sup_{u \in \mathcal{U}_\Gamma} x^\top (Mx + q(u)),
\]
which is equivalent to
\[
\min_{x} \sup_{u \in \mathcal{U}_\Gamma} \{x^\top Mx + x^\top q(u) : x \geq 0, Mx \geq -q(u) \text{ for all } u \in \mathcal{U}_\Gamma\}.
\] (6)

In analogy to Proposition 3.1 in [32], we directly obtain the following proposition.

**Proposition 4.1.** A vector \( x \in \mathcal{X} \) solves
\[
0 \leq x \perp Mx + q(u) \geq 0 \text{ for all } u \in \mathcal{U}_\Gamma
\]
if and only if \( x \) is a solution of (6) with optimal objective function value of zero.

The meaning of this proposition is that a robust feasible point for the LCP is a \( \Gamma \)-robust LCP solution if and only if it satisfies complementarity for every realization of the uncertainty. In other words, we face the standard \( \exists \forall \) quantifier structure of robust optimization. Unfortunately, it is unlikely that there exists a point that is a \( \Gamma \)-robust LCP solution, i.e., a solution of (7). The same has already been commented on for the case of strict robustness in [32] as well. Thus, in what follows, we consider the global optima of the worst-case gap minimization problem (6) instead of the original problem (7). To this end, we choose different uncertainty sets \( \mathcal{U}_\Gamma \) and study whether (6) has a tractable convex counterpart. Moreover, we investigate the feasibility of this counterpart and the existence and, if possible, uniqueness of its solution.

### 4.1. Box Uncertainty \( \mathcal{U}^{\text{box}}_{\Gamma, \bar{u}} \)

In this section, we consider \( \mathcal{U}_\Gamma \) to be the box uncertainty set
\[
\mathcal{U}^{\text{box}}_{\Gamma, \bar{u}} := \{u \in \mathbb{R}^n : -\bar{u}_i \leq u_i \leq \bar{u}_i \forall i \in [n] \text{ and } \{|i \in [n] : u_i \neq 0|\} \leq \Gamma\}
\]
with \( \bar{u}_i \geq 0 \) for all \( i \in [n] \). Hence, we write \( q(u) := \bar{q} + u \) with \( u \in \mathcal{U}^{\text{box}}_{\Gamma, \bar{u}} \) for the uncertain LCP vector. The robust counterpart (6) in this case reads
\[
\min_{x \geq 0} x^\top Mx + x^\top \bar{q} + \max_{\{I \subseteq [n] : |I| \leq \Gamma\}} \sum_{i \in I} \bar{u}_i x_i
\]
\[
\text{s.t. } Mx \geq -\bar{q} + \sum_{i \in I} \bar{u}_i e_i \text{ for all } I \subseteq [n], |I| \leq \Gamma,
\]
(8a)

(8b)

where \( e_i \) is the ith unit vector in \( \mathbb{R}^n \). Note first that we—in contrast to [29]—do not need absolute values of \( x \) in the last sum of the objective function because we restrict \( x \) to be non-negative. Second, we can write “max” instead of “sup” because of the boundedness of all \( u_i \). The hardness of Problem (8) stems from the \( \min \max \) objective function and the fact that it is made up of exponentially many constraints in (8b). Fortunately, the robust counterpart (8) can be reformulated in a tractable way. First, we derive an equivalent problem of polynomial size and without an inner maximization problem.
Theorem 4.2. The robust counterpart (8) is equivalent to

\[
\begin{align*}
\min_{x, \alpha, \beta} & \quad x^\top M x + x^\top \bar{q} + \alpha \Gamma + \sum_{i=1}^{n} \beta_i \\
\text{s.t.} & \quad M_i x \geq -\bar{q}_i + \bar{u}_i, \quad i \in [n], \\
& \quad \alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n], \\
& \quad \alpha \geq 0, \\
& \quad x_i \geq 0, \quad \beta_i \geq 0, \quad i \in [n].
\end{align*}
\]

Before we prove the theorem, we notice that a variable without index denotes the vector containing all corresponding variables with indices, e.g., \( \beta := (\beta_i)_{i \in [n]} \).

Proof. First, we rewrite the robust counterpart (8) as

\[
\begin{align*}
\min_{x, \eta} & \quad \eta \\
\text{s.t.} & \quad \eta \geq x^\top M x + x^\top \bar{q} + \max_{|I| \leq \Gamma} \sum_{i \in I} \bar{u}_i x_i, \\
& \quad M x \geq -\bar{q} + \sum_{i \in I} \bar{u}_i e_i \quad \text{for all } I \subseteq [n], \quad |I| \leq \Gamma,
\end{align*}
\]

so that all uncertainties are moved to the constraints. The number of constraints in (10c) is exponential, so we first reformulate these constraints. For each realization of the uncertainty and each row \( i \in [n] \) of \( M \), we need to satisfy

\[
M_i x + \bar{q}_i (u) = \begin{cases} 
M_i x + \bar{q}_i - \bar{u}_i, & \text{if } i \in I, \\
M_i x + \bar{q}_i, & \text{if } i \notin I.
\end{cases}
\]

As \( \Gamma \geq 1 \), for each index \( i \in [n] \) exists at least one realization \( I \subseteq [n], \ |I| \leq \Gamma \), is given by

\[
M_i x + \bar{q}_i - \bar{u}_i \geq 0, \quad i \in [n].
\]

Next, we reformulate the inner maximization problem that is part of Constraint (10b), state its dual problem, and use strong duality to replace the inner maximization problem with its dual minimization problem. An equivalent formulation of

\[
\max_{|I| \leq \Gamma} \sum_{i \in I} \bar{u}_i x_i
\]

is given by

\[
\begin{align*}
\max_{x \in \mathbb{R}^n} & \quad \sum_{i=1}^{n} \bar{u}_i x_i z_i \quad (11a) \\
\text{s.t.} & \quad \sum_{i=1}^{n} z_i \leq \Gamma, \quad (11b) \\
& \quad 0 \leq z_i \leq 1, \quad i \in [n]; \quad (11c)
\end{align*}
\]
where $\alpha$

Remark 4.3

set. Thus, we obtain holds, it also holds for the minimum value of the constraints in (11c). We can now apply the strong duality theorem and replace the inner maximization problem in (10b) with the corresponding dual minimization problem. We notice that we do not need the minimum here because if

\[ \eta \geq x^\top M x + x^\top \bar{q} + \alpha \Gamma + \sum_{i=1}^n \beta_i, \quad \alpha \geq 0, \quad \beta_i \geq 0, \quad i \in [n], \]

holds, it also holds for the minimum value of $\alpha \Gamma + \sum_{i=1}^n \beta_i$ over the dual feasible set. Thus, we obtain

\[
\min_{x \geq 0, \eta, \alpha, \beta} \eta \\
\text{s.t.} \quad \eta \geq x^\top M x + x^\top \bar{q} + \alpha \Gamma + \sum_{i=1}^n \beta_i, \\
\qquad M_i, x \geq -\bar{q}_i + \bar{u}_i, \quad i \in [n], \\
\qquad \alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n], \\
\qquad \alpha \geq 0, \\
\qquad \beta_i \geq 0, \quad i \in [n],
\]

and the claim follows by eliminating $\eta$. \qed

Remark 4.3

(i) For the nominal model we know that $x = 0$ is a solution of the LCP($q, M$) if $q \geq 0$ holds. A generalization for the robust LCP (4) for the box uncertainty set is the following: If $\bar{q} - \bar{u} \geq 0$ holds, then $x = 0$ is a robust solution. The reason is the following. With the definition of $q(u)$ and $U_{box}$ one has $\bar{q}_i - \bar{u}_i \leq q_i + u_i \leq \bar{q}_i + \bar{u}_i$ for all $i \in [n]$. This, together with the assumption $\bar{q}_i - \bar{u}_i \geq 0$ for all $i \in [n]$, yields $q_i + u_i \geq 0$ for all $i \in [n]$. Hence, $x = 0$ is a solution of (4) because $M x + q(u) = q(u) \geq \bar{q} - \bar{u} \geq 0$ and complementarity directly follows.

(ii) As already mentioned, we briefly explain why we consider the Relaxation (8), respectively (9), instead of the robust LCP formulation (4). By Proposition 4.1 we know that the worst-case gap has to be zero to guarantee that we also have a robust LCP solution. This means that

\[ x^\top M x + x^\top \bar{q} + \alpha \Gamma + \sum_{i=1}^n \beta_i = 0 \]

holds for a solution $(x, \alpha, \beta)$ of (9). As $\alpha, \Gamma, \text{ and } \beta$ are all non-negative, we have that $x^\top (M x + \bar{q}) \leq 0$ needs to hold. Since every robust solution also needs to be a nominal solution, it follows $x^\top (M x + \bar{q}) \geq 0$ and we obtain $x^\top (M x + \bar{q}) = 0$. Hence, $\alpha$ and $\beta$ have to be zero as well. By Constraints (9e) we get $x = 0$ as the only robust solution—which is a solution of the nominal LCP. Thus, for $x$ being feasible for the nominal LCP this requires $\bar{q} \geq 0$. To sum up, the only robust solution in the case of box uncertainties in $q$ is $x = 0$, which is only a solution for $\bar{q} \geq 0$. 

Given the robust counterpart (9) we can now easily characterize the tractable cases.

**Corollary 4.4.** Suppose that \( M \in \mathbb{R}^{n \times n} \) is positive semidefinite. Then, Problem (9) is a convex optimization problem.

In the case of a positive semidefinite LCP matrix \( M \), we can also show that Problem (9) is again equivalent to a suitably chosen LCP.

**Theorem 4.5.** Suppose that \( M \in \mathbb{R}^{n \times n} \) is positive semidefinite. Then, the \( \Gamma \)-robust counterpart (9) is equivalent to the LCP(\( q', M' \)) with

\[
M' = \begin{bmatrix}
M + M^T & -M^T & \text{diag}(\bar{u}) & 0_{n \times n} & 0_n \\
-M & 0_{n \times n} & 0_{n \times n} & 0_n \\
0_{n \times n} & 0_{n \times n} & I_{n \times n} & \mathbb{1}_n \\
0_n & 0_n & -\mathbb{1}_n & 0_n \\
\end{bmatrix} \in \mathbb{R}^{(4n+1) \times (4n+1)} \tag{12}
\]

and

\[
q' = (\bar{q}^T, (\bar{q} - \bar{u})^T, 0_n^T, \bar{\Gamma}^T) \in \mathbb{R}^{4n+1},
\]

where \( \mathbb{1}_n \) is the vector of all ones in \( \mathbb{R}^n \).

**Proof.** Under the assumption that \( M \) is positive semidefinite, we obtain by Corollary 4.4 that Problem (9) is convex. Since all constraints are linear, no further constraint qualifications are required and the KKT conditions are sufficient and necessary optimality conditions for Problem (9). They imply

\[
0 \leq Mx + \bar{q} - \bar{u} \perp \lambda \geq 0,
\]

\[
0 \leq \alpha + \beta_i - \bar{u}_i x_i \perp \mu_i \geq 0, \quad i \in [n],
\]

\[
0 \leq x \perp M^T x + Mx - M^T \lambda + (\bar{u}_1 \mu_1, \ldots, \bar{u}_n \mu_n)^T + \bar{q} \geq 0,
\]

\[
0 \leq \alpha \perp \Gamma - \sum_{i=1}^n \mu_i \geq 0, \quad i \in [n],
\]

\[
0 \leq \beta_i \perp 1 - \mu_i \geq 0, \quad i \in [n].
\]

The solutions \( x' := (x^T, \lambda^T, \mu^T, \beta^T, \alpha)^T \) of this system are exactly the solutions of the LCP\( (q', M') \). \( \square \)

As mentioned above, existence of solutions for the original robustified LCP cannot be expected in general. Thus, we consider the existence and uniqueness of solutions of the worst-case gap minimization problem (8). Since its tractable version (9) is equivalent to an LCP again, it is natural to study the existence and uniqueness of solutions for this LCP. As usual for LCPs, results on feasibility of an LCP as well as the existence and uniqueness of solutions of LCPs are stated such that they hold for all LCP vectors \( q \). Moreover, both kinds of results depend on the LCP matrix being a \( P \) or \( S \) matrix, respectively.

We start by showing that the LCP matrix \( M' \) is not a \( P \)-matrix.

**Lemma 4.6.** The matrix \( M' \) defined in (12) is not a \( P \)-matrix.

**Proof.** A matrix \( A \in \mathbb{R}^{m \times m} \) is a \( P \)-matrix if and only if for all index sets \( J \subseteq \{1, \ldots, m\} \) it holds \( \det(A_{J,J}) > 0 \). Choosing \( J := \{n + 1\} \), however, yields \( \det(M_{n+1,n+1}') = 0 \). Thus, \( M' \) is not a \( P \)-matrix. \( \square \)

Moreover, we can also show that the matrix \( M' \) is not an \( S \)-matrix.

**Lemma 4.7.** The matrix \( M' \) defined in (12) is not an \( S \)-matrix.
Then, the solution of Problem Theorem 4.8, this guarantees the existence of a solution of Problem defined in (12) holds, we need \( \mu_i < 0 \) for all \( i \in [n] \) to obtain \( M'x' > 0 \).

The two latter results show that the LCP with matrix \( M' \) is neither uniquely solvable nor feasible for all possible LCP vectors \( q' \). However, taking a closer look on the specific vector in (13), we now ask whether a feasible point of (9)—and hence of (8)—exists for all such specific \( q' \).

First, for positive definite matrices \( M \) we are able to prove that Problem (9) is always feasible.

**Theorem 4.8.** Suppose that the matrix \( M \) in Problem (9) is positive definite. Then, Problem (9) is feasible.

**Proof.** As \( M \) is positive definite, there exists a vector \( x > 0 \) with \( Mx > 0 \). Thus, we can choose a scalar \( \lambda > 0 \) sufficiently large such that \( \lambda Mx \geq -q + u \) and we define \( \hat{x} := \lambda x \). Then, \( \hat{u}, \hat{x} \geq 0 \) holds for all \( i \in [n] \) and the right-hand sides of the constraints in (9c) are non-negative and fixed. Hence, \( \beta_i := \hat{u}_i \hat{x}_i \) for all \( i \in [n] \) and \( \alpha = 0 \) is a feasible solution of Problem (9).

The next question is whether Problem (9) (and hence (8)) is solvable if it is feasible. This is equivalent to the question if \( M' \) is a \( Q_0 \)-matrix. In our setting, this is guaranteed if the original LCP matrix \( M \) is positive semidefinite.

**Theorem 4.9.** Let \( M \) be a positive semidefinite matrix. Then, the matrix \( M' \) defined in (12) is positive semidefinite and, thus, a \( Q_0 \)-matrix. Together with Theorem 4.8, this guarantees the existence of a solution of Problem (8) if \( M \) is positive definite.

**Proof.** Let \( x' := (x^T, \lambda^T, \mu^T, \beta^T, \alpha)^T \), then we have

\[
(x')^T M' x' = (x^T, \lambda^T, \mu^T, \beta^T, \alpha)^T \begin{pmatrix}
(M + M^T)x - M^T \lambda + \text{diag}(\bar{u}) \mu \\
Mx \\
- \text{diag}(\bar{u}) x + I_{n \times n} \beta + \mathbb{I}_n \alpha \\
-I_{n \times n} \mu \\
- \sum_{i=1}^n \mu_i
\end{pmatrix}
\]

\[
= x^T (M + M^T)x - x^T M^T \lambda + x^T \text{diag}(\bar{u}) \mu + \lambda^T Mx - \mu^T \text{diag}(\bar{u}) x + \mu^T I_{n \times n} \beta + \mu^T \mathbb{I}_n \alpha - \beta^T I_{n \times n} \mu - \sum_{i=1}^n \alpha \mu_i
\]

Thus, \( M' \) is positive semidefinite.

Finally, we can show \( x \)-uniqueness of Problem (9) for positive definite matrices \( M \). Since feasibility and existence is already shown, the next result is a direct consequence of [24].

**Proposition 4.10.** Suppose that the matrix \( M \) in Problem (9) is positive definite. Then, the solution of Problem (9) is unique in \( x \).
However, the entire primal solution of Problem (9) is not unique because it is not unique in $\alpha$ and $\beta$ in general—even for the case of $M$ being positive definite. This is shown in the following example.

**Example 4.11.** Let $n = 3$, $\Gamma = 1$, $M = I_{3 \times 3}$, $\bar{q} = (-4, 2, 0)^T$, and $\bar{u} = (3, 2, 10)^T$ be the input data of Problem (9). One solution with value 221 is $x = (7, 0, 10)^T$, $\alpha = 76$, and $\beta = (0, 0, 24)^T$. A second solution reads $x = (7, 0, 10)^T$, $\alpha = 80$, and $\beta = (0, 0, 20)^T$. These are two solutions with the same values for $x$ but different values for $\alpha$ and $\beta$.

Next, we briefly discuss the concept of $\rho$-relaxations of robust LCPs, which has been introduced in [31].

**Definition 4.12.** Let $U$ be the uncertainty set of the ULCP (4) and let $\rho \geq 0$ be given. Then, the system

\begin{align*}
x \geq 0, \\
M(u)x + q(u) \geq 0, & \quad u \in U, \\
x^T(M(u)x + q(u)) \leq \rho, & \quad u \in U,
\end{align*}

is called the $\rho$-relaxation of the ULCP (4). Solutions of System (14) are called $\rho$-robust solutions.

As before, we only consider uncertainty in the vector $q$. Thus, System (14) for the box-uncertainty set $U_{\Gamma, \bar{u}}^{\text{box}}$ reads

\begin{align*}
x \geq 0, \\
Mx + \bar{q} + u \geq 0, & \quad u \in U_{\Gamma, \bar{u}}^{\text{box}}, \\
x^T(Mx + \bar{q} + u) \leq \rho, & \quad u \in U_{\Gamma, \bar{u}}^{\text{box}}.
\end{align*}

As in [31], our goal now is to derive a finite system of equations and inequalities that characterizes $\rho$-robust solutions. This is achieved by the following theorem, which is closely related to Theorem 4.2.

**Theorem 4.13.** Let $U = U_{\Gamma, \bar{u}}^{\text{box}}$ be the given uncertainty set. Then, $x$ is a $\rho$-robust LCP solution if and only if there exist $\alpha \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$, $i \in [n]$, that satisfy

\begin{align*}
x^T(Mx + \bar{q}) + \alpha \Gamma + \sum_{i=1}^{n} \beta_i & \leq \rho, \\
Mx + \bar{q} - \bar{u} & \geq 0, \\
\alpha + \beta_i - \bar{u}_i x_i & \geq 0, \quad i \in [n], \\
\alpha & \geq 0, \\
x_i, \beta_i & \geq 0, \quad i \in [n].
\end{align*}

**Proof.** First, assume that $x, \alpha, \beta$ are given that satisfy System (15). This means that $x, \alpha, \beta$ are feasible for Problem (9) and that the corresponding objective function value is not larger than $\rho$. By Theorem 4.2, this is equivalent to $x$ being a $\rho$-robust solution. On the other hand, let $x$ be a $\rho$-robust solution. Again by Theorem 4.2 this means that Problem (9) has a feasible point with objective function value less than $\rho$, which directly implies the existence of $\alpha$ and $\beta$ so that (15) is satisfied. \qed

The proof of the last theorem also directly shows the final result in this section.

**Corollary 4.14.** Let $U = U_{\Gamma, \bar{u}}^{\text{box}}$ be the given uncertainty set. Then, $x$ is a $\rho$-robust LCP solution if and only if the quadratic program (9) has an optimal solution with optimal objective function value not larger than $\rho$. 
We close this section with a remark about the connection of the uncertainty set \( \mathcal{U}_{\text{box}}^\Gamma, \bar{u} \) and the \( \ell_{\infty} \)-norm uncertainty set
\[
\mathcal{U}_{\infty}^\Gamma, \delta := \{ u \in \mathbb{R}^n : \| u \|_{\infty} \leq \delta \text{ and } |\{ i \in [n] : u_i \neq 0 \}| \leq \Gamma \}.
\]
It is easy to see that \( \mathcal{U}_{\text{box}}^\Gamma, \bar{u} = \mathcal{U}_{\infty}^\Gamma, \delta \) if \( \bar{u}_i = \bar{u} = \delta \) holds. That is, all results in this section also hold for \( \mathcal{U}_{\infty}^\Gamma, \delta \).

4.2. Uncertainty set \( \mathcal{U}_{1}^\Gamma, \delta \). In this section, we consider the \( \ell_1 \)-norm uncertainty set
\[
\mathcal{U}_{1}^\Gamma, \delta := \{ u \in \mathbb{R}^n : \| u \|_1 \leq \delta \text{ and } |\{ i \in [n] : u_i \neq 0 \}| \leq \Gamma \}
\]
for a given \( \delta > 0 \). The robust counterpart (6) in this case reads

\[
\begin{align*}
\min_{x \geq 0, t \geq 0} & \quad x^\top M x + x^\top \bar{q} + \max_{u \in \mathcal{U}_{1}^\Gamma, \delta} \sum_{i \in [n]} u_i x_i \\
\text{s.t.} & \quad M_i x + \bar{q}_i + \min_{u \in \mathcal{U}_{1}^\Gamma, \delta} u_i \geq 0, \quad i \in [n].
\end{align*}
\]

This optimization model can be reformulated in a tractable way. To prove this, we apply the strategy used in [32].

**Theorem 4.15.** The robust counterpart (16) is equivalent to

\[
\begin{align*}
\min_{x \geq 0, t \geq 0} & \quad x^\top M x + x^\top \bar{q} + \delta t \\
\text{s.t.} & \quad x_i \leq t, \quad i \in [n], \\
& \quad M_i x + \bar{q}_i - \delta \geq 0, \quad i \in [n].
\end{align*}
\]

**Proof.** We can rewrite the inner maximization problem in the objective function of (16) as \( \delta \| x \|_\infty \). Thus, the robust counterpart (16) is equivalent to

\[
\min_{x \geq 0, t \geq 0} \quad x^\top M x + x^\top \bar{q} + \delta t \\
\text{s.t.} \quad x_i \leq t, \quad i \in [n], \\
& \quad M_i x + \bar{q}_i + \min_{u \in \mathcal{U}_{1}^\Gamma, \delta} u_i \geq 0, \quad i \in [n].
\]

With
\[
\min_{u \in \mathcal{U}_{1}^\Gamma, \delta} u_i = -\delta,
\]
the claim follows. \( \square \)

**Remark 4.16.** Here, something interesting is happening: The \( \Gamma \)-robust counterpart (17) is independent of \( \Gamma \), which is not the case for the uncertainty set \( \mathcal{U}_{\text{box}}^\Gamma, \bar{u} \) discussed in the last section. The reason is that the condition \( \| u \|_1 = \sum_{i=1}^n |u_i| \leq \delta \) aggregates all uncertain components.

Moreover, the \( \ell_1 \)-norm counterpart yields some kind of \( \ell_{\infty} \)-norm regularization of the original LCP since Problem (17) can be rewritten as

\[
\min_{x \geq 0} \quad x^\top M x + x^\top \bar{q} + \delta \| x \|_\infty \quad \text{s.t.} \quad M x + \bar{q} \geq \mathbb{I} \delta.
\]

That is, we have the original gap function extended by an \( \ell_{\infty} \)-regularizer as well as a \( \delta \)-tightened constraint set.

Given the robust counterpart (17) we can now easily characterize the tractable case.

**Corollary 4.17.** Suppose that \( M \in \mathbb{R}^{n \times n} \) is positive semidefinite. Then, the \( \Gamma \)-robust counterpart (17) is a convex optimization problem.
Under the assumption of the last corollary, we can also show that Problem (17) is again equivalent to a suitably chosen LCP.

**Theorem 4.18.** Suppose that $M \in \mathbb{R}^{n \times n}$ is positive semidefinite. Then, the $\Gamma$-robust counterpart (17) is equivalent to the LCP($q', M'$) with

$$M' = \begin{bmatrix} M + M^\top & 0_{n \times 1} & I_{n \times n} & -M^\top \\ 0_{1 \times n} & 0_{1 \times 1} & -I_n & 0_{1 \times n} \\ -I_{n \times n} & I_n & 0_{n \times n} & 0_{n \times n} \\ M & 0_{n \times 1} & 0_{n \times n} & 0_{n \times n} \end{bmatrix} \in \mathbb{R}^{(3n+1) \times (3n+1)}$$

(18)

and

$$q' = (q^\top, 0_n^\top, (q - \delta \mathbb{1}_n)^\top)^\top \in \mathbb{R}^{3n+1}.$$  

**Proof.** Using the assumption that $M$ is positive semidefinite, we obtain by Corollary 4.17 that Problem (17) is convex. Since all constraints are linear, no further constraint qualifications are required and the KKT conditions are sufficient and necessary optimality conditions of Problem (17). They comprise

$$0 \leq x \perp M^\top x + Mx - M^\top \gamma + \beta + q \geq 0,$$

$$0 \leq \delta - \sum_{i \in [n]} \beta_i \perp t \geq 0,$$

$$0 \leq t - x_i \perp \beta_i \geq 0, \quad i \in [n],$$

$$0 \leq Mx + \bar{q} - \delta \mathbb{1}_n \perp \gamma \geq 0.$$  

Here, $\beta_i, i \in [n],$ are the duals of the Constraints (17b) and the vector $\gamma$ contains the dual variables of the constraints in (17c). The solutions $x' := (x^\top, t, \beta^\top, \gamma^\top)^\top$ of this system are solutions of the LCP($q', M'$).

As in Section 4.1 we now investigate existence and uniqueness of solutions of the robust counterpart (17). To this end, we again try to make use of classical LCP theory. However, the LCP matrix $M'$ is neither a $P$ nor an $S$-matrix.

**Theorem 4.19.** The matrix $M'$ defined in (18) is neither an $S$-matrix nor a $P$-matrix.

**Proof.** We proceed as in the proof of Theorem 4.7. We show that $M'$ is not an $S$-matrix by proving that there exists no vector $x' := (x^\top, t, \beta^\top, \gamma^\top)^\top \geq 0_{3n+1}$ with $M'x' > 0_{3n+1}$. From $x' \geq 0$ we especially obtain $\beta_i \geq 0$ for all $i \in [n]$. However, as

$$M'x' = \begin{bmatrix} (M + M^\top)x - M^\top \gamma + \beta \\ -\sum_{i \in [n]} \beta_i \\ -x + \mathbb{1}_n t \\ Mx \end{bmatrix}$$

holds, we see that $-\sum_{i \in [n]} \beta_i \leq 0$ follows and, thus, $-\sum_{i \in [n]} \beta_i > 0$ is impossible and, thus, $M'$ is not an $S$-matrix. For the $P$ property see the proof of Lemma 4.6.  

As in the last section, we can also derive conditions under which Problem (17) is feasible.

**Theorem 4.20.** Suppose that the matrix $M$ in Problem (17) is positive definite. Then, Problem (17) is feasible.

**Proof.** As $M$ is positive definite, there exists a vector $x > 0$ with $Mx > 0$. Hence, we can choose $\lambda > 0$ sufficiently large such that $\lambda Mx \geq \delta \mathbb{1}_n - \bar{q}$ and we define $\hat{x} := \lambda x$. Then, choosing $t := \max\{\hat{x}_i : i \in [n]\}$ yields a feasible solution $(\hat{x}, t)$.  

In addition, we also obtain existence and uniqueness if the original LCP matrix $M$ is positive definite.
Theorem 4.21. Suppose that the matrix $M$ in Problem (17) is positive definite. Then, a solution of the $\Gamma$-robust counterpart (17) exists and is unique.

Proof. First, existence of a solution follows from Theorem 4.20 and the theorem of Frank–Wolfe [13] since the objective function is bounded below on the polyhedral feasible set. The uniqueness of the solution in $x$ again follows from [24]. Then, the solution is also unique in $t$ because $\delta > 0$ holds and we obtain the unique optimal value $t = \max\{x_i : i \in [n]\}$ by Conditions (17b).

Again, we close this section with a brief discussion of $\rho$-relaxations of the uncertain LCPs. Thus, we consider the System (14) for $\mathcal{U}_1^\rho$, which reads

$$x \geq 0,$$

$$Mx + \bar{q} + u \geq 0, \quad u \in \mathcal{U}_1^\rho,$$

$$x^\top (Mx + \bar{q} + u) \leq \rho, \quad u \in \mathcal{U}_1^\rho.$$

Theorem 4.22. Let $\mathcal{U} = \mathcal{U}_1^\rho$ be the given uncertainty set. Then, $x$ is a $\rho$-robust LCP solution if and only if there exists a scalar $t \in \mathbb{R}$ that satisfies

$$x^\top (Mx + \bar{q}) + \delta t \leq \rho,$$

$$x \geq 0,$$

$$x_i \leq t, \quad i \in [n],$$

$$Mx + \bar{q} - \delta \mathbb{I}_n \geq 0.$$

Proof. The proof is completely analogous to the proof of Theorem 4.13.

In addition, we also get the following corollary, which is analogous to Corollary 4.14.

Corollary 4.23. Let $\mathcal{U} = \mathcal{U}_1^\rho$ be the given uncertainty set. Then, $x$ is a $\rho$-robust LCP solution if and only if the quadratic program (17) has an optimal solution with optimal function value not larger than $\rho$.

5. $\Gamma$-Uncertainty in $M$

In this section, we consider the ULCP (4) with uncertainties in the matrix $M$. The entries in $q$ are considered to be certain. That is, the problem now reads

$$\{0 \leq x \perp M(u)x + q \geq 0\}_{u \in \mathcal{U}}.$$

5.1. Box Uncertainty $\mathcal{U}_{\Gamma, \bar{u}, \bar{u}}^{\text{box}}$. We start with a definition of $M(u)$ in analogy to $q(u)$ in the last section. That is, let $M := [\bar{m}_{ij}]_{i,j \in [n]}$ be the matrix containing all nominal values and let $M(u) := [\bar{m}_{ij} + u_{ij}]_{i,j \in [n]}$ with $[u_{ij}]_{i,j \in [n]} \in \mathcal{U}$. In this section, we consider box uncertainties for the entries in $M$. To this end, for every row $i \in [n]$ we define

$$\mathcal{U}_{\Gamma, \bar{u}, \bar{u}}^{\text{box}, i} := \{u_i \in \mathbb{R}^n : -\bar{u}_{ij} \leq u_{ij} \leq \bar{u}_{ij} \forall j \in [n] \text{ and } |\{j \in [n] : u_{ij} \neq 0\}| \leq \Gamma_i\}$$

as the uncertainty set of row $i$ of $M$ and $\Gamma_i \in \{1, \ldots, n\}$. In this case, the robust counterpart (6) for uncertainty in $M$ reads

$$\min_{x \geq 0} \quad x^\top \hat{M}x + x^\top \bar{q} + \sum_{i \in [n]} \max_{|I_i| \leq \Gamma_i} \sum_{j \in I_i} \bar{u}_{ij}x_i \quad \text{ s.t. } \quad \sum_{j \in [n]} \bar{m}_{ij}x_j - \sum_{i \in [n]} \max_{|I_i| \leq \Gamma_i} \sum_{j \in I_i} \bar{u}_{ij}x_j \geq -q_i, \quad i \in [n].$$

As it was the case for uncertainty in $q$, our goal is to derive a tractable robust counterpart. However, the following theorem reveals that this is not possible in general for the case of uncertain $M$. 

Theorem 5.1. Let $\mathcal{U}_{\Gamma_i}^{box,\bar{u},i}$ be the uncertainty set of row $i \in [n]$ in $M(u)x + q \geq 0$. Then, the robust counterpart (19) is equivalent to

$$\min_{x, \alpha, \beta, \gamma, \delta, \epsilon, \xi} \quad x^T Mx + x^T q + \sum_{i \in [n]} \left( \gamma_i \Gamma_i + \sum_{j \in [n]} \delta_{ij} \right)$$

(20a)

s.t. \quad \sum_{j \in [n]} \bar{m}_{ij} x_j - \bar{\epsilon}_i \Gamma_i - \sum_{j \in [n]} \xi_{ij} \geq -q_i, \quad i \in [n],

(20b)

\[ \bar{\epsilon}_i \geq 0, \quad i \in [n], \]

(20c)

\[ \xi_{ij} \geq 0, \quad i, j \in [n], \]

(20d)

\[ \gamma_i \geq 0, \quad i \in [n], \]

(20e)

\[ \delta_{ij} \geq 0, \quad i, j \in [n]. \]

(20f)

Proof. To prove this theorem, we proceed as in the proof of Theorem 4.2. First, we rewrite Problem (19) as

$$\min_{x, \eta} \quad x^T Mx + x^T q + \sum_{i \in [n]} \max_{\{I_i \subseteq [n] : |I_i| \leq \Gamma_i\}} \sum_{j \in I_i} \bar{u}_{ij} x_j x_j$$

(21a)

s.t. \quad \eta \geq x^T \bar{M}x + x^T q + \sum_{i \in [n]} \max_{\{I_i \subseteq [n] : |I_i| \leq \Gamma_i\}} \sum_{j \in I_i} \bar{u}_{ij} x_j x_j,

(21b)

\[ \sum_{j \in [n]} \bar{m}_{ij} x_j - \max_{\{I_i \subseteq [n] : |I_i| \leq \Gamma_i\}} \sum_{j \in I_i} \bar{u}_{ij} x_j x_j \geq -q_i \quad i \in [n]. \]

(21c)

Now, we reformulate the inner maximization problem of Constraint (21b). To this end, we use the equivalent formulation

$$\max_{z_i} \quad \sum_{j \in [n]} \bar{u}_{ij} x_j z_{ij}$$

s.t. \quad \sum_{j \in [n]} z_{ij} \leq \Gamma_i,

(22a)

\[ 0 \leq z_{ij} \leq 1, \quad j \in [n], \]

(22b)

of

$$\max_{\{I_i \subseteq [n] : |I_i| \leq \Gamma_i\}} \sum_{j \in I_i} \bar{u}_{ij} x_j x_j.$$ 

Its dual problem is given by

$$\min_{\gamma_i, \delta_{ij}} \quad \gamma_i \Gamma_i + \sum_{j \in [n]} \delta_{ij}$$

s.t. \quad \gamma_i + \delta_{ij} \geq \bar{u}_{ij} x_j x_j, \quad j \in [n],

(22c)

\[ \gamma_i \geq 0, \]

(22d)

\[ \delta_{ij} \geq 0, \quad j \in [n]. \]

In a second step, we use the equivalent formulation

$$\max_{z_i} \quad \sum_{j \in [n]} \bar{u}_{ij} x_j z_{ij}$$

s.t. \quad \sum_{j \in [n]} z_{ij} \leq \Gamma_i,

(22e)

\[ 0 \leq z_{ij} \leq 1, \quad j \in [n], \]
for each $i \in [n]$ of the inner maximization problem in (21c). We can again replace this maximization problem by its dual, which reads

$$
\min_{\varepsilon_i, \xi_i} \varepsilon_i \Gamma_i + \sum_{j \in [n]} \xi_{ij}
$$

subject to

$$
\varepsilon_i + \xi_{ij} \geq \bar{u}_{ij} x_j, \quad j \in [n],
$$

$$
\varepsilon_i \geq 0,
$$

$$
\xi_{ij} \geq 0, \quad j \in [n].
$$

Using these dual reformulations and the arguments used in the previous proofs, the claim follows.

Unfortunately, Problem (20) is a nonconvex and, thus, intractable optimization problem due the bilinear terms on the right-hand side of the constraints in (20f). To be more specific, Problem (20) is a (nonconvex) quadratically constrained quadratic program. For such problems, even in the case of a convex objective, i.e., for positive semidefinite $\bar{M}$, the existence of solutions cannot be guaranteed in general; see, e.g., [23]. This is the reason why we also consider another definition of $M(u)$, which is the same as in [32]. Let now

$$
M(u) := \bar{M} + \sum_{\ell \in [L]} u_{\ell} M^\ell
$$

with $L \in \mathbb{N}$ and $M^\ell := [m_{ij}]_{i,j \in [n]} \in \mathbb{R}^{n \times n}$. With this, the uncertainty set is defined as

$$
u \in \mathcal{U}_{\Gamma, \bar{u}}^{\text{box}} := \{u \in \mathbb{R}^L : 0 \leq u_{\ell} \leq \bar{u}_{\ell}, \ \ell \in [L], \ |\{\ell \in [L] : u_{\ell} \neq 0\}| \leq \Gamma\}.$$

In this case, the robust counterpart reads

$$
\begin{align*}
\min_{x \geq 0} & \quad x^T M x + x^T q + \max_{u \in \mathcal{U}_{\Gamma, \bar{u}}^{\text{box}}} \sum_{\ell \in [L]} u_{\ell} x^T M^\ell x \\
\text{s.t.} & \quad \bar{M} x + q + \min_{u \in \mathcal{U}_{\Gamma, \bar{u}}^{\text{box}}} \sum_{\ell \in [L]} u_{\ell} M^\ell x \geq 0.
\end{align*}
$$

The following theorem states that we can rewrite this robust counterpart as a tractable one. To prove this theorem, we proceed as in the proof of Theorem 3.4 in [32].

**Theorem 5.2.** Consider the uncertainty set

$$
\mathcal{U}_{\Gamma, \bar{u}}^{\text{box}} := \{u \in \mathbb{R}^L : 0 \leq u_{\ell} \leq \bar{u}_{\ell}, \ \ell \in [L], \ |\{\ell \in [L] : u_{\ell} \neq 0\}| \leq \Gamma\}
$$

and $L \geq \Gamma$. Furthermore, suppose that $\bar{M}$ and $M^\ell$, $\ell \in [L]$, are positive semidefinite. Then, Problem (22) is equivalent to the convex, and thus tractable, problem

$$
\begin{align*}
\min_{x, \alpha, \beta, \gamma, \delta} & \quad x^T \bar{M} x + x^T q + \Gamma \alpha + \sum_{\ell \in [L]} \beta_{\ell} \\
\text{s.t.} & \quad \alpha + \beta_{\ell} \geq \bar{u}_{\ell} x^T M^\ell x, \quad \ell \in [L], \\
& \alpha \geq 0, \\
& \beta_{\ell} \geq 0, \quad \ell \in [L], \\
& \gamma_i \geq 0, \quad i \in [n], \\
& \delta_{\ell} \geq 0, \quad i \in [n], \ \ell \in [L], \\
& x \geq 0,
\end{align*}
$$

$$
\begin{align*}
& \bar{M}_i x + q_i - \gamma_i \Gamma - \sum_{\ell \in [L]} \delta_{\ell} x_i \geq 0, \quad i \in [n],
\end{align*}
$$
\( \gamma_i + \delta_{i\ell} \geq -\bar{u}_{i\ell} M_{i\ell}^\ell x, \quad i \in [n], \ \ell \in [L]. \) (23i)

**Proof.** First, we rewrite Problem (22) as

\[
\begin{align*}
\text{min}_{x \geq 0, \eta} \quad & \eta \\
\text{s.t.} \quad & x^\top M x + x^\top q + \max_{u \in U_{\Gamma}^L} \sum_{\ell \in [L]} u_{i\ell} x^\top M_{i\ell}^\ell x \leq \eta, \quad (24a) \\
& M x + q + \min_{u \in U_{\Gamma}^L} \sum_{\ell \in [L]} u_{i\ell} M_{i\ell}^\ell x \geq 0 \quad (24c)
\end{align*}
\]

and reformulate the inner maximization problem in Constraint (24b). Since all \( M_{i\ell}^\ell, \ell \in [L], \) are positive semidefinite one has

\[
\max_{u \in U_{\Gamma}^L} \sum_{\ell \in [L]} u_{i\ell} x^\top M_{i\ell}^\ell x = \max_{\{I \subseteq [L]: |I| \leq \Gamma\}} \sum_{\ell \in I} \bar{u}_{i\ell} x^\top M_{i\ell}^\ell x,
\]

where we can write the right-hand side as

\[
\begin{align*}
\max_z \quad & \sum_{\ell \in [L]} (\bar{u}_{i\ell} x^\top M_{i\ell}^\ell x) z_{\ell} \\
\text{s.t.} \quad & \sum_{\ell \in [L]} z_{\ell} \leq \Gamma, \\
& 0 \leq z_{\ell} \leq 1, \quad \ell \in [L].
\end{align*}
\]

Its dual problem reads

\[
\begin{align*}
\text{min}_{\alpha, \beta} \quad & \Gamma \alpha + \sum_{\ell \in [L]} \beta_{\ell} \\
\text{s.t.} \quad & \alpha + \beta_{\ell} \geq \bar{u}_{i\ell} x^\top M_{i\ell}^\ell x, \quad \ell \in [L], \\
& \alpha \geq 0, \\
& \beta_{\ell} \geq 0, \quad \ell \in [L],
\end{align*}
\]

and again we can replace the inner maximization problem in Constraint (24b) with this dual. In a second step, we have a closer look at Constraint (24c). We now consider this constraint componentwise,

\[
M_{i\ell} x + q_{i\ell} + \min_{u \in U_{\Gamma}^L} \sum_{\ell \in [L]} u_{i\ell} M_{i\ell}^\ell x \geq 0
\]

and fix \( i \in [n] \) for what follows. Assume now, w.l.o.g., that we have an ascending order of the values \( \bar{u}_{i\ell} M_{i\ell}^\ell x \) for all \( i \in [n], \) i.e.,

\[
\bar{u}_{1\ell} M_{1\ell}^\ell x \leq \bar{u}_{2\ell} M_{2\ell}^\ell x \leq \cdots \leq \bar{u}_{L\ell} M_{L\ell}^\ell x.
\]

Then, we obtain

\[
\min_{u \in U_{\Gamma}^L} \sum_{\ell \in [L]} u_{i\ell} M_{i\ell}^\ell x = \sum_{\ell = 1}^r \min(0, \bar{u}_{i\ell} M_{i\ell}^\ell x).
\]

So, we can rewrite for each \( i \in [n] \) the Constraint (24c) as

\[
M_{i\ell} x + q_{i\ell} + \min_{\{I \subseteq [L]: |I| \leq \Gamma\}} \sum_{\ell \in I} \bar{u}_{i\ell} M_{i\ell}^\ell x \geq 0.
\]
Now, we reformulate the inner minimization problem in this constraint as
\[ \min_{z_i} \sum_{\ell \in [L]} (\bar{u}_{\ell} M^{\ell}_i, x) z_{i\ell} \]
\[ \text{s.t. } \sum_{\ell \in [L]} z_{i\ell} \leq \Gamma, \]
\[ 0 \leq z_{i\ell} \leq 1, \quad \ell \in [L], \]
where its dual reads
\[ \max_{\gamma_i, \delta_{i\ell}} -\gamma_i \Gamma - \sum_{\ell \in [L]} \delta_{i\ell} \]
\[ \text{s.t. } \gamma_i \geq 0, \quad \delta_{i \ell} \geq 0, \quad \ell \in [L], \]
\[ \gamma_i + \delta_{i \ell} \geq -\bar{u}_{\ell} M^{\ell}_i, x, \quad \ell \in [L]. \]
Using this dual problem and the arguments used in the previous proofs, the claim follows. \[ \square \]

Remark 5.3. Let us make two remarks regarding the last theorem. First, note that we only consider the case \( L > \Gamma \) because otherwise we are in the strictly robust case. Second, the latter theorem is qualitatively different to the results that we obtained for uncertain \( q \) in the last section. For uncertain \( q \), we are able to state a finite-dimensional counterpart without inner minimization or maximization problems independent on whether the original LCP matrix \( M \) is positive semidefinite or not. Only the convexity of the counterpart depends on whether \( M \) is positive semidefinite or not. Here, we are only able to state a finite-dimensional counterpart without inner minimization or maximization problems in the case of positive semidefinite \( M \). In Section 4.1 on uncertain \( q \) we have been able to reformulate the robust counterpart as an equivalent LCP again. This is not possible anymore in the case of uncertain \( M \). The reason is the quadratic term on the right-hand side of the constraints in (23b).

Next, we derive conditions for the existence and uniqueness of solutions to (23).

**Theorem 5.4.** Assume that Problem (23) is feasible and that \( \bar{M} \) and \( M_{\ell}, \ell \in [L], \) are positive semidefinite. Then, there exists a solution of Problem (23).

**Proof.** From Constraint (23b) it follows that \( \bar{M} x + q \geq 0 \) holds, which implies \( x^T \bar{M} x + x^T q \geq 0 \) due to \( x \geq 0 \). Thus, the objective function of Problem (23) is bounded below on the feasible set of the problem. We can thus apply Theorem 3 of [23] that ensures the existence of a solution. \[ \square \]

Note that the feasibility of Problem (23) is assumed in the theorem. Unfortunately, we have not been able to prove the feasibility of the problem in general like we did in, e.g., Theorem 4.8.

If the matrix \( \bar{M} \) in Problem (23) is positive definite, we obtain \( x \)-uniqueness of the solution as a consequence of [24].

**Proposition 5.5.** Suppose that the matrix \( \bar{M} \) in (23) is positive definite. Then, the solution of Problem (23) is unique in \( x \).

In the light of Example 4.11, we think that uniqueness of the other variables \( \alpha, \beta, \gamma, \) and \( \delta \) cannot be achieved.

Finally, we again consider \( \rho \)-robust solutions. The results have the same flavor as the corresponding ones in Section 4 and can be proven in the same way.
Theorem 5.6. Let $\mathcal{U} = \mathcal{U}_{\cdot,\cdot}^{\rho,\alpha}$ be the given uncertainty set and let $M, M_\ell, \ell \in [L]$, be positive semidefinite. Then, $x$ is a $\rho$-robust LCP solution if and only if there exist $\alpha, \beta, \gamma, \delta$ such that the system
\[
x^\top M x + x^\top q + \Gamma \alpha + \sum_{\ell \in [L]} \beta_\ell \leq \rho,
\]
with objective function value not larger than
\[
\alpha + \beta_\ell \geq \bar{u}_\ell x^\top M_\ell x, \quad \ell \in [L],
\]
\[
\alpha \geq 0,
\]
\[
\beta_\ell \geq 0, \quad \ell \in [L],
\]
\[
\gamma_i \geq 0, \quad i \in [n],
\]
\[
\delta_\ell \geq 0, \quad i \in [n], \ell \in [L],
\]
\[
x \geq 0,
\]
\[
\bar{M}_i, x + q_i - \gamma_i \Gamma - \sum_{\ell \in [L]} \delta_\ell \geq 0, \quad i \in [n],
\]
\[
\gamma_i + \delta_\ell \geq -\bar{u}_\ell M_\ell^i x, \quad i \in [n], \ell \in [L],
\]
is satisfied. Moreover, this is equivalent to the case that Problem (23) has a solution with objective function value not larger than $\rho$.

5.2. Uncertainty set $\mathcal{U}_{1,\cdot}^{\rho,\alpha}$. As for the case of uncertain $q$, we now also consider $\ell_1$-norm uncertainty in $M$ and we again define
\[
M(u) := M + \sum_{\ell \in [L]} u_\ell M_\ell
\]
with $L \in \mathbb{N}, M_\ell := [m_{ij}^\ell]_{i,j \in [n]} \in \mathbb{R}^{n \times n}$, and
\[
u \in \mathcal{U}_{1,\cdot}^{\rho,\alpha} := \{u \in \mathbb{R}^L : \sum_{\ell \in [L]} u_\ell \leq \delta, 0 \leq u_\ell, \ell \in [L], |\{\ell \in [L] : u_\ell \neq 0\}| \leq \Gamma\}.
\]
The robust counterpart reads
\[
\begin{align*}
\min_{x \geq 0} & \quad x^\top \bar{M} x + x^\top q + \sum_{u \in \mathcal{U}_{1,\cdot}^{\rho,\alpha}} \max_{\ell \in [L]} u_\ell x^\top M_\ell x, \\
\text{s.t.} & \quad \bar{M}_i, x + q_i - \sum_{u \in \mathcal{U}_{1,\cdot}^{\rho,\alpha}} \max_{\ell \in [L]} u_\ell M_\ell^i, x \geq 0, \quad i \in [n].
\end{align*}
\]
Now, we proceed as in the proof of Theorem 3.4 in [32] to obtain for the latter optimization problem an equivalent reformulation.

Theorem 5.7. Let $\mathcal{U}_{1,\cdot}^{\rho,\alpha}$ be the uncertainty set. Furthermore, let $M_\ell, \ell \in [L]$, be positive semidefinite. Then, Problem (25) is equivalent to the problem
\[
\begin{align*}
\min_{x \geq 0, t \geq 0, s \geq 0} & \quad x^\top \bar{M} x + x^\top q + \delta t, \\
\text{s.t.} & \quad x^\top M_\ell x \leq t, \quad \ell \in [L], \\
& \quad \bar{M} x + q \geq \delta s, \\
& \quad s \geq M_\ell x, \quad \ell \in [L].
\end{align*}
\]
Proof. First, we rewrite the inner maximization problem in the objective function of Problem (25). Since
\[
\sum_{u \in \mathcal{U}_{1,\cdot}^{\rho,\alpha}} \max_{\ell \in [L]} u_\ell x^\top M_\ell x = \delta \max_{\ell \in [L]} \{x^\top M_\ell x\}.
\]
holds, the problem can be reformulated as
\[
\min_{x \geq 0, t \geq 0} \quad x^\top \bar{M} x + x^\top q + \delta t \tag{27a}
\]
\[
s.t. \quad x^\top M_\ell x \leq t, \quad \ell \in [L], \tag{27b}
\]
\[
\bar{M}_i, x + q_i - \max_{u \in U^i} \sum_{\ell \in [L]} u_{\ell} M_{i,\ell} x \geq 0, \quad i \in [n]. \tag{27c}
\]

In a second step we eliminate the minimization problem in Constraints (27c). As all \(u_\ell, \ell \in [L]\), are non-negative, we have
\[
\max_{u \in U^i} \sum_{\ell \in [L]} u_{\ell} M_{i,\ell} x = \delta \max_{\ell \in [L]} \{M_{i,\ell} x, 0\}
\]
for all \(i \in [n]\) and, thus, Conditions (27c) can be replaced by
\[
\bar{M}_i, x + q_i \geq \delta \max_{\ell \in [L]} \{M_{i,\ell} x, 0\}, \quad i \in [n].
\]

This last reformulation is equivalent to
\[
\bar{M}_i, x + q_i \geq \delta s_i, \quad i \in [n],
\]
\[
s_i \geq M_{i,\ell} x, \quad i \in [n], \quad \ell \in [L],
\]
\[
s_i \geq 0, \quad i \in [n].
\]

So, the claim of the theorem follows. \(\square\)

We notice that again the equivalent reformulation (26) of the robust counterpart (25) is independent of the parameter \(\Gamma\), as it is the case for the uncertainty set \(U^i_{\delta}\) for uncertainty in \(q\); cf. Section 4.2.

The tractable case is again easy to determine.

**Corollary 5.8.** Let \(\bar{M}, M^\ell, \ell \in [L]\), be positive semidefinite. Then, the \(\Gamma\)-robust counterpart (26) is a convex optimization problem.

**Theorem 5.9.** Assume that Problem (26) is feasible and that \(\bar{M}\) and \(M_\ell, \ell \in [L]\), are positive semidefinite. Then, there exists a solution of Problem (26). If \(\bar{M}\) is, in addition, positive definite, then the solution is unique.

**Proof.** From Constraint (26c) follows \(\bar{M} x + q \geq 0\), which implies \(x^\top \bar{M} x + x^\top q \geq 0\) due to \(x \geq 0\). Thus, the objective function is bounded below on the feasible set of the problem. We can thus apply Theorem 3 of [23] that ensures the existence of a solution. If \(\bar{M}\) is positive definite, the uniqueness of \(x\) again follows from [24]. Using this unique part of the solution, it is easy to see that \(t = \max\{x^\top M^\ell x : \ell \in [L]\}\) and \(s_i = \max\{M_{i,\ell} x : \ell \in [L]\}\) for all \(i \in [n]\) are the unique solutions for the remaining variables. \(\square\)

We close this section with the straightforward result about \(\rho\)-robust solutions. The proof can be done by following the lines of the corresponding proofs in Section 4.

**Theorem 5.10.** Let \(U = U^i_{\delta}\) be the given uncertainty set and assume that all \(M^\ell, \ell \in [L]\), are positive semidefinite. Then, \(x\) is a \(\rho\)-robust LCP solution if and only if there exist \(s, t\) such that the system
\[
x^\top \bar{M} x + x^\top q + \delta t \leq \rho,
\]
\[
x^\top M^\ell x \leq t, \quad \ell \in [L],
\]
\[
\bar{M} x + q \geq \delta s,
\]
\[
s \geq M^\ell x, \quad \ell \in [L],
\]
\[
t, s \geq 0
\]
is satisfied. Moreover, this is equivalent to the case that Problem (26) has a solution with objective function value not larger than $\rho$.

6. Case Study

In this section, we briefly discuss the dependence of the worst-case gap function minima on the considered uncertain data. We try to keep things as simple as possible and, thus, only discuss uncertainty in the LCP vector $q$ of the market equilibrium model discussed in Section 2.1. Here, we consider three goods and production activity levels as well as three technology constraints. The cost vector is $c = (3, 2, 1)^T$ and $A = -I_{3\times 3}$, $b = (-4, -5, -10)^T$ is used. This means that the three productions have capacities 4, 5, and 10. Moreover, we set $B = I_{3\times 3}$, i.e., every production separately yields a certain good. The demand is calibrated by $D = -I_{3\times 3}$ and $d = (6, 9, 3)^T$, i.e., price sensitivity is the same for all demands but consumers have a different maximum willingness to pay. The resulting LCP matrix $M$ is positive semidefinite and considered certain. The uncertain data is the vector $d \in \mathbb{R}^3$ that we parameterize as $d_i + u_i$ with $-\bar{u} \leq u_i \leq \bar{u}$ for all $i \in \{1, 2, 3\}$. Thus, we consider the same box uncertainty for every entry of the vector $d$. In Figure 1 (left) we show the optimal values of the worst-case gap minimization problem for $\Gamma \in \{1, 2, 3\}$ and different values of $\bar{u}$ ranging from 0 (which yields a certain LCP) and 3. We choose 3 here to ensure that also the smallest possible entry in $d$ is still non-negative. First, the dependence of the minimum worst-case gap, i.e., the objective function value of Problem (9), on the value of $\Gamma$ is as expected: Larger values of $\Gamma$, i.e., more data that is allowed to realize in a worst-case way, lead to larger minimum gaps. Moreover, for fixed $\Gamma$, the gaps seem to quadratically depend on the size of the uncertainty boxes.

As it can be seen in Problem (9), feasibility does not depend on the actual value of $\Gamma$ since it only appears in the objective function—which is also the case in classical $\Gamma$-robust optimization; cf., e.g., Theorem 1 in [29]. This can also be seen in Figure 1 (right), where we plotted the $\ell_2$-norm difference of the optimal prices $p^*$ for different box uncertainty sizes $\bar{u}$ and the optimal (equilibrium) prices for the nominal case. It can be seen that all curves are the same for all $\Gamma$. Moreover, the norm of the price differences is linearly increasing in the box uncertainty sizes. This is to be expected since the relation between the variables $x$ and the box uncertainty size $\bar{u}$ is linear.

![Figure 1. Left: Worst-case gap function minimum for box uncertainties in q, for $\Gamma \in \{1, 2, 3\}$, and for varying uncertainty set parameter $\bar{u} \in [0, 3]$. Right: $\ell_2$-norm of the nominal price vector and the robust price vector for box uncertainties in q, for $\Gamma \in \{1, 2, 3\}$, and for varying uncertainty set parameter $\bar{u} \in [0, 3]$.](image)
The same observations can be made for the case of uncertain \( q \) and \( \ell_1 \)-norm uncertainty sets as discussed in Section 4.2. In Figure 2 (left) we again plot the worst-case gap minima, i.e., the optimal objective function values of Problem (17). For the \( \ell_1 \)-norm uncertainty sets we have shown that the robust counterparts are independent of \( \Gamma \). Thus, we have only one curve that shows the optimal gaps for varying values of \( \delta \) (from 0 to 3). The quadratic nature is the same as for the other box uncertainties and also the \( \ell_2 \)-norm of the difference of the nominal and robust price vectors is again linear in \( \delta \); see Figure 2 (right).

7. Conclusion

In this paper, we considered \( \Gamma \)-robustifications of uncertain linear complementarity problems. After defining the problem class we studied both the case of uncertainty in the LCP vector \( q \) and in the LCP matrix \( M \). Moreover, for both cases we considered box as well as \( \ell_1 \)-norm uncertainty sets and discussed that one cannot expect the existence of robust equilibria in a pure sense. Thus, we investigated the global minimizers of the worst-case gap function. For both types of uncertainty sets, we derived conditions (typically monotonicity of the original LCP) for the tractability of the robust counterpart. In the case of uncertain \( q \), this then allows to consider the tractable robust counterpart again as an LCP. For this LCP, desired matrix properties like the membership in the classes \( S \) and \( P \) are shown to be violated in every case. Nevertheless, we derived specific conditions for the feasibility of the robust counterpart as well as for the existence and, if possible, uniqueness of its solution. In the case of uncertain \( M \), the tractable counterparts are convex QCQPs, for which we also study the existence and uniqueness of solutions. Finally, we also characterized \( \rho \)-robust solutions, i.e., solutions to certain relaxations of the uncertain LCP.

Despite the theoretical results obtained in this paper, there still are some open questions in the context of \( \Gamma \)-robust LCPs. Let us briefly discuss three of them. First, other uncertainty sets instead of box and \( \ell_1 \)-norm uncertainties can be discussed. In our opinion, the canonical next step is the study of ellipsoidal, i.e., \( \ell_2 \)-norm uncertainty sets. However, also general polyhedral uncertainty sets might be of
interest. Second, the question remains open whether there exist a-priori conditions on $M$, $q$, and the uncertainty set that ensure the existence of $\rho$-robust solutions for a given $\rho > 0$. Third and finally, the deeper study of relevant applications is of interest. We sketched some possible applications in this paper but many other, e.g., traffic equilibrium problems, might give interesting application-specific insights in this rather new field of robust optimization.

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