Stability Analysis for a Class of Sparse Optimization Problems

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ABSTRACT
The sparse optimization problems arise in many areas of science and engineering, such as compressed sensing, image processing, statistical and machine learning. The \(\ell_0\)-minimization problem is one of such optimization problems, which is typically used to deal with signal recovery. The \(\ell_1\)-minimization method is one of the plausible approaches for solving the \(\ell_0\)-minimization problems, and thus the stability of such a numerical method is vital for signal recovery. In this paper, we establish a stability result for the \(\ell_1\)-minimization problems associated with a general class of \(\ell_0\)-minimization problems. To this goal, we introduce the concept of restricted weak range space property (RSP) of a transposed sensing matrix, which is a generalized version of the weak RSP of the transposed sensing matrix introduced in [Zhao et al., Math. Oper. Res., 44(2019), 175-193]. The stability result established in this paper includes several existing ones as special cases.

KEYWORDS
Sparsity optimization; \(\ell_1\)-minimization; stability; optimality condition; Hoffman theorem; restricted weak range space property.

1. Introduction

The sparsity is a useful assumption under which the sparse optimization models arise frequently in many areas in science and engineering. Let \(A \in \mathbb{R}^{m \times n}(m \ll n)\), \(B \in \mathbb{R}^{l \times n}(l < n)\) and \(U \in \mathbb{R}^{m \times h}(m \ll h)\) be three given full-row-rank matrices. Let \(y \in \mathbb{R}^m\) and \(b \in \mathbb{R}^l\) be given vectors and \(\varepsilon\) be a positive number. Consider the following sparse optimization model:

\[
\min_{x \in \mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad \begin{cases} a_1 \|y - Ax\|_2 + a_2 \|U^T(Ax - y)\|_\infty + a_3 \|U^T(Ax - y)\|_1 \leq \varepsilon \\
Bx \leq b, \end{cases}
\]

where \(\|x\|_0\) is called the ‘\(\ell_0\)-norm’ which counts the number of nonzero components of \(x\), and \(a_1, a_2\) and \(a_3\) are given nonnegative parameters satisfying \(\sum_{i=1}^3 a_i = 1\). Many problems in signal and image processing (see, e.g., [6, 13, 17]) and statistical regressions [23] can be formulated as the form (1) or its special cases. In problem (1), the constraint \(Bx \leq b\) is motivated by some practical applications. For instance, many

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signal recovery models might need to include certain constraints reflecting special structures of the target signal. For simplicity, we define

\[ \phi(x) = U^T(Ax - y), \]

and write the problem (1) as

\[
\min_{x \in \mathbb{R}^n} \{ \|x\|_0 : a_1 \|y - Ax\|_2 + a_2 \|\phi(x)\|_\infty + a_3 \|\phi(x)\|_1 \leq \epsilon, Bx \leq b \}.
\]

The following \( \ell_0 \)-minimization models are clearly the special cases of (1):

(C1) \( \min_x \{ \|x\|_0 : y = Ax \} \); (C2) \( \min_x \{ \|x\|_0 : \|y - Ax\|_2 \leq \epsilon \} \);

(C3) \( \min_x \{ \|x\|_0 : \|U^T(Ax - y)\|_1 \leq \epsilon \} \); (C4) \( \min_x \{ \|x\|_0 : \|U^T(Ax - y)\|_\infty \leq \epsilon \} \).

The problem (C1) is often called the standard \( \ell_0 \)-minimization problem [8, 17, 28]. Two structured sparsity models, called the nonnegative sparsity model [7, 8, 17, 28] and the monotonic sparsity model (isotonic regression) [23, 24], are also the special cases of the model (1).

It is well known that \( \ell_1 \)-minimization is a useful method to solve the \( \ell_0 \)-minimization problem. By replacing the \( \ell_0 \)-norm with the \( \ell_1 \)-norm in problem (1), we immediately obtain the \( \ell_1 \)-minimization problem

\[
\min_x \{ \|x\|_1 : a_1 \|y - Ax\|_2 + a_2 \|\phi(x)\|_\infty + a_3 \|\phi(x)\|_1 \leq \epsilon, Bx \leq b \}.
\]

(2)

Similar to its \( \ell_0 \) counterpart, the problem (2) includes the following special cases:

(D1) \( \min_x \{ \|x\|_1 : y = Ax \} \);

(D2) \( \min_x \{ \|x\|_1 : \|y - Ax\|_2 \leq \epsilon \} \);

(D3) \( \min_x \{ \|x\|_1 : \|U^T(Ax - y)\|_1 \leq \epsilon \} \); (D4) \( \min_x \{ \|x\|_1 : \|U^T(Ax - y)\|_\infty \leq \epsilon \} \).

The problem (D2) is often called quadratically constrained basis pursuit [10, 17, 28], and it reduces to (D1) if \( \epsilon = 0 \), which is called standard \( \ell_1 \)-minimization or the basis pursuit [8, 12, 17, 19, 26]. The problem (D4) is the type of Dantzig Selectors [9, 17].

From both numerical and theoretical viewpoints, it is important to know how close the solutions of \( \ell_0 \)- and \( \ell_1 \)-minimization problems are. To address this question, one needs to study the stability of \( \ell_1 \)-minimization methods. The stability of a sparse optimization method can be described as follows: For any \( x \in \mathbb{R}^n \) in the feasible set of a sparse optimization problem, the solution \( x^\# \) generated by the method satisfies the following bound:

\[
\|x - x^\#\|_2 \leq C_1 \sigma_k(x)_1 + C_2 \epsilon
\]

where \( C_1 \) and \( C_2 \) are constants, and \( \sigma_k(x)_1 \) is called the error of the best \( k \)-term approximation of the vector \( x \) (see, e.g., [12, 17]):

\[
\sigma_k(x)_1 = \min_z \{ \|x - z\|_1 : \|z\|_0 \leq k \}.
\]

In this paper, we establish a stability result for the \( \ell_1 \)-minimization method (2). The stability of (D1) and (D2) has been investigated by Donoho, Candès, Tao, Romberg and others [3, 6–8, 12–14, 16, 25] under various assumptions such as the so-called
restricted isometry property (RIP) of order $k$, mutual coherence, stable null space property (NSP) of order $k$ or robust NSP of order $k$. The RIP of order $k$ was introduced by Candès and Tao [8] to study the stability of $\ell_1$-minimization. The singular-value-property-based stability analysis for (D1), (D2) and the Dantzig Selector have also been performed by Tang and Nehorai in [22].

A new and unified stability analysis for $\ell_1$-minimization methods has been developed by Zhao, Jiang and Luo [29] under the assumption of weak RSP of order $k$, which has been proven as a necessary and sufficient condition for the standard $\ell_1$-minimization to be stable. The main difference between the weak-RSP-based-analysis and existing ones lies in the constants $C_1$ and $C_2$ in (3). Specifically, the constants $C_1$ and $C_2$ in (3) are determined by the RIP or NSP constant in existing analysis [3, 8, 17]. However, in [28, 29], these constants are determined by the so-called Robinson’s constant. Motivated by the new analysis tool introduced in [29], we develop the stability result for the model (2) in this paper under the assumption of restricted weak range space property (RSP) of order $k$ (which will be introduced in next section). Our result extends the stability theorem for $\ell_1$-minimization established by Zhao et al. [28–30].

This paper is organized as follows. In Section 2, we introduce the concept of restricted weak RSP of order $k$. An approximation of the solution set of (2) will be discussed in Section 3. Then, in Section 4, we show the main stability result of this paper. Finally, some special cases are discussed in Section 5.

Notation

The field of real numbers is denoted by $\mathbb{R}$ and the $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$. Let $\mathbb{R}_+^n$ and $\mathbb{R}_-^n$ be the sets of nonnegative and nonpositive vectors, respectively. Unless otherwise stated, the identity matrix of suitable size is denoted by $I$. Given a vector $u \in \mathbb{R}^n$, $|u|$, $(u)^+$ and $(u)^-$ denote the vectors with components $|u|_j = |u_j|$, $[(u)^+]_j = \max \{u_j, 0\}$ and $[(u)^-)_j = \min \{u_j, 0\}$, $j = 1, ..., n$, respectively. The cardinality of the set $\mathcal{S}$ is denoted by $|\mathcal{S}|$ and the complementary set of $\mathcal{S} \subseteq \{1, ..., n\}$ is denoted by $\bar{\mathcal{S}}$, i.e., $\bar{\mathcal{S}} = \{1, ..., n\} \setminus \mathcal{S}$. For a given vector $x \in \mathbb{R}^n$, $x_\mathcal{S}$ denotes the vector supported on $\mathcal{S}$, $a_{i,j}$ denotes the entry of the matrix $A$ in row $i$ and column $j$. For the set $\mathcal{S} \subseteq \{1, ..., n\}$, $A_\mathcal{S}$ denotes the submatrix of $A \in \mathbb{R}^{m \times n}$ obtained by deleting the columns indexed by $\mathcal{S}$. For a matrix $A = (a_{i,j})$, $|A|$ represents the absolute version of $A$, i.e., $|A| = (|a_{i,j}|)$. $\mathcal{R}(A^T) = \{A^T y : y \in \mathbb{R}^n\}$ is the range space of $A^T$. $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, where $p \geq 1$, is a norm, called the $\ell_p$-norm of $x$. $\|x\|_\infty = \max_{i=1}^{n} |x_i|$ is called the $\ell_\infty$-norm of $x$. For $1 \leq p, q \leq \infty$, $\|A\|_{p \rightarrow q} = \sup_{\|x\|_p \leq 1} \|Ax\|_q$ is the matrix norm induced by $\ell_p$- and $\ell_q$-norms.

2. Restricted weak range space property

The RSP of order $k$ of a transposed matrix was first introduced in [26, 27] to develop a necessary and sufficient condition for the uniform recovery of sparse signals via $\ell_1$-minimization. Zhao et al. [29] generalised the RSP of order $k$ to the following weak RSP of order $k$ to develop a stability theory for convex optimization algorithms:

**Definition 2.1** (weak RSP of order $k$). Given a matrix $A \in \mathbb{R}^{m \times n}$, $A^T$ is said to satisfy the weak RSP order $k$ if for any two disjoint sets $J_1, J_2 \subseteq \{1, ..., n\}$ satisfying...
\(|J_1| + |J_2| \leq k\), there exists a vector \(\eta \in \mathcal{R}(A^T)\) such that

\[
\begin{cases}
\eta_i = 1 & \text{if } i \in J_1, \\
\eta_i = -1 & \text{if } i \in J_2, \\
|\eta_i| \leq 1 & \text{if } i \notin J_1 \cup J_2.
\end{cases}
\]

In [28, 29], it was shown that the weak RSP of order \(k\) is a sufficient condition for the stability of many convex optimization methods, and it is also a necessary stability condition for many optimization methods.

Different from the problems (D1)-(D4), the problem (2) is more general than these models. To investigate the stability of the problem (2), we need to extend the notion of weak RSP of order \(k\) to the so-called restricted weak RSP of order \(k\), which is defined as follows:

**Definition 2.2** (Restricted weak RSP of order \(k\)). Given matrices \(A \in \mathbb{R}^{m \times n}\) and \(B \in \mathbb{R}^{l \times n}\), the pair \((A^T, B^T)\) is said to satisfy the restricted weak RSP of order \(k\) if for any two disjoint sets \(J_1, J_2 \subseteq \{1, \ldots, n\}\) satisfying \(|J_1| + |J_2| \leq k\), there exists a vector \(\eta \in \mathcal{R}(A^T, B^T)\) such that \(\eta = (A^T, B^T)\left(\nu \ h\right)\) where \(\nu \in \mathbb{R}^m\), \(h \in \mathbb{R}^l\) and

\[
\begin{cases}
\eta_i = 1 & \text{if } i \in J_1, \\
\eta_i = -1 & \text{if } i \in J_2, \\
|\eta_i| \leq 1 & \text{if } i \notin J_1 \cup J_2.
\end{cases}
\]

It is worth mentioning that a generalized version of the RSP of order \(k\) is also used in [31] to study the exact sign recovery in 1-bit compressive sensing.

### 3. Approximation of (2) and its solution set

By introducing the slack variables \(r, s, \xi, v\), the problem (2) can be rewritten as

\[
\begin{align*}
\min_{(x,r,s,\xi,v)} & \quad \|x\|_1 \\
\text{s.t.} & \quad a_1 s + a_2 \xi + a_3 (e^h)^T v \leq \varepsilon, \\
& \quad r \in s\mathcal{B}, \quad r = y - Ax, \quad (s, \xi, v) \geq 0, \\
& \quad \|\phi(x)\|_{\infty} \leq \xi, \quad |\phi(x)| \leq v, \quad Bx \leq b,
\end{align*}
\]

where \(e^h\) is the vector of ones in \(\mathbb{R}^h\) and \(\mathcal{B}\) is the unit \(\ell_2\)-ball defined as \(\mathcal{B} = \{z \in \mathbb{R}^m : \|z\|_2 \leq 1\}\). The unit ball \(\mathcal{B}\) can be also described as

\[
\mathcal{B} = \bigcap_{\|a\|_2=1} \{z \in \mathbb{R}^m : a^T z \leq 1\}.
\]

Denote the set \(\mathcal{E}\) by

\[
\mathcal{E} = \{(x, s, \xi, v) : a_1 s + a_2 \xi + a_3 (e^h)^T v \leq \varepsilon, \quad Bx \leq b, \quad \|\phi(x)\|_{\infty} \leq \xi, \quad |\phi(x)| \leq v, \quad (s, \xi, v) \geq 0\}.
\]
and hence the solution set of (4) can be represented as

$$\Omega^* = \{(x, r, s, \xi, v) : \|x\|_1 \leq \theta^*, \ r \in s\mathcal{B}, \ r = y - Ax, \ (x, s, \xi, v) \in E\},$$

where $\theta^*$ is the optimal value of (4). By replacing $\mathcal{B}$ in (6) with a polytope $P \supseteq \mathcal{B}$, we can get the relaxation of $\Omega^*$, denoted by $\Omega_P$, i.e.,

$$\Omega_P = \{(x, r, s, \xi, v) : \|x\|_1 \leq \theta^*, \ r \in sP, \ r = y - Ax, \ (x, s, \xi, v) \in E\}. \quad (7)$$

The polytope $\Omega_P$ can approximate $\Omega^*$ to any level of accuracy provided that $P$ is chosen suitably. Recall the Hausdorff metric of two sets $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathbb{R}^m$:

$$\delta^H(\mathcal{M}_1, \mathcal{M}_2) = \max\left\{\sup_{x \in \mathcal{M}_1} \inf_{z \in \mathcal{M}_2} \|x - z\|_2, \sup_{z \in \mathcal{M}_2} \inf_{x \in \mathcal{M}_1} \|x - z\|_2\right\}.$$ 

Following the analysis in [28, 29] (see Lemmas 5.1, 5.2 and 5.3 in [29]), we can obtain the following lemma:

**Lemma 3.1.** Let $\varepsilon$ be the given number in problem (2). Then for any $\varepsilon' \leq \varepsilon$, there exists a polytope approximation $P$ of $\mathcal{B}$ satisfying $P \supseteq \mathcal{B}$ and

$$\delta^H(\Omega^*, \Omega_P) \leq \varepsilon'.$$ 

In the remainder of this paper, we fix $\varepsilon' \in (0, \varepsilon]$ and choose the polytope $P$ such that $\Omega_P$ and $\Omega^*$ satisfy (8). The polytope $P$ can be represented as the intersection of a finite number of half spaces:

$$P = \{z \in \mathbb{R}^m : (a^i)^T z \leq 1, 1 \leq i \leq L\},$$

where $a^i$, $1 \leq i \leq L$ are some unit vectors (i.e., $\|a^i\|_2 = 1$), and $L$ is an integer number. By adding the $2m$ half spaces

$$(\beta^j)^T z \leq 1, \ - (\beta^j)^T z \leq 1, \ j = 1, ..., m$$

to $P$, where $\beta^j$ is the $j$th column of the $m \times m$ identity matrix, we obtain the following polytope:

$$P_0 = P \cap \left\{z \in \mathbb{R}^m : (\beta^j)^T z \leq 1, - (\beta^j)^T z \leq 1, j = 1, ..., m\right\}$$

$$= \left\{z \in \mathbb{R}^m : (a^i)^T z \leq 1, 1 \leq i \leq L; (\beta^j)^T z \leq 1, - (\beta^j)^T z \leq 1, j = 1, ..., m\right\}. \quad (9)$$

We define $T$ as the collection of the vectors $a^i$ and $\pm \beta^j$ in $P_0$, that is,

$$T := \{a^i : 1 \leq i \leq L\} \cup \{\pm \beta^j : 1 \leq j \leq m\}.$$ 

Clearly, $P_0$ still satisfies (8) in Lemma 3.1, i.e.,

$$\delta^H(\Omega^*, \Omega_{P_0}) \leq \varepsilon'.$$
In the remainder of the chapter, we use the above defined polytope $P_0$. Let $N = |T|$, and let $M_{P_0}$ be the matrix with column vectors in $T$. Thus $P_0$ can be written as

$$P_0 = \{ z \in R^n : (M_{P_0})^T z \leq e^N \},$$

where $e^N$ is the vector of ones in $R^N$.

By replacing $B$ by $P_0$, we obtain the following approximation of the optimal value $\theta^*$ of (2):

$$\theta^*_{P_0} : = \min_{(x,r,s,\xi,v)} \{ \|x\|_1 : r \in sP_0, \; r = y - Ax, \; (x, s, \xi, v) \in E \}$$

The associated approximation problem of (2) can be written as

$$\min_{(x,s,\xi,v)} \{ \|x\|_1 : (MP_0)^T (y - Ax) \leq se^N, \; (x, s, \xi, v) \in E \}. \tag{10}$$

The solution set of (10) is

$$\Omega^*_{P_0} = \{ x \in R^n : \|x\|_1 \leq \theta^*_{P_0}, \; r \in sP_0, \; r = y - Ax, \; (x, s, \xi, v) \in E \}. \tag{11}$$

By the definition of $P_0$, we also have $\Omega^* \subseteq \Omega^*_{P_0}$. In the next section, we prove the main result for the problem (2).

4. Main result

Introducing a variable $t$ yields the following equivalent form of (10):

$$\min_{(x,t,s,\xi,v)} e^T t \quad \text{s.t.} \quad a_1 s + a_2 \xi + a_3 (e^h)^T v \leq \epsilon, \; Bx \leq b, \; |x| \leq t, \; (MP_0)^T (y - Ax) \leq se^N, \; (t, s, \xi, v) \geq 0, \; \|\phi(x)\|_\infty \leq \xi, \; |\phi(x)| \leq v. \tag{12}$$

The solution set of (12) is given as (11). Note that the above optimization problem is equivalent to a linear programming problem. In fact, the constraint $||\phi(x)||_\infty \leq \xi$ can be rewritten as $|\phi(x)| \leq \xi e^h$, where $e^h$ is the vector of ones in $R^h$. Thus the model (12) can be rewritten explicitly as the linear programming problem

$$\min_{(x,t,s,\xi,v)} e^T t \quad \text{s.t.} \quad x + t \geq 0, \; -x + t \geq 0, \; -a_1 s - a_2 \xi - a_3 (e^h)^T v \geq -\epsilon, \; M_{P_0}^T Ax + e^N s \geq M_{P_0}^T y, \; U^T Ax + \xi e^h \geq U^T y, \; -U^T Ax + \xi e^h \geq -U^T y, \; U^T Ax + v \geq U^T y, \; -U^T Ax + v \geq -U^T y, \; -Bx \geq -b, \; (t, s, \xi, v) \geq 0. \tag{13}$$
The dual problem of (13) is given as follows:

\[
\begin{align*}
\max \quad & w & -\varepsilon w_3 + y^T M_{3} w_4 + y^T U (w_5 - w_6 + w_7 - w_8) - b^T w_9 \\
\text{s.t.} \quad & w_1 - w_2 + A^T M_{3} w_4 + A^T U (w_5 - w_6 + w_7 - w_8) - B^T w_9 = 0, \\
& w_1 + w_2 \leq e, \\
& -a_1 w_3 + (e^N)^T w_4 \leq 0, \\
& -a_2 w_3 + (e^h)^T (w_5 + w_6) \leq 0, \\
& -a_3 w e^h + w_7 + w_8 \leq 0, \\
& w_1, w_2 \in R^N_+, w_3 \in R_+, w_4 \in R^N_+, w_5 - 8 \in R^h_+, w_9 \in R^l_+.
\end{align*}
\]

The optimality condition yields the following lemma:

**Lemma 4.1.** Denote by \( u = (x, t, s, \xi, v, w) \). Then \( x^* \) is an optimal solution of (10) if and only if there exists a vector \( u^* = (x^*, t^*, s^*, \xi^*, v^*, w^*) \in \Theta \), where \( \Theta \) is the set given as

\[
\Theta = \left\{ u : -x - t \leq 0, \quad x - t \leq 0, \quad a_1 s + a_2 \xi + a_3 (e^h)^T v \leq \varepsilon, \quad-M_{3}^T A x - e^N s \leq -M_{3}^T y, \quad B x \leq b, \quad-U^T A x - \xi e^h \leq -U^T y, \quad U^T A x - \xi e^h \leq U^T y, \quad-w_1 - w_2 + A^T M_{3} w_4 + A^T U (w_5 - w_6 + w_7 - w_8) - B^T w_9 = 0, \quad w_1 + w_2 \leq e, \quad-a_1 w_3 + (e^N)^T w_4 \leq 0, \quad(t, s, \xi, v, w) \geq 0, \quad-a_2 w_3 + (e^h)^T (w_5 + w_6) \leq 0, \quad-a_3 w e^h + w_7 + w_8 \leq 0, \quad e^T t = -\varepsilon w_3 + y^T M_{3} w_4 + y^T U (w_5 - w_6 + w/7 - w_8) - b^T w_9 \right\}.
\]

Clearly, \( |x^*| = t^* \) holds for every \( u^* \in \Theta \). The set \( \Theta \) can be written as the form

\[
\Theta = \left\{ u : M_{1}^T u \leq p', \quad M_{2}^T u = q' \right\}, \tag{15}
\]

where the vectors \( p' = 0 \) and

\[
p' = \begin{bmatrix} 0 & 0 & \varepsilon & -M_{3}^T y & b & -U^T y & U^T y & -U^T y & U^T y & -e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.
\]

The matrices \( M_{1}' \) and \( M_{2}' \) in (15) are given as follows:

\[
M_{1}' = \begin{bmatrix} D^1 & 0 \\ 0 & D^2 \\ 0 & 0 & -\tilde{I} \end{bmatrix}, \quad M_{2}' = \begin{bmatrix} M_{*} & M_{**} \end{bmatrix}, \tag{16}
\]

where the matrices \( M_{*}, M_{**}, D^1, D^2 \) and \( \tilde{I} \) are given as follows:

\[
M_{*} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I & -I & 0 \\ 0 & e^T & 0 & 0 & 0 & 0 & 0 & \epsilon \end{bmatrix},
\]

7
\[ M_{ss} = \begin{bmatrix} A^T M_{P_0} & A^T U & -A^T U & A^T U & -A^T U & -B^T \\ -y^T M_{P_0} & -y^T U & y^T U & -y^T U & y^T U & b^T \end{bmatrix}, \]

\[ D^1 = \begin{bmatrix} -I & -I & 0 & 0 & 0 \\ I & -I & 0 & 0 & 0 \\ -M_{P_0}^T A & 0 & -e^N & 0 & 0 \\ B & 0 & 0 & 0 & 0 \\ -U^T A & 0 & 0 & -e^h & 0 \\ U^T A & 0 & 0 & 0 & -I^h \\ -U^T A & 0 & 0 & 0 & -I^h \end{bmatrix}, \]

\[ D^3 = \begin{bmatrix} 0 & -I & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -I^h \end{bmatrix}, \]

\[ D^2 = \begin{bmatrix} I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_1 (e^N)^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 (e^h)^T & (e^h)^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_3 e^h & 0 & 0 & 0 & I^h & I^h & 0 \end{bmatrix}, \]

\[ \tilde{I} = I^{2n+1+N+4h+l}. \]

In the above matrices, 0’s are zero matrices with suitable sizes and \(I, I^h\) and \(\tilde{I}\) are the \(n \times n\), \(h \times h\) and \((2n+1+N+4h+l) \times (2n+1+N+4h+l)\) identity matrices, respectively.

To prove the main stability result, we also need the next two Lemmas.

**Lemma 4.2** (Hoffman [18, 21]). Let \(M_1 \in \mathbb{R}^{m \times n}\) and \(M_2 \in \mathbb{R}^{l \times n}\) be two given matrices and the set \(Q\) be given as

\[ Q = \{x \in \mathbb{R}^n : M_1 x \leq p, M_2 x = q\}. \]

For any vector \(x \in \mathbb{R}^n\), there exists a vector \(x^* \in Q\) satisfying

\[ \|x - x^*\|_2 \leq \sigma(M_1, M_2) \left\| \begin{bmatrix} (M_1 x - p)^+ \\ M_2 x - q \end{bmatrix} \right\|_1, \]

where \(\sigma(M_1, M_2)\) is a constant determined by \(M_1\) and \(M_2\).

The constant \(\sigma(M_1, M_2)\) is also called the Robinson constant. We also use the following lemma in the proof of the main result in this section.

**Lemma 4.3** ([28, 30]). Let \(\pi_S(x)\) be the projection of \(x\) into the convex set \(S\), i.e., \(\pi_S(x) = \arg \min_{z \in S} \|x - z\|_2\). Let the three convex compact sets \(T_1, T_2\) and \(T_3\) satisfy that \(T_1 \subseteq T_2\) and \(T_3 \subseteq T_2\). Then for any \(x \in \mathbb{R}^n\) and any \(z \in T_3\) the following holds:

\[ \|x - \pi_{T_1}(x)\|_2 \leq \delta^H(T_1, T_2) + 2 \|x - z\|_2. \]

We also define two types of constants. Let

\[ C = [A^T, B^T]^T = \begin{bmatrix} A \\ B \end{bmatrix}, \]
be a matrix with full row rank. Given three positive numbers \( c, d, \hat{d} \in [1, \infty] \), we define the constants \( \Upsilon(d, \hat{d}) \) and \( \vartheta(c) \) as follows:

\[
\Upsilon(d, \hat{d}) = \max_{d \leq \{1, \ldots, h\} \setminus \{j\} = m} \left\| \left\| U_{\hat{d}}^{-1} \right\|_{\hat{d} \to d} \left\| (CC^T)^{-1} C \right\|_{\infty \to \hat{d}} \right\|
\]

\[
\vartheta(c) = \left\| (CC^T)^{-1} C \right\|_{\infty \to c}.
\]

We will use the above constants together with the specific constants \( \Upsilon(1, 1), \Upsilon(\infty, \infty) \) and \( \vartheta(1) \) in the stability analysis of (2). The main result is given as follows.

**Theorem 4.4.** Let the problem data \((U, A, B, \varepsilon, a_1, a_2, a_3, b, y)\) of (2) be given, and the matrix \( C \in \mathbb{R}^{(m+1) \times n} \) be given in (18) with full row rank. Let \( P_0 \) be the polytope given in (9) satisfying (8). If \((A^T, B^T)\) satisfies the restricted weak RSP of order \( k \), then for any \( x \in \mathbb{R}^n \), there is an optimal solution \( x^* \) of (2) satisfying the bound

\[
\|x - x^*\|_2 \leq \varepsilon' + 2\sigma' \left\{ 2\sigma_k(x)_{1} + \varepsilon \hat{\Upsilon} + \| (Bx - b)_{+} \|_1 + + \| Bx - b \|_{c'} \vartheta(c) + \| \varphi(x) \|_{d'} \Upsilon(d, \hat{d}) + (a_1 \| y - Ax \|_2 + a_2 \| \phi(x) \|_\infty + (a_3 \| \varphi(x) \|_1 - \varepsilon)) \right\}
\]

where \( \sigma' \) is the Robinson constant determined by \((M_1', M_2')\) in (16), \( \Upsilon(d, \hat{d}) \) and \( \vartheta(c) \) are the constants given in (19a) and (19b), and \( \hat{\Upsilon} = \max\{\Upsilon(1, 1), \Upsilon(\infty, \infty), \vartheta(1)\} \).

\( \hat{d}, d, c, d', c' \in [1, +\infty] \) are five given positive numbers (allowing to be \( \infty \)) satisfying

\[
\frac{1}{c} + \frac{1}{c'} = 1 \quad \text{and} \quad \frac{1}{d} + \frac{1}{d'} = 1.
\]

In particular, if \( x \) is a feasible solution of (2), then there is an optimal solution \( x^* \) of (2) such that

\[
\|x - x^*\|_2 \leq \varepsilon' + 2\sigma' \left\{ \varepsilon \hat{\Upsilon} + 2\sigma_k(x)_{1} + \| \varphi(x) \|_{d'} \Upsilon(d, \hat{d}) + \| Bx - b \|_{c'} \vartheta(c) \right\}.
\]

**Proof.** Let \( x \) be any given vector in \( \mathbb{R}^n \) and \( P_0 \) be the fixed polytope given in (9) satisfying (8) in Lemma 3.1. We let \( (t, s, \xi, v) \) satisfy that

\[
t = |x|, \quad s = \| (M_{P_0}^T(y - Ax))_{\infty} \|_\infty, \quad \xi = \| U^T(y - Ax) \|_\infty, \quad v = |U^T(y - Ax)|.
\]

With such a choice of \( (t, s, \xi, v) \), we have

\[
\begin{align*}
(x - t)^+ &= 0, \quad (x - t)^- = 0, \quad (M_{P_0}^T(y - Ax) - e^N s)^+ = 0, \\
(U^T(y - Ax) - \xi e^h)^+ &= 0, \quad (U^T(y - Ax) - \xi e^h)^- = 0, \\
(U^T(y - Ax) - v)^+ &= 0, \quad (U^T(y - Ax) - v)^- = 0.
\end{align*}
\]

Let \( J \) be the support set of \( k \) largest absolute entries of \( x \), and \( J_1 \) and \( J_2 \) be the sets
such that

\[ J_1 = \{ i : x_i > 0, \ i \in J \}, \ J_2 = \{ i : x_i < 0, \ i \in J \}. \]

Clearly, \(|J_1 \cup J_2| = |J| = |J_1| + |J_2| \leq k\). Let \( J_3 \) be the complementary set of \( J \). Clearly, \( J_1, J_2 \) and \( J_3 \) are disjoint. Under the assumption of restricted weak RSP of order \( k \), there exists a vector \( \eta \in R (A^T, B^T) \) such that \( \eta = A^T \nu^* + B^T h^* \) for some \( \nu^* \in R^m \) and \( h^* \in R^l \) satisfying

\[ \eta_i = 1 \ \text{for} \ i \in J_1; \ \eta_i = -1 \ \text{for} \ i \in J_2; \ |\eta_i| \leq 1 \ \text{for} \ i \in J_3. \]  

(25)

Now we construct a feasible solution \( w = (w_1, \ldots, w_9) \) to the dual problem (14).

**Constructing \((w_1, w_2)\).** Set \( w_1 \) and \( w_2 \) as follows:

\[
\begin{align*}
(w_1)_i &= 0, \ (w_2)_i = 1, \quad i \in J_1; \\
(w_1)_i &= 1, \ (w_2)_i = 0, \quad i \in J_2; \\
(w_1)_i &= \frac{1-\eta_i}{2}, \ (w_2)_i = \frac{1+\eta_i}{2}, \quad i \in J_3.
\end{align*}
\]

Such \( w_1 \) and \( w_2 \) satisfy that

\[ w_1 + w_2 \leq e, \ w_2 - w_1 = \eta, \ w_1, w_2 \geq 0. \]  

(26)

**Constructing \((w_5 - w_8)\).** Note that \( U \) is a matrix with full row rank. There must exist an invertible \( m \times m \) matrix of \( U \), denoted by \( U_{\bar{\Omega}} \), where \( \bar{\Omega} \subseteq \{1, \ldots, h\} \) with \(|\bar{\Omega}| = m\). Denote the complementary set of \( \bar{\Omega} \) by \( \bar{\Omega} = \{1, \ldots, h\} \setminus \bar{\Omega} \). Then we construct a vector \( g \in R^h \) satisfying \( g_{\bar{\Omega}} = U_{\bar{\Omega}}^{-1} \nu^* \) and \( g_{\bar{\Omega}} = 0 \), which imply that

\[ U g = \nu^*. \]  

(27)

Let \( g^+ \ (g^-) \) be the vector obtained by keeping the positive (negative) components of \( g \) and setting the remaining components to 0. By using the vector \( g \), \( w_5 - w_8 \) can be constructed as follows:

\[ w_5 = a_2 g^+, \ w_6 = -a_2 g^-, \ w_7 = a_3 g^+, \ w_8 = -a_3 g^-, \]  

(28)

which implies that

\[ w_5 - w_6 + w_7 - w_8 = (a_2 + a_3) g, \ w_5, w_6, w_7, w_8 \geq 0. \]  

(29)

**Constructing \(w_4\).** Without loss of generality, we suppose that the first \( m \) columns in \( \bar{M}_{\bar{P}_h} \) are \( \beta_j, \ j = 1, \ldots, m, \) and \( -\beta_j, \ j = 1, \ldots, m, \) are the second \( m \) columns in \( \bar{M}_{\bar{P}_h} \). The components of \( w_4 \) can be assigned as follows:

\[
\begin{align*}
(w_4)_j &= a_1 \nu_j^*, \quad \text{if} \ \nu_j^* > 0, \ j = 1, \ldots, m; \\
(w_4)_{j+m} &= -a_1 \nu_j^*, \quad \text{if} \ \nu_j^* < 0, \ j = 1, \ldots, m; \\
0, \quad \text{otherwise}.
\end{align*}
\]
From this choice of \( w_4 \), we can see that
\[
M_P w_4 = a_1 \nu^*, \quad \| w_4 \|_1 = a_1 \| \nu^* \|_1 \quad \text{and} \quad w_4 \geq 0. \quad (30)
\]

Constructing \( w_3 \). Let \( w_3 = \max \{ \| \nu^* \|_1, \| g \|_1, \| g \|_\infty \} \). Such a choice of \( w_3 \) together with the choice of \( w_4 - w_8 \) implies that
\[
\begin{cases}
-a_1 w_3 + (e^N T w_4)^+ & \leq (a_1 \| \nu^* \|_1 + (e^N T w_4)^+) = 0, \\
-a_2 w_3 + e^T (w_5 + w_6)^+ & \leq (a_2 \| g \|_1 + a_2 \| g \|_1) = 0, \\
-a_3 e w_3 + w_7 + w_8 & \leq (a_3 e \| g \|_\infty + a_3 |g|)^+ = 0.
\end{cases} \quad (31)
\]

Constructing \( w_9 \). Let \( w_9 = -h^* \). Clearly, \( w_9 \geq 0 \) due to \( h^* \leq 0 \).

With the above choice of \( w \), we deduce from (26), (29), (30) and (31) that
\[
\begin{cases}
w_1 - w_2 + A^T M_P w_4 + A^T U (w_5 - w_6 + w_7 - w_8) - B^T w_9 = 0, \\
(w_1 + w_2 - e)^+ = 0, \\
-a_1 w_3 + (e^N T w_4)^+ = 0, \\
-a_2 w_3 + e^T (w_5 + w_6)^+ = 0, \\
t^- = 0, \quad s^- = 0, \quad \xi^- = 0, \quad v^- = 0, \quad w^- = 0.
\end{cases} \quad (32)
\]

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be defined as follows:
\[
\begin{cases}
\mathcal{X} = e^T t + \varepsilon w_3 - y^T M_P w_4 - y^T U (w_5 - w_6 + w_7 - w_8) + B^T w_9, \\
\mathcal{Y} = (a_1 s + a_2 \xi + a_3 e^T v - \varepsilon)^+.
\end{cases}
\]

For the vector \( u = (x, t, s, \xi, \nu, w) \) where \( (t, s, \xi, \nu, w) \) is constructed above, by Lemma 4.2, there exists a vector \( \tilde{u} \in \Theta \), where \( \Theta \) is given in Lemma 4.1 and written as (15), such that
\[
\| u - \tilde{u} \|_1 \leq \sigma'
\]

where \( \sigma' \) is the Robinson constant determined by \( (M'_1, M'_2) \) given by (16). Since the
vector \((x, t, s, \xi, v, w)\) satisfies (24) and (32), the inequality (33) can be simplified to
\[
\|u - \hat{u}\|_2 \leq \sigma(M'_1, M'_2)\{\|\mathcal{Y}\| + \|(Bx - b)^+\|_1 + |\mathcal{X}|\}.
\] (34)

In the reminder of the proof, we estimate the terms on the right-hand side of (34). Note that the vectors in \(T\) are unit vectors. It is easy to see that
\[
\max_{1 \leq i \leq N} |(M_{P_i})^T(Ax - y)|_1 \leq \|y - Ax\|_2.
\]
The value of \(s\) in (23) implies that \(s \leq \|y - Ax\|_2\). Therefore we have
\[
\mathcal{Y} \leq (a_1 \|y - Ax\|_2 + a_2 \|U^T(y - Ax)\|_\infty + a_3 \|U^T(y - Ax)\|_1 - \varepsilon)^+.
\] (35)

Due to (27), (29) and (30), we have
\[
|\mathcal{X}| = \|e^T t + \varepsilon w_3 - y^T \nu^* - B^T h^*\| = \|e^T t + \varepsilon w_3 - x^T A^T \nu^* + (\phi(x))^T g + (Bx - b)^T h^* - x^T B^T h^*\|.
\]
The fact \(A^T \nu^* + B^T h^* = \eta\) (due to the restricted weak RSP of order \(k\)) and the triangle inequality imply that
\[
|\mathcal{X}| \leq \|e^T t - x^T \eta\| + \varepsilon |w_3| + \|\phi(x)^T g\| + \|(Bx - b)^T h^*\|.
\] (36)

Now we deal with the right-hand side of the above inequality. First, by using the index sets \(J\) and \(J_3\), we have
\[
|e^T t - x^T \eta| = |e^T_{J_t} t_J + e^T_{J_s} t_J - x^T_{J_t} \eta_J - x^T_{J_s} \eta_J|.
\]
It follows from \(t = |x|\) and (25) that
\[
|e^T_{J_t} t_J + e^T_{J_s} t_J - x^T_{J_t} \eta_J - x^T_{J_s} \eta_J| = |e^T_{J_t} t_J - x^T_{J_t} \eta_J| \leq |e^T_{J_t} t_J| + |x^T_{J_t} \eta_J| \leq \|x_{J_t}\|_1 + |x^T_{J_t} \eta_J| \leq \|x_{J_t}\|_1 + |x^T_{J_t} \eta_J| e \leq 2 \|x_{J_t}\|_1.
\]

Then we obtain
\[
|e^T t - x^T \eta| \leq 2 \|x_{J_t}\|_1 = 2 \sigma_k(x)_1.
\] (37)

By using the restricted weak RSP of order \(k\), we have
\[
\|\nu^*\|_1 \leq \left\|\begin{bmatrix} \nu^* \\ h^* \end{bmatrix}\right\|_1 \leq \left\|\left(CC^T\right)^{-1} C\eta\right\|_1 \leq \left\|\left(CC^T\right)^{-1} C\right\|_{\infty \rightarrow 1} \|\eta\|_{\infty} \leq \vartheta(1),
\]
where \(C = [A^T, B^T]^T \in \mathbb{R}^{(m+l) \times n}\) and \(\vartheta(1)\) is defined in (19b). Moreover, we have
\[
\|g\|_1 = \|g_3\|_1 = \|U^{-1}_G \nu^*\|_1 \leq \|U^{-1}_G\|_{1 \rightarrow 1} \|\nu^*\|_1 \leq \|U^{-1}_G\|_{1 \rightarrow 1} \vartheta(1)
\]
Recall that $\Upsilon(1,1)$ is determined in (19a). Then $\|g\|_1 \leq \Upsilon(1,1)$. Similarly, $\|g\|_\infty \leq \Upsilon(\infty,\infty)$ can be obtained. Due to $w_3 = \max\{\|v^*\|_1, \|g\|_1, \|g\|_\infty\}$, we have

$$\varepsilon \|w_3\| \leq \varepsilon \max\{\Upsilon(1,1), \Upsilon(\infty,\infty), \vartheta(1)\}. \quad (38)$$

Let $c, d, \hat{d} \in [1, +\infty]$ be three given positive numbers and $d, d'$ be two given numbers satisfying (21). For the term $\|(\phi(x))^T g\|$ in (36), it follows from Hölder inequalities that

$$\|(\phi(x))^T g\| \leq \|(\phi(x))^T g\|_d \leq \|\phi(x)\|_d \|g\|_d \leq \|\phi(x)\|_d \|U_\infty^{-1}v^*\|_d \leq \|\phi(x)\|_d \|U_\infty^{-1}\|_{d \to \hat{d}} \|v^*\|_{\hat{d}} \leq \|\phi(x)\|_d \|U_\infty^{-1}\|_{d \to \hat{d}} \left(\mathbb{C}C^T\right)^{-1} C \|_{\infty \to \hat{d}}. \quad (39)$$

Let $\Upsilon(d, \hat{d})$ be given as (19a), i.e.,

$$\Upsilon(d, \hat{d}) = \max_{\Omega \subseteq \{1, \ldots, h\}, (\Omega) = \{m\}} \|U_\infty^{-1}\|_{d \to \hat{d}} \left(\mathbb{C}C^T\right)^{-1} C \|_{\infty \to \hat{d}}. \quad (40)$$

Thus we have

$$\|(\phi(x))^T g\| \leq \Upsilon(d, \hat{d}) \|\phi(x)\|_{d'}. \quad (41)$$

Similarly, the following inequalities holds

$$\|(B x - b)^T h^*\| \leq \|B x - b\|_{c'} \|h^*\|_{c'} \leq \|B x - b\|_{c'} \left(\mathbb{C}C^T\right)^{-1} C \|_{\infty \to c} \|\eta\|_{\infty} \leq \vartheta(c) \|B x - b\|_{c'} \quad (42)$$

Due to (37), (38), (40) and (41), the inequality (36) is reduced to

$$|X| \leq \varepsilon \hat{\Upsilon} + 2\sigma_k(x)_1 + \|\phi(x)\|_{d'} \Upsilon(d, \hat{d}) + \|B x - b\|_{c'} \vartheta(c), \quad (43)$$

where $\hat{\Upsilon} = \max\{\Upsilon(1,1), \Upsilon(\infty,\infty), \vartheta(1)\}$.

Note that $\|x - \hat{x}\|_2 \leq \|u - \hat{u}\|_2$. It follows from (34), (35) and (42) that

$$\|x - \hat{x}\|_2 \leq \sigma' \left\{2\sigma_k(x)_1 + \|(B x - b)^+\|_1 + \varepsilon \hat{\Upsilon} + \|\phi(x)\|_{d'} \Upsilon(d, \hat{d}) + \|B x - b\|_{c'} \vartheta(c) \right\}. \quad (44)$$

We recall the three sets $\Omega^*$, $\Omega^P_a$ and $\Omega^P_b$, where $\Omega^*$ and $\Omega^P_a$ are the solution sets of (2) and (10), given as (6) and (11), respectively, and $\Omega^P_b$ is given as (7) with $P = P_b$. Clearly, $\hat{x} \in \Omega^P_a$. Let $x^*$ denote the projection of $x$ onto $\Omega^*$, that is,

$$x^* = \pi_{\Omega^*}(x).$$

Note that the three sets are compact convex sets satisfying $\Omega^* \subseteq \Omega^P_a$ and $\Omega^P_b \subseteq \Omega^P_b$. 

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Then by applying Lemma 4.3 with $T_1 = \Omega^\ast$, $T_2 = \Omega_{P_0}$ and $T_3 = \Omega_{P_0}^\ast$, we have

$$
\|x - \pi_{\Omega^\ast}(x)\|_2 = \|x - x^\ast\|_2 \leq \delta^\mathcal{H}(\Omega^\ast, \Omega_{P_0}) + 2 \|x - \hat{x}\|_2.
$$

Since $P_0$ satisfies (8), it implies that

$$
\|x - x^\ast\|_2 \leq \varepsilon^\prime + 2 \|x - \hat{x}\|_2.
$$

Let $\hat{\Upsilon} = \max\{\Upsilon(1, 1), \Upsilon(\infty, \infty), \vartheta(1)\}$. Combination of the above inequality and (43) yields the desired results (20). If $x$ is the feasible solution of (2), then $\|(Bx - b)^\dagger\|_1 = 0$ and

$$(a_1 \|y - Ax\|_2 + a_2 \|\phi(x)\|_\infty + a_3 \|\phi(x)\|_1 - \varepsilon)^\dagger = 0,$$

and thus the desired error bound (22) is also obtained.

Based on Theorem 4.4, the error bound for the solutions of (1) and (2) can be stated as follows.

**Corollary 4.5.** For any optimal solution $x$ of (1), there is an optimal solution $x^\ast$ of (2) estimating $x$ with the error:

$$
\|x - x^\ast\|_2 \leq \varepsilon^\prime + 2\sigma^\prime \left\{\varepsilon \hat{\Upsilon} + \sigma_k(x)_1 + \|\phi(x)\|_{d^\dagger} \Upsilon(d, \hat{d}) + \|Bx - b\|_{c^\dagger} \vartheta(c)\right\},
$$

where the constants $\varepsilon^\prime$, $\hat{\Upsilon}$, $\sigma^\prime$, $\Upsilon(d, \hat{d})$ and $\vartheta(c)$ are given as in Theorem 4.4.

5. Special cases

Firstly, by setting different values of $a_1, a_2$ and $a_3$, the problem (2) can reduce to several special cases, and the corresponding stability results for these special cases can be obtained from (20) and (22) immediately. Note that if any of $a_1, a_2$ and $a_3$ is zero, the constant $\hat{\Upsilon} = \max\{\Upsilon(1, 1), \Upsilon(\infty, \infty), \vartheta(1)\}$ in (20) and (22) will be simplified as well. For example, if $a_1 = 0$, the constant $\hat{\Upsilon}$ is reduced to $\max\{\Upsilon(1, 1), \Upsilon(\infty, \infty)\}$. The following table shows the form of the constant $\hat{\Upsilon}$ for different choices of $a_1, a_2$ and $a_3$. Note that for any case with $a_1 = 0$, we have $\Omega^\ast = \Omega_{P_0} = \Omega_{P_0}^\ast$ so that $\hat{x} = x^\ast$ where $\hat{x} \in \Omega_{P_0}^\ast$ and $x^\ast \in \Omega^\ast$. Thus instead of using Lemma 4.3, the stability results can be immediately obtained from (43).

<table>
<thead>
<tr>
<th>$a_i$</th>
<th>$\hat{\Upsilon}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 + a_2 = 0$</td>
<td>$\Upsilon(\infty, \infty)$</td>
</tr>
<tr>
<td>$a_1 + a_3 = 0$</td>
<td>$\Upsilon(1,1)$</td>
</tr>
<tr>
<td>$a_2 + a_3 = 0$</td>
<td>$\vartheta(1)$</td>
</tr>
<tr>
<td>$a_1 = 0$</td>
<td>$\max{\Upsilon(1,1), \Upsilon(\infty, \infty)}$</td>
</tr>
<tr>
<td>$a_2 = 0$</td>
<td>$\max{\Upsilon(\infty, \infty), \vartheta(1)}$</td>
</tr>
<tr>
<td>$a_3 = 0$</td>
<td>$\max{\Upsilon(1,1), \vartheta(1)}$</td>
</tr>
<tr>
<td>$a_1, a_2, a_3 \neq 0$</td>
<td>$\max{\Upsilon(1,1), \Upsilon(\infty, \infty), \vartheta(1)}$</td>
</tr>
</tbody>
</table>
Secondly, without matrix $B$, the problem (2) is reduced to

$$\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad a_1 \|y - Ax\|_2 + a_2 \|\phi(x)\|_\infty + a_3 \|\phi(x)\|_1 \leq \varepsilon.$$  

In this case, the restricted weak RSP of order $k$ is reduced to the standard weak RSP of order $k$, which means $A^T \nu^* = \eta$. In fact, the upper bound of $|((\phi(x))^T g| in (39) can be improved to

$$\begin{align*}
|((\phi(x))^T g| & \leq \|\phi(x)\|_d \|g\|_d \\
& = \|\phi(x)\|_d \left\| U_{\Omega}^{-1} \nu^* \right\|_d \\
& \leq \|\phi(x)\|_d \left\| U_{\Omega}^{-1} (AA^T)^{-1} A \eta \right\|_d \\
& \leq \|\phi(x)\|_d \left\| U_{\Omega}^{-1} (AA^T)^{-1} A \right\|_{\infty \to d}.
\end{align*}$$

Then in order to obtain a tighter bound, $\Upsilon(d, \hat{d})$ can be replaced by

$$\Upsilon'(d) = \max_{\forall i \subseteq \{1, \ldots, h\}, |\Omega| = m} \left\| U_{\Omega}^{-1} (AA^T)^{-1} A \right\|_{\infty \to d}.$$  

Thus we have $|((\phi(x))^T g| \leq \|\phi(x)\|_d \Upsilon'(d)$. Similarly, the constants $\Upsilon(1, 1)$ and $\Upsilon(\infty, \infty)$ are replaced by $\Upsilon'(1)$ and $\Upsilon'(\infty)$, respectively. Clearly, in this case, $\vartheta(c) = \left\| (AA^T)^{-1} A \right\|_{\infty \to c}$. Let $\tilde{\Upsilon}' = \max\{\Upsilon'(1), \Upsilon'(\infty), \vartheta(1)\}$. Then the bound (22) is reduced to

$$\|x - x^*\|_2 \leq \varepsilon' + 2\sigma' \left\{ \varepsilon \tilde{\Upsilon}' + 2\sigma_k(x_1) + \|\phi(x)\|_d \Upsilon'(d) \right\}.$$  

Similarly, we list the constants $\tilde{\Upsilon}'$ for different choices of $a_i$, $i = 1, 2, 3$ in the following table. Note that when $a_1 = 0$, we have $\tilde{\Upsilon}' = \Upsilon'(1)$ due to the fact

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$\Upsilon'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 + a_2 = 0$</td>
<td>$\Upsilon'(\infty)$</td>
</tr>
<tr>
<td>$a_1 + a_3 = 0$</td>
<td>$\Upsilon'(1)$</td>
</tr>
<tr>
<td>$a_2 + a_3 = 0$</td>
<td>$\vartheta(1)$</td>
</tr>
<tr>
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</tr>
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<td>$a_1, a_2, a_3 \neq 0$</td>
<td>$\max{\Upsilon'(1), \Upsilon'(\infty), \vartheta(1)}$</td>
</tr>
</tbody>
</table>

$$\left\| U_{\Omega}^{-1} (AA^T)^{-1} A \right\|_{\infty \to 1} \geq \left\| U_{\Omega}^{-1} (AA^T)^{-1} A \right\|_{\infty \to \infty}.$$  

Moreover, in this case, setting $d = 1$ yields

$$\|x - x^*\|_2 \leq \sigma' \left\{ \varepsilon \Upsilon'(1) + 2\sigma_k(x_1) + \|\phi(x)\|_\infty \Upsilon'(1) \right\},$$  

which is the bound for the following $\ell_1$-minimization established by Zhao and Li [30] (see also in Zhao [28]):

$$\min\{\|x\|_1 : a_2 \|\phi(x)\|_\infty + a_3 \|\phi(x)\|_1 \leq \varepsilon\}.$$  

$$15$$
Last but not least, our analysis can also apply to 1-bit basis pursuit [31], which can be viewed as a special case of our model (2). The stability result for the 1-bit basis pursuit in [31] can be obtained immediately from Theorem 4.4 by setting \(a_2 = a_3 = 0\).

6. Conclusion

In this paper, we have studied the stability issue of the \(\ell_1\)-minimization method (2). To establish our results, we introduced the restricted weak RSP of order \(k\) which is a mild assumption governing the stability of sparsity-seeking algorithms. Under this assumption, we use the classic Hoffman theorem and Lemma 4.3 to show that the \(\ell_1\)-minimization method (2) is stable and thus the error between the solutions of the problems (1) and (2) can be measured in terms of the best \(k\) term approximation and the problem data (see Theorem 4.4). The result developed in this paper can apply to a range of problems with constraints defined by \(\ell_1\)-, \(\ell_2\)-, and \(\ell_\infty\)-norms.

References


