Using Single-Scenario Relaxations to Solve Stochastic Mixed-Integer Programs

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Abstract

Solving the extensive form of a stochastic mixed-integer program can be improved by including strong valid inequalities for the convex hull of feasible solutions. An important class of such valid inequalities arise from (often well-studied) deterministic versions of the stochastic program. In this work, we analyze the strength of such valid inequalities when solving stochastic mixed-integer programs. To illustrate this, we introduce a stochastic variant of the well-known single node fixed-charge flow polytope, in which the capacity and demand values are uncertain. In particular, we provide necessary and sufficient conditions for inequalities derived from a single-scenario relaxation to be facet-defining for the extensive-form polytope. Under mild assumptions (such as those arising in stochastic network design problems), we show that all inequalities that are facet-defining for a single-scenario relaxation are also facet-defining in the stochastic case. We empirically assess the performance of single-scenario valid inequalities for a particular stochastic network design problem, and find that the addition of single-scenario valid inequalities can close between 15% and 70% of the optimality gap. Our results lead to a better understanding of the interaction between single-scenario relaxations and general, two-stage stochastic mixed-integer programs.

1. Introduction and Background

Stochastic mixed-integer programs (SMIPs) with recourse are a powerful modeling tool for optimization under uncertainty. SMIPs with recourse have found broad application in areas
including healthcare (Rais and Viana, 2011), disaster relief (Grass and Fischer, 2016) and more (Birge, 1997; Sahinidis, 2004; Wallace and Ziemba, 2005). In a two-stage SMIP with recourse, a decision maker must make a set of (linearly constrained mixed-integer) decisions before the values of a set of unknown parameters are revealed. Once these first-stage decisions are made, the values of the unknown parameters are realized, at which point the decision maker solves a recourse mixed-integer program (MIP). When the unknown parameters have only a finite number of possible realizations (as is typical in the literature), a two-stage SMIP with recourse may be expressed as a single, large-scale, deterministic MIP, known as the extensive form of the SMIP (Birge and Louveaux, 2011), given by

$$\min \quad c^T x + \sum_{k \in K} p_k (q^k)^T y^k$$

s.t. $Ax = b$

$$ T^k x + W^k y^k = h^k \text{ for all } k \in K $$

$$ x \geq 0, \quad y^k \geq 0 \text{ for all } k \in K $$

$$ x_i \in \mathbb{Z}_+ \text{ for all } i \in J^1 \subseteq N^1 $$

$$ y^k_i \in \mathbb{Z}_+ \text{ for all } i \in J^2 \subseteq N^2, \quad k \in K $$

(1)

where $K := \{1, \ldots, K\}$ is the finite set of realizations of the unknown parameters (known as scenarios), the vector $x$ (indexed by $N^1 := \{1, \ldots, N_1\}$) represents the first-stage decisions, and the vector $y^k$ (indexed by $N^2 := \{1, \ldots, N_2\}$) represents the second-stage decisions under scenario $k$. The scalars $p_k$ denote the probability that scenario $k$ occurs. The values $T^k, W^k, q^k$ and $h^k$ are the realizations of the unknown parameters under scenario $k$, and $A$, $b$ and $c$ are matrices of appropriate size. We denote the feasible region of (1) by $X^{\text{ext}}$.

There are many approaches to solve problems of the form (1) (Yuan and Sen, 2009; Gade et al., 2014). Popular decomposition approaches include Benders (1962) decomposition (also known as the L-shaped method in the context of stochastic programming (Van Slyke and Wets, 1969)), and its generalizations (Laporte and Louveaux, 1993; Sherali and Fraticelli, 2002; Ralphs and Hassanzadeh, 2014; Qi and Sen, 2017), disjunctive decomposition (Ntaimo and Sen, 2008; Ntaimo, 2010; Keller and Bayraksan, 2012) and Fenchel decomposition (Ntaimo, 2013). Heuristic procedures include progressive hedging (Rockafellar and Wets, 1991; Watson and Woodruff, 2011; Atakan and Sen, 2018; Boland et al., 2018). Other approaches, both exact and heuristic, include Carøe and Tind (1997), Carøe and Schultz (1999), Ahmed et al. (2004), Lulli and Sen (2004), Sen and Higle (2005), Kong et al. (2006), Sherali and Smith (2009), Trapp et al. (2013) and Escudero et al. (2016). Introductions and summaries of the SMIP literature can be found in Sen and Sherali (2005), Sen (2005) and Küçükyavuz and Sen (2017). A recent taxonomy of SMIP approaches is given in Tavashoḡlu et al. (2019).

In addition to these approaches, the extensive form (1) can be solved directly using commercial MIP solvers. Although this approach does not scale well as the number of scenarios and
uncertain parameters increases (Parija et al., 2004), it has the (non-trivial) advantage of being particularly easy to implement, as it does not require specialized knowledge or techniques (Watson et al., 2012). Moreover, in contrast to many of the approaches described above, solving (1) directly requires virtually no structural assumptions (e.g., fixed recourse, pure-binary first-stage, etc.) on the underlying SMIP. Consequently, the deterministic equivalent is still often solved directly in practice (e.g., Rath et al. (2016), Badri et al. (2017), Dillon et al. (2017), Hamdan and Diabat (2019), Weskamp et al. (2019)), and is incorporated as an option in various SMIP modeling software tools (Thénié et al., 2007; Watson et al., 2012; Roelofs and Bisschop, 2019).

A critical tool in the solution of MIPs (both deterministic and stochastic) is the inclusion of strong valid inequalities for the convex hull of feasible solutions. For the remainder of the paper we consider inequalities of the form

\[ \alpha^T x + \beta^T y^k \geq \tau, \]  

(FI)

for fixed scenario \( k \in K \) where \( \alpha \in \mathbb{R}^{N_1} \), \( \beta \in \mathbb{R}^{N_2} \), and \( \tau \in \mathbb{R} \). An advantage of inequalities of the form (FI) is that they maintain the “L-shaped” structure of the constraints in (1), and are thus more amenable to use within decomposition-based algorithms than inequalities that combine information from multiple scenarios. Another advantage is that for many structured problems, a wide range of valid inequalities of the form (FI) are known for the deterministic version of (1) (this notion will be made more precise in Section 2). We will see that such valid inequalities can be used directly when solving the extensive form (1). We wish to assess the strength of such valid inequalities.

More specifically, our research is guided by the following question: under what conditions are inequalities of the form (FI) facet-defining for the convex hull of the feasible region of (1)? To partially answer this question, we focus our study on a stochastic version of the single node fixed-charge flow (SFCF) polytope (see Section 3). We study the SFCF polytope because of its many important applications, including fixed-charge transportation problems, facility location problems, and lot-sizing problems (e.g., Louveaux (1993); Guan et al. (2006); Roberti et al. (2015)).

The use of single-scenario inequalities of the form (FI) to solve stochastic mixed-integer programs has been studied throughout the SMIP literature. Many approaches use inequalities of the form (FI) in the context of Benders-like decomposition methods—see Gade et al. (2014), Zhang and Küçükyavuz (2014) and Bodur et al. (2017) for recent examples. Dey et al. (2018) analyze the strength of inequalities of a slightly more general form than (FI) in the context of SMIP. Another general approach focuses on combining inequalities of the form (FI) for different scenarios to create strong valid inequalities for the stochastic problem. For example, Günlük and Pochet (2001) combine valid inequalities for a mixed-integer region into strong valid inequalities. This technique was applied in the stochastic setting.
by Riis and Andersen (2002), who combine single-scenario inequalities of the form (FI) for a multicommodity capacitated network design problem first put forward by Bienstock and Günlük (1996). Guan et al. (2006) later created valid inequalities for the stochastic lot-sizing problem by combining information from multiple scenarios. This procedure was expanded beyond the stochastic lot-sizing problem in Guan et al. (2007), generalizing the work of Günlük and Pochet (2001).

In contrast, this paper analyzes the strength of single-scenario valid inequalities (FI) alone, without combining them with other scenarios. In particular, we seek to provide theoretical justification for the use of single-scenario valid inequalities of the form (FI) when solving stochastic mixed-integer programs.

The main contributions of this paper are as follows:

1. We present general results connecting the convex hull of feasible solutions to (1) with polyhedra derived from (1) that include information from only a single scenario.

2. We introduce the stochastic SFCF polytope (denoted S-SFCF), and provide necessary and sufficient conditions for inequalities of the form (FI) to be facet-defining for the S-SFCF polytope (Theorem 1). We demonstrate that, under mild assumptions (naturally satisfied in settings such as stochastic facility location), all facet-defining inequalities for a single-scenario relaxation of the S-SFCF polytope are facet-defining for the S-SFCF polytope itself (Proposition 10). This result provides theoretical justification for the use of single-scenario valid inequalities when solving SMIPs.

3. Through an extensive computational study, we demonstrate the efficacy of using single-scenario valid inequalities for SMIPs with S-SFCF substructure. We show that inequalities of the form (FI) are able to close between 15% and 70% of the optimality gap for the extensive form of a stochastic network design problem. Our results appear to be very helpful for general SMIPs with variable upper bound constraints.

The remainder of this paper is organized as follows: Section 2 presents general results relating the feasible region of (1) with various single-scenario versions of the same problem. Section 3 introduces the stochastic SFCF polytope, and Section 4 contains our main result. Section 5 contains a series of numerical experiments that quantify the effect of inequalities of the form (FI) on the LP relaxation of (1) for a stochastic network design problem. Section 6 contains concluding remarks.

2. The Relationship Between General Deterministic and Stochastic Polyhedra

We associate four polyhedra with SMIP (1):
1. The extensive polyhedron \( S^{\text{ext}} := \text{conv}(X^{\text{ext}}) \), where \( X^{\text{ext}} \) is the feasible region of (1).

2. For each scenario \( k \in \mathcal{K} \), the \( k \)th single-scenario relaxation, \( S^k \), is obtained by relaxing the constraints in \( S^{\text{ext}} \) associated with every scenario other than scenario \( k \). It is given by

\[
S^k := \text{conv} \left\{ x \in \mathbb{R}^N_+ \mid \begin{array}{l}
Ax = b \\
T^k x + W^k y^k = h^k \\
x_i \in \mathbb{Z}_+ \text{ for all } i \in J^1 \subseteq N^1 \\
y_i^k \in \mathbb{Z}_+ \text{ for all } i \in J^2 \subseteq N^2
\end{array} \right\}.
\]

3. For each scenario \( k \in \mathcal{K} \), the \( k \)th deterministic polyhedron \( D^k \) is given by

\[
D^k := \text{conv} \left\{ x \in \mathbb{R}^N_+ \mid \begin{array}{l}
Ax = b \\
T^k x + W^k y^k = h^k \\
x_i \in \mathbb{Z}_+ \text{ for all } i \in J^1 \subseteq N^1 \\
y_i^k \in \mathbb{Z}_+ \text{ for all } i \in J^2 \subseteq N^2
\end{array} \right\}.
\]

Note that \( D^k \) is the projection of the \( k \)th single-scenario relaxation on to the \((x, y^k)\) coordinates.

4. For each scenario \( k \in \mathcal{K} \), the \( k \)th projected single-scenario polyhedron, \( P^k \), is the projection of \( S^{\text{ext}} \) onto the \((x, y^k)\) coordinates, i.e., \( P^k := \text{proj}_{(x,y^k)}(S^{\text{ext}}) \).

Note that both \( S^{\text{ext}} \) and \( S^k \) for all \( k \in \mathcal{K} \) are contained in the higher-dimensional ambient space \( \mathbb{R}^{N_1 + KN_2} \), while both \( D^k \) and \( P^k \) are contained in the lower-dimensional space \( \mathbb{R}^{N_1 + N_2} \). By a slight abuse of terminology, we will refer to the inequality (FI) as being valid and/or facet-defining for both the polyhedra \( S^{\text{ext}} \) and \( S^k \) in \( \mathbb{R}^{N_1 + KN_2} \), as well as the lower-dimensional polyhedra \( D^k \) and \( P^k \) in \( \mathbb{R}^{N_1 + N_2} \). For the higher-dimensional polyhedra, we assume that the coefficients of the variables \( y^\ell \) for scenarios all scenarios \( \ell \neq k \) are equal to zero. Hence, we describe the same inequality (FI) as being simultaneously valid and/or facet-defining for polyhedra contained in different spaces.

Lemma 1, adapted from Balas and Oosten (1998) (Corollary 3.6), connects the facets of a polyhedron with the facets of its projections.

**Lemma 1.** Let \( Q = \{(u, z) \in \mathbb{R}^p \times \mathbb{R}^q \mid Au + Bz \leq b \} \) be a non-empty polyhedron, and let \((A^=, B^=, b^=)\) denote the equality subsystem of \( Q \) (i.e., the set of inequalities satisfied with equality by all points in \( Q \)). Let \( F = \{(u, z) \in Q \mid \eta^T u + \gamma^T z = \gamma_0 \} \) be a facet of \( Q \). Then \( \text{proj}_z(F) \) is a facet of \( \text{proj}_z(Q) \) if and only if

\[
\text{rank}(A^=) = \text{rank} \left( \begin{bmatrix} \eta^T \\ A^= \end{bmatrix} \right).
\]
In particular, if \( \eta = 0 \), then \( \text{proj}_z(F) \) is always a facet of \( \text{proj}_z(Q) \).

We next present some general results relating the four polyhedra \( S^{\text{ext}} \), \( S^k \), \( D^k \) and \( P^k \).

**Proposition 1.** \( S^{\text{ext}} = \bigcap_{k \in \mathcal{K}} S^k \), and for all scenarios \( k \in \mathcal{K} \), \( S^{\text{ext}} \subseteq S^k \).

*Proof.** Follows from the definition of \( S^k \) and \( S^{\text{ext}} \). \( \square \)

Proposition 1 is well known in the literature (e.g., Riis and Andersen (2002)), and is related to the “possibility interpretation” of the SMIP (1) (Birge and Louveaux, 2011, Chapter 3). A similar result holds for the sets \( P^k \) and \( D^k \):

**Proposition 2.** \( P^k \subseteq D^k \) for all scenarios \( k \in \mathcal{K} \).

*Proof.** Follows because \( P^k \) (resp. \( D^k \)) is the projection of \( S^{\text{ext}} \) (resp. \( S^k \)) onto the \((x, y^k)\) coordinates, and \( S^{\text{ext}} \subseteq S^k \). \( \square \)

The following results relate the facial structures of the four main polyhedra.

**Proposition 3.** The following hold for any scenario \( k \in \mathcal{K} \):

1. (FI) is valid for \( S^k \) if and only if it is valid for \( D^k \).
2. (FI) defines a supporting hyperplane for \( S^k \) if and only if it defines a supporting hyperplane for \( D^k \).
3. (FI) defines a facet for \( S^k \) if and only if it defines a facet for \( D^k \).

*Proof.** We prove part 3—the other parts are similar. The “\( \Rightarrow \)” direction follows from Lemma 1 with \( Q = S^k \), \( z = (x, y^k) \) and \( \eta = 0 \). To prove the “\( \Leftarrow \)” direction, suppose that \( \text{dim}(S^k) = N_1 + KN_2 \) (the non-full-dimensional case is similar). Because (FI) is facet-defining for \( D^k \), there exist \( N_1 + N_2 \) affinely independent points \( \{(x^{(i)}(j), y^{(j),k})\}_{i=1}^{N_1+N_2} \) in \( D^k \) that satisfy (FI) with equality. If we let \( e_j \) denote the \( j \)th standard basis vector in \( \mathbb{R}^{(K-1)N_2} \), and \( 0 \) denote the zero vector in \( \mathbb{R}^{(K-1)N_2} \), then the \( N_1 + KN_2 \) points \( \{(x^{(i)}(j), y^{(j),k}, e_j)\}_{j=1}^{(K-1)N_2} \cup \{(x^{(i)}(j), y^{(j),k}, 0)\}_{i=1}^{N_1+N_2} \), are affinely independent, lie in \( S^k \), and satisfy (FI) with equality. \( \square \)

**Proposition 4.** If, for some scenario \( k \in \mathcal{K} \), (FI) is valid for \( D^k \), then (FI) is valid for \( S^{\text{ext}} \).

*Proof.** If (FI) is valid for \( D^k \), then by Proposition 3 it is valid for \( S^k \). The result follows from Proposition 1. \( \square \)
Proposition 2 shows that \( \mathcal{P}^k \subseteq \mathcal{D}^k \) for all scenarios \( k \in K \). Proposition 5 characterizes precisely when this relation holds with equality. This result will be useful in Section 4, which relates supporting hyperplanes for \( S^k \) with supporting hyperplanes for \( S^\text{ext} \).

**Proposition 5.** For any scenario \( k \in K \) for which \( \mathcal{D}^k \) is bounded, \( \mathcal{P}^k = \mathcal{D}^k \) if and only if every inequality of the form (FI) that defines a supporting hyperplane of \( S^k \) also defines a supporting hyperplane of \( S^\text{ext} \).

**Proof.** “\( \Rightarrow \)” Suppose that, for some \( k \in K \), (FI) defines a supporting hyperplane for \( S^k \), and that \( \mathcal{P}^k = \mathcal{D}^k \). By Proposition 3, (FI) defines a supporting hyperplane for \( \mathcal{D}^k \), and thus \( \mathcal{P}^k \). Because \( \mathcal{P}^k := \text{proj}_{(x,y)}(S^\text{ext}) \), it follows that (FI) defines a supporting hyperplane for \( S^\text{ext} \) as well.

“\( \Leftarrow \)” Fix \( k \in K \) such that \( \mathcal{D}^k \) is bounded, and suppose that every inequality of the form (FI) which defines a supporting hyperplane of \( S^k \) also defines a supporting hyperplane of \( S^\text{ext} \). By Proposition 3, every supporting hyperplane of the form (FI) for \( \mathcal{D}^k \) also defines a supporting hyperplane for \( S^\text{ext} \). By Proposition 2, \( \mathcal{P}^k \subseteq \mathcal{D}^k \). To prove that \( \mathcal{D}^k \subseteq \mathcal{P}^k \), suppose for contradiction that there exists a point \( (\hat{x}, \hat{y}^k) \) of \( \mathcal{D}^k \) that does not lie in \( \mathcal{P}^k \).

Without loss of generality, because \( \mathcal{D}^k \) is bounded, we may assume that \( (\hat{x}, \hat{y}^k) \) is a vertex of \( \mathcal{D}^k \) (if every vertex of \( \mathcal{D}^k \) lies in \( \mathcal{P}^k \), then, because any polytope is the convex hull of its vertices, \( \text{conv}(\mathcal{D}^k) = \mathcal{D}^k \subseteq \mathcal{P}^k \), completing the proof). Then there exists an inequality \( \alpha^T x + \beta^T y^k \geq \tau \) that defines a supporting hyperplane for \( \mathcal{D}^k \) such that \( (\hat{x}, \hat{y}^k) \) is the unique point in \( \mathcal{D}^k \) satisfying \( \alpha^T x + \beta^T y^k = \tau \). From above, \( \alpha^T x + \beta^T y^k \geq \tau \) must define a supporting hyperplane of \( S^\text{ext} \) as well. Hence, there exists \( (\bar{x}, \bar{y}^1, \ldots, \bar{y}^K) \in S^\text{ext} \) such that \( \alpha^T \bar{x} + \beta^T \bar{y}^k = \tau \). By construction, \( (\bar{x}, \bar{y}^k) \in \mathcal{P}^k \), which implies \( (\hat{x}, \hat{y}^k) \notin \mathcal{D}^k \). Because \( (\hat{x}, \hat{y}^k) \) is unique, we have that \( \bar{x} = \hat{x} \) and \( \bar{y}^k = \hat{y}^k \), and thus \( (\hat{x}, \hat{y}^k) \in \mathcal{P}^k \), a contradiction. Because \( \mathcal{D}^k \) is bounded, and every vertex of \( \mathcal{D}^k \) lies in \( \mathcal{P}^k \), it follows that \( \mathcal{D}^k \subseteq \mathcal{P}^k \), and thus \( \mathcal{D}^k = \mathcal{P}^k \). □

Example 1 shows that, if we drop the assumption that \( \mathcal{D}^k \) is bounded, the reverse direction of Proposition 5 does not hold.

**Example 1.** Consider a stochastic linear program (with no integer variables) with \( K = 2 \), where \( \mathcal{D}^1 = \{x, y^1 \in \mathbb{R} \mid x \geq 0, y^1 \geq 0\} \) and \( \mathcal{D}^2 = \{x, y^2 \in \mathbb{R} \mid x \geq 0, y^2 \geq 0, x + y^2 \leq 1\} \) (note that \( \mathcal{D}^1 \) is not bounded). The extensive polyhedron is \( S^\text{ext} = \{x, y^1, y^2 \in \mathbb{R} \mid x + y^2 \leq 1, x \geq 0, y^1 \geq 0, y^2 \geq 0\} \). It is easy to verify that every supporting hyperplane of \( S^1 = \{x, y^1, y^2 \in \mathbb{R} \mid x \geq 0, y^1 \geq 0, y^2 \text{ free}\} \) of the form \( \alpha x + \beta y^1 \geq \tau \) for \( \alpha, \beta, \tau \in \mathbb{R} \) is also a supporting hyperplane of \( S^\text{ext} \). However, \( \mathcal{D}^1 \neq \mathcal{P}^1 \), because the point \( x = 2, y^1 = 0 \) lies in \( \mathcal{D}^1 \) but not in \( \mathcal{P}^1 = \text{proj}_{(x,y^1)}(S^\text{ext}) \).

**Proposition 6.** If (FI) is facet-defining for \( S^\text{ext} \), then it is facet-defining for \( \mathcal{P}^k \). Moreover, if
(FI) is facet-defining for $S^{\text{ext}}$ and $\beta = 0$, then (FI) is facet-defining for $P^k$ for all scenarios $k \in \mathcal{K}$.

**Proof.** Follows from Lemma 1 with $Q = S^{\text{ext}}$, $z = (x, y^k)$, $u = (y^1, \ldots, y^{k-1}, y^{k+1}, \ldots, y^K)$, $\gamma = (\alpha, \beta)$ and $\eta = 0$. \qed

Example 2 in Section 4 will show that the converse of Proposition 6 is not true—that is, a facet-defining inequality for the projection $P^k$ of the form (FI) is not necessarily facet-defining for $S^{\text{ext}}$.

### 3. The SFCF Polytope and its Stochastic Extension

We now introduce a stochastic extension of one of the most widely-studied polyhedra in the operations research literature: the single node fixed-charge (SFCF) polytope (Padberg et al., 1985; Van Roy and Wolsey, 1986; Stallaert, 1997; Gu et al., 1999; Atamtürk, 2001). We first define the SFCF polytope in the deterministic setting. The SFCF polytope models a node in a network with a fixed demand value $d \in \mathbb{R}$, a set of arcs $\mathcal{N}^-$ entering the node, and a set of arcs $\mathcal{N}^+$ leaving the node (Figure 1). Each arc $i \in \mathcal{N} := \mathcal{N}^+ \cup \mathcal{N}^-$ has an associated capacity $u_i$. The decision maker must select a subset of arcs $\mathcal{C} \subseteq \mathcal{N}$ to activate (indicated by a binary vector $x$), then select the amount of flow $y_i$ to send on each arc $i \in \mathcal{C}$, as long as $y_i$ does not exceed the capacity $u_i$ of the activated arcs.

More formally, the SFCF polytope is given by $S_{\text{FC}} := \text{conv}(X_{\text{FC}})$, where

$$X_{\text{FC}} := \left\{ x \in \{0,1\}^N \left| \begin{array}{c} y \in \mathbb{R}_+^N \\ \sum_{i \in \mathcal{N}^+} y_i - \sum_{i \in \mathcal{N}^-} y_i \geq d \\ 0 \leq y_i \leq u_i x_i, \text{ for all } i \in \mathcal{N} \end{array} \right. \right\}.$$  

We assume that $\mathcal{N}^+ = \{1, \ldots, N^+\}$ and $\mathcal{N}^- = \{N^++1, \ldots, N\}$, where $|\mathcal{N}| = N$, $|\mathcal{N}^+| = N^+$ and $|\mathcal{N}^-| = N^-$. The constraint $\sum_{i \in \mathcal{N}^+} y_i - \sum_{i \in \mathcal{N}^-} y_i \geq d$ is the flow constraint, and the constraints $0 \leq y_i \leq u_i x_i$ are variable upper bound constraints.

**Remark 1.** The choice of a greater-than ($\geq$) inequality in the flow constraint (rather than $\leq$) is without loss of generality.

Although the SFCF polytope can be interpreted as representing a node within a network, it
Demand

$$\mathcal{N}^-$$ arcs

$$u_1$$

$$u_2$$

$$u_3$$

$$\text{Demand } d$$

$$u_4$$

$$u_5$$

$$\mathcal{N}^+$$ arcs

Figure 1: Interpretation of the deterministic SFCF polytope $\mathcal{S}$ as a node in a network (cf. Nemhauser and Wolsey (1988, Sec. II.2.4)).

is more general than it appears. Any feasible set of the form

$$\left\{ \begin{array}{l}
\sum_{j \in J_1} (\alpha_j y_j + \beta_j x_j) + \sum_{j \in J_2} \alpha_j y_j + \sum_{j \in J_3} \beta_j x_j \leq b \\
0 \leq y_j \leq k_j x_j \text{ for } j \in J_1 \\
0 \leq y_j \leq k_j \text{ for } j \in J_2 \\
x_j \in \{0, 1\} \text{ for } j \in J_1 \cup J_3,
\end{array} \right.$$

is contained within an SFCF polytope (Nemhauser and Wolsey, 1988, Sec. II.2.4). Hence, by studying the SFCF polytope, we aim to present results that may be generalized to any SMIP model with variable upper bounds on all (or a subset) of the continuous variables.

A wide body of valid inequalities is known for the SFCF polytope, although a complete description for its convex hull is unknown. Padberg et al. (1985) developed the flow cover (FC) inequalities for $\mathcal{S}_{fc}$ in the case $\mathcal{N}^- = \emptyset$, and proved necessary and sufficient conditions for these inequalities to be facet-defining. Van Roy and Wolsey (1986) extended this to generalized flow cover (GFC) valid inequalities (a special case of submodular inequalities (Wolsey, 1989)). Subsequently, Gu et al. (1999) strengthened both the FC and GFC cuts by applying a sequence-independent lifting procedure. Using a similar idea, Stallaert (1997) introduced the flow pack valid inequalities, which were strengthened by Atamtürk (2001).

A natural extension of the SFCF polytope is allowing the capacity parameters $u_i$ and demand parameter $d$ to be uncertain. We model a decision-maker who must select some subset of arcs $\mathcal{C} \subseteq \mathcal{N}$ to turn “on” (set $x_i = 1$), while the capacity of the arcs $u_i$ and demand $d$ are still unknown. Once $\mathcal{C}$ is selected, the capacities and demands are realized, at which point the decision-maker selects the amount of flow $y_i$ to use on each arc $i \in \mathcal{C}$ ($y_i = 0$ for $i \notin \mathcal{C}$).

As discussed in Section 1, we assume that the parameters $u_i$ and $d$ can have only a finite number of different realizations, represented as $u_i^k, d^k$ for $k \in \mathcal{K} := \{1, \ldots, K\}$ with $K < \infty$. We define the stochastic extension of the SFCF polytope (which we denote S-SFCF) as

$$\sum_{i \in \mathcal{N}} (\alpha_i y_i + \beta_i x_i) + \sum_{i \in \mathcal{N}} \alpha_i y_i + \sum_{i \in \mathcal{N}} \beta_i x_i \leq b$$

with

$$0 \leq y_i \leq k_i x_i \text{ for } i \in \mathcal{N}^-$$

and

$$0 \leq y_i \leq k_i \text{ for } i \in \mathcal{N}^+$$

for $i \in \mathcal{N}$.

A natural extension of the SFCF polytope is allowing the capacity parameters $u_i$ and demand parameter $d$ to be uncertain. We model a decision-maker who must select some subset of arcs $\mathcal{C} \subseteq \mathcal{N}$ to turn “on” (set $x_i = 1$), while the capacity of the arcs $u_i$ and demand $d$ are still unknown. Once $\mathcal{C}$ is selected, the capacities and demands are realized, at which point the decision-maker selects the amount of flow $y_i$ to use on each arc $i \in \mathcal{C}$ ($y_i = 0$ for $i \notin \mathcal{C}$).

As discussed in Section 1, we assume that the parameters $u_i$ and $d$ can have only a finite number of different realizations, represented as $u_i^k, d^k$ for $k \in \mathcal{K} := \{1, \ldots, K\}$ with $K < \infty$. We define the stochastic extension of the SFCF polytope (which we denote S-SFCF) as
follows. Let
\[
X^{\text{ext}}_{\text{fc}} := \left\{ x \in \{0,1\}^N \right. \\
y^k \in \mathbb{R}^N_+, \quad k \in \mathcal{K} \left. \quad \sum_{i \in \mathcal{N}^+} y_i^k - \sum_{i \in \mathcal{N}^-} y_i^k \geq d^k, \quad \text{for all } k \in \mathcal{K} \right. \\
0 \leq y_i^k \leq u_i^k x_i, \quad \text{for all } i \in \mathcal{N}, \quad k \in \mathcal{K} \right\},
\]
where, for all scenarios \( k \in \mathcal{K} \), \( y_i^k \) and \( u_i^k \) are the flow and the capacity, respectively, of channel \( i \in \mathcal{N} \), and \( d^k \) denotes the demand value of the node. The S-SFCF polytope is given by \( S^{\text{ext}}_{\text{fc}} := \text{conv}(X^{\text{ext}}_{\text{fc}}) \). Example 3 in Section 4 demonstrates that the S-SFCF polytope is a relaxation of a particular stochastic facility location problem.

For each scenario \( k \in \mathcal{K} \), the deterministic polytope is given by \( D^k_{\text{fc}} := \text{conv}(X^k_{\text{fc}}) \) where
\[
X^k_{\text{fc}} := \left\{ x \in \{0,1\}^N \right. \\
y^k \in \mathbb{R}^N_+ \left. \quad \sum_{i \in \mathcal{N}^+} y_i^k - \sum_{i \in \mathcal{N}^-} y_i^k \geq d^k \right. \\
0 \leq y_i^k \leq u_i^k x_i, \quad \text{for all } i \in \mathcal{N} \right\}.
\]

The sets \( P^k_{\text{fc}} \) and \( S^k_{\text{fc}} \) are defined as in Section 2. For the remainder of the paper, we use the subscript \( \text{fc} \) (for \( \text{F} \)ixed \( \text{C} \)harge) to distinguish the polytopes associated with the SFCF polytope from the polyhedra associated with the general SMIP (1).

### 4. The Relationship Between the Deterministic and Stochastic SFCF Polytopes

The remainder of the paper derives conditions under which facet-defining inequalities for the deterministic polyhedron \( D^k_{\text{fc}} \) (and/or the single-scenario relaxation \( S^k_{\text{fc}} \)) are also facet-defining for the extensive polyhedron \( S^{\text{ext}}_{\text{fc}} \).

To reach this goal, we focus now on the SFCF polytope introduced in Section 3. Although the results in this section do not directly apply in the full generality of the SMIP (1), as discussed in Section 3, the choice of the SFCF polytope is still quite general.

We begin by providing some results on the dimension of the polytope \( S^{\text{ext}}_{\text{fc}} \), which generalize well-known results from the deterministic setting.

**Proposition 7.** \( S^{\text{ext}}_{\text{fc}} \) is full-dimensional if and only if all of the following hold:

\[
u_i^k > 0 \text{ for all } i \in \mathcal{N}, \quad k \in \mathcal{K}. \quad (A1)
\]
\[
\sum_{i \in \mathcal{N}^+ \setminus \{j\}} u_i^k \geq d^k \text{ for all } j \in \mathcal{N}^+, \quad k \in \mathcal{K}. \quad (A2)
\]
\[ \sum_{i \in N^+} u_k^i > d_k \text{ for all } k \in K. \] (A3)

The proof is constructive and is given in the electronic companion. Note that if \( N^+ \) is non-empty, then (A1) and (A2) imply (A3). Otherwise, we interpret (A3) to mean \( d_k < 0 \) for all \( k \in K \). We assume that (A1) through (A3) hold throughout the remainder of the paper.

**Corollary 1.** (Padberg et al., 1985; Atamtürk, 2001) The deterministic polytope \( D_{fc}^k \) is full-dimensional for some \( k \in K \) if and only if \( u_k^i > 0 \) for all \( i \in N^+ \), \( \sum_{i \in N^+ \setminus \{j\}} u_k^i \geq d_k \) for all \( j \in N^+ \), and \( \sum_{i \in N^+} u_k^i > d_k \).

**Corollary 2.** \( S_{fc}^\text{ext} \) is full-dimensional if and only if \( D_{fc}^k \) is full-dimensional for all \( k \in K \).

**Corollary 3.** For all scenarios \( k \in K \), \( \dim(P_{fc}^k) = 2N \).

**Proof.** Follows because the projection of a full-dimensional polyhedron is full dimensional in its ambient space. \( \square \)

We now investigate the relationship between the well-studied facial structure of the deterministic SFCF polytope \( D_{fc}^k \) and the facial structure of the S-SFCF polytope \( S_{fc}^\text{ext} \). As before, we consider inequalities of the form (FI), which may be valid for the low-dimensional polytopes \( D_{fc}^k \) and \( P_{fc}^k \), and/or the high-dimensional polytopes \( S_{fc}^\text{ext} \) and \( S_{fc}^k \). Inequalities of the form (FI) include extended flow cover inequalities, flow pack inequalities, and their lifted variants (Atamtürk, 2001; Gu et al., 1999; Stallaert, 1997).

**Proposition 8** shows that some simple facet-defining inequalities for the deterministic polytope \( D_{fc}^k \) also define facets for the extensive polytope \( S_{fc}^\text{ext} \).

**Proposition 8.** The following inequalities are facet-defining for \( S_{fc}^\text{ext} \):

1. \( x_i \leq 1 \) for all \( i \in N \).
2. \( y_k^i \geq 0 \) for all \( i \in N^- \) and \( k \in K \).
3. \( y_k^i \leq u_i^k x_i \) for all \( i \in N^+ \) and \( k \in K \).

The proof is constructive and is given in the electronic companion. We note that if \( i \in N^+ \), then \( y_k^i \geq 0 \) may not be facet-defining for \( S_{fc}^\text{ext} \) for some \( k \in K \). Similarly, if \( i \in N^- \), then \( y_k^i \leq u_i^k x_i \) may not be facet-defining for \( S_{fc}^\text{ext} \) for some \( k \in K \).

**Proposition 9** characterizes the projected polytope \( P_{fc}^k \).
Proposition 9. For any $k \in \mathcal{K}$,

$$\text{proj}(x, y^k)(X^\text{ext}_{\text{fc}}) = \left\{ (x, y^k) \in X^k | \sum_{j \in \mathcal{N}^+} u_j^k x_j \geq \max\{d^k, 0\} \text{ for all } \ell \in \mathcal{K} \right\}.$$  

Proof. “$\subseteq$” Fix $(x, z^k) \in \text{proj}(x, y^k)(X^\text{ext}_{\text{fc}})$. It is immediate that $(x, z^k) \in X^k$. Furthermore, for all scenarios $\ell \neq k$, there exists a vector $z^\ell \in \mathbb{R}^\mathcal{N}$ such that the point $(x, z^1, \ldots, z^k, \ldots, z^K) \in X^\text{ext}_{\text{fc}}$. In particular, for any $\ell \in \mathcal{K}$, the point $(x, z^\ell) \in X^\ell_{\text{fc}}$, and hence $\sum_{j \in \mathcal{N}^+} u_j^\ell x_j \geq \sum_{j \in \mathcal{N}^+} z_j^\ell \geq \sum_{j \in \mathcal{N}^-} z_j^\ell - \sum_{j \in \mathcal{N}^-} z_j^\ell \geq d^\ell$. Moreover, by (A1), $\sum_{j \in \mathcal{N}^+} u_j^\ell x_j \geq 0$. Hence $\sum_{j \in \mathcal{N}^+} u_j^\ell x_j \geq \max\{d^k, 0\}$.

“$\supseteq$” Fix $(x, z^k) \in X^k$ such that $\sum_{j \in \mathcal{N}^+} u_j^k x_j \geq \max\{d^k, 0\}$ for all $\ell \in \mathcal{K}$. For all scenarios $\ell \neq k$, define $z^\ell \in \mathbb{R}^\mathcal{N}$ by

$$z_j^\ell = \begin{cases} u_j^\ell x_j, & \text{if } j \in \mathcal{N}^+; \\ 0, & \text{otherwise}. \end{cases}$$

We wish to show that $(x, z^1, \ldots, z^k, \ldots, z^K) \in X^\text{ext}_{\text{fc}}$. By definition of $X^\text{ext}_{\text{fc}}$ and $X^\ell_{\text{fc}}$, it suffices to show that $(x, z^\ell) \in X^\ell_{\text{fc}}$ for all $\ell \neq k$. Clearly, $(x, z^\ell)$ satisfies $0 \leq z_j^\ell \leq u_j^\ell x_j$ for all $j \in \mathcal{N}$. The flow constraint is also satisfied: $\sum_{j \in \mathcal{N}^+} z_j^\ell - \sum_{j \in \mathcal{N}^-} z_j^\ell = \sum_{j \in \mathcal{N}^+} u_j^\ell x_j \geq \max\{d^k, 0\} \geq d^\ell$. \hfill \Box

Remark 2. If $d^k \leq 0$ for all scenarios $k \in \mathcal{K}$, then $\mathcal{P}^k = \mathcal{D}^k$ for all $k \in \mathcal{K}$ (this follows because (A1) renders the inequality $\sum_{j \in \mathcal{N}^+} u_j^k x_j \geq \max\{d^k, 0\}$ trivial in the case $d^k \leq 0$).

Remark 2 is critical for characterizing the relationship between facets of $\mathcal{D}^k$ and facets of $\mathcal{S}^\text{ext}$.

Proposition 6 showed that for an inequality of the form (FI) to be facet-defining for $\mathcal{S}^\text{ext}$ it is necessary that it be facet-defining for the projection $\mathcal{P}^k$. Example 2 shows that this condition is not sufficient in general. That is, an inequality which is facet-defining for $\mathcal{P}^k$ for some $k \in \mathcal{K}$ is not necessarily facet-defining for $\mathcal{S}^\text{ext}$.

Example 2. Consider $\mathcal{N} = \mathcal{N}^+ = \{1,2,3,4\}$ and $\mathcal{N}^- = \emptyset$. Let there be two scenarios $(\mathcal{K} = \{1,2\})$, with data given by $u^1 = (1, 1, 1, 1)$, $u^2 = (1.1, 1.1, 1.1, 0.9)$, and $d^1 = d^2 = 3$. It is easy to verify that $\mathcal{D}^k_{\text{fc}} = \mathcal{P}^k_{\text{fc}}$ for $k = 1, 2$. Furthermore, the inequality $\sum_{i=1}^4 x_i \geq 3$ is facet-defining for $\mathcal{D}^2_{\text{fc}} = \mathcal{P}^2_{\text{fc}}$, but it is not facet-defining for $\mathcal{S}^\text{ext}_{\text{fc}}$.

Theorem 1, our main result, shows when an inequality of the form (FI) that is facet-defining for the single-scenario projection $\mathcal{P}^k_{\text{fc}}$ is also facet-defining for the extensive polytope $\mathcal{S}^\text{ext}_{\text{fc}}$.

Theorem 1. Suppose that, for some scenario $k \in \mathcal{K}$, the inequality (FI) is valid for $\mathcal{D}^k_{\text{fc}}$, and both of the following conditions hold:
1. The inequality (FI) is facet-defining for $\mathcal{P}_{\text{FC}}^k$, and

2. For all $i \in \mathcal{N}$ and $\ell \in \mathcal{K} \setminus \{k\}$, there exists a point $(x, y^1, \ldots, y^K) \in X_{\text{FC}}^{\text{ext}}$ that satisfies (FI) with equality such that $x_i = 1$ and $\sum_{j \in \mathcal{N}^+} u^\ell_j x_j > d^\ell$.

Then (FI) is facet-defining for $\mathcal{S}_{\text{FC}}^{\text{ext}}$.

Moreover, if (FI) is not a scalar multiple of $y_i^k \geq 0$ for any $i \in \mathcal{N}^-$ nor a scalar multiple of $y_i^k \leq u_i^k x_i$ for any $i \in \mathcal{N}^+$, then Conditions 1 and 2 above are also necessary.

The following two lemmas will be used to prove Theorem 1. Their proofs can be found in the electronic companion.

**Lemma 2.** Suppose $\{x^{(i)} \in \mathbb{R}^M \mid i = 1, \ldots, n\}$ is a set of $n$ affinely independent points, and let $\lambda_1, \ldots, \lambda_n$ be scalars such that $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_1 \neq 0$. Define $\hat{x} = \sum_{i=1}^n \lambda_i x^{(i)}$. Then $\hat{x}, x^{(2)}, \ldots, x^{(n)}$ are affinely independent.

**Lemma 3.** Suppose $\{(f^{(i)}, g^{(i)}) \in \mathbb{R}^{M_1+M_2} \mid i = 1, \ldots, n\}$ is a set of $n$ points such that $f^{(1)}, \ldots, f^{(n)}$ are affinely independent. Suppose $h^{(1)}, \ldots, h^{(m)}$ are $m$ linearly independent points in $\mathbb{R}^{M_2}$. Then for any $s \in \{1, \ldots, n\}$, the $n + m$ points

\[
\begin{pmatrix}
  f^{(1)} \\
  g^{(1)} \\
  \vdots \\
  f^{(n)} \\
  g^{(n)} \\
  \vdots \\
  f^{(s)} + h^{(1)} \\
  \vdots \\
  f^{(s)} + h^{(m)}
\end{pmatrix}
\]

are affinely independent.

In the proof of Theorem 1, we let $\vec{e}_{ik}$ denote the standard basis vector in $\mathbb{R}^{N(K-1)}$, with $y^k = e_i$ and $y^\ell = 0$ for all scenarios $\ell \neq k$, where $e_i$ is the $i$th standard basis vector in $\mathbb{R}^N$.

**Proof.** Proof of Theorem 1.

**Sufficiency** We will construct $N(K+1)$ affinely independent points in $\mathcal{S}_{\text{FC}}^{\text{ext}}$ that satisfy (FI) with equality. We construct these points in two separate sets, Set 1 containing $N(K-1)$ points, and Set 2 containing $2N$ points. In our construction, we use $\ell \in \mathcal{K} \setminus \{k\}$ to index points in $X_{\text{FC}}^{\text{ext}}$, and use $r \in \mathcal{K}$ to index second-stage variables of points in $X_{\text{FC}}^{\text{ext}}$.

**Set 1:** Fix $(i, \ell) \in \mathcal{N} \times \mathcal{K} \setminus \{k\}$, let $(x^{(i,\ell)}, y^{1,(i,\ell)}, \ldots, y^{K,(i,\ell)}) \in X_{\text{FC}}^{\text{ext}}$ denote the point whose existence is asserted by Condition 2 of the theorem, and define the positive constant $\varepsilon_{it} = \min \{u^\ell_i, \sum_{j \in \mathcal{N}^+} u^\ell_j x^{(i,\ell)}_j - d^\ell \} > 0$ (recall that $u^\ell_i > 0$ by (A1)). For all $r \in \mathcal{K}$, define
the vector $z^{r,(i,\ell)} \in \mathbb{R}^N$ by $z^{r,(i,\ell)} = y^{r,(i,\ell)}$ for $r \neq \ell$, and

$$z_j^{\ell,(i,\ell)} = \begin{cases} 
  u_\ell^i - \varepsilon_{i\ell}, & \text{if } i \in \mathcal{N}^+ \text{ and } j = i, \\
  \varepsilon_{i\ell}, & \text{if } i \in \mathcal{N}^- \text{ and } j = i, \\
  u_j^{x_j^{(i,\ell)}}, & \text{if } j \in \mathcal{N}^+ \text{ and } j \neq i, \\
  0, & \text{otherwise (i.e., } j \in \mathcal{N}^- \text{ and } j \neq i). 
\end{cases}$$

For brevity, let $z^{(i,\ell)} := (z^{1,(i,\ell)}, \ldots, z^{K,(i,\ell)}) \in \mathbb{R}^{NK}$.

We now show that the point $(x^{(i,\ell)}, z^{(i,\ell)})$ satisfies (FI) with equality and lies in $X_{\text{FC}}^{\text{ext}}$. The former is true because $\ell \neq k$ implies $z^{k,(i,\ell)} = y^{k,(i,\ell)}$, and $(x^{(i,\ell)}, y^{k,(i,\ell)})$ satisfies (FI) with equality by Condition 2. Because $x^{(i,\ell)}$ is integral, to show $(x^{(i,\ell)}, z^{(i,\ell)})$ lies in $X_{\text{FC}}^{\text{ext}}$, it suffices to show that $(x^{(i,\ell)}, z^{k,(i,\ell)}) \in X_{\text{FC}}^{r}$ for all scenarios $r \in \mathcal{K}$. If $r \neq \ell$, then $z^{r,(i,\ell)} = y^{r,(i,\ell)}$, and thus $(x^{(i,\ell)}, z^{r,(i,\ell)}) \in X_{\text{FC}}^{r}$ by Condition 2.

Otherwise, if $r = \ell$, we show that $(x^{(i,\ell)}, z^{\ell,(i,\ell)})$ satisfies both the variable upper bound constraints and the flow constraint:

- **Variable upper bound constraints**: For $j \neq i$, $z_j^{\ell,(i,\ell)}$ is either 0 or $u_j^{x_j^{(i,\ell)}}$. In either case, $z_j^{\ell,(i,\ell)} \leq u_j^{x_j^{(i,\ell)}}$. For $j = i$, Condition 2 gives that $x_i^{(i,\ell)} = 1$. The coordinate $z_i^{\ell,(i,\ell)}$ is either $\varepsilon_{i\ell}$ or $u_\ell^i - \varepsilon_{i\ell}$, both of which lie between 0 and $u_\ell^i$.

- **Flow constraint**: If $i \in \mathcal{N}^+$, then $\sum_{j \in \mathcal{N}^+} z_j^{\ell,(i,\ell)} = -\varepsilon_{i\ell} + \sum_{j \in \mathcal{N}^+} u_j^{x_j^{(i,\ell)}}$ and $\sum_{j \in \mathcal{N}^-} z_j^{\ell,(i,\ell)} = 0$. Otherwise, if $i \in \mathcal{N}^-$, then $\sum_{j \in \mathcal{N}^+} z_j^{\ell,(i,\ell)} = \sum_{j \in \mathcal{N}^+} u_j^{x_j^{(i,\ell)}}$, and $\sum_{j \in \mathcal{N}^-} z_j^{\ell,(i,\ell)} = \varepsilon_{i\ell}$. In either case, $\sum_{j \in \mathcal{N}^+} z_j^{\ell,(i,\ell)} \geq -\varepsilon_{i\ell} + \sum_{j \in \mathcal{N}^+} u_j^{x_j^{(i,\ell)}} = d^\ell$. Because $\varepsilon_{i\ell} = \min\{u_\ell^i, \sum_{j \in \mathcal{N}^+} u_j^{x_j^{(i,\ell)}} - d^\ell\}$, it follows that $-\varepsilon_{i\ell} + \sum_{j \in \mathcal{N}^+} u_j^{x_j^{(i,\ell)}} \geq d^\ell$.

We conclude that the point $(x^{(i,\ell)}, z^{(i,\ell)})$ lies in $X_{\text{FC}}^{\text{ext}}$ and satisfies (FI) with equality.

Next, note that if $i \in \mathcal{N}^+$, we may add $\varepsilon_{i\ell}$ to the $i$th entry of $z^{\ell,(i,\ell)}$, and the vector $(x^{(i,\ell)}, z^{(i,\ell)})$ will remain in $X_{\text{FC}}^{\text{ext}}$. Similarly, if $i \in \mathcal{N}^-$, we may subtract $\varepsilon_{i\ell}$ from the $i$th entry of $z^{\ell,(i,\ell)}$ and the vector $(x^{(i,\ell)}, z^{(i,\ell)})$ will remain in $X_{\text{FC}}^{\text{ext}}$. Hence, if $i \in \mathcal{N}^+$, the vector

$$\frac{1}{N(K-1)} \left( x^{(i,\ell)} + \varepsilon_{i\ell} \overline{z}^{(i,\ell)} \right) + \frac{1}{N(K-1)} \sum_{(j,r) \in \mathcal{N} \times \mathcal{K} \setminus \{k\}} \left( x^{(j,r)} + \varepsilon_{j\ell} \overline{z}^{(j,r)} \right)$$

lies in $S_{\text{FC}}^{\text{ext}}$ as a convex combination of points in $X_{\text{FC}}^{\text{ext}}$, and satisfies (FI) with equality, as a
convex combination of points that satisfy (FI) with equality. Similarly, if \( i \in \mathcal{N}^- \), the point

\[
\frac{1}{N(K-1)} \left( x^{(i,\ell)} - \varepsilon_{i\ell} \bar{e}_{i\ell} \right) + \frac{1}{N(K-1)} \sum_{\ell \neq (i,\ell)} \left( x^{(j,\ell)} - \varepsilon_{j\ell} \bar{e}_{j\ell} \right)
\]

lies in \( \mathcal{S}_{\text{ext}} \) and satisfies (FI) with equality. Define the midpoint of the vectors \( (x^{(i,\ell)}, z^{(i,\ell)}) \) as

\[
\left( \hat{x}, \hat{z} \right) := \frac{1}{N(K-1)} \sum_{(i,\ell) \in \mathcal{N} \setminus \{k\}} \left( x^{(i,\ell)}, z^{(i,\ell)} \right).
\]

Set 1 is given by the \( N(K-1) \) points

\[
\left\{ \left( \hat{x} + \frac{1}{N(K-1)} \varepsilon_{i\ell} \bar{e}_{i\ell} \right) : i \in \mathcal{N}^+, \ell \in \mathcal{K} \setminus \{k\} \right\} \cup \left\{ \left( \hat{x} - \frac{1}{N(K-1)} \varepsilon_{i\ell} \bar{e}_{i\ell} \right) : i \in \mathcal{N}^-, \ell \in \mathcal{K} \setminus \{k\} \right\}.
\]

We emphasize that the points (P1) lie in \( \mathcal{S}_{\text{ext}} \) and satisfy (FI) with equality.

**Set 2:** By Condition 1, there exist \( 2N \) affinely independent points \( \{(\hat{x}^{(q)}, \hat{y}^{k,(q)})\}_{q=1}^{2N} \) in \( \text{proj}_{(x,y)}(X_{\text{ext}}) \) that satisfy (FI) with equality. Hence, the affine hull of these points equals \( \{(x, y) \in \mathbb{R}^{2N} | \alpha^T x + \beta^T y = \tau \} \). In particular, if we let \( \hat{z}^k \) denote the scenario \( k \) component of the vector \( \hat{z} \) in (4), then the point \( (\hat{x}, \hat{z}^k) \) can be expressed as an affine combination of the points \( \{(\hat{x}^{(q)}, \hat{y}^{k,(q)})\} \), and thus, by Lemma 2, the \( 2N \) points

\[
\left( \begin{array}{c} \hat{x} \\ \hat{z}^1 \\ \vdots \\ \hat{z}^K \end{array} \right), \left( \begin{array}{c} \hat{x}^{(2)} \\ \hat{y}^{1,(2)} \\ \vdots \\ \hat{y}^{K,(2)} \end{array} \right), \ldots, \left( \begin{array}{c} \hat{x}^{(2N)} \\ \hat{y}^{1,(2N)} \\ \vdots \\ \hat{y}^{K,(2N)} \end{array} \right)
\]

are affinely independent. Moreover, for all \( q = 2, \ldots, 2N \), \( (\hat{x}^{(q)}, \hat{y}^{k,(q)}) \in \text{proj}_{(x,y)}(X_{\text{ext}}) \) implies that, for all \( \ell \in \mathcal{K} \setminus \{k\} \) there exists \( \hat{y}^{\ell,(q)} \in \mathbb{R}^N \) such that \( (\hat{x}^{(q)}, \hat{y}^{1,(q)}, \ldots, \hat{y}^{k,(q)}, \ldots, \hat{y}^{K,(q)}) \in X_{\text{ext}} \). Hence, the \( 2N \) points

\[
\left( \begin{array}{c} \hat{x} \\ \hat{z}^1 \\ \vdots \\ \hat{z}^K \end{array} \right), \left( \begin{array}{c} \hat{x}^{(2)} \\ \hat{y}^{1,(2)} \\ \vdots \\ \hat{y}^{K,(2)} \end{array} \right), \ldots, \left( \begin{array}{c} \hat{x}^{(2N)} \\ \hat{y}^{1,(2N)} \\ \vdots \\ \hat{y}^{K,(2N)} \end{array} \right)
\]

lie in \( \mathcal{S}_{\text{ext}} \) and satisfy (FI) with equality. The points (P2) form Set 2.

The \( N(K-1) \) points (P1) and the \( 2N \) points (P2) all lie in \( \mathcal{S}_{\text{ext}} \) and satisfy (FI) with equality. To show that they are affinely independent, we apply Lemma 3 as follows. Take \( n = M_1 = 2N \), \( m = M_2 = N(K-1) \), \( f^{(1)} = (\hat{x}, \hat{z}^k) \), and \( f^{(q)} = (\hat{x}^{(q)}, \hat{y}^{k,(q)}) \) for \( q = 2, \ldots, 2N \) (note that the points \( f^{(q)} \) are affinely independent for \( q = 1, \ldots, 2N \)). Take the point \( g^{(1)} = (\hat{z}^1, \ldots, \hat{z}^{k-1}, \hat{z}^{k+1}, \ldots, \hat{z}^K) \in \mathbb{R}^{N(K-1)} \) and \( g^{(q)} = (\hat{y}^{1,(q)}, \ldots, \hat{y}^{k-1,(q)}, \hat{y}^{k+1,(q)}, \ldots, \hat{y}^{K,(q)}) \in \mathbb{R}^{N(K-1)} \) for \( q = 2, \ldots, 2N \).
\( \mathbb{R}^{N(K-1)} \) for \( q = 2, \ldots, 2N \). The point \((f^{(s)}, g^{(s)})\) in Lemma 3 is given by \((\hat{x}, \hat{z})\), and the points \( h^{(j)} \) are given by \( \pm \frac{1}{N(K-1)} \epsilon_\ell \epsilon_{\ell'} \) (cf. (P1)), which are non-zero scalar multiples of distinct standard basis vectors, and thus linearly independent. Hence, the \( N(K + 1) \) points (P1) and (P2) are affinely independent, and we conclude that (FI) is facet-defining for \( \mathcal{S}_{\text{ext}}^{\text{ext}} \), completing the proof of sufficiency.

**Necessity:** Suppose that (FI) is facet-defining for \( \mathcal{S}_{\text{ext}}^{\text{ext}} \). Condition 1 follows from Proposition 6. To show that Condition 2 holds, fix \( i \in \mathcal{N} \) and \( \ell \in \mathcal{K} \setminus \{k\} \). Suppose first that \( i \in \mathcal{N}^+ \). Because (FI) is not a scalar multiple of the inequality \( y_i^\ell \leq u_i^\ell x_i \), there exists a point \((\hat{x}, \hat{y}_1^\ell, \ldots, \hat{y}_K^\ell) \in X_{\text{ext}}^{\text{ext}} \) that satisfies (FI) with equality such that \( \hat{y}_i^\ell < u_i^\ell \hat{x}_i \) (in particular, \( \hat{x}_i = 1 \)). Hence

\[
\sum_{j \in \mathcal{N}^+} u_j^\ell \hat{x}_j > \sum_{j \in \mathcal{N}^+} \hat{y}_j \geq \sum_{j \in \mathcal{N}^+} \hat{y}_j^\ell - \sum_{j \in \mathcal{N}^-} \hat{y}_j^\ell \geq d_j^\ell,
\]

as desired. If, on the other hand, \( i \in \mathcal{N}^- \), then, because (FI) is not a scalar multiple of \( y_i^\ell \geq 0 \), there exists a point \((\hat{x}, \hat{y}_1^\ell, \ldots, \hat{y}_K^\ell) \in X_{\text{ext}}^{\text{ext}} \) that satisfies (FI) with equality such that \( \hat{y}_i^\ell > 0 \) (in particular, \( \hat{x}_i = 1 \)). Hence

\[
\sum_{j \in \mathcal{N}^+} u_j^\ell \hat{x}_j > \sum_{j \in \mathcal{N}^+} \hat{y}_j \geq \sum_{j \in \mathcal{N}^+} \hat{y}_j^\ell - \sum_{j \in \mathcal{N}^-} \hat{y}_j^\ell \geq d_j^\ell,
\]

concluding the proof of necessity.

In the case when the inequality (FI) satisfies \( \beta \equiv 0 \)—i.e., the only non-zero coefficients in the inequality are on the binary \( x \) variables—we can say more.

**Corollary 4.** An inequality of the form \( \alpha^T x \geq \tau \) is facet-defining for \( \mathcal{S}_{\text{ext}}^{\text{ext}} \) if and only if both of the following conditions hold:

1. The inequality is facet-defining for \( \text{proj}_x(\mathcal{S}_{\text{ext}}^{\text{ext}}) \).

2. For all \( i \in \mathcal{N} \) and \( \ell \in \mathcal{K} \), there exists a point \((x, y_1^\ell, \ldots, y_K^\ell) \in X_{\text{ext}}^{\text{ext}} \) that satisfies \( \alpha^T x = \tau \), \( x_i = 1 \) and \( \sum_{j \in \mathcal{N}^+} u_j^\ell x_j > d_j^\ell \).

**Proof.** The proof of sufficiency is similar to the proof of Theorem 1, except that the first set of points (P1) now contains \( NK \) points (one for each \( i \in \mathcal{N} \) and each \( \ell \in \mathcal{K} \)), and the second set of points (P2) contains only \( N \) points, because \( \text{proj}_x(\mathcal{S}_{\text{ext}}^{\text{ext}}) \subseteq \mathbb{R}^N \) (rather than \( \mathcal{P}_{\text{PC}}^k \subseteq \mathbb{R}^{2N} \) in Theorem 1). The proof of necessity is also similar to that of Theorem 1. \( \square \)

Proposition 10 is an important implication of Theorem 1.
Proposition 10. Suppose that \( d^k < 0 \) for all \( k \in \mathcal{K} \). Then for any \( k \in \mathcal{K} \), any inequality (FI) that is facet-defining for the deterministic polytope \( \mathcal{D}^k_{\text{FC}} \) is facet-defining for the extensive polytope \( \mathcal{S}^\text{ext}_{\text{FC}} \).

Proof. Fix \( k \in \mathcal{K} \). It suffices to show that Conditions 1 and 2 of Theorem 1 hold. By Remark 2, \( d^k < 0 \) implies \( \mathcal{P}^k_{\text{FC}} = \mathcal{D}^k_{\text{FC}} \) for all \( k \in \mathcal{K} \). So Condition 1 of Theorem 1 is satisfied. Moreover, because \( d^k < 0 \) for all \( k \in \mathcal{K} \), Condition 2 is equivalent to showing that, for all \( i \in \mathcal{N} \), there exists a point \((\hat{x}, \hat{y}^1, \ldots, \hat{y}^K) \in \mathcal{X}^\text{ext}_{\text{FC}} \) that satisfies (FI) with equality such that \( \hat{x}_i = 1 \). Fix \( i \in \mathcal{N} \). Because (FI) is facet-defining for \( \mathcal{P}^k_{\text{FC}} \), there exists a point \((\hat{x}, \hat{y}^k) \in \text{proj}_{(x,y^k)}(\mathcal{X}^\text{ext}_{\text{FC}}) \) that satisfies (FI) with equality such that \( \hat{x}_i = 1 \) (this follows because \( x_i \geq 0 \) can never be facet-defining for \( \text{proj}_{(x,y^k)}(\mathcal{X}^\text{ext}_{\text{FC}}) \) due to the variable upper bound constraints). Because \((\hat{x}, \hat{y}^k) \in \text{proj}_{(x,y^k)}(\mathcal{X}^\text{ext}_{\text{FC}}) \), for all scenarios \( \ell \neq k \) there exists \( \hat{y}^\ell \in \mathbb{R}^N \) such that \((\hat{x}, \hat{y}^1, \ldots, \hat{y}^k, \ldots, \hat{y}^K) \in \mathcal{X}^\text{ext}_{\text{FC}} \). Moreover, this point satisfies (FI) with equality, as well as \( \hat{x}_i = 1 \). Hence, Condition 2 of Theorem 1 is satisfied, and we conclude that (FI) is facet-defining for \( \mathcal{S}^\text{ext}_{\text{FC}} \). \( \square \)

We illustrate the utility of Theorem 1 and Proposition 10 with the following example.

Example 3 (Stochastic Facility Location). Define the stochastic capacitated facility location problem as follows (see Snyder (2006) for a review): consider facilities \( i = 1, \ldots, n \), and customers \( j = 1, \ldots, m \). Let \( b^k_j > 0 \) denote the demand by customer \( j \) under scenario \( k \in \mathcal{K} \), and let \( c^k_i > 0 \) denote the capacity of facility \( i \) under scenario \( k \in \mathcal{K} \), where \( \mathcal{K} \) is finite. Let the variable \( y^k_{ij} \) denote the amount of customer \( j \)'s demand fulfilled by facility \( i \) under scenario \( k \). Let the variable \( x_i \in \{0, 1\} \) denote whether facility \( i \) is opened (a decision made before the facility capacities and customer demands are known). The feasible region of the stochastic (capacitated) facility location problem is defined by the constraints

\[
\begin{align*}
\sum_{i=1}^{n} y^k_{ij} &= b^k_j \quad \text{for all } j = 1, \ldots, m \text{ and for all } k \in \mathcal{K}, \quad (5a) \\
\sum_{j=1}^{m} y^k_{ij} &\leq c^k_i x_i \quad \text{for all } i = 1, \ldots, n \text{ and for all } k \in \mathcal{K}, \quad (5b) \\
y^k_{ij} &\geq 0 \quad \text{for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, m, \text{ and for all } k \in \mathcal{K}, \quad (5c) \\
x_i &\in \{0, 1\} \quad \text{for all } i = 1, \ldots, n. \quad (5d)
\end{align*}
\]

Models related to (5) arise in the literature (e.g., Louveaux and Peeters (1992), Santoso et al. (2005), Snyder (2006)). If we consider only a single customer \( j \), then (5) can be relaxed into a S-SFCF polytope \( \mathcal{S}^\text{ext}_{\text{FC}} \), with \( \mathcal{N} = \mathcal{N}^+ = \{1, \ldots, n\} \) \( (\mathcal{N}^- = \emptyset) \), \( d^k = b^k_j \) for all \( k \in \mathcal{K} \) and \( u^k_i = c^k_i \) for all \( i \in \mathcal{N} \) and \( k \in \mathcal{K} \). Relax the equality constraint \( \sum_i y^k_{ij} = b^k_j \) into the inequality \( \sum_i y^k_{ij} \leq b^k_j \). By Proposition 10 and Remark 1, because
\[ d^k = b^k_j > 0 \] for all scenarios \( k \in \mathcal{K} \), any facet-defining inequality for the \( (\leq\text{-constrained}) \) single-scenario, single-customer SFCF polytope is facet-defining for the S-SFCF polytope. Hence, Proposition 10 gives guarantees about the strength of single-scenario inequalities.

When only the demands \( d^k \) are uncertain, the S-SFCF polytope may be viewed as representing a single node within a larger network facing demand uncertainty (for example, as a single customer in the stochastic facility location problem above). In this setting, Proposition 11 tells us that the facet-defining inequalities corresponding to the most restrictive (i.e., highest demand) scenario are also facet-defining for the extensive polytope.

**Proposition 11.** Consider an S-SFCF polytope \( \mathcal{S}^\text{ext}_{\text{FC}} \) that satisfies \( u_i^k = u_i \) for all \( k \in \mathcal{K} \) and \( i \in \mathcal{N} \), and let \( k_0 \in \arg \max \{ d^k \mid k \in \mathcal{K} \} \). Then every facet-defining inequality for \( \mathcal{D}^k_{\text{FC}} \) is also facet-defining for \( \mathcal{S}^\text{ext}_{\text{FC}} \).

**Proof.** Suppose that (FI) is facet-defining for \( \mathcal{D}^k_{\text{FC}} \) (and hence, by Proposition 3, for \( \mathcal{S}^k_{\text{FC}} \)). If (FI) is a scalar multiple of the inequality \( y_i^k \geq 0 \) (for some \( i \in \mathcal{N}^- \)) or \( y_i^k \leq u_i^k x_i \) (for some \( i \in \mathcal{N}^+ \)), then by Proposition 8, we are done. Hence, for the remainder of the proof we assume that (FI) is not a scalar multiple of the inequalities \( y_i^k \geq 0 \) for any \( i \in \mathcal{N}^- \) or \( y_i^k \leq u_i^k x_i \) for any \( i \in \mathcal{N}^+ \).

We next claim that \( \mathcal{D}^k_{\text{FC}} = \mathcal{P}^k_{\text{FC}} \). By Proposition 2, \( \mathcal{P}^k_{\text{FC}} \subseteq \mathcal{D}^k_{\text{FC}} \). To establish \( \subseteq \)"", fix \( (\bar{x}, \bar{z}^{k_0}) \in X_{\text{FC}}^{k_0} \). Then

\[
\sum_{j \in \mathcal{N}^+} u_j \bar{x}_j \geq \sum_{j \in \mathcal{N}^+} \bar{z}_j^{k_0} \geq \sum_{j \in \mathcal{N}^+} \bar{z}_j^{k_0} - \sum_{j \in \mathcal{N}^-} \bar{z}_j^{k_0} \geq d^{k_0} \geq d^k,
\]

where the last inequality holds for any scenario \( k \in \mathcal{K} \). By Proposition 9, \( (\bar{x}, \bar{z}^{k_0}) \in \text{proj}_{(x,y)^{k_0}}(X_{\text{FC}}^{\text{ext}}) \), and thus \( X_{\text{FC}}^{k_0} \subseteq \text{proj}_{(x,y)^{k_0}}(X_{\text{FC}}^{\text{ext}}) \Rightarrow \mathcal{D}^k_{\text{FC}} \subseteq \mathcal{P}^k_{\text{FC}} \Rightarrow \mathcal{P}^k_{\text{FC}} = \mathcal{D}^k_{\text{FC}} \). Because \( \mathcal{D}^k_{\text{FC}} = \mathcal{P}^k_{\text{FC}} \), Condition 1 of Theorem 1 is satisfied. To show that Condition 2 is also satisfied, it suffices to show that for all \( i \in \mathcal{N} \) there exists a point \( (\hat{x}, \hat{y}_1, \ldots, \hat{y}^K) \in X_{\text{FC}}^{\text{ext}} \) (depending on \( i \)) that satisfies (FI) with equality such that \( \hat{x}_i = 1 \) and \( \sum_{j \in \mathcal{N}^+} u_j \hat{x}_j > d^{k_0} \). Fix \( i \in \mathcal{N} \). If \( i \in \mathcal{N}^+ \), then (because (FI) is facet-defining for \( \mathcal{P}^k_{\text{FC}} \)) and is not a scalar multiple of \( y^{k_0} \leq u_i x_i \), there exists a point \( (\hat{x}, \hat{y}^{k_0}) \in \text{proj}_{(x,y)^{k_0}}(X_{\text{FC}}^{\text{ext}}) \) that satisfies (FI) with equality, and \( \hat{y}^{k_0} < u_i \hat{x}_i \) (in particular, \( \hat{x}_i = 1 \)). Thus

\[
\sum_{j \in \mathcal{N}^+} u_j \hat{x}_j > \sum_{j \in \mathcal{N}^+} \hat{y}_j^{k_0} \geq \sum_{j \in \mathcal{N}^+} \hat{y}_j^{k_0} - \sum_{j \in \mathcal{N}^-} \hat{y}_j^{k_0} \geq d^{k_0}.
\]

Because \( (\hat{x}, \hat{y}^{k_0}) \in \text{proj}_{(x,y)^{k_0}}(X_{\text{FC}}^{\text{ext}}) \), for all scenarios \( \ell \neq k_0 \), there exist \( \hat{y}^\ell \in \mathbb{R}^N \) such that \( (\hat{x}, \hat{y}_1^\ell, \ldots, \hat{y}^K_\ell) \in X_{\text{FC}}^{\text{ext}} \). Moreover, this point satisfies (FI) with equality, \( \hat{x}_i = 1 \), and \( \sum_{j \in \mathcal{N}^+} u_j \hat{x}_j > d^{k_0} \), and thus satisfies Condition 2 of Theorem 1. A similar argument follows if \( i \in \mathcal{N}^- \) (cf. the proof of necessity in Theorem 1), and we conclude by Theorem 1 that (FI)
is facet-defining for $S_{	ext{FC}}^{\text{ext}}$.

Finally, we show that Corollary 4 has implications for the deterministic setting as well. In particular, Corollary 4 generalizes a theorem of Padberg et al. (1985), which relates the deterministic SFCF polytope $\mathcal{S}$ with the so-called “associated knapsack polytope.” In fact, the associated knapsack polytope is simply the set $\text{proj}_x(S_{	ext{FC}}^{\text{ext}})$ in the case when $K = 1$.

**Corollary 5.** (Theorem 2 of Padberg et al. (1985)) Suppose $N^- = 0$, and that the inequality $\alpha^T x \geq \tau$ is facet-defining for $\text{proj}_x(S_{	ext{FC}})$. Then $\alpha^T x \geq \tau$ is facet-defining for $S_{	ext{FC}}$ if and only if for all $i \in N$, there exists a point $(x^{(i)}, y^{(i)}) \in X_{	ext{FC}}$ that satisfies $\alpha^T x^{(i)} = \tau$, $x_i^{(i)} = 1$, and $y_i^{(i)} < u_i$.

**Proof.** Apply Corollary 4 with $K = 1$ and $N^- = 0$. □

## 5. Computational Results

We now illustrate the effect of using single-scenario valid inequalities (FI) when solving the extensive form of a SMIP. Our chosen test problem is a stochastic version of the well-known capacitated fixed-charge network design problem, which we denote by S-CND. Specifically, we will explore how the addition of inequalities of the form (FI) for single-node relaxations of the S-CND problem improve the tightness of the linear programming relaxation of the extensive form.

Consider a directed network with node set $V$ and arc set $A$, and let $K$ be a finite set of scenarios. For each node $i \in V$, denote by $A^-_i$ and $A^+_i$ the set of incoming arcs to node $i$ and the set of outgoing arcs from node $i$, respectively. Let $x_e$ for $e \in A$ be a binary variable indicating whether arc $e$ is activated, and let $y^k_e$ be the flow on arc $e \in A$ under scenario $k \in K$. The S-CND problem is

$$\begin{align*}
\min & \quad \sum_{e \in A} c_e x_e + \sum_{k \in K} p_k \sum_{e \in A} f^k_e y^k_e, \\
\text{s.t.} & \quad \sum_{e \in A^-_i} y^k_e - \sum_{e \in A^+_i} y^k_e = d^k_i \quad \text{for all } i \in V, k \in K, \\
& \quad 0 \leq y^k_e \leq u^k_e x_e \quad \text{for all } e \in A, k \in K, \\
& \quad x_e \in \{0, 1\} \quad \text{for all } e \in A,
\end{align*}$$

where parameters $c_e$ are the activation costs for arc $e \in A$; $f^k_e$ and $u^k_e$ are the unit flow cost and the capacity for arc $e \in A$ under scenario $k \in K$, respectively, and $d^k_i$ is the demand (or supply) of node $i \in V$ under scenario $k \in K$. The value $p_k \in [0, 1]$ denote the probability of scenario $k \in K$ occurring.
We derive single-scenario inequalities (FI) for single-node relaxations of (S-CND). Specifically, for each scenario \( k \in \mathcal{K} \) and node \( i \in \mathcal{V} \) with \( d_i^k \neq 0 \), we derive valid inequalities for the SFCF polytope

\[
\mathcal{R}_k^+(i) = \text{conv}\left\{ x_e \in \{0,1\}, \ e \in \mathcal{A}_i^+ \cup \mathcal{A}_i^- \mid \sum_{e \in \mathcal{A}_i^+} y_e^k - \sum_{e \in \mathcal{A}_i^-} y_e^k \leq d_i^k, \ \text{if} \ d_i^k > 0, \text{and the SFCF polytope} \right. \\
\left. y_e^k \in \mathbb{R}_+, \ e \in \mathcal{A}_i^+ \cup \mathcal{A}_i^- \mid 0 \leq y_e^k \leq u_e^k x_e \text{ for all } e \in \mathcal{A}_i^+ \cup \mathcal{A}_i^- \right\},
\]

if \( d_i^k > 0 \), and the SFCF polytope

\[
\mathcal{R}_k^-(i) = \left\{ x_e \in \{0,1\}, \ e \in \mathcal{A}_i^+ \cup \mathcal{A}_i^- \mid \sum_{e \in \mathcal{A}_i^+} y_e^k - \sum_{e \in \mathcal{A}_i^-} y_e^k \leq -d_i^k, \ \text{if} \ d_i^k < 0, \text{and the SFCF polytope} \right. \\
\left. y_e^k \in \mathbb{R}_+, \ e \in \mathcal{A}_i^+ \cup \mathcal{A}_i^- \mid 0 \leq y_e^k \leq u_e^k x_e \text{ for all } e \in \mathcal{A}_i^+ \cup \mathcal{A}_i^- \right\},
\]

if \( d_i^k < 0 \).

We compare the following four well-known families of valid inequalities for the relaxations \( \mathcal{R}_k^+(i) \) and \( \mathcal{R}_k^-(i) \). For each family of inequalities, we describe the inequalities for the polytope \( \mathcal{R}_k^+(i) \) (the polytope \( \mathcal{R}_k^-(i) \) is similar).

1. **Cover inequalities** (Balas, 1975; Hammer et al., 1975; Wolsey, 1975). A set \( C \subseteq \mathcal{A}_i^+ \) is a *cover* if \( \sum_{e \in C} u_e^k > d_i^k \). For any cover \( C \), the inequality \( \sum_{e \in C} x_e \leq |C| - 1 \) is valid for \( \mathcal{R}_k^+(i) \). We separate the cover inequalities exactly by solving a knapsack problem (Crowder et al., 1983). Note that the cover inequalities are of the form (FI) with \( \beta = 0 \) (cf. Corollary 4).

2. **Generalized flow cover (GFC) inequalities** (Van Roy and Wolsey, 1986). A pair \( (C^+, C^-) \) with \( C^+ \subseteq \mathcal{A}_i^+ \) and \( C^- \subseteq \mathcal{A}_i^- \) is a *flow cover* if \( \sum_{e \in C^+} u_e^k - \sum_{e \in C^-} u_e^k = d_i^k + \lambda \) for \( \lambda > 0 \). For any \( L^- \subseteq \mathcal{A}_i^- \setminus C^- \), the generalized flow cover (GFC) inequality

\[
\sum_{e \in C^+} [y_e^k + (u_e^k - \lambda)(1 - x_e)] \leq d_i^k + \sum_{e \in C^-} u_e^k + \sum_{e \in L^-} \lambda x_e + \sum_{e \in \mathcal{A}_i^- \setminus (L^- \cup C^-)} y_e^k,
\]

is valid for \( \mathcal{R}_k^+(i) \). We separate the GFC inequalities using the heuristic procedure outlined in Van Roy and Wolsey (1987).

3. **Extended generalized flow cover (ExGFC) inequalities** (Van Roy and Wolsey, 1986). Given a flow cover \( (C^+, C^-) \) and sets \( L^+ \subseteq \mathcal{A}_i^+ \setminus C^+, L^- \subseteq \mathcal{A}_i^- \setminus C^- \), the extended
generalized flow cover (ExGFC) inequality

\[
\sum_{e \in C^+ \cup L^+} y_e + \sum_{e \in C^+} (u^k_e - \lambda)^+(1 - x_e) - \sum_{e \in L^+} (\bar{u}^k_e - \lambda)^+ x_e - \sum_{e \in A_i^- \setminus (L^- \cup C^-)} y_e \\
\leq d^k_i + \sum_{e \in C^-} u^k_e - \sum_{e \in C^-} \min \{\lambda, (u^k_e - (\bar{u}^k_e - \lambda))^+\}(1 - x_e) + \sum_{e \in L^-} \max \{\lambda, u^k_e - (\bar{u} - \lambda)\} x_e,
\]

where \(\bar{u} = \max_{e \in C^+} \{u^k_e\}\), \(\bar{u}^k_e = \max \{\bar{u}, u^k_e\}\) and \(\bar{u} > \lambda > 0\) is valid for \(\mathcal{R}^k_+(i)\). We separate the ExGFC inequalities using the GFC separation heuristic mentioned above (Van Roy and Wolsey, 1987).

4. Flow pack inequalities (Stallaert, 1997; Atamtürk, 2001). A pair \((C^+, C^-)\) with \(C^+ \subseteq \mathcal{A}_i^+\) and \(C^- \subseteq \mathcal{A}_i^-\) is a flow pack if \(\sum_{e \in C^+} u^k_e - \sum_{e \in C^-} u^k_e + \mu = d^k_i\) for \(\mu > 0\). For any \(L^+ \subseteq \mathcal{A}_i^+ \setminus C^+\), the flow pack inequality

\[
\sum_{e \in C^+} y^k_e + \sum_{e \in L^+} [y^k_e - \min \{\mu, u^k_e\} x_e] + \sum_{e \in C^-} (u^k_e - \mu)^+(1 - x_e) - \sum_{e \in A_i^- \setminus C^-} y_e \leq \sum_{e \in C^+} u^k_e.
\]

is valid for \(\mathcal{R}^k_+(i)\). The flow pack inequalities are separated heuristically in much the same way as the GFC inequalities.

To assess the effect these inequalities have on the LP relaxation of \((S-CND)\), we perform two experiments. Experiment 1 tests inequalities for only a single scenario \(k \in \mathcal{K}\), while Experiment 2 uses inequalities from all scenarios \(k \in \mathcal{K}\). We repeat each experiment separately for each family of inequalities described above.

**Experiment 1**: This experiment is uses inequalities from all scenarios \(k \in \mathcal{K}\). Begin by solving the LP relaxation of \((S-CND)\). For each node \(i \in \mathcal{V}\) and scenario \(k \in \mathcal{K}\) with \(d^k_i \neq 0\), attempt to separate an inequality from the family under consideration, as described above. All of the \((\text{at most } |\mathcal{V}||\mathcal{K}|)\) separated inequalities are added to the LP relaxation, which is then re-solved. This process is repeated until no violated inequalities are found.

**Experiment 2**: This experiment is similar to Experiment 1, but considers each scenario \(k \in \mathcal{K}\) separately. Begin by fixing a single scenario \(k \in \mathcal{K}\). Next, repeat the following process. First, solve the LP relaxation of \((S-CND)\). For each node \(i \in \mathcal{V}\) with \(d^k_i \neq 0\), attempt to separate an inequality from the family under consideration, as described above. All of the \((\text{at most } |\mathcal{V}|)\) separated inequalities are added to the LP relaxation, which is then re-solved. This process is repeated (keeping \(k\) fixed) until no violated inequalities are found. Experiment 1 is repeated once for each scenario \(k \in \mathcal{K}\). We report the improvement in the optimality gap for the worst scenario (i.e., the scenario that improved the optimality gap the least), the best scenario, and the mean across all scenarios.
<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Network edge density ((\rho))</td>
<td>({30%, 60%})</td>
</tr>
<tr>
<td>Demands (d^k_i)</td>
<td>({10, 11, 12, \ldots, 30})</td>
</tr>
<tr>
<td>Capacities (u^k_e)</td>
<td>({20, 21, 22, \ldots, 40})</td>
</tr>
<tr>
<td>Unit flow costs (f^k_e)</td>
<td>({20, 21, 22, \ldots, 40})</td>
</tr>
<tr>
<td>Fraction of transhipment nodes</td>
<td>60%</td>
</tr>
<tr>
<td>Fraction of source nodes</td>
<td>20%</td>
</tr>
<tr>
<td>Fraction of sink nodes</td>
<td>20%</td>
</tr>
</tbody>
</table>

Table 1: Modified NETGEN algorithm: parameters used to generate test instances.

Problem instances are generated using the NETGEN algorithm (Klingman et al., 1974). The NETGEN algorithm is modified to ensure feasibility of the resulting model (S-CND). Parameters for the network edge density \(\rho\), demand values \(d^k_i\), capacities \(u^k_e\) and unit flow costs \(f^k_e\) are shown in Table 1. To generate the activation cost \(c_e\) for each arc \(e \in A\), we take the average of the unit flow costs \(f^k_e\) across all scenarios and multiply by 200.

We randomly generate instances by varying \(|\mathcal{V}| \in \{10, 30, 50\}\) and \(|\mathcal{K}| \in \{10, 20, 30\}\). For each setting of \(|\mathcal{V}|\) and \(|\mathcal{K}|\), we generate 10 instances with edge density \(\rho = 30\%\) and 10 instances with edge density \(\rho = 60\%\) (for a total of 180 instances). For each tuple \((|\mathcal{V}|, |\mathcal{K}|, \rho)\), the reported values are averaged over all 10 instances. Instances are available from the authors upon request.

In order to obtain integer optimal solutions, we solve the extensive form directly using CPLEX 12.7.0 on the NEOS server (Gropp and Moré, 1997; Czyzyk et al., 1998; Dolan, 2001). All instances with \(|\mathcal{V}| = 10\) were solved to optimality. For \(|\mathcal{V}| \in \{30, 50\}\), the reported values are computed using the best integer solutions obtained by CPLEX within a time limit of 1800s.

Tables 2—5 contain our results. The values in each table are the fraction of the optimality gap closed by the addition of single-scenario inequalities of each type (a fraction of 1 indicates that the integer-optimal solution was found). For instances which were not solved to optimality, the best-know optimal solution is used—consequently, the reported values are actually lower bounds on the fraction of the optimality gap closed. In each table, the column \(K\) scen. indicates the results of Experiment 1, i.e., the fraction of the optimality gap closed by adding inequalities of the form (FI) for all nodes \(i \in \mathcal{V}\) and scenarios \(k \in \mathcal{K}\). The remaining columns contain the results of Experiment 2, including the minimum, maximum and mean fraction of the gap closed by adding inequalities (FI) for only a single scenario \(k \in \mathcal{K}\).

These results indicate that the addition of single-scenario inequalities closes between 4\% (for large instances) and 25\% (for small instances) of the optimality gap for the extensive form of
Table 2: Percent of gap closed by cover inequalities. The column \( K \) scen." contains the result of Experiment 1; the remaining columns contain the results of Experiment 2, with values reported for the best, worst and average scenario.

| \(|\mathcal{V}|\) | \(|\mathcal{K}|\) | \(\rho = 30\%\) | \(\rho = 60\%\) |
|---|---|---|---|
|    | \(K\) scen. | min scen. | max scen. | mean scen. | \(K\) scen. | min scen. | max scen. | mean scen. |
| 10 | 68.2 | 13.1 | 50.5 | 26.2 | 56.1 | 6.9 | 39.5 | 16.1 |
| 20 | 63.8 | 9.5 | 51.1 | 20.6 | 61.3 | 15.2 | 45.9 | 20.8 |
| 30 | 52.1 | 13.2 | 43.9 | 21.7 | 60.0 | 9.4 | 43.1 | 18.6 |
| 30 | 45.4 | 16.9 | 30.2 | 22.0 | 41.0 | 15.0 | 27.5 | 19.1 |
| 50 | 34.4 | 8.5 | 19.0 | 11.8 | 35.5 | 9.2 | 19.5 | 12.0 |
| 30 | 39.1 | 8.8 | 20.3 | 12.7 | 35.8 | 14.7 | 21.7 | 17.5 |
| 50 | 28.3 | 11.6 | 17.5 | 13.8 | 28.3 | 11.9 | 17.9 | 14.2 |
| 30 | 29.7 | 9.7 | 16.4 | 12.3 | 25.9 | 12.0 | 15.4 | 13.3 |
| 30 | 23.8 | 7.1 | 12.2 | 9.0 | 24.9 | 11.1 | 15.3 | 13.0 |

(S-CND), when only one scenario is considered at a time. When all scenarios are aggregated, between 15% and 70% of the optimality gap is closed, on average.

In particular, we see that flow pack inequalities are the most effective at closing the optimality gap for the LP relaxation of (S-CND), with more than 30% of the gap closed on average for even the most difficult instances considered (when using inequalities from all \( K \) scenarios), and 14% when only inequalities from a single scenario are used. In contrast, (plain) GFC inequalities are the least effective, closing between 15% and 50% of the optimality gap when using inequalities from all \( K \) scenarios, and less than 1% when inequalities from only a single scenario are used. We note that the results given here consider adding only one type of inequality at a time. We find that, when adding all four kinds of inequalities simultaneously (results not shown here), we see improvements of approximately 1% in the optimality gap beyond the results of flow pack inequalities on their own—consistent with the fact that the flow pack inequalities appear to be the strongest inequalities we consider for our problem.

These results are consistent with Theorem 1 and Corollary 4, which indicate that single-scenario valid inequalities can be very strong for the extensive form of the stochastic problem. These promising results illustrate the potential for leveraging well-known inequalities of the form (FI) for the deterministic problem when solving the extensive form of the stochastic problem.

### 6. Conclusion and Future Work

In this work, we relate the convex hull of the feasible region of a two-stage stochastic mixed-integer program to the convex hulls of the single-scenario subproblems thereof. In particular, we provide necessary and sufficient conditions for single-scenario inequalities of the form (FI)
### Table 3: Percent of gap closed by GFC inequalities (see Table 2 for details).

| $|\mathcal{V}|$ | $|\mathcal{K}|$ | $\rho = 30\%$ | $\rho = 60\%$ |
|---|---|---|---|
|   |   | $K$ min max mean | $K$ min max mean |
| 10 | 10 | 42.8 0.6 18.9 5.3 | 27.2 0.3 10.7 3.7 |
| 20 |   | 51.6 0.8 16.7 3.4 | 33.8 0.0 13.9 4.5 |
| 30 |   | 34.6 0.0 15.6 2.9 | 21.2 0.0 9.6 2.1 |
| 30 | 10 | 26.6 0.5 9.1 3.5 | 24.0 0.3 8.0 2.7 |
| 20 |   | 21.6 0.0 5.5 1.4 | 27.3 0.0 9.6 2.0 |
| 30 |   | 20.6 0.0 4.7 0.8 | 20.4 0.0 4.5 1.1 |
| 50 | 10 | 23.3 0.8 6.6 3.0 | 19.5 0.8 5.1 2.5 |
| 20 |   | 19.4 0.1 4.1 1.3 | 15.8 0.1 3.0 1.1 |
| 30 |   | 17.2 0.0 3.1 0.8 | 14.5 0.0 3.2 0.7 |

### Table 4: Percent of gap closed by ExGFC inequalities (see Table 2 for details).

| $|\mathcal{V}|$ | $|\mathcal{K}|$ | $\rho = 30\%$ | $\rho = 60\%$ |
|---|---|---|---|
|   |   | $K$ min max mean | $K$ min max mean |
| 10 | 10 | 58.1 6.1 41.6 18.0 | 38.8 2.2 24.8 9.9 |
| 20 |   | 52.4 1.2 34.3 11.8 | 46.0 0.4 30.0 11.4 |
| 30 |   | 47.2 1.9 36.1 17.3 | 37.8 0.2 24.8 8.1 |
| 30 | 10 | 36.5 5.2 19.8 11.2 | 35.2 2.9 16.1 9.2 |
| 20 |   | 33.1 1.5 15.4 6.8 | 35.0 1.0 14.2 5.8 |
| 30 |   | 30.1 0.2 12.5 3.8 | 33.2 0.6 10.7 4.6 |
| 50 | 10 | 31.8 5.8 15.8 10.1 | 28.3 5.6 12.6 8.9 |
| 20 |   | 27.7 1.2 10.1 5.2 | 24.2 1.9 8.8 4.9 |
| 30 |   | 24.3 1.4 8.0 4.1 | 24.1 1.2 8.2 3.8 |

### Table 5: Percent of gap closed by flow pack inequalities (see Table 2 for details).

| $|\mathcal{V}|$ | $|\mathcal{K}|$ | $\rho = 30\%$ | $\rho = 60\%$ |
|---|---|---|---|
|   |   | $K$ min max mean | $K$ min max mean |
| 10 | 10 | 73.7 14.7 52.3 26.4 | 64.2 7.2 43.2 18.2 |
| 20 |   | 71.4 10.0 56.3 21.6 | 71.2 15.9 48.0 29.5 |
| 30 |   | 61.6 13.7 45.1 31.4 | 64.6 6.6 43.8 21.1 |
| 30 | 10 | 54.3 17.7 32.9 24.2 | 48.9 16.1 30.2 21.8 |
| 20 |   | 42.0 8.9 20.6 13.4 | 44.5 9.9 22.0 14.0 |
| 30 |   | 46.6 8.7 21.7 13.6 | 42.4 14.5 23.4 18.4 |
| 50 | 10 | 37.6 13.3 21.0 16.7 | 35.7 14.2 19.9 16.9 |
| 20 |   | 36.3 10.7 17.8 13.9 | 32.2 13.2 17.3 15.0 |
| 30 |   | 30.2 7.5 13.5 10.1 | 31.0 12.1 17.1 14.5 |
to be facet-defining for the convex hull of a stochastic extension of the well-known single node fixed-charge flow (SFCF) polytope. Under mild assumptions, our results show that every facet-defining inequality for the single-scenario SFCF polytope is facet-defining for the extensive polytope.

These results are supported by the computational results in Section 5, which illustrate the potential of using single-scenario valid inequalities when solving structured two-stage SMIPs with recourse. This work has (at least) two natural future directions. The first is to extend our results to structured polyhedra beyond the S-SFCF polytope. In particular, results of the form of Theorem 1 may improve the understanding of the polyhedral structure of general two-stage SMIPs, as well as providing theoretical justification for the use of valid inequalities of the form (FI) in the solution of such SMIPs.

Second, it remains an open question as to how single-scenario inequalities of the form (FI) can be effectively exploited in the solution of two-stage (and multi-stage) SMIPs. While our results have clear implications for the direct solution of the extensive form (1), leveraging these results in the context of the approaches discussed in Section 1 is less straightforward. Future work may explore the combination of such approaches with valid inequalities of the form (FI) to improve the solution of two-stage SMIPs.

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7. Proof of Proposition 7

Proposition 7. \( S_{fc}^{ext} \) is full-dimensional if and only if all of the following hold:

\[
\begin{align*}
u^k_i &> 0 \text{ for all } i \in \mathcal{N}, k \in \mathcal{K}. \quad (A1) \\
\sum_{i \in \mathcal{N}^+ \setminus \{j\}} u^k_i &\geq d^k \text{ for all } j \in \mathcal{N}^+, k \in \mathcal{K}. \quad (A2) \\
\sum_{i \in \mathcal{N}^+} u^k_i &> d^k \text{ for all } k \in \mathcal{K}. \quad (A3)
\end{align*}
\]

Proof. “⇒” Suppose \( S_{fc}^{ext} \) is full-dimensional, i.e. \( \dim(S_{fc}^{ext}) = N(K+1) \). First, suppose that \( u^k_i \leq 0 \) for some \( i \in \mathcal{N}, k \in \mathcal{K} \). If \( u^k_i < 0 \), then \( S_{fc}^{ext} = \emptyset \), a contradiction. Otherwise, \( u^k_i = 0 \), which implies \( y^k_i = 0 \) for any point in \( S_{fc}^{ext} \). Hence, \( S_{fc}^{ext} \) lies in a subspace of dimension \( N(K+1) - 1 \), a contradiction. Second, suppose \( \sum_{i \in \mathcal{N}^+ \setminus \{j\}} u^k_i < d^k \) for some \( j \in \mathcal{N}^+ \) and \( k \in \mathcal{K} \). Then \( x_j = 1 \) for any point in \( S_{fc}^{ext} \), again a contradiction. Third, if \( \mathcal{N}^+ \neq \emptyset \), then (A1) and (A2) imply (A3), and we are done. Otherwise, \( \mathcal{N}^+ = \emptyset \), and suppose \( d^k \geq 0 \) for some \( k \in \mathcal{K} \). If \( d^k > 0 \) then \( S_{fc}^{ext} = \emptyset \), a contradiction. Otherwise, \( d^k = 0 \), and every point in \( S_{fc}^{ext} \) satisfies \( y^k_i = 0 \) for all \( i \in \mathcal{N} \), again a contradiction.

(⇐) Now suppose (A1), (A2) and (A3) hold. The following argument holds regardless of whether \( \mathcal{N}^+ \) is empty—if \( \mathcal{N}^+ = \emptyset \), simply skip the corresponding points. We construct \( N(K+1)+1 \) affinely independent points in \( X_{fc}^{ext} \), defined in three sets as follows (cf. Figure 2):

Set 1: The first set contains \( N \) points. For each \( j \in \mathcal{N}^+ \) and \( k \in \mathcal{K} \), define a point by

first stage: \( x^j_i = \begin{cases} 1, & \text{if } i \in \mathcal{N}^+ \setminus \{j\}, \\ 0, & \text{otherwise,} \end{cases} \)
second stage: \( y^k_i = \begin{cases} u^k_i, & \text{if } i \in \mathcal{N}^+ \setminus \{j\}, \\ 0, & \text{otherwise,} \end{cases} \)

Similarly, for each \( j \in \mathcal{N}^- \) and \( k \in \mathcal{K} \), define a point by

first stage: \( x^j_i = \begin{cases} 1, & \text{if } i \in \mathcal{N}^- \cup \{j\}, \\ 0, & \text{otherwise,} \end{cases} \)
second stage: \( y^k_i = \begin{cases} u^k_i, & \text{if } i \in \mathcal{N}^- \cup \{j\}, \\ 0, & \text{otherwise,} \end{cases} \)

for all \( k \in \mathcal{K} \). That these points lie in \( X_{fc}^{ext} \) follows from (A2).

Set 2: The second set contains \( NK \) points. For each \( j \in \mathcal{N}^+ \) and \( k \in \mathcal{K} \), define a point
Figure 2: $N(K + 1) + 1$ affinely independent points in $X_{\text{ext}}$ used in the proofs of Proposition 7 and Proposition 8. The characters in the top row ($\mu, \lambda, \sigma$) correspond to the multipliers in (6). Blank entries are equal to zero.
with first stage given by

\[ x^{(j,k)}_i = \begin{cases} 
1, & \text{if } i \in \mathcal{N}^+, \\
0, & \text{otherwise}. 
\end{cases} \]

The second stage is given by

\[ \text{scenario } k : \ y^{k,(j,k)}_i = \begin{cases} 
\mu^k_i, & \text{if } i \in \mathcal{N}^+ \setminus \{ j \}, \\
0, & \text{otherwise}, 
\end{cases} \quad \text{scenario } \ell \neq k : \ y^{\ell,(j,k)}_i = \begin{cases} 
\mu^\ell_i, & \text{if } i \in \mathcal{N}^+, \\
0, & \text{otherwise}. 
\end{cases} \]

Similarly, for each \( j \in \mathcal{N}^- \) and \( k \in \mathcal{K} \), define a point with first stage given by

\[ x^{(j,k)}_i = \begin{cases} 
1, & \text{if } i \in \mathcal{N}^+ \cup \{ j \}, \\
0, & \text{otherwise}. 
\end{cases} \]

For all \( j \in \mathcal{N}^- \), define \( \varepsilon^k_j = \min \left\{ \mu^k_j, d^k + \sum_{i \in \mathcal{N}^-} \mu^k_i \right\} \). By (A1) and (A3), \( \varepsilon^k_j > 0 \). The second stage is given by

\[ \text{scenario } k : \ y^{k,(j,k)}_i = \begin{cases} 
\varepsilon^k_j, & \text{if } i = j, \\
\mu^k_i, & \text{if } i \in \mathcal{N}^+, \\
0, & \text{otherwise}, 
\end{cases} \quad \text{scenario } \ell \neq k : \ y^{\ell,(j,k)}_i = \begin{cases} 
\mu^\ell_i, & \text{if } i \in \mathcal{N}^+, \\
0, & \text{otherwise}. 
\end{cases} \]

These points lie in \( X_{\text{FC}}^{\text{ext}} \) by (A2).

**Set 3:** The third set contains only a single point, with

\[ \text{first stage: } \hat{x}_i = \begin{cases} 
1, & \text{if } i \in \mathcal{N}^+, \\
0, & \text{otherwise}, 
\end{cases} \quad \text{second stage: } \hat{y}^k_i = \begin{cases} 
\mu^k_i, & \text{if } i \in \mathcal{N}^+, \\
0, & \text{otherwise}, 
\end{cases} \]

for all \( k \in \mathcal{K} \). This point lies in \( X_{\text{FC}}^{\text{ext}} \) by (A2).

With the first set of points, we associate the scalar multipliers \( \mu_j \) for all \( j \in \mathcal{N} \). With the second set of points, we associate the scalar multipliers \( \lambda^k_j \) for all \( j \in \mathcal{N} \) and \( k \in \mathcal{K} \). With the single point in the third set, we associate the scalar multiplier \( \sigma \).

We now suppose that the zero vector can be expressed as an affine combination of the
\(N(K + 1) + 1\) points above. This reduces to the system of equations (cf. Figure 2)

\[
\begin{align*}
\sigma + \sum_{i \in \mathcal{N}} \mu_i + \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{N}} \lambda_i^k &= \mu_j \quad \forall j \in \mathcal{N}^+, \\
\mu_j + \sum_{k \in \mathcal{K}} \lambda_j^k &= 0 \quad \forall j \in \mathcal{N}^-, \\
\sum_{i \in \mathcal{N}} \mu_i + \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{N}} \lambda_i^k &= \sum_{i \in \mathcal{N}} \mu_i + \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{N}} \lambda_i^k = 0.
\end{align*}
\]

Combining (6a) and (6e), we obtain \(\mu_j = 0\) for all \(j \in \mathcal{N}^+\). By (6d), we have that \(\lambda_j^k = 0\) for all \(j \in \mathcal{N}^-, k \in \mathcal{K}\). Using these facts in (6b), we obtain \(\mu_j = 0\) for all \(j \in \mathcal{N}^+\). Combining (6c) and (6e), it follows that \(\lambda_j^k = 0\) for all \(j \in \mathcal{N}^+\) and \(k \in \mathcal{K}\). Using this fact in (6a), we obtain \(\mu_j^+ = 0\) for all \(j \in \mathcal{N}^+\). Combining all of these deductions, (6e) gives that \(\sigma = 0\). Hence, the given points are affinely independent, completing the proof.

\section*{8. Proof of Proposition 8}

**Proposition 8.** The following inequalities are facet-defining for \(S_{pc}^{ext}\):

1. \(x_i \leq 1\) for all \(i \in \mathcal{N}\).
2. \(y_i^k \geq 0\) for all \(i \in \mathcal{N}^-\) and \(k \in \mathcal{K}\).
3. \(y_i^k \leq u_i^k x_i\) for all \(i \in \mathcal{N}^+\) and \(k \in \mathcal{K}\).

**Proof.**

1. Fix \(i \in \mathcal{N}\). If \(i \in \mathcal{N}^+\), the \(N(K + 1)\) required points in \(X_{pc}^{ext}\) are given by the \(N(K + 1) + 1\) points in Figure 2, with the point \((x^{(i)}, y^{(i)})\) removed. Otherwise, if \(i \in \mathcal{N}^-\), the \(N(K + 1)\) points are constructed using the \(N(K + 1) + 1\) points in Figure 2 as follows. First, remove the point \((x^{(i)}, y^{(i)})\). Next, for any point in the set with \(x_i = 0\), set instead \(x_i = 1\). An argument similar to that in the proof of Proposition 7 shows that the resulting points are affinely independent.

2. and 3. The required \(N(K + 1)\) points in \(X_{pc}^{ext}\) are given by the \(N(K + 1) + 1\) points in Figure 2, with the point \((x^{(i,k)}, y^{(i,k)})\) removed. 

\[\square\]
9. Proofs of Lemma 2 and Lemma 3

Lemma 2. Suppose \( \{x^{(i)} \in \mathbb{R}^M \mid i = 1, \ldots, n\} \) is a set of \( n \) affinely independent points, and let \( \lambda_1, \ldots, \lambda_n \) be scalars such that \( \sum_{i=1}^n \lambda_i = 1 \) and \( \lambda_1 \neq 0 \). Define \( \hat{x} = \sum_{i=1}^n \lambda_i x^{(i)} \). Then \( \hat{x}, x^{(2)}, \ldots, x^{(n)} \) are affinely independent.

Proof. Suppose that the scalars \( \{\mu_i\}_{i=1}^n \) satisfy \( \mu_1 \hat{x} + \sum_{i=2}^n \mu_i x^{(i)} = 0 \) and \( \sum_{i=1}^n \mu_i = 0 \). We will show that \( \mu_i = 0 \) for all \( i = 1, \ldots, n \). Using the definition of \( \hat{x} \), the first equality becomes

\[
0 = \mu_1 \hat{x} + \sum_{i=2}^n \mu_i x^{(i)} = \mu_1 \sum_{i=1}^n \lambda_i x^{(i)} + \sum_{i=2}^n \mu_i x^{(i)} = \mu_1 \lambda_1 x^{(1)} + \sum_{i=2}^n (\mu_1 \lambda_i + \mu_i) x^{(i)}. \tag{7}
\]

Note that the coefficients of the \( x^{(i)} \) in (7) sum to zero:

\[
\mu_1 \lambda_1 + \sum_{i=2}^n (\mu_1 \lambda_i + \mu_i) = \mu_1 \lambda_1 + \mu_1 \sum_{i=2}^n \lambda_i + \sum_{i=2}^n \mu_i = \mu_1 \lambda_1 + \mu_1 (1 - \lambda_1) - \mu_1 = 0,
\]

where the second equality follows from the fact that \( \sum_{i=1}^n \lambda_i = 1 \) and \( \sum_{i=1}^n \mu_i = 0 \). Because the \( x^{(i)} \) are affinely independent by assumption, we conclude that all the coefficients of the \( x^{(i)} \) in (7) equal zero. In particular, \( \mu_1 \lambda_1 = 0 \), which implies \( \mu_1 = 0 \), because \( \lambda_1 \neq 0 \). Furthermore, \( \mu_1 \lambda_i + \mu_i = 0 \) for all \( i = 2, \ldots, n \), which implies \( \mu_i = 0 \) for all \( i = 2, \ldots, n \). \( \square \)

Lemma 3. Suppose \( \{(f^{(i)}, g^{(i)}) \in \mathbb{R}^{M_1+M_2} \mid i = 1, \ldots, n\} \) is a set of \( n \) points such that \( f^{(1)}, \ldots, f^{(n)} \) are affinely independent. Suppose \( h^{(1)}, \ldots, h^{(m)} \) are \( m \) linearly independent points in \( \mathbb{R}^{M_2} \). Then for any \( s \in \{1, \ldots, n\} \), the \( n + m \) points

\[
\left( f^{(1)} \right), \ldots, \left( f^{(n)} \right), \left( f^{(s)} \right), \left( \begin{array}{c} f^{(s)} \\ g^{(s)} + h^{(1)} \end{array} \right), \ldots, \left( \begin{array}{c} f^{(s)} \\ g^{(s)} + h^{(m)} \end{array} \right)
\]

are affinely independent.

Proof. Suppose without loss of generality that \( s = 1 \). It suffices to show that the \( n + m - 1 \) points

\[
\left( f^{(2)} - f^{(1)} \right), \ldots, \left( f^{(n)} - f^{(1)} \right), \left( \begin{array}{c} 0 \\ h^{(1)} \end{array} \right), \ldots, \left( \begin{array}{c} 0 \\ h^{(m)} \end{array} \right)
\]

are linearly independent. Suppose there exist constants \( \lambda_i \) for \( i = 2, \ldots, n \) and \( \mu_j \) for \( j = 1, \ldots, m \) such that

\[
\sum_{i=2}^n \lambda_i \left( f^{(i)} - f^{(1)} \right) + \sum_{j=1}^m \mu_j \left( \begin{array}{c} 0 \\ h^{(j)} \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right). \tag{8}
\]
Because the points \( \{ f^{(i)} \}_{i=1}^n \) are affinely independent, the points \( \{ f^{(i)} - f^{(1)} \}_{i=2}^n \) are linearly independent. Hence, the first \( M_1 \) rows of the system (8) give that \( \lambda_i = 0 \) for \( i = 2, \ldots, n \), and the system (8) reduces to \( \sum_{j=1}^m \mu_j h^{(j)} = 0 \). Because the points \( \{ h^{(j)} \}_{j=1}^m \) are linearly independent, we conclude that \( \mu_j = 0 \) for \( j = 1, \ldots, m \). \qed