NONLINEAR TRANSVERSALITY OF COLLECTIONS OF SETS: DUAL SPACE NECESSARY CHARACTERIZATIONS

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Dedicated to Professor Alexander Ioffe on the occasion of his eightieth birthday

ABSTRACT. This paper continues the study of 'good arrangements' of collections of sets in normed vector spaces near a point in their intersection. Our aim is to study general nonlinear *transversality* properties. We focus on dual space (subdifferential and normal cone) necessary characterizations of these properties. As an application, we provide dual necessary conditions for the nonlinear extensions of the new *transversality properties of a set-valued mapping to a set in the range space* due to Ioffe.

1. INTRODUCTION

In this paper we continue studying 'good arrangements' of collections of sets in normed vector spaces near a point in their intersection, known as *transversality* (*regularity*) properties (cf. Ioffe [14, Section 7.1]) and playing an important role in optimization and variational analysis, e.g., as constraint qualifications in optimality conditions, and qualification conditions in subdifferential, normal cone and coderivative calculus, and convergence analysis of computational algorithms [1–5, 7, 13, 14, 16–18, 20–30, 32–34, 36, 37, 41, 42].

Up until recently, mostly 'linear' transversality properties have been studied, although it has been observed that such properties often fail in very simple situations, for instance, when it comes to convergence analysis of computational algorithms. Moreover, even in the linear setting, dual characterizations of semitransversality have not been studied.

Our aim in this paper is to develop dual space (subdifferential and normal cone) necessary conditions of the general nonlinear transversality properties of collections of sets in normed vector spaces, utilizing the corresponding slope characterizations of these properties established in our recent paper [10]. Dual space sufficient characterizations for nonlinear transversality properties are going to appear in [11].

Our working model in this paper is a collection of $n \ge 2$ arbitrary subsets $\Omega_1, \ldots, \Omega_n$ of a normed vector space X, having a common point $\bar{x} \in \cap_{i=1}^n \Omega_i$. When formulating dual necessary conditions in Section 4, the sets are assumed closed and convex.

The next definition from [10] introduces nonlinear transversality properties extending the definitions of the corresponding Hölder transversality properties in [26, Definition 1]. The nonlinearity in the definitions of the properties is determined by a continuous strictly increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(0) = 0$ and $\lim_{t\to+\infty} \varphi(t) = +\infty$. The family of all such functions is denoted by $\mathscr{C}$. We denote by $\mathscr{C}^1$ the subfamily of functions from $\mathscr{C}$ which are differentiable on $]0, +\infty[$ with $\varphi'(t) > 0$ for all $t > 0$. Obviously, if $\varphi \in \mathscr{C}$ ($\varphi \in \mathscr{C}^1$), then $\varphi^{-1} \in \mathscr{C}$ ($\varphi^{-1} \in \mathscr{C}^1$). Observe that, for any $\alpha > 0$ and $q > 0$, the function $t \mapsto \alpha t^q$ on $\mathbb{R}_+$ belongs to $\mathscr{C}^1$. In

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Date: Received: date / Accepted: date.

2010 *Mathematics Subject Classification.* Primary 49J52; 49J53; Secondary 49K40; 90C30; 90C46.

The research was supported by the Australian Research Council, project DP160100854. The second author benefited from the support of the FMJH Program PGMO and from the support of EDF..

Section 4 and some statements in Section 3, φ is assumed convex, and $\varphi'_+(0)$ stands for the right derivative of φ at 0.

Definition 1. Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$.

(i) $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ if there exists a $\delta>0$ such that

$$\bigcap_{i=1}^n(\Omega_i-x_i)\cap B_\rho(\bar{x})\neq\emptyset$$

for all $\rho\in]0,\delta[$ and $x_i\in X$ with $\varphi(\|x_i\|)<\rho$ $(i=1,\ldots,n)$.

(ii) $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ if there exist $\delta_1>0$ and $\delta_2>0$ such that

$$\bigcap_{i=1}^n\Omega_i\cap B_\rho(x)\neq\emptyset$$

for all $\rho\in]0,\delta_1[$ and $x\in B_{\delta_2}(\bar{x})$ with $\varphi(d(x,\Omega_i))<\rho$ $(i=1,\ldots,n)$.

(iii) $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if there exist $\delta_1>0$ and $\delta_2>0$ such that

$$\bigcap_{i=1}^n(\Omega_i-\omega_i-x_i)\cap(\rho\mathbb{B})\neq\emptyset$$

for all $\rho\in]0,\delta_1[$, $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ and $x_i\in X$ with $\varphi(\|x_i\|)<\rho$ $(i=1,\ldots,n)$.

The most important realization of the three properties in Definition 1 corresponds to the Hölder setting, i.e. φ being a power function, given for all $t\geq 0$ by $\varphi(t):=\alpha^{-1}t^q$ with $\alpha>0$ and $q>0$. In this case, condition $\varphi(\|x_i\|)<\rho$ involved in parts (i) and (iii) of the definition becomes $\|x_i\|^q<\alpha\rho$, while in part (ii), $\varphi^{-1}(\rho)=(\alpha\rho)^{\frac{1}{q}}$, and Definition 1 sharpens [26, Definition 1]. In the Hölder setting, we refer to the three properties above as $\alpha-$(semi-, sub-)transversality of order q. With $q=1$ (linear case), the properties were discussed, e.g., in [17, 18, 27].

The rest of the paper is organized as follows. Section 2 introduces notation and presents some basic facts from variational analysis and generalized differentiation used in the formulations and proofs of the results. In Section 3, we briefly discuss the nonlinear transversality properties of finite collections of sets in normed vector spaces, and recall their metric and slope characterizations as well as their dual sufficient characterizations from [6, 10, 11]. Section 4 is dedicated to dual necessary conditions of nonlinear transversality in the convex setting. As an application, we provide in Section 5 dual necessary conditions for nonlinear extensions of the new *transversality properties of a set-valued mapping to a set in the range space* due to Ioffe [14].

2. Notation and Preliminaries

Our basic notation is standard, see, e.g., [12, 31, 38]. Throughout the paper, X and Y are normed vector spaces. The topological dual of the space X is denoted by X^*, while $\langle\cdot,\cdot\rangle$ denotes the bilinear form defining the pairing between the two spaces. The open unit balls in X and X^* are denoted by $\mathbb{B}$ and $\mathbb{B}^*$, respectively, and $B_\delta(x)$ stands for the open ball with center x and radius $\delta>0$. If not explicitly stated otherwise, products of normed vector spaces are assumed to be equipped with the maximum norm $\|(x,y)\|:=\max\{\|x\|,\|y\|\}$, $(x,y)\in X\times Y$. $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ denote the real line (with the usual norm), the set of all nonnegative real numbers and the set of all positive integers, respectively.

For a set $\Omega\subset X$, its boundary is denoted by $\operatorname{bd}\Omega$. The distance from a point $x\in X$ to a set $\Omega\subset X$ is defined by $d(x,\Omega):=\inf_{u\in\Omega}\|u-x\|$, and we use the convention $d(x,\emptyset)=+\infty$.

For an extended-real-valued function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ on a normed vector space X, its domain and epigraph are defined, respectively, by $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x,\alpha) \in X \times \mathbb{R} \mid f(x) \le \alpha\}$. The inverse of f (if it exists) is denoted by f^{-1}. Note that f is allowed to take the value $-\infty$. This convention is needed only to accommodate for the general chain rule in Proposition 4. Throughout the paper, we employ the conventional definitions of the lower and upper limits:

$$\liminf_{x\to\bar{x}} f(x) := \sup_{\varepsilon>0} \inf_{0<\|x-\bar{x}\|<\varepsilon} f(x) \quad \text{and} \quad \limsup_{x\to\bar{x}} f(x) := \inf_{\varepsilon>0} \sup_{0<\|x-\bar{x}\|<\varepsilon} f(x).$$

Note that both definitions exclude the reference point $\bar{x}$ when computing the respective inf and sup.

A set-valued mapping $F : X \rightrightarrows Y$ between two sets X and Y is a mapping, which assigns to every $x \in X$ a subset (possibly empty) $F(x)$ of Y. We use the notations $\operatorname{gph} F := \{(x,y) \in X \times Y \mid y \in F(x)\}$ and $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$ for the graph and the domain of F, respectively, and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse (which always exists with possibly empty values at some y) is defined by $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$, $y \in Y$. Obviously $\operatorname{dom} F^{-1} = F(X)$.

Dual characterizations of transversality properties require dual tools – normal cones and subdifferentials, usually in the Fréchet or Clarke sense. Given a subset Ω of a normed vector space X and a point $\bar{x} \in \Omega$, the sets (cf. [9, 15])

$$N^F_\Omega(\bar{x}) := \left\{ x^* \in X^* \mid \limsup_{x \overset{\Omega}{\to} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \le 0 \right\}, \tag{1}$$

$$N^C_\Omega(\bar{x}) := \left\{ x^* \in X^* \mid \langle x^*, z\rangle \le 0 \quad \text{for all} \quad z \in T^C_\Omega(\bar{x}) \right\} \tag{2}$$

are the *Fréchet* and *Clarke normal cones* to Ω at $\bar{x}$. The notation $x \overset{\Omega}{\to} \bar{x}$ in (1) means $x \to \bar{x}$ and $x \in \Omega \setminus \{\bar{x}\}$, and $T^C_\Omega(\bar{x})$ in (2) stands for the *Clarke tangent cone* to Ω at $\bar{x}$:

$$T^C_\Omega(\bar{x}) := \Big\{ z \in X \mid \forall x_k \overset{\Omega}{\to} \bar{x},\ \forall t_k \downarrow 0,\ \exists z_k \to z \quad \text{such that} \quad x_k + t_k z_k \in \Omega \quad \text{for all} \quad k \in \mathbb{N} \Big\}.$$

The sets (1) and (2) are nonempty closed convex cones satisfying $N^F_\Omega(\bar{x}) \subset N^C_\Omega(\bar{x})$. If Ω is a convex set, they reduce to the normal cone in the sense of convex analysis:

$$N_\Omega(\bar{x}) := \{x^* \in X^* \mid \langle x^*, x-\bar{x}\rangle \le 0 \quad \text{for all} \quad x \in \Omega\}.$$

Given a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ and a point $\bar{x} \in X$ with $|f(\bar{x})| < +\infty$, the *Fréchet* and *Clarke subdifferentials* of f at $\bar{x}$ can be defined via the corresponding normal cones to its epigraph at $(\bar{x}, f(\bar{x}))$ as follows (cf. [9, 15]):

$$\partial^F f(\bar{x}) = \left\{ x^* \in X^* \mid (x^*, -1) \in N^F_{\operatorname{epi} f}(\bar{x}, f(\bar{x})) \right\}, \tag{3}$$

$$\partial^C f(\bar{x}) = \left\{ x^* \in X^* \mid (x^*, -1) \in N^C_{\operatorname{epi} f}(\bar{x}, f(\bar{x})) \right\}. \tag{4}$$

The sets are closed and convex, and satisfy $\partial^F f(\bar{x}) \subset \partial^C f(\bar{x})$. If f is convex, then (3) and (4) reduce to the subdifferential in the sense of convex analysis:

$$\partial f(\bar{x}) := \{x^* \in X^* \mid f(x) - f(\bar{x}) - \langle x^*, x-\bar{x}\rangle \ge 0 \quad \text{for all} \quad x \in X\}.$$

By convention, we set $N^F_\Omega(\bar{x}) = N^C_\Omega(\bar{x}) := \emptyset$ if $\bar{x} \notin \Omega$ and $\partial^F f(\bar{x}) = \partial^C f(\bar{x}) := \emptyset$ if $|f(\bar{x})| = +\infty$. It is easy to check that $N^F_\Omega(\bar{x}) = \partial^F i_\Omega(\bar{x})$ and $N^C_\Omega(\bar{x}) = \partial^C i_\Omega(\bar{x})$, where i_Ω is the *indicator function* of Ω: $i_\Omega(x) = 0$ if $x \in \Omega$ and $i_\Omega(x) = +\infty$ if $x \notin \Omega$.

The proofs of the main results in this paper make use of the conventional subdifferential sum rule of convex analysis; see e.g. [40, Theorem 2.8.7].

Lemma 2. *Suppose X is a normed vector space, $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ are convex, and $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. If f_1 is continuous at a point in $\operatorname{dom} f_2$, then*

$$\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

The following fact is an immediate consequence of the definitions of the Fréchet and Clarke normal cones (cf. e.g. [9, 15, 31]).

Lemma 3. *Let Ω_1,Ω_2 be subsets of a normed vector space X, and $\omega_i \in \Omega_i$ $(i=1,2)$. Then*

$$N_{\Omega_1\times\Omega_2}(\omega_1,\omega_2) = N_{\Omega_1}(\omega_1)\times N_{\Omega_2}(\omega_2),$$

where in both parts of the equality N stands for either the Fréchet or the Clarke normal cone.

Next we formulate a chain rule for Fréchet subdifferentials proved in [11], which is going to be used in the sequel. Such rules are extensively used when proving dual characterizations of nonlinear transversality, regularity and error bound properties. The next statement seems to present the chain rule under the weakest assumptions, compared to the existing assertions of this type, cf. [9, Thereom 2.3.9], [15, Corollary 1.14.1], [35, Lemma 1], [8, Thereom 10.20], [37, Proposition 4.39], [39, Proposition 2.1].

In the statement below, X is a general normed vector space, $\psi: X\to\mathbb{R}\cup\{+\infty\}$, and $\varphi:\mathbb{R}\to\mathbb{R}\cup\{\pm\infty\}$. The composition function $\varphi\circ\psi$ is defined as follows:

$$(\varphi\circ\psi)(x) := \begin{cases} \varphi(\psi(x)) & \text{if } x\in \operatorname{dom}\psi,\\ +\infty & \text{if } x\notin \operatorname{dom}\psi. \end{cases} \tag{5}$$

The outer function φ is assumed differentiable at the reference point, while the inner function ψ is arbitrary and does not have to be even (semi-)continuous. Note that $\varphi\circ\psi$ can take the value $-\infty$.

Proposition 4. *Let X be a normed vector space, $\psi: X\to\mathbb{R}\cup\{+\infty\}$, $\varphi:\mathbb{R}\to\mathbb{R}\cup\{\pm\infty\}$ and $\bar x\in\operatorname{dom}\psi$. Suppose that φ is nondecreasing on $\mathbb{R}$, finite and differentiable at $\psi(\bar x)$ with $\varphi'(\psi(\bar x))>0$. Then*

$$\partial^F(\varphi\circ\psi)(\bar x) = \varphi'(\psi(\bar x))\partial^F\psi(\bar x).$$

Remark 5. (i) The chain rule in Proposition 4 is a local result. Instead of assuming that φ is defined on the whole real line with possibly infinite values, one can assume that φ is defined and finite on a closed interval $[\alpha,\beta]$ around the point $\psi(\bar x)$: $\alpha<\psi(\bar x)<\beta$. The statement above remains valid if the definition (5) of the composition function is slightly modified. The 'if' condition in the first line should be changed to $x\in\operatorname{dom}\psi$ and $\psi(x)\in\operatorname{dom}\varphi$, and another two lines should be added: $(\varphi\circ\psi)(x):=\varphi(\alpha)$ if $\psi(x)<\alpha$, and $(\varphi\circ\psi)(x):=\varphi(\beta)$ if $\beta<\psi(x)<+\infty$.

(ii) If, additionally, ψ is assumed lower semicontinuous at $\bar x$, then it is sufficient to assume that φ is nondecreasing only on $[\psi(\bar x),+\infty[$, and there is no need to allow φ to take the value $-\infty$.

We are going to use a representation of the subdifferential of a special convex function on X^{n+1} given in the next lemma; cf. [23, Lemma 3].

Lemma 6. *Let X be a normed vector space, and a function $\psi: X^{n+1}\to\mathbb{R}_+$ be defined by*

$$\psi(u_1,\ldots,u_n,u) := \max_{1\le i\le n}\|u_i-a_i-u\|,\quad u_1,\ldots,u_n,u\in X, \tag{6}$$

where $a_i\in X$ $(i=1,\ldots,n)$. Let $x_1,\ldots,x_n,x\in X$ and $\max_{1\le i\le n}\|x_i-a_i-x\|>0$. Then

$$\partial\psi(x_1,\ldots,x_n,x) = \Big\{ (x_1^*,\ldots,x_n^*,x^*)\in(X^*)^{n+1} \mid x^*+\sum_{i=1}^n x_i^*=0,$$
$$\sum_{i=1}^n\|x_i^*\|=1,\ \sum_{i=1}^n\langle x_i^*,x_i-a_i-x\rangle = \max_{1\le i\le n}\|x_i-a_i-x\| \Big\}. \tag{7}$$

Remark 7. (i) It is easy to notice that in the representation (7), for any $i=1,\ldots,n$, either $\langle x_i^*, x_i-a_i-x\rangle = \max_{1\le j\le n}\|x_j-a_j-x\|$ or $x_i^*=0$.

(ii) The maximum norm on X^n used in (6) is a composition of the given norm on X and the maximum norm on $\mathbb{R}^n$. The corresponding dual norm produces the sum of the norms in (7). Any other norm on $\mathbb{R}^n$ can replace the maximum in (6) as long as the corresponding dual norm is used to replace the sum in (7).

3. Nonlinear Transversality Properties of Collections of Sets

In this section, we briefly discuss the properties in Definition 1. More details and discussions can be found in [10, 11].

Each of the properties in Definition 1 is determined by a function $\varphi\in\mathscr{C}$, and either a number $\delta>0$ in item (i) or two numbers $\delta_1>0$ and $\delta_2>0$ in items (ii) and (iii). The function plays the role of a kind of rate or modulus of the respective property, while the role of the δ's is more technical: they control the size of the interval for the values of ρ and, in the case of $\varphi-$subtransversality and $\varphi-$transversality in parts (ii) and (iii), the size of the neighbourhoods of $\bar{x}$ involved in the respective definitions. Of course, if a property is satisfied with some $\delta_1>0$ and $\delta_2>0$, it is also satisfied with the single $\delta:=\min\{\delta_1,\delta_2\}$ in place of both δ_1 and δ_2. We consider in the current paper two different parameters to emphasise their different roles in the definitions and the corresponding characterizations. Moreover, we are going to provide quantitative estimates for the values of these parameters.

Note that the definitions of the nonlinear transversality properties and their primal space characterizations discussed in this section do not require φ to be differentiable.

The next proposition provides several simple facts about the properties in Definition 1. For the proofs, more primal space characterizations and discussions, we refer the readers to [10].

Proposition 8. *Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$.*

(i) *If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then it is $\varphi-$semitransversal at $\bar{x}$ with δ_1 and $\varphi-$subtransversal at $\bar{x}$ with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ such that $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$.*

(ii) *Suppose $\Omega_1,\ldots,\Omega_n$ are closed and $\bar{x}\in\operatorname{bd}\cap_{i=1}^n\Omega_i$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal (particularly, if it is $\varphi-$transversal) at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then there exists a $\bar{t}\in]0,\min\{\delta_2,\varphi^{-1}(\delta_1)\}[$ such that $\varphi(t)\ge t$ for all $t\in]0,\bar{t}]$. As a consequence, $\liminf_{t\downarrow0}\varphi(t)/t\ge1$.*

Remark 9. In the Hölder setting, i.e. when $\varphi(t)=\alpha^{-1}t^q$ with $\alpha>0$ and $q>0$, the conditions on φ in Proposition 8(ii) can only be satisfied if either $q<1$, or $q=1$ and $\alpha\le1$. This reflects the well known fact that the Hölder subtransversality and transversality properties are only meaningful when $q\le1$ and, moreover, the linear case ($q=1$) is only meaningful when $\alpha\le1$; cf. [23, p. 705], [20, p. 118]. Note that the Hölder semitransversality can hold also with $q>1$; see an example in [10].

The nonlinear transversality properties of collections of sets in Definition 1 can be characterized in metric terms; cf. [10].

Theorem 10. *Let $\Omega_1,\ldots,\Omega_n$ be subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$.*

(i) *$\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$ if and only if*

$$d\left(\bar{x},\bigcap_{i=1}^n(\Omega_i-x_i)\right)\le\varphi\left(\max_{1\le i\le n}\|x_i\|\right)$$

for all $x_i\in X$ with $\varphi(\|x_i\|)<\delta$ $(i=1,\ldots,n)$.

(ii) $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$ *with some* $\delta_1>0$ *and* $\delta_2>0$ *if and only if*

$$d\left(x,\bigcap_{i=1}^{n}\Omega_i\right)\le\varphi\left(\max_{1\le i\le n}d(x,\Omega_i)\right)$$

for $x\in B_{\delta_2}(\bar{x})$ *with* $\varphi(d(x,\Omega_i))<\delta_1$ $(i=1,\ldots,n)$.

(iii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*transversal at* $\bar{x}$ *with some* $\delta_1>0$ *and* $\delta_2>0$, *then*

$$d\left(x,\bigcap_{i=1}^{n}(\Omega_i-x_i)\right)\le\varphi\left(\max_{1\le i\le n}d(x,\Omega_i-x_i)\right)\tag{8}$$

for all $x,x_i\in X$ *such that* $x+x_i\in B_{\delta_2'}(\bar{x})$ *and* $\varphi(d(x,\Omega_i-x_i))<\delta_1'$ $(i=1,\ldots,n)$, *and all* $\delta_1'\in]0,\delta_1]$ *and* $\delta_2'>0$ *satisfying* $\delta_2'+\varphi^{-1}(\delta_1')\le\delta_2$.

Conversely, if there exist numbers $\delta_1>0$ *and* $\delta_2>0$ *such that inequality* (8) *holds for all* $x,x_i\in X$ *satisfying* $x+x_i\in B_{\delta_2}(\bar{x})$ *and* $\varphi(d(x,\Omega_i-x_i))<\delta_1$ $(i=1,\ldots,n)$, *then* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*transversal at* $\bar{x}$ *with any* $\delta_1'\in]0,\delta_1]$ *and* $\delta_2'>0$ *satisfying* $\delta_2'+\varphi^{-1}(\delta_1')\le\delta_2$.

As shown in [10], a variable point x in the metric characterization (8) of $\varphi-$transversality can be replaced by the fixed given point $\bar{x}$.

Proposition 11. *Let* $\Omega_1,\ldots,\Omega_n$ *be subsets of a normed vector space* X, $\bar{x}\in\cap_{i=1}^{n}\Omega_i$, $\varphi\in\mathscr{C}$, $\delta_1>0$ *and* $\delta_2>0$. *Condition* (8) *is satisfied for all* $x,x_i\in X$ *such that* $x+x_i\in B_{\delta_2}(\bar{x})$ *and* $\varphi(d(x,\Omega_i-x_i))<\delta_1$ $(i=1,\ldots,n)$ *if and only if*

$$d\left(\bar{x},\bigcap_{i=1}^{n}(\Omega_i-x_i)\right)\le\varphi\left(\max_{1\le i\le n}d(\bar{x},\Omega_i-x_i)\right)$$

for all $x_i\in\delta_2\mathbb{B}$ *such that* $\varphi(d(\bar{x},\Omega_i-x_i))<\delta_1$ $(i=1,\ldots,n)$.

Remark 12. In the Hölder setting, i.e. when $\varphi(t)=\alpha^{-1}t^q$ with $\alpha>0$ and $q>0$, Theorem 10 provides metric characterizations of the corresponding Hölder transversality properties; cf. [26]. We refer the readers to [20,23,24] for more discussions and historical comments.

The next three propositions established in [10] provide 'slope' necessary conditions of the three nonlinear transversality properties in Definition 1. They employ the following norm on X^{n+1} depending on a parameter $\gamma>0$:

$$\|(x_1,\ldots,x_n,x)\|_\gamma:=\max\left\{\|x\|,\gamma\max_{1\le i\le n}\|x_i\|\right\},\quad x_1,\ldots,x_n,x\in X.\tag{9}$$

Proposition 13. *Let* $\Omega_1,\ldots,\Omega_n$ *be closed subsets of a normed vector space* X, $\bar{x}\in\cap_{i=1}^{n}\Omega_i$, *and* $\varphi\in\mathscr{C}$. *Suppose* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*semitransversal at* $\bar{x}$ *with some* $\delta>0$.

(i) *If there exists an* $\alpha>0$ *such that* $\varphi(t)\ge\alpha t$ *for all* $t\in]0,\varphi^{-1}(\delta)[$, *then, with* $\gamma:=(\alpha^{-1}+1)^{-1}$,

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\(u_1,\ldots,u_n,u)\ne(\bar{x},\ldots,\bar{x},\bar{x})}}\frac{\varphi\left(\max\limits_{1\le i\le n}\|x_i\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma}\ge1\tag{10}$$

for all $x_i\in X$ $(i=1,\ldots,n)$ *satisfying*

$$0<\max_{1\le i\le n}\|x_i\|<\varphi^{-1}(\delta).\tag{11}$$

(ii) *If $\Omega_1,\ldots,\Omega_n$ and φ are convex, and $\varphi'_+(0)>0$, then, with $\gamma := ((\varphi'_+(0))^{-1}+1)^{-1}$,*

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar{x}\ (i=1,\ldots,n),\ u\to\bar{x}\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|x_i\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma}\ge 1 \tag{12}$$

for all $x_i\in X$ $(i=1,\ldots,n)$ satisfying (11).

Proposition 14. *Let $\Omega_1,\ldots,\Omega_n$ be closed subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$. Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$.*

(i) *If there exists an $\alpha>0$ such that $\varphi(t)\ge\alpha t$ for all $t\in]0,\varphi^{-1}(\delta_1)[$, then, with $\gamma := (\alpha^{-1}+1)^{-1}$,*

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma}\ge 1 \tag{13}$$

for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying

$$\|x-\bar{x}\|<\delta_2,\quad 0<\max_{1\le i\le n}\|\omega_i-x\|<\varphi^{-1}(\delta_1). \tag{14}$$

(ii) *If $\Omega_1,\ldots,\Omega_n$ and φ are convex, and $\varphi'_+(0)>0$, then, with $\gamma := ((\varphi'_+(0))^{-1}+1)^{-1}$,*

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to x\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-u\|\right)}{\|(u_1,\ldots,u_n,u-(\omega_1,\ldots,\omega_n,x)\|_\gamma}\ge 1 \tag{15}$$

for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (14).

Proposition 15. *Let $\Omega_1,\ldots,\Omega_n$ be closed subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$. Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$.*

(i) *If there exists an $\alpha>0$ such that $\varphi(t)\ge\alpha t$ for all $t\in]0,\varphi^{-1}(\delta_1)[$, then, with $\gamma := (\alpha^{-1}+1)^{-1}$,*

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,\bar{x})}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma}\ge 1 \tag{16}$$

for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying

$$\max_{1\le i\le n}\|\omega_i-\bar{x}\|<\delta_2,\quad 0<\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|<\varphi^{-1}(\delta_1). \tag{17}$$

(ii) *If $\Omega_1,\ldots,\Omega_n$ and φ are convex, and $\varphi'_+(0)>0$, then, with $\gamma := ((\varphi'_+(0))^{-1}+1)^{-1}$,*

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to\bar{x}\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,\bar{x})}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma}\ge 1 \tag{18}$$

for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (17).

Remark 16. (i) In view of Proposition 8(ii), one can suppose in Propositions 14 and 15 that $\alpha \geq 1$ and $\varphi'_+(0) \geq 1$.

(ii) The equalities $\gamma := (\alpha^{-1}+1)^{-1}$ and $\gamma := ((\varphi'_+(0))^{-1}+1)^{-1}$ in the above three propositions can be replaced by the inequalities $0 < \gamma \leq (\alpha^{-1}+1)^{-1}$ and $0 < \gamma \leq ((\varphi'_+(0))^{-1}+1)^{-1}$, respectively.

(iii) The expressions in the left-hand sides of the inequalities (10), (13) and (16) are the *nonlocal* γ*-slopes* and in the left-hand sides of the inequalities (12), (15) and (18) are the γ*-slopes* (cf. [19, p. 60 and 61]) computed at the respective points of the extended real-valued function

$$\widehat{f} := f + i_{\Omega_1\times\ldots\times\Omega_n}, \tag{19}$$

where $f : X^{n+1} \to \mathbb{R}_+$ is a continuous function defined for all $u_1,\ldots,u_n,u \in X$ by one of the expressions:

$$f(u_1,\ldots,u_n,u) := \varphi\left(\max_{1\leq i\leq n}\|u_i-u\|\right), \tag{20}$$

$$f(u_1,\ldots,u_n,u) := \varphi\left(\max_{1\leq i\leq n}\|u_i-x_i-u\|\right), \tag{21}$$

and $x_1,\ldots,x_n$ in (21) are given vectors in X, and $i_{\Omega_1\times\ldots\times\Omega_n} : X^n \to \mathbb{R}_+ \cup \{+\infty\}$ is the indicator function of the set $\Omega_1\times\ldots\times\Omega_n$: $i_{\Omega_1\times\ldots\times\Omega_n}(x) = 0$ if $x \in \Omega_1\times\ldots\times\Omega_n$, and $i_{\Omega_1\times\ldots\times\Omega_n}(x) = +\infty$ otherwise. Note that (20) is a particular case of (21) corresponding to setting $x_i := 0$ $(i = 1,\ldots,n)$.

The next three theorems provide dual sufficient conditions for the three nonlinear transversality properties in Definition 1 in terms of Clarke normals in Banach spaces. We refer the readers to [11] for more dual sufficient conditions, also in terms of Fréchet normals in Asplund spaces.

The statements below employ the following dual norm on $(X^*)^{n+1}$ corresponding to (9):

$$\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma = \|x^*\| + \frac{1}{\gamma}\sum_{i=1}^n\|x_i^*\|, \quad x_1^*,\ldots,x_n^*,x^* \in X^*. \tag{22}$$

We denote by d_γ the distance in $(X^*)^{n+1}$ determined by (22).

Theorem 17. *Let $\Omega_1,\ldots,\Omega_n$ be closed subsets of a Banach space X, $\bar{x} \in \cap_{i=1}^n\Omega_i$, and $\varphi \in \mathscr{C}^1$. $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$ if, for some $\mu > 0$ and any $x_i \in X$ $(i = 1,\ldots,n)$ satisfying (11), there exists a $\lambda \in]\varphi(\max_{1\leq i\leq n}\|x_i\|),\delta[$ such that*

$$\varphi'\left(\max_{1\leq i\leq n}\|\omega_i-x_i-x\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\| + \mu\sum_{i=1}^n d\left(x_i^*, N^C_{\Omega_i}(\omega_i)\right)\right) \geq 1 \tag{23}$$

for all $x \in X$ and $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ satisfying

$$\|x-\bar{x}\| < \lambda, \quad \max_{1\leq i\leq n}\|\omega_i-\bar{x}\| < \mu\lambda,$$
$$0 < \max_{1\leq i\leq n}\|\omega_i-x_i-x\| \leq \max_{1\leq i\leq n}\|x_i\|, \tag{24}$$

and all $x_i^* \in X^*$ $(i = 1,\ldots,n)$ satisfying

$$\sum_{i=1}^n\|x_i^*\| = 1, \tag{25}$$

$$\sum_{i=1}^n\langle x_i^*, x+x_i-\omega_i\rangle = \max_{1\leq i\leq n}\|x+x_i-\omega_i\|. \tag{26}$$

Theorem 18. *Let $\Omega_1,\ldots,\Omega_n$ be closed subsets of a Banach space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$. $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$ if, for some $\mu>0$ and any $x'\in X$ satisfying*

$$\|x'-\bar{x}\|<\delta_2,\quad 0<\max_{1\le i\le n} d(x',\Omega_i)<\varphi^{-1}(\delta_1),$$

there exists a $\lambda\in]\varphi\left(\max_{1\le i\le n} d(x',\Omega_i)\right),\delta_1[$ such that

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\|+\mu\sum_{i=1}^n d\left(x_i^*,N^C_{\Omega_i}(\omega_i)\right)\right)\ge 1 \tag{27}$$

for all $x\in X$ and $\omega_i,\omega_i'\in\Omega_i$ $(i=1,\ldots,n)$ satisfying

$$\|x-x'\|<\lambda,\quad \max_{1\le i\le n}\|\omega_i-\omega_i'\|<\mu\lambda,$$
$$0<\max_{1\le i\le n}\|\omega_i-x\|\le\max_{1\le i\le n}\|\omega_i'-x'\|<\varphi^{-1}(\lambda),$$

and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and

$$\sum_{i=1}^n\langle x_i^*,x-\omega_i\rangle=\max_{1\le i\le n}\|x-\omega_i\|. \tag{28}$$

Theorem 19. *Let $\Omega_1,\ldots,\Omega_n$ be closed subsets of a Banach space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$. $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$ if, for some $\mu>0$ and any $\omega_i'\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\xi\in]0,\varphi^{-1}(\delta_1)[$, there exists a $\lambda\in]\varphi(\xi),\delta_1[$ such that inequality (23) holds for all $x,x_i\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying*

$$\|x-\bar{x}\|<\lambda,\quad \max_{1\le i\le n}\|\omega_i-\omega_i'\|<\mu\lambda,$$
$$0<\max_{1\le i\le n}\|\omega_i-x_i-x\|\le\max_{1\le i\le n}\|\omega_i'-x_i-\bar{x}\|=\xi,$$

and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and (26).

Remark 20. The dual sufficient conditions in Theorems 17, 18 and 19 can be reformulated in terms of the *G-normal cones* by Ioffe [14], corresponding to the *approximate G-subdifferentials*, which, similar to the Clarke ones, possess an exact sum rule on general Banach spaces; cf. [14, Theorem 4.69].

4. Dual Necessary Conditions

In this section, the sets $\Omega_1,\ldots,\Omega_n$ and function $\varphi\in\mathscr{C}$ are assumed to be convex.

4.1. Dual Characterizations of Nonlinear Semitransversality.

In this subsection, we use the function $\widehat{f}$ given by (19) with $f:X^{n+1}\to\mathbb{R}_+$ defined by (21). The next statement provides a dual necessary condition of $\varphi-$semitransversality in terms of subdifferentials of $\widehat{f}$.

Proposition 21. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$, then, with $\gamma:=((\varphi'_+(0))^{-1}+1)^{-1}$,*

$$d_\gamma\left(0,\partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})\right)\ge 1 \tag{29}$$

for all $x_i\in X$ $(i=1,\ldots,n)$ satisfying (11).

Proof. Under the assumptions made, the function $\widehat{f}$ is convex. The assertion follows from Proposition 13(ii) since condition (29) is a direct consequence of (12). Indeed, for all $x_i \in X$ $(i = 1,\ldots,n)$ satisfying (11) and all $(x_1^*,\ldots,x_n^*,x^*) \in \partial \widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})$, we have:

$$\begin{aligned}
\|(x_1^*,\ldots,x_n^*,x^*)\|_\gamma &= \sup_{(u_1,\ldots,u_n,u)\neq 0} \frac{\langle (x_1^*,\ldots,x_n^*,x^*),(u_1,\ldots,u_n,u)\rangle}{\|(u_1,\ldots,u_n,u)\|_\gamma} \\
&= \limsup_{\substack{u_i\to\bar{x}\ (i=1,\ldots,n),\ u\to\bar{x}\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{-\langle (x_1^*,\ldots,x_n^*,x^*),(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\rangle}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} \\
&\geq \limsup_{\substack{u_i\to\bar{x}\ (i=1,\ldots,n),\ u\to\bar{x}\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})-\widehat{f}(u_1,\ldots,u_n,u)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} \\
&= \limsup_{\substack{u_i\xrightarrow{\Omega_i}\bar{x}\ (i=1,\ldots,n),\ u\to\bar{x}\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\varphi\left(\max\limits_{1\leq i\leq n}\|x_i\|\right)-\varphi\left(\max\limits_{1\leq i\leq n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} \geq 1.
\end{aligned}$$

□

Remark 22. Condition (29) in Proposition 21 is required to hold for all $x_i \in X$ $(i = 1,\ldots,n)$ satisfying (11). Observe that vectors x_i $(i = 1,\ldots,n)$ are not explicitly present in (29); they are involved in the definition (21) of the function f.

The key condition (29) in Proposition 21 involves a subdifferential of the function $\widehat{f}$ given by (19) with $f : X^{n+1} \to \mathbb{R}_+$ defined by (21). Subgradients of $\widehat{f}$ belong to $(X^*)^{n+1}$ and have $n+1$ component vectors $x_1^*,\ldots,x_n^*,x^*$. As it can be seen from the representation (22) of the dual norm, the contribution of the vectors $x_1^*,\ldots,x_n^*$ on one hand, and x^* on the other hand to condition (29) is different. The next corollary exposes this difference.

Corollary 23. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x} \in \cap_{i=1}^n \Omega_i$, and $\varphi \in \mathscr{C}$ be convex with $\varphi'_+(0) > 0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semi-transversal at $\bar{x}$ with some $\delta > 0$, then, for all $x_i \in X$ $(i = 1,\ldots,n)$ satisfying (11), and all $(x_1^*,\ldots,x_n^*,x^*) \in \partial \widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})$, it holds*

$$\|x^*\| \geq 1 - ((\varphi'_+(0))^{-1} + 1)\sum_{i=1}^n \|x_i^*\|. \tag{30}$$

As a consequence,

$$\liminf_{\substack{x_i^*\to 0\ (i=1,\ldots,n)\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})}} \|x^*\| \geq 1.$$

Proof. The assertion is a direct consequence of Proposition 21 and the representation (22) of the dual norm. □

The next 'δ-free' statement is a direct consequence of Corollary 23.

Corollary 24. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x} \in \cap_{i=1}^n \Omega_i$, and $\varphi \in \mathscr{C}$ be convex with $\varphi'_+(0) > 0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semi-transversal at $\bar{x}$, then*

$$\liminf_{\substack{x_i\to 0,\ x_i^*\to 0\ (i=1,\ldots,n)\\ \max\limits_{1\leq i\leq n}\|x_i\|>0,\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})}} \|x^*\| \geq 1.$$

In the convex setting, a partial converse to Theorem 17 is possible.

Theorem 25. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$, then, with $\mu:=(\varphi'_+(0))^{-1}+1$,*

$$\varphi'\left(\max_{1\le i\le n}\|x_i\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\|+\mu\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\bar{x})\right)\right)\ge 1 \tag{31}$$

for all $x_i\in X$ $(i=1,\ldots,n)$ satisfying (11) and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and

$$\sum_{i=1}^n\langle x_i^*,x_i\rangle=\max_{1\le i\le n}\|x_i\|. \tag{32}$$

Proof. Let $\{\Omega_1,\ldots,\Omega_n\}$ be $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$. Let $\mu:=(\varphi'_+(0))^{-1}+1$ and $\gamma:=\mu^{-1}$. By Proposition 21, condition (29) is satisfied for all $x_i\in X$ $(i=1,\ldots,n)$ satisfying (11). Observe that $\widehat{f}$ is a sum of the function f given by (21) and the indicator function of the set $\Omega_1\times\ldots\times\Omega_n$. Since $\max_{1\le i\le n}\|x_i\|>0$, f is locally Lipschitz continuous near $(\bar{x},\ldots,\bar{x},\bar{x})$. It is a composition of φ and the function

$$\psi(u_1,\ldots,u_n,u):=\max_{1\le i\le n}\|u_i-x_i-u\|,\quad u_1,\ldots,u_n,u\in X. \tag{33}$$

By Lemmas 2 and 3, Proposition 4 and Remark 5,

$$\begin{aligned}\partial\widehat{f}(\bar{x},\ldots,\bar{x},\bar{x})&=\partial f(\bar{x},\ldots,\bar{x},\bar{x})+N_{\Omega_1\times\ldots\times\Omega_n}(\bar{x},\ldots,\bar{x})\times\{0\}\\&=\varphi'\left(\max_{1\le i\le n}\|x_i\|\right)\left(\partial\psi(\bar{x},\ldots,\bar{x},\bar{x})+N_{\Omega_1}(\bar{x})\times\ldots\times N_{\Omega_n}(\bar{x})\times\{0\}\right).\end{aligned}$$

By Lemma 6, $(-x_1^*,\ldots,-x_n^*,x^*)\in\partial\psi(\bar{x},\ldots,\bar{x},\bar{x})$ if and only if conditions (25) and (32) are satisfied and

$$x^*=\sum_{i=1}^n x_i^*. \tag{34}$$

Hence, condition (31) is a consequence of (29). □

Remark 26. (i) The equality $\mu:=(\varphi'_+(0))^{-1}+1$ in Theorem 25 can be replaced by the inequality $\mu\ge(\varphi'_+(0))^{-1}+1$.

(ii) Conditions (31) and (32) in Theorem 25 are particular cases of conditions (23) and (26) in Theorem 17, respectively, corresponding to setting $\omega_1=\ldots=\omega_n=x:=\bar{x}$.

From Theorems 17 and 25, we deduce the following corollary.

Corollary 27. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a Banach space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$.*

(i) *$\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ if inequality (23) holds with some $\mu>0$ for all $x\in X$ and $\omega_i\in\Omega_i$ near $\bar{x}$, and $x_i\in X$ near 0 $(i=1,\ldots,n)$ satisfying (24), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and (26).*

(ii) *If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$, then inequality (31) holds with $\mu:=(\varphi'_+(0))^{-1}+1$ for all $x_i\in X$ near 0 $(i=1,\ldots,n)$ satisfying $\max_{1\le i\le n}\|x_i\|>0$, and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and (32).*

A decomposition of the dual necessary transversality condition (31) in Theorem 25 can be easily obtained.

Corollary 28. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$, then, for all $x_i\in X$ $(i=1,\ldots,n)$ satisfying (11), the following conditions hold true:*

(i) *for all* $x_i^* \in N_{\Omega_i}(\bar{x})$ $(i=1,\ldots,n)$ *satisfying* (25) *and* (32), *it holds*

$$\varphi'\left(\max_{1\le i\le n}\|x_i\|\right)\left\|\sum_{i=1}^n x_i^*\right\|\ge 1;$$

(ii) *for all* $x_i^* \in X^*$ $(i=1,\ldots,n)$ *satisfying* (25), (32) *and* $\sum_{i=1}^n x_i^*=0$, *it holds*

$$\varphi'\left(\max_{1\le i\le n}\|x_i\|\right)\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\bar{x})\right)\ge\left((\varphi'_+(0))^{-1}+1\right)^{-1}.$$

4.2. **Dual Characterizations of Nonlinear Subtransversality.** In this subsection, we use the function $\widehat{f}$ given by (19) with $f: X^{n+1}\to\mathbb{R}_+$ defined by (20). The next statement provides a dual necessary condition of $\varphi-$subtransversality in terms of subdifferentials of $\widehat{f}$.

Proposition 29. *Let* $\Omega_1,\ldots,\Omega_n$ *be closed convex subsets of a normed vector space* X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, *and* $\varphi\in\mathscr{C}$ *be convex with* $\varphi'_+(0)>0$. *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$ *with some* $\delta_1>0$ *and* $\delta_2>0$, *then, with* $\gamma:=((\varphi'_+(0))^{-1}+1)^{-1}$,

$$d_\gamma\left(0,\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)\right)\ge 1 \tag{35}$$

for all $x\in X$ *and* $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ *satisfying* (14).

Proof. Under the assumptions made, the function $\widehat{f}$ is convex. The assertion follows from Proposition 14(ii) since condition (35) is a direct consequence of (15); cf. the proof of Proposition 21. □

Similar to the case of nonlinear semitransversality, the difference in the contribution of components of subgradients of $\widehat{f}$ to the key dual condition (35) in Proposition 29 can be exposed.

Corollary 30. *Let* $\Omega_1,\ldots,\Omega_n$ *be closed convex subsets of a normed vector space* X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, *and* $\varphi\in\mathscr{C}$ *be convex with* $\varphi'_+(0)>0$. *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$ *with some* $\delta_1>0$ *and* $\delta_2>0$, *then, for all* $x\in X$ *and* $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ *satisfying* (14), *and all* $(x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)$, *condition* (30) *holds. As a consequence,*

$$\liminf_{\substack{x_i^*\to 0\ (i=1,\ldots,n)\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)}}\|x^*\|\ge 1.$$

Proof. The assertion is a direct consequence of Proposition 29 and the representation (22) of the dual norm. □

The next 'δ-free' statement is a direct consequence of Corollary 30.

Corollary 31. *Let* $\Omega_1,\ldots,\Omega_n$ *be closed convex subsets of a normed vector space* X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, *and* $\varphi\in\mathscr{C}$ *be convex with* $\varphi'_+(0)>0$. *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$, *then*

$$\liminf_{\substack{x\to\bar{x},\ \omega_i\xrightarrow{\Omega_i}\bar{x},\ x_i^*\to 0\ (i=1,\ldots,n)\\ \max\limits_{1\le i\le n}\|\omega_i-x\|>0,\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)}}\|x^*\|\ge 1.$$

In the convex setting, a partial converse to Theorem 18 is possible.

Theorem 32. *Let* $\Omega_1,\ldots,\Omega_n$ *be closed convex subsets of a normed vector space* X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, *and* $\varphi\in\mathscr{C}^1$ *be convex with* $\varphi'_+(0)>0$. *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$ *with some* $\delta_1>0$ *and* $\delta_2>0$, *then, with* $\mu:=(\varphi'_+(0))^{-1}+1$, *inequality* (27) *holds for all* $x\in X$ *and* $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ *satisfying* (14), *and all* $x_i^*\in X^*$ $(i=1,\ldots,n)$ *satisfying* (25) *and* (28).

Proof. Let $\{\Omega_1,\ldots,\Omega_n\}$ be $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$. Let $\mu:=(\varphi'_+(0))^{-1}+1$ and $\gamma:=\mu^{-1}$. By Proposition 29, condition (35) is satisfied for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (14). Observe that $\widehat{f}$ is a sum of the function f given by (20) and the indicator function of the set $\Omega_1\times\ldots\times\Omega_n$. Since $\max_{1\le i\le n}\|\omega_i-x\|>0$, f is locally Lipschitz continuous near $(\omega_1,\ldots,\omega_n,x)$. It is a composition of φ and the function

$$\psi(u_1,\ldots,u_n,u):=\max_{1\le i\le n}\|u_i-u\|,\quad u_1,\ldots,u_n,u\in X.$$

By Lemmas 2 and 3, Proposition 4 and Remark 5,

$$\begin{aligned}\partial\widehat{f}(\omega_1,\ldots,\omega_n,x)&=\partial f(\omega_1,\ldots,\omega_n,x)+N_{\Omega_1\times\ldots\times\Omega_n}(\omega_1,\ldots,\omega_n)\times\{0\}\\&=\varphi'\left(\max_{1\le i\le n}\|\omega_i-x\|\right)\left(\partial\psi(\omega_1,\ldots,\omega_n,x)+N_{\Omega_1}(\omega_1)\times\ldots\times N_{\Omega_n}(\omega_n)\times\{0\}\right).\end{aligned}$$

By Lemma 6, $(-x_1^*,\ldots,-x_n^*,x^*)\in\partial\psi(\omega_1,\ldots,\omega_n,x)$ if and only if conditions (25), (28) and (34) are satisfied. Hence, condition (27) is a consequence of (35). □

Remark 33.
(i) The equality $\mu:=(\varphi'_+(0))^{-1}+1$ in Theorem 32 can be replaced by the inequality $\mu\ge(\varphi'_+(0))^{-1}+1$.
(ii) The necessary conditions in Theorem 32 correspond to setting $\omega_i=\omega'_i$ $(i=1,\ldots,n)$ and $x=x'$ in the sufficient conditions in Theorem 18.

From Theorems 18 and 32, we deduce the following corollary.

Corollary 34. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a Banach space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ if and only if inequality (27) holds with $\mu:=(\varphi'_+(0))^{-1}+1$ for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ with $\max_{1\le i\le n}\|\omega_i-x\|>0$, and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and (28).*

Proof. The necessity part is exactly the 'δ-free' version of Theorem 32. To show the sufficiency, observe that the conditions in Theorem 18 are satisfied (for some $\delta_1>0$ and $\delta_2>0$) with $\mu:=(\varphi'_+(0))^{-1}+1$, any sufficiently large $\lambda>0$, and $x'=x$ and $\omega'_i=\omega_i$ $(i=1,\ldots,n)$. □

A decomposition of the dual necessary transversality condition in Theorem 32 can be easily obtained.

Corollary 35. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, for all $x\in X$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfying (14), the following conditions hold true:*

(i) *for all $x_i^*\in N_{\Omega_i}(\omega_i)$ $(i=1,\ldots,n)$ satisfying (25) and (28), it holds*

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x\|\right)\left\|\sum_{i=1}^n x_i^*\right\|\ge 1;$$

(ii) *for all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25), (28) and $\sum_{i=1}^n x_i^*=0$, it holds*

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x\|\right)\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\omega_i)\right)\ge((\varphi'_+(0))^{-1}+1)^{-1}.$$

4.3. **Dual Characterizations of Nonlinear Transversality.** In this subsection, we use the function $\widehat{f}$ given by (19) with $f:X^{n+1}\to\mathbb{R}_+$ defined by (21). The next statement provides a dual necessary condition of $\varphi-$transversality in terms of subdifferentials of $\widehat{f}$.

Proposition 36. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, with $\gamma:=((\varphi'_+(0))^{-1}+1)^{-1}$,*

$$d_\gamma\left(0,\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})\right)\geq 1 \tag{36}$$

for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (17).

Proof. Under the assumptions made, the function $\widehat{f}$ is convex. The assertion follows from Proposition 15 since condition (36) is a direct consequence of (18); cf. the proof of Proposition 21. □

Similar to the case of the other two nonlinear transversality properties, the difference in the contribution of components of subgradients of $\widehat{f}$ to the key dual condition (36) in Proposition 36 can be exposed.

Corollary 37. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (17), and all $(x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})$, condition (30) holds. As a consequence,*

$$\liminf_{\substack{x_i^*\to 0\ (i=1,\ldots,n)\\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})}}\|x^*\|\geq 1.$$

Proof. The assertion is a direct consequence of Proposition 36 and the representation (22) of the dual norm. □

The next 'δ-free' statement is a direct consequence of Corollary 37.

Corollary 38. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$, then*

$$\liminf_{\substack{\omega_i\xrightarrow{\Omega_i}\bar{x},\ x_i\to 0,\ x_i^*\to 0\ (i=1,\ldots,n)\\ \max\limits_{1\leq i\leq n}\|\omega_i-x_i-\bar{x}\|>0,\ (x_1^*,\ldots,x_n^*,x^*)\in\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x})}}\|x^*\|\geq 1.$$

In the convex setting, a partial converse to Theorem 19 is possible.

Theorem 39. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, with $\mu:=(\varphi'_+(0))^{-1}+1$,*

$$\varphi'\left(\max_{1\leq i\leq n}\|\omega_i-x_i-\bar{x}\|\right)\left(\left\|\sum_{i=1}^n x_i^*\right\|+\mu\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\omega_i)\right)\right)\geq 1 \tag{37}$$

for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (17), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and

$$\sum_{i=1}^n\langle x_i^*,\bar{x}+x_i-\omega_i\rangle=\max_{1\leq i\leq n}\|\bar{x}+x_i-\omega_i\|. \tag{38}$$

Proof. Let $\{\Omega_1,\ldots,\Omega_n\}$ be $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$. Let $\mu:=(\varphi'_+(0))^{-1}+1$ and $\gamma:=\mu^{-1}$. By Proposition 36, condition (36) is satisfied for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (17). Observe that $\widehat{f}$ is a sum of the function f given by (21) and the indicator function of the set $\Omega_1\times\ldots\times\Omega_n$. Since $\max_{1\leq i\leq n}\|\omega_i-x_i-\bar{x}\|>0$, f is locally Lipschitz continuous near $(\omega_1,\ldots,\omega_n,\bar{x})$. It is a composition of φ and the function ψ defined by (33). By Lemmas 2 and 3, Proposition 4 and Remark 5,

$$\partial\widehat{f}(\omega_1,\ldots,\omega_n,\bar{x}) = \partial f(\omega_1,\ldots,\omega_n,\bar{x}) + N_{\Omega_1\times\ldots\times\Omega_n}(\omega_1,\ldots,\omega_n)\times\{0\}$$
$$= \varphi'\left(\max_{1\le i\le n}\|\omega_i - x_i - \bar{x}\|\right)\left(\partial\psi(\omega_1,\ldots,\omega_n,\bar{x}) + N_{\Omega_1}(\omega_1)\times\ldots\times N_{\Omega_n}(\omega_n)\times\{0\}\right).$$

By Lemma 6, $(-x_1^*,\ldots,-x_n^*,x^*)\in\partial\psi(\omega_1,\ldots,\omega_n,\bar{x})$ if and only if conditions (25), (34) and (38) are satisfied. Hence, condition (37) is a consequence of (36). □

Remark 40. (i) The equality $\mu := (\varphi'_+(0))^{-1}+1$ in Theorem 39 can be replaced by the inequality $\mu \ge (\varphi'_+(0))^{-1}+1$.
(ii) Conditions (37) and (38) in Theorem 39 are particular cases of conditions, respectively, (23) and (26) in Theorem 19, corresponding to setting $x := \bar{x}$.
(iii) The necessary conditions in Theorems 25 and 32 correspond to setting, respectively, $\omega_i := \bar{x}$ $(i=1,\ldots,n)$ and $x_1 = \ldots = x_n$ in the necessary conditions in Theorem 39.

From Theorems 19 and 39, we deduce the following corollary.

Corollary 41. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a Banach space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$.*
(i) *$\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if inequality (23) holds with some $\mu>0$ for all $x\in X$ and $\omega_i\in\Omega_i$ near $\bar{x}$ and $x_i\in X$ near 0 $(i=1,\ldots,n)$ with $\max_{1\le i\le n}\|\omega_i-x_i-x\|>0$, and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and (26).*
(ii) *If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$, then inequality (37) holds with $\mu := (\varphi'_+(0))^{-1}+1$ for all $\omega_i\in\Omega_i$ near $\bar{x}$ and $x_i\in X$ near 0 $(i=1,\ldots,n)$ with $\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|>0$, and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and (38).*

A decomposition of the dual transversality condition (31) in Theorem 39 can be easily obtained.

Corollary 42. *Let $\Omega_1,\ldots,\Omega_n$ be closed convex subsets of a normed vector space X, $\bar{x}\in\cap_{i=1}^n\Omega_i$, and $\varphi\in\mathscr{C}^1$ be convex with $\varphi'_+(0)>0$. If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then, for all $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfying (17), and all $x_i^*\in X^*$ $(i=1,\ldots,n)$ satisfying (25) and (38), the following conditions hold true:*
(i) *if $x_i^*\in N_{\Omega_i}(\bar{x})$ $(i=1,\ldots,n)$, then*
$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)\left\|\sum_{i=1}^n x_i^*\right\|\ge 1;$$
(ii) *if $\sum_{i=1}^n x_i^*=0$, then*
$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)\sum_{i=1}^n d\left(x_i^*,N_{\Omega_i}(\bar{x})\right)\ge\left((\varphi'_+(0))^{-1}+1\right)^{-1}.$$

5. Nonlinear Transversality of a Set-Valued Mapping to a Set

In this section, we provide applications of the dual necessary conditions of nonlinear transversality properties of collections of sets established in Section 4 to nonlinear extensions of the new *transversality properties of a set-valued mapping to a set in the range space* due to Ioffe [14].

In the linear case, i.e. when $\varphi(t)=\alpha t$ for some $\alpha>0$ and all $t\ge 0$, the properties in parts (ii) and (iii) of the next definition reduce, respectively, to the ones in [14, Definitions 7.11 and 7.8]; cf. the metric characterizations of the properties in [6, 10]. The property in part (i) is new.

Definition 43. Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed vector spaces, $S \subset Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\bar{y} \in S$, and $\varphi \in \mathscr{C}$.

(i) F is $\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ if $\{\operatorname{gph} F, X \times S\}$ is $\varphi-$semitransversal at $(\bar{x},\bar{y})$, i.e. there exists a $\delta > 0$ such that
$$(\operatorname{gph} F - (u_1,v_1)) \cap (X \times (S - v_2)) \cap B_\rho(\bar{x},\bar{y}) \neq \emptyset$$
for all $\rho \in]0,\delta[$, $u_1 \in X$, $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}) < \rho$.

(ii) F is $\varphi-$subtransversal to S at $(\bar{x},\bar{y})$ if $\{\operatorname{gph} F, X \times S\}$ is $\varphi-$subtransversal at $(\bar{x},\bar{y})$, i.e. there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that
$$\operatorname{gph} F \cap (X \times S) \cap B_\rho(x,y) \neq \emptyset$$
for all $\rho \in]0,\delta_1[$ and $(x,y) \in B_{\delta_2}(\bar{x},\bar{y})$ with $\varphi(\max\{d((x,y),\operatorname{gph} F), d(y,S)\}) < \rho$.

(iii) F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ if $\{\operatorname{gph} F, X \times S\}$ is $\varphi-$transversal at $(\bar{x},\bar{y})$, i.e. there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that
$$(\operatorname{gph} F - (x_1,y_1) - (u_1,v_1)) \cap (X \times (S - y_2 - v_2)) \cap (\rho\mathbb{B}) \neq \emptyset$$
for all $\rho \in]0,\delta_1[$, $(x_1,y_1) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2 \in S \cap B_{\delta_2}(\bar{y})$, $u_1 \in X$, $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}) < \rho$.

When S is a singleton, i.e. $S = \{\bar{y}\}$, the properties in Definition 43 have strong connections with the corresponding nonlinear regularity properties of set-valued mappings; cf. [10].

The next three statements are direct consequences of Theorems 25, 32 and 39, respectively.

Proposition 44. *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed vector spaces with closed convex graph, S be a closed convex subset of Y, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\bar{y} \in S$, and $\varphi \in \mathscr{C}^1$ be convex with $\varphi'_+(0) > 0$. If F is $\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ with some $\delta > 0$, then, with $\mu := (\varphi'_+(0))^{-1} + 1$,*
$$\varphi'(\max\{\|u_1\|,\|v_1\|,\|v_2\|\})\Big(\|x_1^*\| + \|y_1^* + y_2^*\| \\ + \mu\big(d((x_1^*,y_1^*),N_{\operatorname{gph} F}(\bar{x},\bar{y})) + d(y_2^*,N_S(\bar{y}))\big)\Big) \geq 1$$
for all $u_1 \in X$, $v_1, v_2 \in Y$ satisfying
$$0 < \max\{\|u_1\|,\|v_1\|,\|v_2\|\} < \varphi^{-1}(\delta),$$
and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying
$$\|x_1^*\| + \|y_1^*\| + \|y_2^*\| = 1, \tag{39}$$
$$\langle x_1^*,u_1\rangle + \langle y_1^*,v_1\rangle + \langle y_2^*,v_2\rangle = \max\{\|u_1\|,\|v_1\|,\|v_2\|\}.$$

Proposition 45. *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed vector spaces with closed convex graph, S be a closed convex subset of Y, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\bar{y} \in S$, and $\varphi \in \mathscr{C}^1$ be convex with $\varphi'_+(0) > 0$. If F is $\varphi-$subtransversal to S at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then, with $\mu := (\varphi'_+(0))^{-1} + 1$,*
$$\varphi'(\max\{\|x_1 - x\|,\|y_1 - y\|,\|y_2 - y\|\})\Big(\|x_1^*\| + \|y_1^* + y_2^*\| \\ + \mu\big(d((x_1^*,y_1^*),N_{\operatorname{gph} F}(x_1,y_1)) + d(y_2^*,N_S(y_2))\big)\Big) \geq 1$$
for all $(x,y) \in B_{\delta_2}(\bar{x},\bar{y})$ and $(x_1,y_1) \in \operatorname{gph} F$, $y_2 \in S$ satisfying
$$0 < \max\{\|x_1 - x\|,\|y_1 - y\|,\|y_2 - y\|\} < \varphi^{-1}(\lambda),$$
and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying (39) and
$$\langle x_1^*,x - x_1\rangle + \langle y_1^*,y - y_1\rangle + \langle y_2^*,y - y_2\rangle = \max\{\|x - x_1\|,\|y - y_1\|,\|y - y_2\|\}.$$

Proposition 46. *Let $F : X \rightrightarrows Y$ be a set-valued mapping between normed vector spaces with closed convex graph, S be a closed convex subset of Y, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, $\bar{y} \in S$, and $\varphi \in \mathscr{C}^1$ be convex with $\varphi'_+(0) > 0$. If F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then, with $\mu := (\varphi'_+(0))^{-1} + 1$,*

$$\varphi'\left(\max\{\|x_1-u_1-\bar{x}\|,\|y_1-v_1-\bar{y}\|,\|y_2-v_2-\bar{y}\|\}\right)\Big(\|x_1^*\|+\|y_1^*+y_2^*\| \\ +\mu\left(d((x_1^*,y_1^*),N_{\operatorname{gph}F}(x_1,y_1))+d(y_2^*,N_S(y_2))\right)\Big) \geq 1$$

for all $(x_1,y_1) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2 \in S \cap B_{\delta_2}(\bar{y})$ and $u_1 \in X$, $v_1, v_2 \in Y$ with

$$0 < \max\{\|x_1-u_1-\bar{x}\|,\|y_1-v_1-\bar{y}\|,\|y_2-v_2-\bar{y}\|\} < \varphi^{-1}(\delta_1),$$

and all $x_1^* \in X^*$, $y_1^*, y_2^* \in Y^*$ satisfying (39) and

$$\langle x_1^*,\bar{x}+u_1-x_1\rangle + \langle y_1^*,\bar{y}+v_1-y_1\rangle + \langle y_2^*,\bar{y}+v_2-y_2\rangle \\ = \max\{\|x_1-u_1-\bar{x}\|,\|y_1-v_1-\bar{y}\|,\|y_2-v_2-\bar{y}\|\}.$$

Remark 47. Decompositions of the combined dual necessary transversality conditions in Propositions 44, 45 and 46 can be easily obtained; cf. Corollaries 28, 35 and 42.

Acknowledgements

We wish to thank the anonymous referee for their comments and suggestions.

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