RELATIONS BETWEEN ABS-NORMAL NLPS AND MPECs UNDER STRONG CONSTRAINT QUALIFICATIONS

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Abstract. This work is part of an ongoing effort of comparing non-smooth optimization problems in abs-normal form to MPECs. We study the general abs-normal NLP with equality and inequality constraints in relation to an equivalent MPEC reformulation. We show that kink qualifications and MPEC constraint qualifications of linear independence type and Mangasarian-Fromovitz type are equivalent. Then we consider strong stationarity concepts with first and second order optimality conditions, which again turn out to be equivalent for the two problem classes. Throughout we also consider specific slack reformulations suggested in [6], which preserve constraint qualifications of linear independence type but not of Mangasarian-Fromovitz type.

1. Introduction

We take interest in non-smooth nonlinear optimization problems of the form

\[
\min_x f(x) \quad \text{s.t.} \quad g(x) = 0, \quad h(x) \geq 0, \tag{NLP}
\]

where \(D^x \subset \mathbb{R}^n\) is open, the objective \(f \in C^d(D^x, \mathbb{R})\) is a smooth function (\(d \geq 1\)) and the equality and inequality constraints \(g \in C^d_{\text{abs}}(D^x, \mathbb{R}^{m_1})\) and \(h \in C^d_{\text{abs}}(D^x, \mathbb{R}^{m_2})\) are non-smooth functions with the non-smoothness exposed in abs-normal form [1]. Thus, there exist functions \(c_E \in C^d(D^x, |z|, \mathbb{R}^{m_1})\), \(c_I \in C^d(D^x, |z|, \mathbb{R}^{m_2})\) and \(c_Z \in C^d(D^x, |z|, \mathbb{R}^s)\) with \(\partial_2 c_Z(x, |z|)\) strictly lower triangular such that

\[
g(x) = c_E(x, |z|), \quad h(x) = c_I(x, |z|), \quad z = c_Z(x, |z|) \quad \text{with} \quad \partial_2 c_Z(x, |z|) \quad \text{strictly lower triangular}.
\]

Note that we introduce one joint switching constraint \(c_Z\) for \(g\) and \(h\) and reuse switching variables \(z\) if the same argument occurs inside an absolute value in \(g\) and \(h\). Here, components of \(z\) can be computed one by one from \(x\) and \(z_i, i < j\), since \(\partial_2 c_Z(x, |z|)\) is strictly lower triangular. In the following we write \(z(x)\) to denote this dependence explicitly. However, \(z\) is implicitly defined by \(z = c_Z(x, |z|)\). To consider solvability of this system, we use the reformulation \(|z_i| = \text{sign}(z_i)z_i\).

Definition 1 (Signature of \(z\)). Let \(x \in D^x\). We define the signature \(\sigma(x)\) and the associated signature matrix \(\Sigma(x)\) as

\[
\sigma(x) := \text{sign}(z(x)) \in \{-1, 0, 1\}^s, \quad \Sigma(x) := \text{diag}(\sigma(x)).
\]

A signature vector \(\sigma(x) \in \{-1, 1\}^s\) is called definite, otherwise indefinite.

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With Definition 1, we can write $|z(x)| = \Sigma(x)z(x)$ and consider the system $z = c_x(x, \Sigma z)$ for fixed $\Sigma = \Sigma(\hat{x})$. Application of the implicit function theorem leads to the existence of a locally unique solution $z(x)$ with Jacobian

$$\partial_z z(\hat{x}) = [I - \partial_2 c_x(\hat{x}, |z(\hat{x})|)]\Sigma^{-1}\partial_1 c_x(\hat{x}, |z(\hat{x})|) \in \mathbb{R}^{s \times n}.$$

**Definition 2 (Active Switching Set).** We call the switching variable $z_i$ active if $z_i(x) = 0$. The active switching set $\alpha$ consists of all indices of active switching variables, i.e.

$$\alpha(x) := \{ i \in \{1, \ldots, s\} : z_i(x) = 0 \}.$$

We denote the number of active switching variables by $|\alpha(x)|$ and the number of inactive ones by $|\sigma(x)| := s - |\alpha(x)|$.

1.1. **Literature, Contributions, and Structure.** The abs-normal NLPs considered here are a direct generalization of unconstrained abs-normal problems developed by Griewank and Walther [1, 2]. These problems offer particularly attractive theoretical features when generalizing KKT theory and stationarity concepts, and they are tractable by sophisticated algorithms with guaranteed convergence based on piecewise linearizations and using algorithmic differentiation techniques [3, 4].

In [5] we have shown that unconstrained optimization problems in abs-normal form are a subclass of MPECs and we have studied regularity concepts of linear independence type, Mangasarian-Fromovitz type and Abadie type. We have also shown that abs-normal NLPs with general constraints are equivalent to the class of MPECs. In [6] we have generalized optimality conditions of unconstrained abs-normal problems to the case with equality and inequality constraints under the linear independence kink qualification.

In this article we provide a detailed comparative study of general abs-normal NLPs and MPECs, considering constraint qualifications of linear independence type and Mangasarian Fromovitz type for the standard formulation and for a reformulation with absolute value slacks that was suggested in [6]. In particular, we show that corresponding constraint qualifications of abs-normal NLPs and MPECs are equivalent and that the linear independence type constraint qualifications are preserved by the slack reformulation whereas Mangasarian-Fromovitz type constraint qualifications are not. Then we compare optimality conditions of first and second order for abs-normal NLPs and MPECs under the respective linear independence type constraint qualifications. We show equivalence of kink stationarity and strong stationarity and thus of first order necessary conditions. Under additional assumptions, we then prove equivalence of positive (semi-)definiteness of the associated reduced Hessians, which gives correspondences of second order necessary and sufficient conditions.

The remainder of this article is structured as follows. In Section 2 we present the general abs-normal NLP and its slack reformulation, and we formulate the associated kink qualifications and compare them. In Section 3 we introduce counterpart MPECs for the two formulations of abs-normal NLPs and compare the associated MPEC-constraint qualifications. Then, we show equivalence of the regularity concepts for abs-normal NLPs and MPECs in Section 4. Finally, in Section 5 we state optimality conditions of first and second-order for abs-normal NLPs and MPECs and prove equivalence relations between them. We conclude in Section 6 and give a brief outlook.

2. **Abs-Normal NLPs**

In this section we consider two formulations for non-smooth NLPs in abs-normal form that differ in the treatment of inequality constraints.
2.1. General Abs-Normal NLPs. In this paragraph we consider abs-normal NLPs with equalities and inequalities, obtained by substituting the constraints representation in abs-normal form (ANF) into the general non-smooth problem (NLP). Note that we use the variables \((t, z')\) instead of \((x, z)\).

**Definition 3** (Abs-Normal NLP). Let \(D^t\) be an open subset of \(\mathbb{R}^m\). A non-smooth NLP is called an abs-normal NLP if functions \(f \in C^d(D^t, \mathbb{R}), c_\Sigma \in C^d(D^t, \mathbb{R}^m_1), c_{\Sigma} \in C^d(D^t, \mathbb{R}^m_z)\), and \(c_{\Xi} \in C^d(D^t, \mathbb{R}^n)\) with \(d \geq 1\) exist such that the NLP reads

\[
\begin{align*}
\min_{t, z'} f(t) \quad \text{s.t.} \quad & c_\Sigma(t, |z'|) = 0, \\
& c_{\Xi}(t, |z'|) \geq 0, \\
& c_{\Xi}(t, |z'|) - z' = 0,
\end{align*}
\]

where \(0 \in D^{|z|}\) and \(\partial_2 c_{\Xi}(x, |z'|)\) is strictly lower triangular. The feasible set of \((I-NLP)\) is denoted by

\[
F_{\text{abs}} := \left\{ (t, z') \mid c_\Sigma(t, |z'|) = 0, c_{\Xi}(t, |z'|) \geq 0, \quad c_{\Xi}(t, |z'|) - z' = 0 \right\}.
\]

In contrast to standard NLP theory, we do not count equalities as active constraints.

**Definition 4** (Active Set). Let \((t, z'(t)) \in F_{\text{abs}}\). We call the constraint \(i \in I\) active if \(c_i(t, |z'(t)|) = 0\). The active set \(A(t)\) consists of all indices of active constraints,

\[
A(t) = \{ i \in I : c_i(t, |z'(t)|) = 0 \}.
\]

We denote the number of active inequality constraints by \(|A(t)|\).

To define the linear independence kink qualification as well as the interior direction kink qualification for \((I-NLP)\) we need its Jacobians.

**Definition 5** (Jacobians). Consider the abs-normal NLP \((I-NLP)\). For \(t \in D^t\) set \(A = A(t), \alpha = \alpha(t), \sigma = \sigma(t), \Sigma = \text{diag}(\sigma)\), and \(c_A = [c_i]_{i \in A}\). The equality Jacobian is

\[
J_\Sigma(t) := \partial_1 c_\Sigma(t, \Sigma z'(t)) = \partial_1 c_\Sigma(t, \Sigma z'(t)) + \partial_2 c_\Sigma(t, \Sigma z'(t)) \Sigma \partial_3 z'(t)
\]

\[
= \partial_1 c_\Sigma(t, |z'(t)|) + \partial_2 c_\Sigma(t, |z'(t)|) \Sigma \partial_3 z'(t),
\]

the active inequality Jacobian is

\[
J_A(t) := \partial_1 c_A(t, \Sigma z'(t)) = \partial_1 c_A(t, \Sigma z'(t)) + \partial_2 c_A(t, \Sigma z'(t)) \Sigma \partial_3 z'(t)
\]

\[
= \partial_1 c_A(t, |z'(t)|) + \partial_2 c_A(t, |z'(t)|) \Sigma \partial_3 z'(t),
\]

and the active switching Jacobian is

\[
J_\alpha(t) := [c_i^T \partial_3 z'(t)]_{i \in \alpha} = [c_i^T \Sigma^{-1} \partial_3 c_\Xi(t, |z'(t)|)]_{i \in \alpha}.
\]

**Definition 6** (Linear Independence Kink Qualification (LIKQ)). We say that the linear independence kink qualification (LIKQ) holds for \((I-NLP)\) at a feasible point \(t \in D^t\) if

\[
J_{\text{abs}}(t) = \begin{bmatrix} J_\Sigma(t) \\ J_A(t) \\ J_\alpha(t) \end{bmatrix} = \begin{bmatrix} \partial_1 c_\Sigma(t, |z'(t)|) \\ \partial_1 c_A(t, |z'(t)|) \\ \Sigma^{-1} c_i^T \partial_3 z'(t) \end{bmatrix}_{i \in \alpha} \in \mathbb{R}^{(m_1 + |A| + |\alpha|) \times n_z}
\]

has full row rank \(m_1 + |A| + |\alpha|\).
Definition 7 (Interior Direction Kink Qualification (IDKQ)). We say that the interior direction kink qualification (IDKQ) holds for (I-NLP) at a feasible point \( t \in D_t \) if
\[
\begin{bmatrix}
J_E(t) \\
J_\alpha(t)
\end{bmatrix} = \begin{bmatrix}
\partial_t c_E(t, |z^t(t)|) \\
[e^T_\alpha \partial_t z^t(t)]_{E\alpha}
\end{bmatrix} \in \mathbb{R}^{(m_1 + |\alpha|) \times n_t}
\]
has full row rank \( m_1 + |\alpha| \) and if there exists a vector \( d \in \mathbb{R}^{n_t} \) such that
\[
J_E(t) d = 0, \quad J_\alpha(t) d = 0, \quad \text{and} \quad J_A(t) d > 0.
\]

For the general abs-normal NLP (I-NLP) considered here, IDKQ actually generalizes MFCQ from the smooth case and corresponds to MPEC-MFCQ, as we will show below. We cannot use the canonical name MFKQ, however, since Griewank and Walther have already defined MFKQ as a different weakening of LIKQ in [4]. We believe that other possible names like “Abs-normal MFKQ” or “Constrained MFKQ” would produce confusion rather than clarification and hence suggest the descriptive name “Interior Direction KQ”.

The following example from [7] (converted from MPEC form to abs-normal NLP form) shows that IDKQ is weaker than LIKQ in the presence of inequality constraints.

Example 8 (IDKQ is weaker than LIKQ). Consider the problem
\[
\min_{t \in \mathbb{R}^3, z^t \in \mathbb{R}} f(t) \quad \text{s.t.} \quad \begin{align*}
t_1 + t_2 - t_3 &= 0, \\
4t_1 - t_3 &\geq 0, \\
4t_2 - t_3 &\geq 0, \\
t_1 - t_2 - z^t &= 0
\end{align*}
\]
with solution \( t^* = (0, 0, 0) \) and \( (z^t)^* = 0 \). We compute
\[
J_A(t^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J_E(t^*) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad J_\alpha(t^*) = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\]

Here, LIKQ is not satisfied but IDKQ is satisfied with \( d = (0, 0, -1) \).

2.2. Abs-Normal NLPs with Inequality Slacks. In this paragraph we consider abs-normal NLPs with slack variables introduced for all inequalities. We make use of the absolute value of a slack variable, an idea due to Griewank. This results in a class of purely equality-constrained abs-normal NLPs, which simplifies the derivation of optimality conditions under the LIKQ, see [6] and Section 5.

Using slack variables \( w \in \mathbb{R}^{m_2} \), we obtain the following reformulation of (NLP):
\[
\min_{t, w} f(t) \quad \text{s.t.} \quad \begin{align*}
g(t) &= 0, \\
h(t) - |w| &= 0.
\end{align*}
\]
Then, we express \( g \) and \( h \) in abs-normal form as in (ANF) and introduce additional switching variables \( z^w \) to handle \( |w| \). This approach leads to the next definition.

Definition 9 (Abs-Normal NLP with Inequality Slacks). An abs-normal NLP posed in the following form is called an abs-normal NLP with inequality slacks:
\[
\min_{t, w, z^t, z^w} f(t) \quad \text{s.t.} \quad \begin{align*}
c_E(t, |z^t|) &= 0, \\
c_I(t, |z^t|) - |z^w| &= 0, \\
c_Z(t, |z^t|) - z^t &= 0, \\
w - z^w &= 0,
\end{align*}
\]
where \( 0 \in D|z^t| \) and \( \partial z E(x, |z^t|) \) is strictly lower triangular. The feasible set of (E-NLP) is a lifting of \( F_{abs} \),
\[
F_{e-abs} := \left\{ (t, w, z^t, z^w) \left| \begin{array}{l}
  c_E(t, |z^t|) = 0, \ c_T(t, |z^t|) - |z^w| = 0, \\
  c_T(t, |z^t|) - z^t = 0, \ w - z^w = 0
\end{array} \right. \right\}
\]
\[
= \left\{ (t, w, z^t, z^w) : (t, z^t) \in F_{abs}, \ w = z^w, \ |z^w| = c_Z(t, |z^t|) \right\}.
\]

We split the active switching set into components for variables \( t \) and \( w \), i.e. \( \alpha = (\alpha^t, \alpha^w) \).

**Remark 10.** Note that introducing \( |w| \) converts inequalities to pure equalities without a nonnegativity condition for the slack variables \( w \). However, the slack reformulation has some subtle issues. Subsequently we will show that, in contrast to LIKQ, IDKQ is not preserved. Moreover, one cannot eliminate the equation \( w - z^w = 0 \) (and hence \( z^w \) or \( w \)) in (E-NLP) since this would destroy the abs-normal form. Finally, the slack \( w \) is not uniquely determined since the signs of nonzero components \( w_i \) can be chosen arbitrarily, yielding a set of \( 2^{n_w-\alpha^w} \) choices, \( W(t) := \{w : |w| = c_Z(t, |z^t(t)|)\} \).

**Lemma 11.** LIKQ for (E-NLP) at \((t, w) \in D^t \times \mathbb{R}^{m_z}\) is full row rank of
\[
J_{e-abs}(t, w) = \begin{bmatrix}
\partial z E(t, |z^t(t)|) & 0 \\
0 & -\Sigma^w \\
[c^i \partial z E(t, |z^t(t)|)_{i \in \alpha^w}] & 0 \\
0 & [c^i \partial z E(t, |z^t(t)|)_{i \in \alpha^w}]
\end{bmatrix} \in \mathbb{R}^{(m_1 + m_2 + |\alpha^t| + |\alpha^w|) \times (m_z + m_2)}.
\]

**Proof.** Set \( x = (t, w), \ z = (z^t, z^w), \ f(x) = f(t), \ \tilde{c}_E(x, |z|) = (c_E(t, |z^t|), c_T(t, |z^t|) - |z^w|), \) and \( \tilde{c}_Z(x, |z|) = (c_Z(t, |z^t|), w) \). Then, we can write (E-NLP) compactly as
\[
\min_{x, z} f(x) \text{ s.t. } \tilde{c}_E(x, |z|) = 0, \quad \tilde{c}_Z(x, |z|) - z = 0,
\]
and compute \( \bar{J}_E \) and \( \bar{J}_\alpha \) from Definition 5 using the special structure of (E-NLP).

The resulting matrix \( J_{e-abs}(x) = [J_E(x)^T \ J_\alpha(x)^T]^T \) in Definition 6 has the form above.

**Remark 12.** Clearly, the rank of \( J_{e-abs}(x) \) does not depend on the signs of \( \pm 1 \) entries in \( \Sigma^w \) but only on their positions. Hence, LIKQ does not depend on the particular choice of \( w \). Otherwise it would not make sense to consider (E-NLP).

Note that, since the abs-normal NLP (E-NLP) does not contain any inequalities, the concept of IDKQ is equivalent to LIKQ here. This is in contrast to the standard reformulation of smooth NLP inequalities as equalities with nonnegative slacks where the validity of LICQ and MFCQ are both unaffected.

### 2.3. Relations of Kink Qualifications for Abs-Normal NLPs

In this paragraph we discuss the relations of kink qualifications for the two different formulations of abs-normal NLPs. We use the set \( W(t) \) from above.

**Theorem 13.** LIKQ for (I-NLP) holds at \( t \in D^t \) if and only if LIKQ for (E-NLP) holds at \((t, w) \in D^t \times \mathbb{R}^{m_z}\) for any \( w \in W(t) \), and hence for all \( w \in W(t) \) by Remark 12.

**Proof.** This follows immediately by comparison of \( J_{abs} \) and \( J_{e-abs} \) using that
\[
\alpha^w(w) = \{i \in I : w_i = 0\} = \{i \in I : c_t(t, z^t(t)) = 0\} = \mathcal{A}(t)
\]
and
\[
\Sigma^w = \text{diag}(\sigma^w) \quad \text{with} \quad \sigma^w_i = \text{sign}(w_i) = \begin{cases}
0, & i \in \mathcal{A}(t), \\
\pm 1, & i \notin \mathcal{A}(t).
\end{cases}
\]
Theorem 14. IDKQ for (I-NLP) holds at \( t \in D^t \) if IDKQ for (E-NLP) holds at \((t, w) \in D^t \times \mathbb{R}^{m_2} \) for any (and hence all) \( w \in W(t) \). The converse is not true.

Proof. Since (E-NLP) has no inequalities, the concepts of IDKQ and LIKQ coincide. LIKQ for (E-NLP) is equivalent to LIKQ for (I-NLP) by Theorem 13, and LIKQ for (I-NLP) implies IDKQ for (I-NLP). The converse does not hold since LIKQ for (I-NLP) is stronger then IDKQ as we have shown in Example 8. \( \square \)

3. Counterpart MPECs

In this section we introduce MPEC counterpart problems for the two formulations (I-NLP) and (E-NLP). Then, we have a quick look at relations between them.

3.1. Counterpart MPEC for the General Abs-Normal NLP. To reformulate (I-NLP) as an MPEC, we partition \( z^t \) into its nonnegative part and the modulus of its nonpositive part, \( u^t := [z^t]^+ := \max(z^t, 0) \) and \( v^t := [z^t]^− := \max(−z^t, 0) \). Then, we require complementarity of these two variables to replace \( |z^t| \) by \( u^t + v^t \) and \( z^t \) itself by \( u^t − v^t \).

Definition 15 (Counterpart MPEC of (I-NLP)). The counterpart MPEC of the abs-normal NLP (I-NLP) reads

\[
\begin{align*}
\min_{t,u^t,v^t} f(t) \quad \text{s.t.} \quad & \ c_\mathcal{E}(t, u^t + v^t) = 0, \\
& \ c_\mathcal{T}(t, u^t + v^t) \geq 0, \\
& \ c_\mathcal{Z}(t, u^t + v^t) − (u^t − v^t) = 0, \\
& \ 0 \leq u^t \perp v^t \geq 0,
\end{align*}
\]

(I-MPEC)

where \( u^t, v^t \in \mathbb{R}^{n_t} \). The feasible set of (I-MPEC) is denoted by

\[
\mathcal{F}_{\text{mpec}} := \left\{ (t, u^t, v^t) \mid c_\mathcal{E}(t, u^t + v^t) = 0, \ c_\mathcal{T}(t, u^t + v^t) \geq 0, \ c_\mathcal{Z}(t, u^t + v^t) = u^t − v^t, \ 0 \leq u^t \perp v^t \geq 0 \right\}.
\]

Lemma 16. Given an abs-normal NLP (I-NLP) and its counterpart MPEC (I-MPEC), we have a homeomorphism \( \phi: \mathcal{F}_{\text{mpec}} \to \mathcal{F}_{\text{abs}} \) defined as

\[
\phi(t, u^t, v^t) = (t, u^t − v^t), \quad \phi^{-1}(t, z^t) = (t, [z^t]^+, [z^t]^−).
\]

Proof. Obvious. \( \square \)

Just like the active switching set of the abs-normal NLP we define index sets of the counterpart MPEC.

Definition 17 (Index Sets). We denote by \( \mathcal{U}_0^t := \{ i \in \{1, \ldots, s_t \} : u^t_i = 0 \} \) the set of indices of active inequalities \( u^t_i \geq 0 \), and by \( \mathcal{U}_0^t := \{ i \in \{1, \ldots, s_t \} : u^t_i > 0 \} \) the set of indices of inactive inequalities \( u^t_i \geq 0 \). Analogous definitions hold of \( \mathcal{V}_0^t \) and \( \mathcal{V}_0^t \). By \( \mathcal{D}^t := \mathcal{U}_0^t \cap \mathcal{V}_0^t \) we denote the set of indices of non-strict (degenerate) complementarity pairs. Thus we have the partitioning \( \{1, \ldots, s_t\} = \mathcal{U}_0^t \cup \mathcal{V}_0^t \cup \mathcal{D}^t \).

In the following we define constraint qualifications for the counterpart MPEC. The standard definitions say that MPEC-LICQ and MPEC-MFCQ are LICQ and MFCQ, respectively, for the so-called tightened NLP (see [5, 8]) with associated Jacobian

\[
J(y) = \begin{bmatrix}
\partial_1 c_\mathcal{E} & \partial_2 c_\mathcal{E} P_T^{\mathcal{U}_0^t} & \partial_2 c_\mathcal{E} P_T^{\mathcal{V}_0^t} & \partial_2 c_\mathcal{E} P_T^{\mathcal{V}_0^t} & \partial_2 c_\mathcal{E} P_T^{\mathcal{V}_0^t} \\
\partial_1 c_\mathcal{T} & \partial_2 c_\mathcal{T} P_T^{\mathcal{U}_0^t} & \partial_2 c_\mathcal{T} P_T^{\mathcal{V}_0^t} & \partial_2 c_\mathcal{T} P_T^{\mathcal{V}_0^t} & \partial_2 c_\mathcal{T} P_T^{\mathcal{V}_0^t} \\
\partial_1 c_\mathcal{Z} & \partial_2 c_\mathcal{Z} P_T^{\mathcal{U}_0^t} & \partial_2 c_\mathcal{Z} P_T^{\mathcal{V}_0^t} & \partial_2 c_\mathcal{Z} P_T^{\mathcal{V}_0^t} & \partial_2 c_\mathcal{Z} P_T^{\mathcal{V}_0^t} \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I
\end{bmatrix}
\]
at \( y = (t, u', v') \), where \( P_S \in \mathbb{R}^{s \times s} \) denotes the projector onto the subspace defined by \( S \subseteq \{1, \ldots, s_t\} \) and all partial derivatives are evaluated at \( (t, u' + v') \). This Jacobian will be needed in section 5.2 to formulate second order conditions. Here we exploit the two unit blocks to state constraint qualifications in a more compact form.

**Definition 18 (MPEC-LICQ for (I-MPEC)).** We say that the MPEC-LICQ holds for (I-MPEC) at a feasible point \((t, u', v')\) if

\[
J_{\text{mpec}}(t, u', v') = \begin{bmatrix}
\partial_1 c_E & \partial_2 c_E P^T_{U_E} & \partial_2 c_E P^T_{V_E} \\
\partial_1 c_A & \partial_2 c_A P^T_{U_A} & \partial_2 c_A P^T_{V_A} \\
\partial_1 c_Z & [\partial_2 c_Z - I] P^T_{U_Z} & [\partial_2 c_Z + I] P^T_{V_Z}
\end{bmatrix}
\in \mathbb{R}^{(m_1 + |\mathcal{A}| + s_t) \times (m_1 + |\mathcal{U}_E| + |\mathcal{V}_E|)}
\]

has full row rank \( m_1 + |\mathcal{A}| + s_t \). Here all partial derivatives are evaluated at \((t, u' + v')\).

**Definition 19 (MPEC-MFCQ for (I-MPEC)).** We say that the MPEC-MFCQ holds for (I-MPEC) at a feasible point \((t, u', v')\) if

\[
\begin{bmatrix}
\partial_1 c_E & \partial_2 c_E P^T_{U_E} & \partial_2 c_E P^T_{V_E} \\
\partial_1 c_A & \partial_2 c_A P^T_{U_A} & \partial_2 c_A P^T_{V_A} \\
\partial_1 c_Z & [\partial_2 c_Z - I] P^T_{U_Z} & [\partial_2 c_Z + I] P^T_{V_Z}
\end{bmatrix}
\in \mathbb{R}^{(m_1 + s_t) \times (n_1 + |\mathcal{U}_E| + |\mathcal{V}_E|)}
\]

has full row rank \( m_1 + s_t \) and if there exists a vector \( d \in \mathbb{R}^{n_1 + |\mathcal{U}_E| + |\mathcal{V}_E|} \) such that

\[
\begin{bmatrix}
\partial_1 c_E & \partial_2 c_E P^T_{U_E} & \partial_2 c_E P^T_{V_E} \\
\partial_1 c_A & \partial_2 c_A P^T_{U_A} & \partial_2 c_A P^T_{V_A} \\
\partial_1 c_Z & [\partial_2 c_Z - I] P^T_{U_Z} & [\partial_2 c_Z + I] P^T_{V_Z}
\end{bmatrix}
d = 0,
\]

\[
\begin{bmatrix}
\partial_1 c_E & \partial_2 c_E P^T_{U_E} & \partial_2 c_E P^T_{V_E} \\
\partial_1 c_A & \partial_2 c_A P^T_{U_A} & \partial_2 c_A P^T_{V_A} \\
\partial_1 c_Z & [\partial_2 c_Z - I] P^T_{U_Z} & [\partial_2 c_Z + I] P^T_{V_Z}
\end{bmatrix}
d \geq 0.
\]

Again all partial derivatives are evaluated at \((t, u' + v')\).

As with LIKQ and IDKQ for (I-NLP), MPEC-MFCQ is weaker than MPEC-LICQ for the counterpart MPEC of (I-NLP). This can be easily seen by rewriting Example 8 as the counterpart MPEC and checking the above conditions.

### 3.2. Counterpart MPEC for the Abs-Normal NLP with Inequality Slacks

Using the same approach as in the preceding paragraph, we formulate the counterpart MPEC of (E-NLP).

**Definition 20 (Counterpart MPEC of (E-NLP)).** The counterpart MPEC of the abs-normal NLP (E-NLP) reads:

\[
\min_{t, w, w', w''} f(t) \quad \text{s.t.} \quad c_E(t, u' + v') = 0,
\]

\[
c_Z(t, u' + v') - (u'' + v'') = 0,
\]

\[
c_Z(t, u' + v') - (u' - v') = 0,
\]

\[
w - (u'' - v'') = 0,
\]

\[
0 \leq u' \perp v' \geq 0,
\]

\[
0 \leq u'' \perp v'' \geq 0,
\]

(E-MPEC)
where \( u', v'\in \mathbb{R}^n\) and \( w^w, w^v\in \mathbb{R}^m\). The feasible set is a lifting of \( F_{e-mpec}\):

\[
F_{e-mpec} = \begin{cases} (t, w, u', v', w^w, v^w) & \begin{cases} c_E(t, u'^t + v'^t) = 0, & c_Z(t, u'^t + v'^t) = w^w + v^w, \\ c_E(t, u'^t) = u'^t - v'^t, & w = u^w - v^w, \\ 0 \leq u'^t \perp v'^t \geq 0, \quad 0 \leq u^w \perp v^w \geq 0 \end{cases} \\
= \begin{cases} (t, w, u', v', w^w, v^w) & \begin{cases} (t, u', v') \in F_{mpec}, & c_T(t, u'^t + v'^t) = w^w + v^w, \\ w = w^w - v^w, \quad 0 \leq u^w \perp v^w \geq 0 \end{cases} \end{cases}
\]

Clearly, the homeomorphism between \( F_{mpec}\) and \( F_{abs}\) extends to \( F_{e-mpec}\) and \( F_{e-abs}\).

**Lemma 21.** Given an abs-normal NLP (E-NLP) and its counterpart MPEC (E-MPEC), we have a homeomorphism \( \phi: F_{e-mpec} \rightarrow F_{e-abs} \) defined as

\[
\phi(t, w, u', v', w^w, v^w) = (t, w, u'^t - v'^t, w^w - v^w),
\]

\[
\phi^{-1}(t, w, z', z^w) = (t, w, [z']^+, [z']^-, [z^w]^+, [z^w]^-) = (w^w + v^w).
\]

**Proof.** Obvious. \(\Box\)

**Lemma 22.** MPEC-LICQ for (E-MPEC) at a feasible point \( y = (t, w, u', v', u^w, v^w) \) is full row rank of

\[
J_{e-mpec}(y) = \begin{bmatrix}
\partial c_E & 0 & \partial c_E P_T^{u'_t} & \partial c_E P_T^{v'_t} & 0 & 0 \\
\partial c_E & 0 & \partial c_E P_T^{u'_t} & \partial c_E P_T^{v'_t} & -P_T^{u'_t} & -P_T^{v'_t} \\
\partial c_E & 0 & |\partial c_E - I| P_T^{u'_t} & |\partial c_E + I| P_T^{v'_t} & 0 & 0 \\
0 & I & 0 & 0 & -P_T^{u'_t} & -P_T^{v'_t} \\
\end{bmatrix}
\in \mathbb{R}^{(m_1 + m_2 + m_3 + m_4) \times (n_1 + m_2 + |\gamma'_{u'_t}| + |\gamma'_{v'_t}| + |\gamma'_v| + |\gamma'_w|)}
\]

where all partial derivatives are evaluated at \((t, u'^t + v'^t)\).

**Proof.** We set \( x = (t, w) \), \( u = (u', u^w) \), \( v = (v', v^w) \) as well as \( f(x) = f(t) \),

\[
\tilde{c}_E(x, u + v) = \begin{cases} c_E(t, u'^t + v'^t) \end{cases} \quad \text{and} \quad \tilde{c}_Z(x, u + v) = \begin{cases} c_Z(t, u'^t + v'^t) \end{cases}.
\]

Then, (E-MPEC) becomes

\[
\min_{x, u, v} f(x) \quad \text{s.t.} \quad \tilde{c}_E(x, u + v) = 0, \quad \tilde{c}_Z(x, u + v) - (u - v) = 0,
\]

\[
0 \leq u \perp v \geq 0,
\]

and we can compute the Jacobian from Definition 18 using the special structure of (E-MPEC). The resulting matrix has the stated form, except that the last four columns belong to variables \((u', v', u^w, v^w)\) rather than \((u, v) = (u', u^w, v', v^w)\). \(\Box\)

Like LIKQ for (E-NLP), MPEC-LICQ for (E-MPEC) does not depend on the particular choice of \( w \), and like IDKQ for (E-NLP), the concept of MPEC-MFCQ for (E-MPEC) is equivalent to MPEC-LICQ since no inequalities are present besides the complementarities.

### 3.3. Relations of MPEC Constraint Qualifications

In this paragraph we state the relations of constraint qualifications for the two different formulations introduced in the previous paragraphs. They follow from the results in the previous section and in the two following sections. For an illustration see Fig. 1 below. We set \( W(t, u', v') := \{(w, w^w, v^w) : |w| = c_Z(t, u'^t + v'^t), u^w = |u|^+ + v^w = |v|^+ \} \).

**Theorem 23.** MPEC-LICQ for (I-MPEC) holds at \( (t, u', v') \) if and only if MPEC-LICQ for (E-MPEC) holds at \( (t, w, u', v', u^w, v^w) \) for any \((w, u^w, v^w) \in W(t, u', v')\).
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Proof. This follows directly from Theorem 13, Theorem 25 and Theorem 27. □

Theorem 24. MPEC-MFCQ for (I-MPEC) holds at \((t, u^t, v^t)\) if MPEC-MFCQ for (E-MPEC) holds at \((t, w, u^w, v^w, v^w)\) for any \((w, u^w, v^w) \in W(t, u^t, v^t)\). The converse is not true.

Proof. This follows directly from Theorem 14, Theorem 26 and Corollary 28. □

4. Kink Qualifications and MPEC Constraint Qualifications

In this section we have a closer look at relations between abs-normal NLPS and counterpart MPECS in both formulations.

4.1. Relations of General Abs-Normal NLP and MPEC. Here we use the variables \(x\) and \(z\) instead of \(t\) and \(z^t\) to shorten notation because we do not consider inequality slacks. Then the general abs-normal NLP (I-NLP) becomes:

\[
\begin{align*}
\min_{x,z} & \ f(x) \quad \text{s.t.} \quad c_E(x, |z|) = 0, \\
& \quad c_Z(x, |z|) \geq 0, \\
& \quad c_Z(x, |z|) - z = 0.
\end{align*}
\]

The counterpart MPEC (I-MPEC) reads:

\[
\begin{align*}
\min_{x,u,v} & \ f(x) \quad \text{s.t.} \quad c_E(x, u + v) = 0, \\
& \quad c_Z(x, u + v) \geq 0, \\
& \quad c_Z(x, u + v) - (u - v) = 0, \\
& \quad 0 \leq u \perp v \geq 0.
\end{align*}
\]

We obtain the following relations of kink qualifications and MPEC constraint qualifications.

Theorem 25 (Equivalence of LIKQ and MPEC-LICQ). LIKQ for (I-NLP) holds at \(x \in D^x\) if and only if MPEC-LICQ for (I-MPEC) holds at \((x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^x \times \mathbb{R}^+ \times \mathbb{R}^+\).

Proof. Setting \(y := (x, u + v)\) and \(r := m_1 + |A| + s\), MPEC-LICQ for the counterpart MPEC is

\[
\begin{bmatrix}
\partial c_E(y) & \partial c_A(y)P_{U+}^T & \partial c_A(y)P_{V+}^T \\
\partial c_A(y) & \partial c_A(y)P_{U+}^T & \partial c_A(y)P_{V+}^T \\
\partial c_Z(y) & [\partial c_Z(y) - I]P_{U+}^T & [\partial c_Z(y) + I]P_{V+}^T
\end{bmatrix}
\]

By negating the second column and combining it with the third column, this is equivalent to

\[
\begin{bmatrix}
\partial c_E(y) & -\partial c_E(y)\Sigma P_{U+\cup V+}^T \\
\partial c_A(y) & -\partial c_A(y)\Sigma P_{U+\cup V+}^T \\
\partial c_Z(y) & [I - \partial c_Z(y)\Sigma]P_{U+\cup V+}^T
\end{bmatrix}
\]

and, by non-singularity of \(I - \partial c_Z(y)\Sigma\), to

\[
\begin{bmatrix}
\partial c_E(y) & -\partial c_E(y)\Sigma P_{U+\cup V+}^T \\
\partial c_A(y) & -\partial c_A(y)\Sigma P_{U+\cup V+}^T \\
[I - \partial c_Z(y)\Sigma]^{-1}\partial c_Z(y) & P_{U+\cup V+}^T
\end{bmatrix}
\]

The converse is not true.
Next, we use the third row to eliminate the entries above $P^T_{\mathcal{U}_+ \cup \mathcal{V}_+}$ to obtain
\[
\begin{bmatrix}
\partial c_E(y) + \partial c_E(y)\Sigma[I - \partial c_E(y)\Sigma]^{-1}\partial c_E(y) \\
\partial c_A(y) + \partial c_A(y)\Sigma[I - \partial c_E(y)\Sigma]^{-1}\partial c_E(y) \\
\partial c(z)(x)
\end{bmatrix}
\begin{bmatrix}
P^{T}_{\mathcal{U}_+ \cup \mathcal{V}_+}
\end{bmatrix} = r,
\]
which we can write with $u + v = z = z(x)$ and $\Sigma z(x) = |z(x)|$ as
\[
\begin{bmatrix}
\partial c_E(x, |z(x)|) \\
\partial c_A(x, |z(x)|) \\
\partial c(z)(x)
\end{bmatrix}
\begin{bmatrix}
P^{T}_{\mathcal{U}_+ \cup \mathcal{V}_+}
\end{bmatrix} = r.
\]
Finally, since $\alpha = \mathcal{D}$ is the complement of $\mathcal{U}_+ \cup \mathcal{V}_+$, this is equivalent to
\[
\begin{bmatrix}
\partial c_E(x, |z(x)|) \\
\partial c_A(x, |z(x)|) \\
[c^T \partial c(z)(x)]_{i \in \alpha}
\end{bmatrix}
= m_1 + |A| + |\alpha|,
\]
which is LIKQ for the abs-normal NLP.

**Theorem 26** (Equivalence of IDKQ and MPEC-MFCQ). IDKQ for (I-NLP) holds at $x \in D^x$ if and only if MPEC-MFCQ for (I-MPEC) holds at $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^x \times \mathbb{R}^u \times \mathbb{R}^v$.

**Proof.** Again with $y := (x, u + v)$, MPEC-MFCQ for the counterpart MPEC is

1. full row rank of
\[
\begin{bmatrix}
\partial c_E(y) \\
\partial c_E(y)P^T_{\mathcal{U}_+} \\
\partial c_E(y)P^T_{\mathcal{V}_+}
\end{bmatrix} \in \mathbb{R}^{(m_1 + u) \times n}.
\]
As in the proof of Theorem 25, this is seen to be full row rank of
\[
\begin{bmatrix}
\partial c_E(x, |z(x)|) \\
[c^T \partial c(z)(x)]_{i \in \alpha}
\end{bmatrix} \in \mathbb{R}^{(m_1 + |\alpha|) \times n}.
\]

2. the existence of a vector $d = (d_x, d_u, d_v) \in \mathbb{R}^{n+|\mathcal{U}_+\cup\mathcal{V}_+|}$ such that
\[
\begin{bmatrix}
\partial c_E(y) \\
\partial c_E(y)P^T_{\mathcal{U}_+} \\
\partial c_E(y)P^T_{\mathcal{V}_+}
\end{bmatrix} d = 0,
\]
\[
\begin{bmatrix}
\partial c_E(y) \\
\partial c_E(y)P^T_{\mathcal{U}_+} \\
\partial c_E(y)P^T_{\mathcal{V}_+}
\end{bmatrix} d > 0.
\]
We combine $d_u$ and $-d_v$ to $d_{uv} \in \mathbb{R}^{|\mathcal{U}_+\cup\mathcal{V}_+|}$. Then this is equivalent to
\[
\partial c_E(y)d_x + \partial c_E(y)\Sigma P^T_{\mathcal{U}_+\cup\mathcal{V}_+}d_{uv} = 0,
\]
\[
\partial c_E(y)d_x - [I - \partial c_E(y)\Sigma] P^T_{\mathcal{U}_+\cup\mathcal{V}_+}d_{uv} = 0,
\]
\[
\partial c_A(y)d_x + \partial c_A(y)\Sigma P^T_{\mathcal{U}_+\cup\mathcal{V}_+}d_{uv} > 0.
\]
The second condition can be written as
\[
[I - \partial c_E(y)\Sigma]^{-1}\partial c_E(y)d_x = P^T_{\mathcal{U}_+\cup\mathcal{V}_+}d_{uv}.
\]
Multiplying this by $P^T_{\mathcal{U}_+\cup\mathcal{V}_+}$ yields
\[
[e^T[I - \partial c_E(y)\Sigma]^{-1}\partial c_E(y)]_{i \in \alpha} d_x = [e^T \partial c(z)(x)]_{i \in \alpha} d_x = 0.
\]
With $u + v = z = z(x)$ as well as $\Sigma z(x) = |z(x)|$, substituting the right-hand side of (3) into the first and third condition finally gives
\[
\partial c_E(x, |z(x)|)d_x = 0,
\]
\[
[e^T \partial c(z)(x)]_{i \in \alpha} d_x = 0,
\]
\[
\partial c_A(x, |z(x)|)d_x > 0.
\]
4.2. Relations of Abs-Normal NLP and MPEC with Inequality Slacks.

As the reformulation with inequality slacks is just a specialization of the general case, we do without proofs and give remarks where differences occur.

Using the short notation \((E-NLP)\) for \((E-NLP)\) (see proof of Lemma 11) and similarly \((E-MPEC)\) for the counterpart MPEC \((E-MPEC)\) (see proof of Lemma 22), we obtain the same relation between LIKQ and MPEC-LICQ as in the previous paragraph.

**Theorem 27** (Equivalence of LIKQ and MPEC-LICQ). LIKQ for \((E-NLP)\) holds at \(x \in D\) if and only if MPEC-LICQ for \((E-MPEC)\) holds at \((x,u,v) = (x, [z(x)]^+, [z(x)]^-) \in D^x \times \mathbb{R}^{s_2+m_2} \times \mathbb{R}^{s_2+m_2}\).

**Proof.** This follows as in the proof of Theorem 25. \(\square\)

Note that this directly implies the next result since LIKQ and IDKQ as well as MPEC-LICQ and MPEC-MFCQ coincide in the purely equality constrained setting.

**Corollary 28** (Equivalence of IDKQ and MPEC-MFCQ). IDKQ for \((E-NLP)\) holds at \(x \in D^x\) if and only if MPEC-MFCQ for \((E-MPEC)\) holds at \((x,u,v) = (x, [z(x)]^+, [z(x)]^-) \in D^x \times \mathbb{R}^{s_2+m_2} \times \mathbb{R}^{s_2+m_2}\).

5. Optimality Conditions

In this section we consider first and second order optimality conditions for \((I-MPEC)\) under MPEC-LICQ and for \((I-NLP)\) under LIKQ, respectively, and discuss their relations. Since both regularity conditions are invariant under the slack reformulation by Theorem 13 and Theorem 23, the results hold also for \((E-MPEC)\) and \((E-NLP)\). Conditions for general MPECs can be found in the literature; in case of first order conditions for example in [8]. Second order conditions are stated in [7] but have to be adapted to our different setting. For the abs-normal NLP \((E-NLP)\) we have derived first and second order conditions in [6]. Since LIKQ is preserved under the slack reformulation by Theorem 13, we can transfer these results directly to \((I-NLP)\).
5.1. First Order Conditions. In this paragraph, we compare stationarity concepts and first order conditions for (I-MPEC) and (I-NLP). First, we define strong stationarity for (I-MPEC) and state the corresponding first order conditions.

**Definition 29** (S-Stationarity). A feasible point \( y^* = (t^*, (u^i)^*, (v^i)^*) \) of (I-MPEC) is strongly stationary (S-stationary) if there exist multipliers \( \lambda = (\lambda_E, \lambda_I, \lambda_Z) \) and \( \mu = (\mu_u, \mu_v) \) such that the following conditions are satisfied:

\[
\partial_t \mathcal{L}_{\perp}(y^*, \lambda, \mu) = 0, \quad \text{(4a)}
\]

\[
(\mu_u)_i \geq 0, \quad (\mu_v)_i \geq 0, \quad i \in \mathcal{D}^t(t^*), \quad \text{(4b)}
\]

\[
(\mu_u)_i = 0, \quad i \in \mathcal{U}_i^t(t^*), \quad \text{(4c)}
\]

\[
(\mu_v)_i = 0, \quad i \in \mathcal{V}_i^t(t^*), \quad \text{(4d)}
\]

\[
\lambda_I \geq 0, \quad \text{(4e)}
\]

\[
\lambda_T^2 c_Z(t^*, (u^i)^* + (v^i)^*) = 0. \quad \text{(4f)}
\]

Herein, \( \mathcal{L}_{\perp} \) is the MPEC-Lagrangian function associated with (I-MPEC):

\[
\mathcal{L}_{\perp}(y, \lambda, \mu) := f(t) + \lambda_T^2 c_E(t, u^i + v^i) - \lambda_T^2 c_Z(t, u^i + v^i)
\]

\[+ \lambda_T^2 c_Z(t, u^i + v^i) - (u^i - v^i)] - \mu_u u^i - \mu_v v^i.
\]

**Theorem 30** (First Order Conditions for (I-MPEC)). Assume that \( (t^*, (u^i)^*, (v^i)^*) \) is a local minimizer of (I-MPEC) and that MPEC-LICQ holds at \( (t^*, (u^i)^*, (v^i)^*) \). Then, \( (t^*, (u^i)^*, (v^i)^*) \) is an S-stationary point.

**Proof.** A proof may be found in [8].

Now, kink stationarity is defined and the matching first order conditions are formulated.

**Definition 31** (Kink Stationarity). A feasible point \( (t^*, (z^i)^*) \) of (I-NLP) is kink stationary if there exist multipliers \( \lambda = (\lambda_E, \lambda_I, \lambda_Z) \) such that the following conditions are satisfied:

\[
f'(t^*) + \lambda_T^2 \partial_t c_E - \lambda_T^2 \partial_t c_I + \lambda_T^2 \partial_t c_Z = 0, \quad \text{(5a)}
\]

\[
[\lambda_T^2 \partial_t c_E - \lambda_T^2 \partial_t c_I + \lambda_T^2 \partial_t c_Z]_i \geq |(\lambda_Z)_i|, \quad i \in \alpha'(t^*), \quad \text{(5b)}
\]

\[
[\lambda_T^2 \partial_{z_i} c_E - \lambda_T^2 \partial_{z_i} c_I]_i \geq |(\lambda_Z)_i(\sigma^i)_*|, \quad i \notin \alpha'(t^*), \quad \text{(5c)}
\]

\[
\lambda_I \geq 0, \quad \text{(5d)}
\]

\[
\lambda_T^2 c_I = 0. \quad \text{(5e)}
\]

Here, the constraints and the partial derivatives are evaluated at \( (t^*, |(z^i)^*|) \).

**Theorem 32** (First Order Conditions for (I-NLP)). Assume that \( (t^*, (z^i)^*) \) is a local minimizer of (I-NLP) and that LIKQ holds at \( t^* \). Then, \( (t^*, (z^i)^*) \) is a kink stationary point.

**Proof.** By Lemma 13 we may consider the slack reformulation (E-NLP) instead of (I-NLP). In [6, Theorem 5.10], conditions (5) were proven for (E-NLP) using a splitting of the switching variables \( z \) and the switching constraints \( c_Z \). Without the splitting they read:

\[
f'(x^*) + \lambda_T^2 \partial_t \hat{c}_E(x^*, |z^*|) + \lambda_T^2 \partial_t \hat{c}_I(x^*, |z^*|) = 0,
\]

\[
[\lambda_T^2 \partial_{z_i} \hat{c}_E(x^*, |z^*|) + \lambda_T^2 \partial_{z_i} \hat{c}_Z(x^*, |z^*|)]_i \geq |(\lambda_Z)_i|, \quad i \in \alpha(x^*),
\]

\[
[\lambda_T^2 \partial_{z_i} \hat{c}_E(x^*, |z^*|) + \lambda_T^2 \partial_{z_i} \hat{c}_Z(x^*, |z^*|)]_i = (\lambda_Z)_i(\sigma^i)_*, \quad i \notin \alpha(x^*).\]
We rewrite these conditions in the original notation of (I-NLP) with \( \lambda^*_E = (\lambda_Z, -\lambda_I) \) and \( \lambda_Z = (\lambda_Z, \lambda_Z^\infty) \), where all derivatives are evaluated at \((t^*, (z^*)^+)\):

\[
f'(t^*) + \lambda^*_T \partial_t c_E - \lambda^*_T \partial_t c_Z + \lambda^*_T \partial_{t}Z = 0,
(\lambda^*_Z)^T = 0,
[\lambda^*_T \partial z c_E - \lambda^*_T \partial z c_Z + \lambda^*_T \partial z c_Z]_{i} \geq |(\lambda_Z)_i|, \quad i \in \alpha^i(t^*),
[\lambda^*_T \partial z c_E - \lambda^*_T \partial z c_Z + \lambda^*_T \partial z c_Z]_{\sigma i} = (\lambda_Z)_i (\sigma^i)_i, \quad i \notin \alpha^i(t^*),
\lambda_Z \geq |(\lambda^*_Z)_i|, \quad i \in \alpha^v(w^*),
\lambda_I = (\lambda^*_Z^\infty), \quad i \notin \alpha^w(w^*),
\]

The claim follows by eliminating \( \lambda^*_Z = 0 \) and noting that \( \alpha^w(w^*) = \mathcal{A}(t^*) \). \( \square \)

The next theorem shows that the two stationarity concepts coincide.

**Theorem 33** (S-Stationarity is Kink Stationarity). A feasible point \((t^*, (z^*)^+\) of (I-NLP) is kink stationary if and only if \((t^*, (u^*)^+, (v^*)^+) = (t^*, [z^*(t^*)]^+, [z^*(t^*)]^-)\) of (I-MPEC) is S-stationary.

**Proof.** Comparison of the stationarity conditions of (I-NLP) and (I-MPEC) shows directly that (4e) and (5d) as well as (4f) and (5e) coincide. Thus, we have to check the remaining conditions (4a) to (4d) for (I-MPEC) and (5a) to (5c) for (I-NLP). Condition (4a) of (I-MPEC) is equivalent to checking all derivatives are evaluated at \((t^*, (u^*)^+ + (v^*)^+)\), is

\[
f'(t^*) + \lambda^*_T \partial_t c_E - \lambda^*_T \partial_t c_Z + \lambda^*_T \partial_{t}Z = 0,
\lambda^*_T \partial z c_E - \lambda^*_T \partial z c_Z + \lambda^*_T \partial z c_Z - I - \mu^*_T = 0,
\lambda^*_T \partial z c_E - \lambda^*_T \partial z c_Z + \lambda^*_T \partial z c_Z + I - \mu^*_T = 0.
\]

The first condition coincides with (5a). We combine the second and the third condition with conditions (4b) to (4d), yielding

\[
[\lambda^*_T \partial z c_E - \lambda^*_T \partial z c_Z + \lambda^*_T \partial z c_Z]_{i} = +(\lambda_Z)_i, \quad i \in U^c_+(t^*),
[\lambda^*_T \partial z c_E - \lambda^*_T \partial z c_Z + \lambda^*_T \partial z c_Z]_{i} = -(\lambda_Z)_i, \quad i \in U^c_+(t^*),
[\lambda^*_T \partial z c_E - \lambda^*_T \partial z c_Z + \lambda^*_T \partial z c_Z + I]_{i} \geq 0, \quad i \in \mathcal{D}^c(t^*).
\]

These are exactly conditions (5b) and (5c) for (I-NLP) by definition of the index sets and of \( \sigma^* \). \( \square \)

As LKQ for (I-NLP) is equivalent to MPEC-LICQ for (I-MPEC), the previous theorem provides a different perspective on Theorem 32 and Theorem 30: one can be obtained from the other directly via Theorem 33 and vice versa.

5.2. Second Order Conditions. In this paragraph, we compare second-order conditions for MPECs and abs-normal NLPs.

First, we formulate them for (I-MPEC). This is based on [7] but some additional assumptions on the Lagrange multipliers are made. These are given in the next definition.

**Definition 34** (MPEC-Strict Complementarity). Consider a strongly stationary point \((t^*, (u^*)^+, (v^*)^+)\) with Lagrange multipliers \((\lambda^*_i, \mu^*_i)\). We say that MPEC-strict complementarity holds if \( \lambda^*_i > 0 \) for all \( i \in \mathcal{A} \) as well as \( (\mu^*_i)_i > 0 \) and \( (\mu^*_i)_i > 0 \) for all \( i \in \mathcal{D}^c \).
We will show in the next lemma that under MPEC-LICQ and MPEC-strict complementarity the critical cone reduces to the nullspace of the Jacobian of the tightened NLP (with columns reordered according to the index sets \(U_+^e, U_+^o, V_+^e, V_+^o\)).

\[
J(y^*) = \begin{bmatrix}
\partial_1 c_E & \partial_2 c_E P^T_{U_+^e} & \partial_2 c_E P^T_{U_+^o} & \partial_2 c_E P^T_{V_+^e} & \partial_2 c_E P^T_{V_+^o} \\
\partial_1 c_A & \partial_2 c_A P^T_{U_+^e} & \partial_2 c_A P^T_{U_+^o} & \partial_2 c_A P^T_{V_+^e} & \partial_2 c_A P^T_{V_+^o} \\
\partial_2 c_Z - I P^T_{U_+^e} & \partial_2 c_Z - I P^T_{U_+^o} & \partial_2 c_Z + I P^T_{V_+^e} & \partial_2 c_Z + I P^T_{V_+^o} \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I \\
\end{bmatrix},
\]

as introduced in Section 3.1. Here, all partial derivatives are evaluated at \((t^*, u^* + v^*).\) It is readily verified that the nullspace of \(J(y^*)\) is spanned by the matrix

\[
U_{mpec}(y^*) = \begin{bmatrix} I \\ -P_{U_+^e}(I - \partial c_E \Sigma')^{-1} \partial_1 c_E \\ 0 \\ \partial_1 c_A + \partial_2 c_A \Sigma'(I - \partial c_E \Sigma')^{-1} \partial_1 c_E \\ [v^T \partial c_E(I - \partial c_E \Sigma')^{-1} \partial_1 c_E, i \in D'] \end{bmatrix} U_1(y^*)
\]

where \(U_1(y^*)\) spans the nullspace of

\[
\begin{bmatrix}
\partial_1 c_E + \partial_2 c_E \Sigma'(I - \partial c_E \Sigma')^{-1} \partial_1 c_E \\
\partial_1 c_A + \partial_2 c_A \Sigma'(I - \partial c_E \Sigma')^{-1} \partial_1 c_E \\
[v^T \partial c_E(I - \partial c_E \Sigma')^{-1} \partial_1 c_E, i \in D'] \end{bmatrix}.
\]

Second order necessary and sufficient conditions for a slightly more general class of MPECs are given in [7, Theorem 7] using the concept of critical directions. We first specialize the definition from [7] to our setting.

**Definition 35 (Critical Direction).** A vector \(d = (dt, du^t, dv^t) \in \mathbb{R}^{n_t} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_v}\) is called a critical direction at a weakly stationary point \(y^*\) of (I-MPEC) if

\[
\begin{align*}
\min(dt_i^t, dv_i^t) &= 0, \quad i \in D^t, \\
dt_i^t &= 0, \quad i \in V_+^t, \\
dv_i^t &= 0, \quad i \in U_+^t, \\
\partial_1 c_E dt^t + \partial_2 c_E (du^t + dv^t) &\geq 0, \\
\partial_1 c_A dt^t + \partial_2 c_A (du^t + dv^t) &= 0, \\
\partial_1 c_Z dt^t + [\partial_2 c_Z - I]du^t + [\partial_2 c_Z + I]dv^t &= 0, \\
f'(t^*)dt &= 0,
\end{align*}
\]

where all constraint derivatives are evaluated at \((t^*, u^* + v^*).\)

The set of critical directions is just the nullspace of \(J(y^*)\) under stronger assumptions.

**Lemma 36.** Assume that MPEC-LICQ and MPEC-strict complementarity hold at an S-stationary point \(y^* = (t^*, u^* + v^*)\) of (I-MPEC) with Lagrange multipliers \((\lambda^*, \mu^*).\) Then, the set of critical directions is \(\ker J(y^*).\)

**Proof.** First consider a critical direction \(d = (dt, du^t, dv^t)\) at the strongly (hence weakly) stationary point \(y^*.\) Then, (8e) and (8f) imply that rows one and three of
$J(y^\ast)d$ vanish, and by (4a) and (8g) we further have
\[
0 = \partial_{t,u,v} L_\perp(y^\ast, \lambda^\ast, \mu^\ast)d \\
= f'(t^\ast)dt + (\lambda^\ast)^T[\partial_1 c t dt + \partial_2 c t (du^t + dv^t)] \\
- (\lambda^\ast)^T[\partial t c z dt + \partial_2 c z (du^t + dv^t)] \\
+ (\lambda^\ast)^T[\partial t c z dt + (\partial_2 c z - I)du^t + (\partial_2 c z + I)dv^t] \\
- (\mu^\ast)^Tdu^t - (\mu^\ast)^Tdv^t \\
= -(\lambda^\ast)^T[\partial t c z dt + \partial_2 c z (du^t + dv^t)] - (\mu^\ast)^Tdu^t - (\mu^\ast)^Tdv^t.
\]
With $(\lambda^\ast)^Tc_z = 0$ (4f), $(\mu^\ast)_i = 0$ for $i \in U^+_0$ (4c), $(\mu^\ast)_i = 0$ for $i \in V^+_0$ (4d), and (8b), (8c) we obtain
\[
0 = (\lambda^\ast)^T[\partial t c_A du^t + \partial_2 c_A (du^t + dv^t)] + \sum_{i \in D^\ast} [(\mu^\ast)_i du^t + (\mu^\ast)_i dv^t].
\]
All factors in this sum of products are nonnegative by (4b), (4e), (8d), and (8a), which implies
\[
0 = (\lambda^\ast)^T[\partial t c_A du^t + \partial_2 c_A (du^t + dv^t)], \\
0 = (\mu^\ast)_i du^t = (\mu^\ast)_i dv^t, \quad i \in D^\ast.
\]
Finally, by MPEC-strict complementarity we have
\[
0 = \partial t c_A du^t + \partial_2 c_A (du^t + dv^t), \\
0 = du^t = dv^t, \quad i \in D^\ast,
\]
and $du^t = 0$ for $i \in U^+_0$ as well as $dv^t = 0$ for $i \in V^+_0$ follow since $U^+_0 = V^+_0 \cup D^t$ and $V^+_0 = U^+_0 \cup D^t$. Thus $d$ is a nullspace vector of $J(y^\ast)$.

Conversely, given a nullspace vector $d = (dt, du^t, dv^t)$, the first three rows of $J(y^\ast)d = 0$ yield conditions (8c), (8d), and (8f), with equality “= 0” in case of (8d). The last two rows yield $du^t = 0$ for $i \in U^+_0$ and $dv^t = 0$ for $i \in V^+_0$, hence (8b), (8c), and $du^t = dv^t = 0$ for $i \in D^t$ (8a). Moreover, we have $(\mu^\ast)_i = 0$ for $i \in U^+_0$ (4c), $(\mu^\ast)_i = 0$ for $i \in V^+_0$ (4d), and $(\lambda^\ast)_i = 0$ for $i \notin A$ (4f), so that (4a) becomes (8g):
\[
0 = \partial_{t,u,v} L_\perp(y^\ast, \lambda^\ast, \mu^\ast)d = f'(t^\ast)dt.
\]
Thus $d$ is a critical direction. \[\square\]

Now we use [7, Theorem 7] to prove second order necessary and sufficient conditions for our setting.

**Theorem 37** (Second Order Necessary Conditions for (I-MPEC)). Assume that $y^\ast = (t^\ast, (u^\ast)^t, (v^\ast)^t)$ is a local minimizer of (I-MPEC) and that MPEC-LICQ holds at $y^\ast$. Denote by $(\lambda^\ast, \mu^\ast)$ the unique Lagrange multiplier vector and assume further that MPEC-strict complementarity holds. Then,
\[
U_mpec(y^\ast)^T H_mpec(y^\ast, \lambda^\ast) U_mpec(y^\ast) \geq 0
\]
where $H_mpec(y^\ast, \lambda^\ast) = \partial_{yy}^2 L_\perp(y^\ast, \lambda^\ast, \mu^\ast)$. (Note that $\partial_{yy}^2 L_\perp$ does not depend on $\mu^\ast$.)

**Proof.** The first part of Theorem 7 in [7] asserts that every critical direction $d$ satisfies $d^T H_mpec(y^\ast, \lambda^\ast)d \geq 0$ at a local minimizer $y^\ast$ if MPEC-SMFCQ holds at $y^\ast$. Since MPEC-LICQ implies MPEC-SMFCQ and the set of critical directions is $\ker J(y^\ast)$ under our stronger assumptions, the claim follows directly from [7, Theorem 7]. \[\square\]
Remark 38. Here we have simplified the exposition by making the assumption of MPEC-strict complementarity, so that we can directly rely on [7, Theorem 7]. However, the second order necessary conditions can also be proved without MPEC-strict complementarity by considering branch problems of (I-MPEC). The corresponding approach for (I-NLP) has been taken in [6], so that Theorem 40 below does not require strict complementarity.

**Theorem 39** (Second Order Sufficient Conditions for (I-MPEC)). Assume that \( y^* = (t^*, (u^t)^*, (v^t)^*) \) is strongly stationary for (I-MPEC) with Lagrange multiplier vector \((\lambda^*, \mu^*)\) satisfying MPEC-strict complementarity. Assume further that MPEC-LICQ holds at \( y^* \), and that

\[
U_{\text{mpec}}(y^*)^T H_{\text{mpec}}(y^*, \lambda^*) U_{\text{mpec}}(y^*) > 0.
\]

Then, \( y^* \) is a strict local minimizer of (I-MPEC).

**Proof.** In the second part of [7, Theorem 7], our assertion is proved under the weaker assumption that \( y^* \) is strongly stationary for every critical direction \( d \neq 0 \) there exists a Lagrange multiplier vector \((\lambda^*, \mu^*)\) such that \( d^T H_{\text{mpec}}(y^*, \lambda^*) d > 0 \). Under our additional assumptions of MPEC-LICQ and MPEC-strict complementarity, the set of critical directions is spanned by the matrix \( U_{\text{mpec}}(y^*) \), see proof of the previous lemma. Thus, the claim follows directly from [7, Theorem 7]. \( \square \)

We proceed by formulating second-order conditions for (I-NLP). To this end, we need the Lagrangian

\[
\mathcal{L}(t, z^t, \lambda) = f(t) + \lambda^T_1 c_E(t, |z^t|) - \lambda^T_2 c_Z(t, |z^t|) + \lambda^T_2 [c_Z(t, |z^t|) - z^t]
\]

and the matrix

\[
U_{\text{abs}}(t) := \begin{bmatrix} U(t) \\ e_1^T \Sigma \partial_t z^t(t) U(t) \end{bmatrix}_{\not\in \alpha^t},
\]

where \( U(t) \) spans the nullspace of \( J_{\text{abs}}(t) \). We also use the Lagrangian of (E-NLP),

\[
\bar{\mathcal{L}}(x, z, \lambda) = f(x) + \lambda^T_1 c_E(x, |z|) + \lambda^T_2 [c_Z(x, |z|) - z],
\]

\[
= f(t) + \lambda^T_1 c_E(t, |z^t|) - \lambda^T_2 [c_Z(t, |z^t|) - |z^w|] + \lambda^T_2 [c_Z(t, |z^t|) - z^t] + (\lambda^w_2)^T [w - z^w].
\]

**Theorem 40** (Second Order Necessary Conditions for (I-NLP)). Assume that \( y^* = (t^*, (z^t)^*) \) is a local minimizer of (I-NLP) and that LIKQ holds at \( t^* \). Denote by \( \lambda^* \) the unique Lagrange multiplier and set \( \alpha^t = \alpha^t(t^*) \). Then,

\[
U_{\text{abs}}(t^*)^T H_{\text{abs}}(y^*, \lambda^*) U_{\text{abs}}(t^*) \geq 0
\]

where \( H_{\text{abs}}(y^*, \lambda^*) = \begin{bmatrix} I & 0 \\ 0 & P_{\alpha^t} \end{bmatrix} \left[ \begin{bmatrix} \partial_{11} \mathcal{L}(y^*, \lambda^*) & \partial_{12} \mathcal{L}(y^*, \lambda^*) \\ \partial_{21} \mathcal{L}(y^*, \lambda^*) & \partial_{22} \mathcal{L}(y^*, \lambda^*) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P_{\alpha^t} \end{bmatrix} \right].
\]

**Proof.** As in Theorem 32, we can consider (E-NLP) instead of (I-NLP) by Theorem 13. In [6, Theorem 5.15] the second order necessary conditions for (E-NLP) have been derived using a variable splitting. Without the splitting they read

\[
\dot{U}_{\text{c-abs}}(x^*)^T H_{\text{c-abs}}(y^*, \lambda^*) U_{\text{c-abs}}(x^*) \geq 0
\]

with \( \dot{y}^* = (x^*, z^*) \), \( \dot{\lambda}_E = (\lambda_E, -\lambda_I) \), \( \dot{\lambda}_Z = (\lambda_Z, \lambda_Z^w) \), and the Hessian

\[
H_{\text{c-abs}}(y^*, \lambda^*) = \begin{bmatrix} I & 0 \\ 0 & P_{\alpha^c} \end{bmatrix} \left[ \begin{bmatrix} \partial_{11} \mathcal{L}(y^*, \lambda^*) & \partial_{12} \mathcal{L}(y^*, \lambda^*) \\ \partial_{21} \mathcal{L}(y^*, \lambda^*) & \partial_{22} \mathcal{L}(y^*, \lambda^*) \end{bmatrix} I & 0 \\ 0 & P_{\alpha^c} \end{bmatrix},
\]

where \( \alpha^c \) is the complement of \( \alpha \) and the matrix \( U_{\text{c-abs}} \) is defined as

\[
U_{\text{c-abs}}(x^*) = \begin{bmatrix} U(x^*) \\ e_1^T \Sigma \partial_x z(x^*) U(x^*) \end{bmatrix}_{\not\in \alpha}.\]
with \( \bar{U}(x) \) spanning \( \ker(J_{e,\text{abs}}(x)) \). Using the special structure of \((E\text{-NLP})\) and comparing the derivatives of \( \mathcal{L}(\bar{y}^*, \lambda^*) \) and \( \mathcal{L}(y^*, \lambda^*) \), the Hessian becomes

\[
\bar{H}_{e,\text{abs}}(\bar{y}^*, \lambda^*) = P^T \begin{bmatrix}
\partial_{11} \mathcal{L} & 0 & \partial_{12} \mathcal{L} & 0 \\
0 & 0 & 0 & 0 \\
\partial_{21} \mathcal{L} & 0 & \partial_{22} \mathcal{L} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} P \text{ with } P := \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & P_{(\alpha^*)'} & 0 \\
0 & 0 & 0 & P_{(\alpha^*)'}^T
\end{bmatrix}.
\]

All partial derivatives of \( \mathcal{L} \) are evaluated at \((y^*, \lambda^*) \). Moreover, \( J_{e,\text{abs}}(x) = J_{e,\text{abs}}(t, w) \) has the form derived in Lemma 11, and thus its nullspace is spanned by

\[
\bar{U}(x) = \begin{bmatrix}
U(t) \\
\Sigma^w \partial_t c_T
\end{bmatrix},
\]

where \( U(t) \) spans the nullspace of \( J_{abs}(t) \) from Definition 6. Using this and \( \partial_t z^w(w) = I \), the matrix \( U_{e,\text{abs}} \) reads

\[
U_{e,\text{abs}}(x) = \begin{bmatrix}
U(t) \\
\Sigma^w \partial_t c_T \\
[e^T \Sigma^w \partial_t z^w(U(t)) | \bar{e} | \bar{x} | \bar{y} \\
[e^T \partial_t c_T | \bar{e} \bar{y}]
\end{bmatrix}.
\]

Finally, we have

\[
0 \leq U_{e,\text{abs}}(x^*)^T H_{e,\text{abs}}(\bar{y}^*, \lambda^*) U_{e,\text{abs}}(x^*) = U_{abs}(t^*)^T H_{abs}(y^*, \lambda^*) U_{abs}(t^*)
\]

with \( U_{abs}(t) \) from (9) and

\[
H_{abs}(y, \lambda) = \begin{bmatrix}
I & 0 & 0 \\
0 & P_{(\alpha^*)'} \end{bmatrix} \begin{bmatrix}
\partial_{11} \mathcal{L}(y, \lambda) & \partial_{12} \mathcal{L}(y, \lambda) \\
\partial_{21} \mathcal{L}(y, \lambda) & \partial_{22} \mathcal{L}(y, \lambda)
\end{bmatrix} \begin{bmatrix}
I & 0 \\
P_{(\alpha^*)'}
\end{bmatrix}.
\]

This proves the claim.

\( \square \)

**Theorem 41** (Second Order Sufficient Conditions for \((I\text{-NLP})\)). Assume that \( y^* = (t^*, (z^*)^*) \) is kink stationary for \((I\text{-NLP})\) with a Lagrange multiplier vector \( \lambda^* \) that satisfies strict complementarity for \( \lambda^*_2 \) and strict normal growth, 

\[
[\lambda^*_2 \partial_{2c_T} - \lambda^*_2 \partial_{2c_T} + \lambda^*_2 \partial_{2c_Z}]_i \geq |(\lambda_Z)_i|, \quad i \in \alpha^i(t^*),
\]

Assume further that LIKQ holds at \( t^* \), and that

\[
U_{abs}(t^*)^T H_{abs}(y^*, \lambda^*) U_{abs}(t^*) > 0.
\]

Then, \((t^*, (z^*)^*)\) is a strict local minimizer of \((I\text{-NLP})\).

**Proof.** As before we consider the slack reformulation \((E\text{-NLP})\) of \((I\text{-NLP})\). The assumption of strict complementarity for \( \lambda^*_2 \) and strict normal growth for \((I\text{-NLP})\) implies strict normal growth for \((E\text{-NLP})\). Moreover, the previous proof shows that the condition

\[
U_{abs}(t^*)^T H_{abs}(y^*, \lambda^*) U_{abs}(t^*) > 0
\]

is equivalent to

\[
U_{e,\text{abs}}(x^*)^T H_{e,\text{abs}}(\bar{y}^*, \lambda^*) U_{e,\text{abs}}(x^*) > 0,
\]

which can be reformulated using the variable splitting of \([6]\). Then, \([6, \text{Theorem 5.19}]\) can be applied, which gives the assertion.

\( \square \)

**Theorem 42.** Assume that \((t^*, (z^*)^*)\) is kink stationary for \((I\text{-NLP})\) with Lagrange multiplier vector \( \lambda^* \) such that strict complementarity and strict normal growth are satisfied. Assume further that LIKQ holds at \( t^* \). Then,

\[
U_{\text{mpec}}(y^*)^T H_{\text{mpec}}(y^*, \lambda^*) U_{\text{mpec}}(y^*) \geq 0
\]

\( \iff \)

\[
U_{abs}(t^*)^T H_{abs}(t^*, (z^*)^*, \lambda^*) U_{abs}(x^*) \geq 0,
\]
where $y^* = (t^*, (u')^*, (v')^*) = (t^*, [(z')^*]^+, [(z')^*]^-)$. The equivalence holds also with strict inequalities.

**Proof.** The Lagrangians of (I-MPEC) and (I-NLP), respectively, are

\[
\mathcal{L}_\perp(y, \lambda) = f(t) + \lambda_2^T \mathcal{C}_2(t, u' + v') - \lambda_2^T \mathcal{C}_2(t, u' + v') + \lambda_2^T [\mathcal{C}_2(t, u' + v') - (u' - v')],
\]

\[
\mathcal{L}(t, z', \lambda) = f(t) + \lambda_2^T \mathcal{C}_2(t, |z'|) - \lambda_2^T \mathcal{C}_2(t, |z'|) + \lambda_2^T [\mathcal{C}_2(t, |z'|) - z'].
\]

Thus we can write $H_{\text{mpec}}(y^*, \lambda^*) = \nabla y y^*, \lambda^*)$ in terms of $\mathcal{L}$ with $(z')^* = (u')^* - (v')^*$.

\[
H_{\text{mpec}} = \begin{bmatrix}
H_{11} & H_{21} P_{U_{t}}^{T} & H_{21} P_{V_{t}}^{T} & H_{21} P_{V_{t}}^{T} \\
U_{t}^{T} H_{12} & U_{t}^{T} H_{22} P_{U_{t}}^{T} & U_{t}^{T} H_{22} P_{V_{t}}^{T} & U_{t}^{T} H_{22} P_{V_{t}}^{T} \\
U_{v_{t}}^{T} H_{12} & U_{v_{t}}^{T} H_{22} P_{U_{t}}^{T} & U_{v_{t}}^{T} H_{22} P_{U_{t}}^{T} & U_{v_{t}}^{T} H_{22} P_{V_{t}}^{T} \\
U_{v_{t}}^{T} H_{12} & U_{v_{t}}^{T} H_{22} P_{U_{t}}^{T} & U_{v_{t}}^{T} H_{22} P_{U_{t}}^{T} & U_{v_{t}}^{T} H_{22} P_{V_{t}}^{T}
\end{bmatrix}
\]

with $H_{\text{mpec}} = H_{\text{mpec}}(y^*, \lambda^*)$, where $H_{ij} := \partial_i \partial_j \mathcal{L}(t^*, (z')^*, \lambda^*)$ and the rows and columns have been ordered like the columns of $J(y^*)$. From (6) we further have

\[
U_{\text{mpec}}(y^*) = \begin{bmatrix}
I & \lambda_1 H_{11} - \lambda_2 P_{U_{t}} - \lambda_3 P_{V_{t}} \\
\lambda_1 U_{t}^{T} & \lambda_2 U_{v_{t}}^{T} - \lambda_3 U_{v_{t}}^{T} \\
\lambda_1 U_{v_{t}}^{T} & \lambda_2 U_{v_{t}}^{T} - \lambda_3 U_{v_{t}}^{T} \\
\lambda_1 U_{v_{t}}^{T} & \lambda_2 U_{v_{t}}^{T} - \lambda_3 U_{v_{t}}^{T}
\end{bmatrix}
\]

with $U_{1}(y^*)$ defined in (7). Now, since $U_{t}^{T} \cup V_{t}^{T} = (\alpha')^{c}$, the left-hand inequality of the claim reads

\[
\begin{bmatrix}
U_{t}^{T} H_{11} & U_{t}^{T} H_{21} P_{\alpha'}^{T} & U_{t}^{T} H_{21} P_{\alpha'}^{T} & U_{t}^{T} H_{21} P_{\alpha'}^{T} \\
\end{bmatrix}
\begin{bmatrix}
U_{1} \\
U_{1}
\end{bmatrix}
\geq 0.
\]

with $U_{1} = U_{1}(y^*)$. This is $U_{\text{abs}}(t^*)^{T} H_{\text{abs}}(t^*, (z')^*, \lambda^*) U_{\text{abs}}(t^*) \geq 0$. \(\square\)

Note that the previous theorem can be used to transfer the second order conditions for (I-NLP) and (I-MPEC) into each other. This follows from the equivalence of LIKQ and MPEC-LIQK by Theorem 25 and from the equivalence of stationarity concepts by Theorem 33.

**6. Conclusions and Outlook**

We have shown that general abs-normal NLPs are essentially the same problem class as MPECs. The two problem classes have corresponding constraint qualifications, stationarity concepts, and optimality conditions of first and second order. We have also shown that the slack reformulation from [6], which is useful to simplify derivations under LIKQ, does not preserve IDKQ and has other subtle drawbacks like non-uniqueness of slack variables. We have not considered counterpart abs-normal NLPs of general MPECs as in [5]. This would provide a different perspective on the equivalence of the two problem classes but no additional insight. Relations between the two problem classes under weaker constraint qualifications of Abadie type and Guignard type are the subject of current research.
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