NEAR-OPTIMAL ROBUST BILEVEL OPTIMIZATION

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Abstract. Bilevel optimization problems embed the optimality conditions of a sub-problem into the constraints of another optimization problem. We introduce near-optimal robustness for bilevel problems, protecting the upper-level solution feasibility from limited deviations at the lower level and show it is a restriction of the corresponding pessimistic bilevel problem. General properties and necessary conditions for the existence of solutions are derived for near-optimal robust versions of generic bilevel problems. A duality-based solution method is defined when the lower level is convex, leveraging the methodology from the robust and bilevel literature. Numerical results assess the efficiency of the proposed algorithm.

Key words. bilevel optimization, robust optimization, game theory, bounded rationality, duality, bilinear constraints, extended formulation

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1. Introduction. Bilevel optimization problems embed the optimality conditions of a sub-problem into the constraints of another problem. They can model various decision-making problems such as Stackelberg or leader-follower games, market equilibria or price-setting problems. A review of methods and applications of bilevel problems is presented in [11]. In the classical setting of bilevel problems, when optimizing their objective function, the upper level (leader) anticipates an optimal reaction of the lower level (follower) to their decisions. However, in many practical cases, the follower makes near-optimal decisions ([38]). An important issue in this setting is the definition of the robustness of the upper-level decisions with respect to near-optimal lower-level solutions.

For example, in some engineering applications [10, 36, 27], the decision-maker optimizes an outcome over a dynamical system (modelled as the lower level). For stable systems, the rate of change of the state variables decreases as the system converges towards a minimum of its potential function. If the system is stopped before reaching the minimum, the designer of the system would require that the upper-level constraints be feasible for near-optimal lower-level solutions.

The concept of bounded rationality, initially proposed in [35], sometimes referred as $\varepsilon$-rationality [2], defines an economic and behavioural interpretation of a decision-making process where an agent aims to take any solution associated with a “satisfactory” objective value instead of the optimal one.

Protecting the upper level from a violation of its constraints by deviations of the lower level is a form of robust optimization, as a protection of some constraints against uncertain parameters of the problem. Therefore, we use the terms “near-optimal robustness” and “near-optimal robust bilevel problem” or NORBiP in the rest of the paper.

The introduction of robustness in games or more broadly decision-making under uncertainty in game theory has been thoroughly explored in the literature. In [1], the authors prove the existence of robust counterparts of Nash equilibria under standard
assumptions for simultaneous games without knowledge of probability distributions associated with the uncertainty. In [28], the robust version of a network congestion problem is developed. Users are assumed to make decisions under bounded rationality, leading to a robust Wardrop equilibrium. A column generation scheme is designed to build path candidates. Robust versions of bilevel problems modelling specific Stackelberg games have been studied in [18, 31], using robust formulations to protect the leader against non-rationality or partial rationality of the follower.

Solving bilevel problems under limited deviations of the lower-level variables was introduced in [38] under the term “ε-approximation” of the pessimistic bilevel problem. The authors derive properties of interest and define a solution method for this variant in the so-called independent case, i.e., where the lower-level feasible set is independent of the upper-level decision. We generalize the approach of [38] to problems involving upper- and lower-level variables in the constraints at both levels. Since this variant consists in protecting the upper-level feasibility against uncertainty of near-optimal solutions of the lower-level, we next use the terms near-optimal robustness and near-optimal robust bilevel problem (NORBiP) to qualify this extension.

The main contributions of the paper are:

1. The generalization of the “ε-approximation” from [38] to generic bilevel problems as NORBiP in section 2, resulting in a generalized semi-infinite problem and its interpretation as a special case of robust optimization applied to bilevel problems.
2. The study of duality-based reformulations of NORBiP where the lower-level problem is convex conic, or linear in section 4, resulting in a finite-dimensional single-level reformulation.
3. An extended formulation for the linear-linear NORBiP in section 5, linearizing the single-level model with disjunctive constraints.
4. A solution algorithm for the linear-linear NORBiP in section 6 using the extended formulation.

The paper is organized as follows. In section 2, we define the concepts of near-optimal set and near-optimal robust bilevel problem. In section 3 we study special properties of near-optimal robust bilevel problems in the general setting while we consider the cases of convex and linear lower-level problems in section 4 and section 5 respectively. These cases where the follower problems are either convex or linear respectively allow the reformulation of the bilevel problem as a single level problem. A solution algorithm and computational experiments are defined and commented for the linear case in section 6. Finally, in section 7 we draw some conclusions and highlight research perspectives on near-optimal robust bilevel problems.

2. Near-optimal set and near-optimal robust bilevel problem. In this section we first define the near-optimal set of the lower level and their extensions to near-optimal robust bilevel problems. Next we illustrate the concepts on an example.

The generic bilevel problem is classically defined as:

\[
\begin{align*}
\min_x & \quad F(x, v) \\
\text{s.t.} & \quad G_k(x, v) \leq 0 \quad \forall k \in [m_u] \\
& \quad x \in \mathcal{X} \\
& \quad v \in \arg\min_{y \in \mathcal{D}} \{f(x, y) \text{ s.t. } g_i(x, y) \leq 0 \quad \forall i \in [m_l]\}.
\end{align*}
\]
The upper- and lower-level objective functions are noted $F, f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$ respectively. Constraint (2.1b) and $g_i(x, y) \leq 0 \forall i \in [m_l]$ are the upper- and lower-level constraints respectively. In this section, we assume that $\mathcal{Y} = \mathbb{R}^{n_l}$ in order that the lower-level feasible set can be only determined by the $g_i$ functions. The optimal value function $\phi(x)$ is defined as follows:

$$\phi : \mathbb{R}^{n_u} \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

$$\phi(x) = \min_y \{f(x, y) \text{ s.t. } g(x, y) \leq 0\}.$$  

(2.2)

To keep the notation succinct, the indices of the lower-level constraints $g_i$ are omitted when not needed as in (2.2). Throughout the paper, it is assumed that the lower-level problem is feasible and bounded for any given upper-level decision.

When, for a feasible upper-level decision, the solution to the lower-level problem is not unique, the bilevel problem is not well-defined and further assumptions are required [11]. In the optimistic case we assume that the follower selects the optimal solution favouring the leader while it is the contrary for the pessimistic case. We refer the reader to [13, Chapter 1] for further details on these two approaches.

The near-optimal set of the lower level $Z(x; \delta)$ is defined for a given upper-level decision $x$ and tolerance $\delta$ as:

$$Z(x; \delta) = \{y \text{ s.t. } g(x, y) \leq 0, f(x, y) \leq \phi(x) + \delta\}.$$  

The special case $Z(x; 0)$ corresponds to the set of optimal solutions to the original lower-level problem and therefore to the pessimistic bilevel problem as formulated in [38]:

$$f(x, y) \leq \phi(x) \forall y \in Z(x; 0).$$

A Near-Optimal Robust Bilevel Problem, NORBiP, of parameter $\delta$ is defined as a bilevel problem where the upper-level constraints are satisfied for any lower-level solution $z$ in the near-optimal set $Z(x; \delta)$.

$$\min_{x,v} F(x,v)$$

s.t.

(2.3a)

$$G_k(x,v) \leq 0 \forall k \in [m_u]$$

(2.3b)

$$f(x,v) \leq \phi(x)$$

(2.3c)

$$g(x,v) \leq 0$$

(2.3d)

$$G_k(x,z) \leq 0 \forall z \in Z(x; \delta) \forall k \in [m_u]$$

(2.3e)

$$x \in \mathcal{X}.$$  

(2.3f)

Each $k$ constraint in (2.3b) is satisfied if the corresponding constraint set in (2.3e) holds and is therefore redundant, since $v \in Z(x; \delta)$. However, we mention (2.3b) in the formulation to highlight the structure of the initial bilevel problem in the near-optimal robust formulation. Constraint (2.3c) is a generalized semi-infinite constraint, based on the terminology from [37]. The dependence of the set of constraints $Z(x; \delta)$ on the decision variable leads to the characterization of the problem as a robust problem.
with decision-dependent uncertainty [17]. Each constraint in the set (2.3e) can be replaced by the corresponding worst-case second-level decision $z_k$ obtained by solving the adversarial problem, parameterized by $(x, v, \delta)$:

\begin{align}
(2.4a) & \quad z_k \in \arg \max_y G_k(x, y) \\
(2.4b) & \quad f(x, y) \leq \phi(x) + \delta \\
(2.4c) & \quad g(x, y) \leq 0.
\end{align}

Finally, the near-optimal robust bilevel optimization problem can be expressed as:

\begin{align}
(2.5a) & \quad \min_{x, v} F(x, v) \\
(2.5b) & \quad f(x, v) \leq \phi(x) \\
(2.5c) & \quad g(x, v) \leq 0 \\
(2.5d) & \quad 0 \geq \max_y \{G_k(x, y) \text{ s.t. } g(x, y) \leq 0, f(x, y) \leq f(x, v) + \delta \} \quad \forall k \in [m_u]. \\
(2.5e) & \quad x \in X.
\end{align}

The literature on robust optimization distinguishes uncertainty on constraints and on the objective function [15]. The first class of problems requires solution to remain feasible for any value of the uncertain parameter, the second corresponds to optimizing for the worst case, with respect to the objective, of the uncertain parameter. In the case of bilevel optimization, the first case corresponds to NORBiP, studying the impact of near-optimal lower-level solutions on the upper-level constraints. The second case corresponds to the impact of near-optimal lower-level decisions on the upper-level objective value. We next prove that the model including uncertainty on the objective, named Objective-Robust Near-Optimal Bilevel Problem (ORNObiP), is a special case of NORBiP.

ORNObiP is defined as:

\begin{align}
(2.6a) & \quad \min_{x \in X} \sup_{z \in Z(x; \delta)} F(x, z) \\
(2.6b) & \quad Z(x; \delta) = \{y \text{ s.t. } g(x, y) \leq 0, f(x, y) \leq \phi(x) + \delta \}.
\end{align}

In contrast with most objective-robust problem formulations, the uncertainty set $Z$ depends on the upper-level solution $x$, qualifying (2.6) as a problem with decision-dependent uncertainty.

**Proposition 2.1.** ORNObiP is a special case of NORBiP.

**Proof.** The reduction of the objective-uncertain robust problem to a constraint-uncertain robust formulation is detailed in [4]. In particular, Problem (2.6) is equiv-
alent to:

\[
\begin{align*}
\min_{x,\tau} & \quad \tau \\
\text{s.t.} & \quad x \in \mathcal{X} \\
& \quad \tau \geq F(x, z) \forall z \in \mathcal{Z}(x, \delta),
\end{align*}
\]

this formulation is a special case of \textit{NORBiP}.

\[\Box\]

\textbf{Proposition 2.2.} The pessimistic bilevel optimization problem defined in [24] is both a special case and a relaxation of \textit{ORNObiP}.

\textit{Proof.} With the special case \(\delta = 0\), the inner problem of \textit{ORNObiP} is equivalent to finding the worst lower-level decision with respect to the upper-level objective amongst the lower-level-optimal solutions. For any \(\delta > 0\), the inner problem can select worse or equal solutions with respect to the upper-level. The pessimistic bilevel problem is therefore a relaxation of \textit{ORNObiP}. \[\Box\]

We next illustrate the concept of near-optimal set and near-optimal robust solution on an example. Let us consider the next linear bilevel problem, represented in Figure 1.

\begin{align}
(2.7a) & \quad \min_{x,v} x \\
(2.7b) & \quad x \geq 0 \\
(2.7c) & \quad v \geq 1 - \frac{x}{10} \\
(2.7d) & \quad v \in \arg \max_y \{y \text{ s.t. } y \leq 1 + \frac{x}{10}\}.
\end{align}

The high-point relaxation of Problem (2.7), obtained by relaxing the optimality constraint of the lower-level, while maintaining feasibility, is:

\begin{align}
\min_{x,v} x \\
\text{s.t.} & \quad x \geq 0 \\
& \quad v \geq 1 - \frac{x}{10} \\
& \quad v \leq 1 + \frac{x}{10}.
\end{align}

The shaded area in Figure 1 represents the interior of the polytope which is feasible for the high-point relaxation. The induced set, obtained by taking into account the optimal lower-level reaction, is given by:

\[
\{(x, y) \in (\mathbb{R}_+, \mathbb{R}) \text{s.t. } y = 1 + \frac{x}{10}\}.
\]

The unique optimal point is \((\hat{x}, \hat{y}) = (0, 1)\).
Let us now consider a near-optimal tolerance of the follower with $\delta = 0.1$. If the upper-level decision is $\hat{x}$, then the lower level can take any value between $1 - \delta = 0.9$ and 1 leading to infeasible upper-level except for 1. The problem can be reformulated as:

$$\min_{x, v} x$$

subject to:

$$x \geq 0$$
$$v \geq 1 - \frac{x}{10}$$
$$v \in \arg \max_y \{ y \text{ s.t. } y \leq 1 + \frac{x}{10} \}$$
$$z \geq 1 - \frac{x}{10} \forall z \text{ s.t. } \{ z \leq 1 + \frac{x}{10}, z \geq v - \delta \}.$$

Figure 2 illustrates the near-optimal equivalent of the problem.
The dashed line represents the constraint of robustness to near-optimality. The optimal upper-level decision is \( x = 0.5 \), for which the optimal lower-level reaction is \( y = 1 + 0.1 - 0.5 = 1.05 \). The boundary of the near-optimal set is \( y = 1 - 0.1 - 0.5 = 0.95 \).

3. Special properties of near-optimal robust bilevel optimization problems. In this section, we define some properties of the near-optimal robust problem.

**Proposition 3.1.** If the second-level optimization problem is convex, then the near-optimal set \( Z(x; \theta) \) is convex.

**Proof.** \( Z(x; \delta) \) is the intersection of two convex sets:
- \( F = \{ y, g(x, y) \leq 0 \} \)
- \( N = \{ y, f(x, y) \leq \phi(x) + \delta \} \)

\( F \) is the intersection of sublevel sets of convex functions \( g_i(x, \cdot) \), \( N \) is the sublevel set of a convex function \( f \).

In robust optimization, the characteristics of the uncertainty set sharply impacts the difficulty solving the problem. The near-optimal set of the lower-level is not systematically bounded; this can lead to infeasible or ill-defined near-optimal robust counterparts of bilevel problems. In the next proposition we define conditions under which the uncertainty set \( Z(x; \delta) \) is bounded.

**Proposition 3.2.** For a given pair \( (x, \delta) \), any of the following properties is sufficient for \( Z(x; \delta) \) to be a bounded set:
1. The lower-level feasible domain is bounded.
2. \( f(x, \cdot) \) is radially unbounded with respect to \( y \).
3. \( f(x, \cdot) \) is radially bounded, such that:
   \[ \lim_{r \in \mathbb{R}, r \to +\infty} f(x, rs) > f(x, v) + \delta \forall s \in S, \]
   with \( S \) the unit sphere in the space of lower-level decisions.

**Proof.** The first case is trivially satisfied since \( Z(x; \delta) \) is the intersection of sets including the lower-level feasible set. If \( f(x, \cdot) \) is radially unbounded, for any finite \( \delta > 0 \), there is a maximum radius around \( v \) beyond which any value of the objective function is greater than \( f(x, v) + \delta \). The third case follows the same line of reasoning as the second, with a lower bound in any direction \( \|y\| \to \infty \), such that this lower bound is above \( f(x, v) + \delta \).

The radius of robust feasibility is defined as the maximum “size” of the uncertain set \([23, 25]\). In the case of near-optimal robustness, the radius can be interpreted as the maximum deviation of the lower-level objective from its optimal value.

**Definition 3.3.** For a given optimization problem \( BiP \), let \( NO(BiP; \delta) \) be the optimum value of the near-optimal robust problem constructed from \( BiP \) with a tolerance \( \delta \). The radius of near-optimal feasibility \( \hat{\delta} \) is defined by:

\[
\hat{\delta} = \arg\max_{\delta} \delta \quad \text{s.t.} \quad NO(BiP; \delta) < \infty.
\]

**Proposition 3.4.** The standard optimistic bilevel problem \( BiP \) is a relaxation of the equivalent near-optimal robust bilevel problem for any \( \delta > 0 \).
Additional variables $z_{jk}, j \in [n_l], k \in [m_u]$ are introduced in the optimistic bilevel problem, resulting in the following model:

$$\begin{align*}
\min_{x,v,z} F(x,v) \\
\text{s.t.} \\
G_k(x,v) \leq 0 \quad \forall k \in [m_u] \\
f(x,v) \leq \phi(x) \\
g(x,v) \leq 0 \\
x \in \mathcal{X}, v \in \mathbb{R}^{n_l}, z \in \mathbb{R}^{n_l \times m_u}.
\end{align*}$$

This model is strictly equivalent to the optimistic bilevel problem with additional variables not used in the objective or constraints. Furthermore, it is a relaxation of Problem (2.5), which has similar variables but additional constraints (2.5d). At each point where the bilevel problem is feasible, either the objective value of the two problems are the same or NORBiP is infeasible.

**Proposition 3.5.** The pessimistic bilevel problem formulated in [38] as:

$$\begin{align*}
\min_{x \in \mathcal{X}} F(x) \\
G(x, z) \leq 0 \quad \forall z \in \mathcal{Z}(x) = \arg \min_y \{f(x,y), y \in \mathcal{Y}(x)\}
\end{align*}$$

is both a relaxation and a special case of NORBiP, with $\mathcal{Y}(x)$ the feasible set of the lower-level problem, depending on the upper-level decision $x$.

**Proof.** For $\delta = 0$, NORBiP can be re-written as:

$$\begin{align*}
\min_{x \in \mathcal{X}} F(x,v) \\
f(x,v) \leq \min_y \{f(x,y), y \in \mathcal{Y}(x)\} \\
v \in \mathcal{Y}(x) \\
G(x,v) \leq 0 \\
G(x, z) \leq 0 \quad \forall z \in \mathcal{Z}(x; \delta = 0) \\
\mathcal{Z}(x; \delta = 0) = \{y, f(x,y) \leq f(x,v) + 0, y \in \mathcal{Y}(x)\} \\
g(x,y) \leq 0 \iff y \in \mathcal{Y}(x).
\end{align*}$$

For any $\delta > 0$, the feasible domain of the adversarial problem includes all solutions that are optimal at the lower-level feasible, but also solutions that are feasible and near-optimal. The generalized semi-infinite constraint is therefore equivalent or more restrictive in the near-optimal case than in the pessimistic bilevel case, in the sense of [38].

**Proposition 3.6.** If the bilevel problem is feasible, then the adversarial problem (2.4) is feasible.

**Proof.** If the bilevel problem is feasible, then the solution $z = v$ is feasible for the primal adversarial problem.

**Proposition 3.7.** If $G_k$ is $K_k$-Lipschitz continuous for a given $k \in [m_u]$, and $(\hat{x}, \hat{y})$ is a bilevel-feasible point, such that:

$$G_k(\hat{x}, \hat{y}) < 0,$$
then the constraint $G_k(\hat{x}, y) \leq 0$ is satisfied for all $y \in \mathcal{F}_L^{(k)}$ such that:

$$\mathcal{F}_L^{(k)}(\hat{x}, \hat{y}) = \{y \in \mathbb{R}^{n_l} \text{ s.t. } \|y - \hat{y}\| \leq \frac{|G_k(\hat{x}, \hat{y})|}{K_k}\}.$$ 

Proof. As $G_k(\hat{x}, \hat{y}) < 0$, and $G_k(\hat{x}, \cdot)$ is continuous, there exists a ball $B_r(\hat{y})$ in $\mathbb{R}^{n_l}$ centered on $(\hat{y})$ of radius $r > 0$, such that

$$G(\hat{x}, y) \leq 0 \forall y \in B_r(\hat{y}).$$ 

Let us define:

$$r_0 = \arg \max_r \quad \text{s.t.} \quad G(\hat{x}, y) \leq 0 \quad \forall y \in B_r(\hat{y}).$$ 

By continuity, this problem always admits a feasible solution. If the feasible set is bounded, there exists a point $y_0$ on the boundary of the ball, such that $G_k(\hat{x}, y_0) = 0$. It follows from the Lipschitz continuity property that:

$$|G_k(\hat{x}, \hat{y}) - G_k(\hat{x}, y_0)| \leq K_k\|y_0 - \hat{y}\|$$

$$\frac{|G_k(\hat{x}, \hat{y})|}{K_k} \leq \|y_0 - \hat{y}\|.$$ 

$G_k(\hat{x}, y) \leq G_k(\hat{x}, y_0) \forall y \in B_{r_0}(\hat{y})$, therefore all lower-level solutions in the set:

$$\mathcal{F}_L^{(k)}(\hat{x}, \hat{y}) = \{y \in \mathbb{R}^{n_l} \text{ s.t. } \|y - \hat{y}\| \leq \frac{|G_k(\hat{x}, \hat{y})|}{K_k}\}$$

satisfy constraint $k$. 

**Corollary 3.8.** Let $(\hat{x}, \hat{y})$ be a bilevel-feasible solution of a near-optimal robust bilevel problem of tolerance $\delta$, and

$$\mathcal{F}_L(\hat{x}, \hat{y}) = \bigcap_{k=1}^{m_u} \mathcal{F}_L^{(k)}(\hat{x}, \hat{y}),$$

then $Z(x; \delta) \subseteq \mathcal{F}_L(\hat{x}, \hat{y})$ is a sufficient condition for near-optimal robustness of $(\hat{x}, \hat{y})$. 

Proof. Any lower-level solution $y \in \mathcal{F}_L(\hat{x}, \hat{y})$ satisfies all $m_u$ upper-level constraints, $Z(x; \delta) \subseteq \mathcal{F}_L(\hat{x}, \hat{y})$ is therefore a sufficient condition for the near-optimality robustness of $(\hat{x}, \hat{y})$. 

4. Near-optimal robust bilevel problems with a convex lower level. In this section, we study near-optimal robust bilevel problems where the lower-level problem (2.1d) is a parametric convex optimization problem with both a differentiable objective function and differentiable constraints. If Slater’s constraint qualifications hold, the KKT conditions are necessary and sufficient for the optimality of the lower-level problem and strong duality holds for the adversarial subproblems. These two properties are leveraged to reformulate NORBiP as a one-level closed-form problem.
Given a bilevel solution \((x, v)\), the adversarial problem associated with constraint \(k\) can be formulated as:

\begin{align}
\text{(4.1a)} & \quad \max_y G_k(x, y) \\
\text{s.t.} & \quad g(x, y) \leq 0 \\
\text{(4.1b)} & \quad f(x, y) \leq f(x, v) + \delta.
\end{align}

Even if the upper-level constraints are convex with respect to \(y\), Problem (4.1) is in general non-convex since the function to maximize is convex over a convex set. First-order optimality conditions may induce several non-optimal critical points and the definition of a solution method needs to rely on global optimization techniques [29, 3].

By assuming that the constraints of the upper-level problem \(G_k(x, y)\) can be decomposed and that the projection onto the lower variable space is affine, the adversarial problem:

\[ G_k(x, y) \leq 0 \iff G_k(x) + H^T y \leq q_k, \]

is convex. The \(k\)-th adversarial problem is then expressed as:

\begin{align}
\text{(4.3a)} & \quad \max_y H^T_k y \\
\text{s.t.} & \quad g_i(x, y) \leq 0 \quad \forall i \in [m_l] \quad (\alpha_i) \\
\text{(4.3b)} & \quad f(x, y) \leq f(x, v) + \delta \quad (\beta)
\end{align}

and is convex for a fixed pair \((x, v)\). Satisfying the upper-level constraint in the worst-case requires that the objective value of Problem (4.3) be lower than \(q_k - G_k(x)\). We denote by \(A_k\) and \(D_k\) the objective values of the adversarial problem (4.3) and its dual or dual adversarial problem respectively. \(D_k\) takes values in the extended real set to account for infeasible and unbounded cases. Proposition 3.6 holds for Problem (4.3). The feasibility of the upper-level constraint with the dual adversarial objective value as formulated in (4.4) is, by weak duality of convex problems, a sufficient condition for the feasibility of the near-optimal solution. If Slater’s constraint qualifications hold, it is also by strong duality a necessary condition [7].

\[ A_k \leq D_k \leq q_k - G_k(x) \]

The generic form for the single-level reformulation of the near-optimal robust problem can then be expressed as:

\begin{align}
\text{(4.5a)} & \quad \min_{x, v, \alpha, \beta} F(x, v) \\
\text{s.t.} & \quad G(x) + Hv \leq q \\
\text{(4.5b)} & \quad f(x, v) \leq \phi(x) \\
\text{(4.5c)} & \quad g(x, v) \leq 0 \\
\text{(4.5d)} & \quad D_k \leq q_k - G_k(x) \quad \forall k \in [m_u] \\
\text{(4.5e)} & \quad x \in \mathcal{X}.
\end{align}
(α, β) are certificates for the near-optimality robustness of the solution. In order to write Problem (4.5) in a closed form, the lower-level problem (4.5d)-(4.5e) is reduced to its KKT conditions:

\[ \nabla_v f(x, v) - \sum_{i=1}^{m_l} \lambda_i \cdot \nabla_v g_i(x, v) = 0 \]  
\[ g_i(x, v) \leq 0 \quad \forall i \in [m_l] \]  
\[ \lambda_i \geq 0 \quad \forall i \in [m_l] \]  
\[ \lambda_i \cdot g_i(x, v) = 0 \quad \forall i \in [m_l]. \]

The constraints derived from the KKT conditions, especially Constraint (4.6d), cannot be tackled directly by non-linear solvers [12]. Specific reformations must be leveraged, such as relaxations of the equality constraints (4.6d) into inequalities or branching on combinations of variables (as developed in [34, 33]).

We consider in the rest of this section problems such that the lower level is a conic convex optimization problem:

\[ \min_y \langle d, y \rangle \]  
\[ \text{s.t.} \]  
\[ Ax + By = b \]  
\[ y \in K \]

where \( \langle \cdot, \cdot \rangle \) is the inner product associated with the space of the lower-level variables. This class encompasses a broad class of convex optimization problems of practical interest [26, Chapter 4], while the dual problem can be written in a closed-form if the dual cone is known. \( K \) is considered to be a proper cone in the sense of [7, Chapter 2]. The \( k \)-th adversarial problem is given by:

\[ \max_y \langle H_k, y \rangle \]  
\[ \text{s.t.} \]  
\[ By = b - Ax \]  
\[ \langle d, y \rangle + r = \langle d, v \rangle + \delta \]  
\[ y \in K \]  
\[ r \geq 0 \]

where \( r \) is a slack variable used to formulate the near-optimality constraint in standard form. With the following change of variables:

\[ \hat{y} = \begin{bmatrix} y \\ r \end{bmatrix} \quad \hat{B} = \begin{bmatrix} B & 0 \end{bmatrix} \quad \hat{d} = \begin{bmatrix} d & 1 \end{bmatrix} \quad \hat{H}_k = \begin{bmatrix} H_k \\ 0 \end{bmatrix} \]

\( \hat{K} = \{ (y, r), \ y \in K, \ r \geq 0 \} \).

\( \hat{K} \) is a cone as the Cartesian product of \( K \) and the nonnegative orthant. Problem
(4.8) is reformulated as:

\[
\max_{\tilde{y}} \langle \tilde{H}_k, \tilde{y} \rangle \\
\text{s.t.} \\
(\tilde{B}\tilde{y})_i = b_i - (Ax)_i, \quad \forall i \in [m_l] \quad (\alpha_i) \\
\langle \tilde{d}, \tilde{y} \rangle = \langle d, v \rangle + \delta \quad (\beta) \\
\tilde{y} \in \hat{K}
\]

which is a conic optimization problem, for which the dual problem is:

\[
\begin{align*}
\text{(4.9a)} & \quad \min_{\alpha, \beta, s} \langle (b - Ax), \alpha \rangle + (\langle d, v \rangle + \delta)\beta \\
\text{(4.9b)} & \quad \text{s.t.} \\
\text{(4.9c)} & \quad \tilde{B}^T \alpha + \beta \tilde{d} + s = \tilde{H}_k \\
\text{(4.9d)} & \quad s \in \hat{K}^*,
\end{align*}
\]

with \(\hat{K}^*\) the dual cone of \(\hat{K}\). In the worst case (maximum number of non-zero coefficients), there are \((m_l \cdot n_u + n_l)\) of these terms in \(m_u\) non-linear non-convex constraints. This number of bilinear terms can be reduced by introducing the following variables \((p, o)\), along with the corresponding constraints:

\[
\begin{align*}
\text{(4.10a)} & \quad \min_{\alpha, \beta, s, p, o} \langle p, \alpha \rangle + (o + \delta)\beta \\
\text{(4.10b)} & \quad p = b - Ax \\
\text{(4.10c)} & \quad o = \langle d, v \rangle \\
\text{(4.10d)} & \quad \tilde{B}^T \alpha + \beta \tilde{d} + s = \tilde{H}_k \\
\text{(4.10e)} & \quad s \in \hat{K}^*.
\end{align*}
\]

The number of bilinear terms in the set of constraints is thus reduced from \(n_u \cdot m_l + n_l\) to \(m_l + 1\) terms in (4.10a). Problem (4.9) or equivalently Problem (4.10) have a convex feasible set but a bilinear non-convex objective function. The KKT conditions of the follower problem (4.7a)-(4.7d) are given for the primal-dual pair \((x, \lambda)\):

\[
\begin{align*}
\text{(4.11a)} & \quad By = b - Ax \\
\text{(4.11b)} & \quad y \in K \\
\text{(4.11c)} & \quad d - B^T \lambda \in K^* \\
\text{(4.11d)} & \quad \langle d - B^T \lambda, y \rangle = 0.
\end{align*}
\]
The upper-level problem is thus expressed as:

\begin{align}
\text{(4.12a)} & \quad \min_{x,v,\lambda,\alpha,\beta,s} F(x,v) \\
\text{s.t.} & \quad G(x) + Hv \leq q \\
\text{(4.12b)} & \quad Ax + Bv = b \\
\text{(4.12c)} & \quad d - BT\lambda \in \mathcal{K}^* \\
\text{(4.12d)} & \quad \langle d - BT\lambda, v \rangle = 0 \\
\text{(4.12e)} & \quad \langle Ax - b, \alpha_k \rangle + \beta_k \cdot \delta \leq q_k - (Gx)_k \quad \forall k \in [m_u] \\
\text{(4.12f)} & \quad \hat{B}^T\alpha_k + \hat{d} \cdot \beta_k + s_k = \hat{H}_k \quad \forall k \in [m_u] \\
\text{(4.12g)} & \quad x \in \mathcal{X}, v \in \mathcal{K} \\
\text{(4.12h)} & \quad s_k \in \hat{\mathcal{K}}^* \quad \forall k \in [m_u].
\end{align}

The Mangasarian-Frolovitz constraint qualification is violated at every feasible point of Constraint (4.12e) [32]. In non-linear approaches to complementarity constraints [34, 12], parameterized successive relaxations of the complementarity constraints are used:

\begin{align}
\text{(4.13a)} & \quad \langle d - BT\lambda, v \rangle \leq \varepsilon \\
\text{(4.13b)} & \quad -\langle d - BT\lambda, v \rangle \leq \varepsilon.
\end{align}

For \( \varepsilon > 0 \), the Mangasarian-Frolovitz constraint qualification is respected. A sequence of solutions of the relaxed problem with \( \varepsilon \to 0 \) converges to a stationary point of the initial problem. Constraints (4.12f) and (4.13) are both bilinear non-convex inequalities, the other ones added by the near-optimal robust model are conic and linear constraints. Near-optimal robustness has thus only added a finite number of constraints of the same nature (bilinear inequalities) to the solution method proposed in [12].

5. Linear near-optimal robust bilevel problem. In this section, we focus our study on near-optimal robust linear-linear bilevel problems. More precisely, the structure of the lower-level problem is exploited to derive an extended formulation leading to an efficient solution algorithm. We consider in this section that all vector spaces are subspaces of \( \mathbb{R}^n \), with appropriate dimensions. The inner product of two vectors \( \langle a, b \rangle \) is equivalently written \( a^Tb \).
For a given pair \((x, v)\), each semi-infinite robust constraint \((5.1e)\) can be reformulated as the objective value of the following adversarial problem:

\[
\begin{align*}
\text{(5.2a)} & & \max_y H_k^T y \\
\text{s.t.} & & (By)_i \leq b_i - (Ax)_i \quad \forall i \in [m_l] \\
& & d^T y \leq d^T v + \delta \\
& & y \in \mathbb{R}^n_+.
\end{align*}
\]

Let \((\alpha, \beta)\) be the dual variables associated with each group of constraints \((5.2b)-(5.2c)\). The near-optimal robust version of Problem \((5.1)\) is feasible only if the objective value of each \(k\)-th adversarial subproblem \((5.2)\) is lower than \(q_k - (Gx)_k\). The dual of problem \((5.2)\) is defined as:

\[
\begin{align*}
\text{(5.3a)} & & \min_{\alpha, \beta} \alpha^T (b - Ax) + \beta \cdot (d^T v + \delta) \\
\text{s.t.} & & B^T \alpha + \beta d \geq H_k \\
& & \alpha \in \mathbb{R}^{m_l}_+ \beta \in \mathbb{R}_+.
\end{align*}
\]

Based on Proposition 3.6 and weak duality results, the dual problem is either infeasible or feasible and bounded. By strong duality, the objective value of the dual and primal problems are equal. This value must be smaller than \(q_k - (Gx)_k\) to satisfy constraint \((5.1e)\). This is equivalent to the existence of a feasible dual solution \((\alpha, \beta)\) certifying the feasibility of \((x, v)\) within the near-optimal set \(Z(x; \delta)\). We obtain one pair of certificates \((\alpha, \beta)\) for each upper-level constraint in \([m_u]\), resulting in the following problem:

\[
\begin{align*}
\text{(5.4a)} & & \min_{x, v, \alpha, \beta} c^T x + c^T v \\
\text{s.t.} & & Gx + Hv \leq q \\
& & d^T v \leq \phi(x) \\
& & Ax + Bv \leq b \\
& & \alpha_k^T (b - Ax) + \beta_k \cdot (d^T v + \delta) \leq q_k - (Gx)_k \quad \forall k \in [m_u] \\
& & B^T \alpha_k + \beta_k d \geq H_k \quad \forall k \in [m_u] \\
& & \alpha_k \in \mathbb{R}^{m_u}_+ \beta_k \in \mathbb{R}_+ \quad \forall k \in [m_u] \\
& & v \in \mathbb{R}^{n_l}_+ \\
& & x \in \mathcal{X}.
\end{align*}
\]

The number of bilinear terms can be reduced by introducing variables \((p, o)\) as detailed in section 4. Lower-level optimality is guaranteed by the corresponding KKT conditions:

\[
\begin{align*}
\text{(5.5a)} & & d_j + \sum_i B_{ij} \lambda_i - \sigma_j = 0 \quad \forall j \in [n_l] \\
\text{(5.5b)} & & 0 \leq b_i - (Ax)_i - (Bv)_i - \lambda_i \geq 0 \quad \forall i \in [m_l] \\
\text{(5.5c)} & & 0 \leq v_j - \sigma_j \geq 0 \quad \forall j \in [n_l] \\
\text{(5.5d)} & & \sigma \geq 0, \lambda \geq 0
\end{align*}
\]
where \( \bot \) defines a complementarity constraint. Constraints (5.5b)–(5.5c) are non-linear, non-convex constraints for which every feasible point violates the Mangasarian–Fromovitz constraint qualification [32]. A common technique to linearize them is the “big-M” reformulation, introducing auxiliary binary variables with primal and dual upper bounds. The resulting formulation has a weak continuous relaxation. Furthermore, the correct choice of bounds is itself a NP-hard problem [19], and the introduction of these bounds can lead to cutting valid and potentially optimal solutions [30]. Other modelling and solution approaches, such as special ordered sets of type 1 (SOS1) or indicator constraints avoid the need to specify such bounds.

The aggregated formulation of the linear near-optimal robust bilevel problem is:

\[
\begin{align*}
\min_{x,v,\lambda,\sigma,\alpha,\beta} & \quad c^T x + c^T_y v \\
\text{s.t.} & \quad Gx + Hv \leq q \\
& \quad Ax + Bv \leq b \\
& \quad d_j + \sum_i \lambda_i B_{ij} - \sigma_j = 0 \quad \forall j \in [n_l] \\
& \quad 0 \leq \lambda_i \bot A_i x + B_i v - b_i \leq 0 \quad \forall i \in [m_l] \\
& \quad 0 \leq \sigma_j \bot v_j \geq 0 \quad \forall j \in [n_l] [n_l] \\
& \quad x \in \mathcal{X} \\
& \quad \sum_{i=1}^{m_l} \alpha_{ki}(b - Ax)_i + \beta_k \cdot (d^T v + \delta) \leq q_k - (Gx)_k \quad \forall k \in [m_u] \\
& \quad \sum_{i=1}^{m_l} B_{ij} \alpha_{ki} + \beta_k d_j \geq H_{kj} \quad \forall k \in [m_u], \forall j \in [n_l] \\
& \quad \alpha_k \in \mathbb{R}^+_m, \beta_k \in \mathbb{R}^+_+ \quad \forall k \in [m_u].
\end{align*}
\]

Problem (5.6) is single-level and has a closed form. However, constraints (5.6h) contain bilinear terms, which cannot be tackled as efficiently as convex constraints by branch-and-cut based solvers. Several solution approaches have been developed for these constraints including linear inequalities iteratively tightened using variable bounds or mixed-integer formulations [9, 8, 20].

The other notably hard constraints from Problem (5.6) are complementarity constraints, for which one solution approach is to apply the same relaxation from \( \langle a, b \rangle = 0 \) to \( \langle a, b \rangle \leq t \) as in section 4. This tightening generates a sequence of non-linear problems converging to a local minimum. Another method is developed in the following subsection, exploiting properties of the dual feasible space of the lower level to construct a completely linear formulation.

Extended formulation. The bilinear constraints (5.6h) involve products of variables from the upper and lower level formulation \((x, v)\) with dual variables of each of the \(k\) dual-adversarial problems. For fixed values of \((x, v)\), \(m_u\) dual adversarial sub-problems \(k\) (5.3) are defined. The optimal value of each of these subproblem must be lower than \(q_k - (Gx)_k\). Their feasible region is defined by (5.3b)–(5.3c)
and is independent of \((x, v)\). The objective functions are linear in \((\alpha, \beta)\). Following Proposition 3.6, Problem (5.3) is bounded. If, moreover, (5.3) is feasible, a vertex of the polytope (5.3b)-(5.3c) is an optimal solution. Constraints (5.6h)-(5.6j) can be replaced by disjunctive constraints, such that for each constraint \((k)\), at least one extreme vertex is feasible. Let \(\mathcal{N}_k\) be the number of vertices of the \(k\)-th sub-problem and \(\alpha^k_l, \beta^k_l\) be the \(l\)-th vertex of the \(k\)-th sub-problem. The constraints (5.6h)-(5.6j) can be written as:

\[
\bigvee_{i=1}^{\mathcal{N}} \sum_{i=1}^{m} \alpha^k_i (b - Ax)_i + \beta^k_i \cdot (d^Tv + \delta) \leq q_k - (Gx)_k \quad \forall k \in [m_u] 
\]

where \(\bigvee_{i=1}^{\mathcal{N}} C_i\) is the disjunction (logical “OR”) operator, expressing the constraint that at least one of the constraints \(C_i\) must be respected. These disjunctions are equivalent to indicator constraints [6].

The previous reformulation of bilinear constraints based on the polyhedral description of the \((\alpha, \beta)\) feasible space is similar to the Benders decomposition. Indeed in the near-optimal robust extended formulation, at least one of the vertices must satisfy a constraint (a disjunction) while Benders decomposition consists in satisfying a set of constraints for all extreme vertices and rays of the dual polyhedron (a constraint described as for all elements in set). Disjunctive constraints (5.7) are equivalent to the following formulation, using set cover and SOS1 constraints:

\[
\begin{align*}
\theta^k_l & \in \mathbb{B} \quad \forall k, \forall l \\
\omega^k_l & \geq 0 \quad \forall k, \forall l \\
(b - Ax)^T \alpha^k + \beta^k (d^Tv + \delta) - \omega^k \leq q_k - (Gx)_k \quad \forall k, \forall l \\
\sum_{i=1}^{\mathcal{N}} \theta^k_i & \geq 1 \quad \forall k \\
SOS1(\theta^k, \omega^k) & \quad \forall k \forall l.
\end{align*}
\]

In conclusion, using disjunctive constraints over the extreme vertices of each dual polyhedron, along with SOS1 constraints for the complementarity constraints leads to an equivalent reformulation of problem (5.6). This reformulation can be tackled by standard MILP solvers. In order to accelerate the resolution and prove infeasibility of some instances to terminate early in the process, we have designed a specific algorithm based on necessary conditions for existence of a solution.

Before defining the algorithm, we first illustrate the extended formulation on an example.

**Bounded example.** Consider the bilevel linear problem defined by the following data:

\[
x \in \mathbb{R}_+, y \in \mathbb{R}_+ \\
G = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad q = \begin{bmatrix} 11 \\ 13 \end{bmatrix} \quad c_x = [1] \quad c_y = [-10]
\]
The optimal solution of the high-point relaxation is \((x, v) = (5, 4)\) which is not bilevel-feasible. The optimal value of the optimistic bilevel problem is reached at \((x, v) = (1, 3)\). These two points are respectively represented by the blue diamond and red cross on Figure 3. The dotted segments represent the upper-level constraints and the solid lines represent the lower-level constraints.

The feasible space is defined as:

\[
\begin{align*}
-1\alpha_{11} - 4\alpha_{12} + \beta_1 & \geq 4 \\
-1\alpha_{21} - 4\alpha_{22} + \beta_2 & \geq 2 \\
\alpha_{ki} & \geq 0, \beta_k & \geq 0.
\end{align*}
\]

This feasible space can be described as a set of extreme points and rays. It consists in this case of one extreme point \((\alpha_{ki} = 0, \beta_1 = 4, \beta_2 = 2)\) and 4 extreme rays. The
Solution algorithm. We next define the solution procedure based on the structure of the extended formulation. One central principle in its design is to prove optimality or infeasibility early in the resolution process, and only then build and solve the extended formulation model. \( P_0(BiP), P_1(BiP), \text{FEAS}_k((BiP)), \mathcal{P}_{no}(BiP; \delta) \) denote respectively the high-point relaxation, optimistic bilevel problem, dual feasibility and near-optimal robust problem. \( C_k \) is the list of extreme vertices of the \( k \)-th dual adversarial polyhedron.

**Algorithm 5.1 Near-Optimal Robust Vertex Enumeration Procedure (NORVEP)**

1: function near_optimal_bilevel(BiP, \( \delta \))
2: {First phase: dual subproblems expansion & pre-solving}
3:    \( C_k \leftarrow (\alpha_k', \beta_k') \in \mathcal{V} \forall k \in \llbracket m_u \rrbracket \)
4:    for \( k \in \llbracket m_u \rrbracket \) do
5:       feas\(_k \leftarrow \text{solve(\text{FEAS}_k((BiP)));} \)
6:       if feas\(_k = \text{Infeasible} \) then
7:          return DualAdversarialInfeasible\(_k \)
8:    end if
9: end for
10: s\(_0 \leftarrow \text{solve}(P_0(BiP)) \)
11: if s\(_0 = \text{Infeasible} \) then
12:    return HighPointInfeasible
13: end if
14: s\(_1 \leftarrow \text{solve}(P_1(BiP)) \)
15: if s\(_1 = \text{Infeasible} \) then
16:    return OptimisticInfeasible
17: end if
18: {Second phase: solving the extended formulation}
19: s\(_{no}(x, v, \alpha, \beta) \leftarrow \text{solve}(\mathcal{P}_{no}(BiP, (C_k)_{k \in \llbracket m_u \rrbracket}; \delta)) \)
20: return s\(_{no} \)
21: end function

Each \textit{solve} function returns an algebraic data type describing the solving status for the given problem (either infeasible, optimal with a given solution, unbounded, etc). The algorithm is split in a first and second phase as indicated by two comments; these two phases correspond respectively to a pre-solve (solving relaxations and enumerating vertices) and solve (solve the complete near-optimal robust problem) step.

6. Computational experiments. In this section, we demonstrate the applicability of our approach through numerical experiments on instances of the linear-linear

\((x,v)\) solution needs to be valid for the corresponding near-optimality conditions:

\[
\begin{align*}
\beta_1 (v + \delta) & \leq 11 + x \\
\beta_2 (v + \delta) & \leq 13 - x.
\end{align*}
\]

The cuts generated are represented in Figure 4 for \( \delta = 0.5 \) and \( \delta = 1.0 \) in dotted blue and dashed orange respectively. The radius of near-optimal feasibility can be computed using the formulation provided in Definition 3.3, a radius of \( \delta = 5 \) can be computed, for which the feasible domain at the upper-level is reduced to the point \( x = 5 \), for which \( v = 0 \), represented as a green circle at \((5,0)\).
near-optimal robust bilevel problem. A total number of 1000 small, 100 medium and 100 large random instances are considered and characterized as follows:

\[
(m_u, m_l, n_l, n_u) = (5, 5, 5) \quad \text{(small)}
\]
\[
(m_u, m_l, n_l, n_u) = (10, 10, 10) \quad \text{(medium)}
\]
\[
(m_u, m_l, n_l, n_u) = (20, 10, 20, 20) \quad \text{(large)}
\]

All matrices are randomly generated with uniform coefficients in [0, 1] with a sparsity of 60% (all matrix entries have a 0.4 probability of being non-zero). High-point feasibility and the vertex enumeration procedures are run after generating each tuple of random parameters to discard infeasible problems. Collecting 1000 small instances required generating 10532 trials, the 100 medium-sized instances were obtained with 8830 trials and the 100 large instances after 90855 trials.

In order to get a comparison baseline for Algorithm 5.1, we first solved the non-extended formulation including bilinear constraints (5.1) with the SCIP and CPLEX solvers. As a general rule, both of them fail to converge even for small problems, reporting an error during the solution process. With the combination of SOS1 constraints encoding the complementarity and inequalities including bilinear terms, both solvers report an error instead of infeasibility of the instance.

The configuration used in the rest of the computational experiments is described below. Algorithm 5.1 is implemented in Julia [3] using the JuMP modelling framework [14]; the MILP solver used is SCIP 6.0 [16] with SoPlex 4.0 as inner LP solver, both with default solving parameters. SCIP handles indicator constraints in the form of linear inequality constraints activated only if a binary variable is equal to one. Polyhedra.jl [21] is used to model the dual subproblem polyhedra with CDDLib [22] as a solver running the double-description algorithm, producing the list of extreme vertices and rays from the constraint-based representation. The exact rational representation of numbers is used in CDDLib instead of floating-point types to avoid rounding errors. All experiments are performed on a consumer-end laptop with 15.5GB of RAM and an Intel i7 1.9GHz CPU running Ubuntu 18.04 LTS.

Table 1 summarizes the number of infeasible instances for different values of \( \delta \). As \( \delta \) increases, so does the proportion of infeasible problems. This is due to the increase in the left-hand side in constraints (5.7).

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>0.01</th>
<th>0.1</th>
<th>0.2</th>
<th>1.0</th>
<th>3.0</th>
<th>5.0</th>
<th>7.0</th>
<th>10.0</th>
<th>12.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small (/1000)</td>
<td>0</td>
<td>8</td>
<td>22</td>
<td>86</td>
<td>185</td>
<td>236</td>
<td>248</td>
<td>257</td>
<td>260</td>
</tr>
<tr>
<td>Medium (/100)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>16</td>
<td>17</td>
<td>17</td>
<td>17</td>
<td>17</td>
</tr>
</tbody>
</table>

Table 1

Number of infeasible problems for various tolerance levels \( \delta \)

Statistics on the computation times of the two phases of Algorithm 5.1 for each instance size are provided in Table 2 and Table 3. The solving time, corresponding to phase 2 of Algorithm 5.1, is greater than the vertex enumeration phase, corresponding to the first phase of the algorithm, but does not dominate it completely for any of the problem sizes.

Out of the 100 large instances, one was stopped before completion for time limit purposes and was not included in the runtime statistics of Table 3. Figure 5 shows
<table>
<thead>
<tr>
<th>Size</th>
<th>mean</th>
<th>10% quant.</th>
<th>50% quant.</th>
<th>90% quant.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>0.023</td>
<td>0.014</td>
<td>0.019</td>
<td>0.046</td>
</tr>
<tr>
<td>Medium</td>
<td>1.098</td>
<td>0.424</td>
<td>0.956</td>
<td>2.148</td>
</tr>
<tr>
<td>Large</td>
<td>44.011</td>
<td>18.740</td>
<td>39.144</td>
<td>76.903</td>
</tr>
</tbody>
</table>

Table 2

Runtime statistics for the vertex enumeration (s).
tion is provided, leveraging the boundedness of the dual to re-write the problem as a MIP with disjunctive constraints.

Future work will tackle the design of efficient solution methods to solve more general near-optimal robust convex bilevel problems.

REFERENCES


