RELATIONS BETWEEN ABS-NORMAL NLPS AND MPECS
UNDER WEAK CONSTRAINT QUALIFICATIONS

L.C. HEGERHORST-SCHULTCHEN, C. KIRCHES, AND M.C. STEINBACH

Abstract. This work continues an ongoing effort of comparing non-smooth optimization problems in abs-normal form to MPECs. We continue our study of general NLPs with equality and inequality constraints in abs-normal form, and their relation to equivalent MPEC reformulations. We introduce Abadie’s and Guignard’s kink qualification and prove relations to ACQ and GCQ for MPEC reformulations. Due to non-uniqueness of a specific slack reformulation, suggested in [8], the relations are non-trivial. It turns out that constraint qualifications of Abadie type are preserved. As we cannot show this for Guignard type we introduce branch formulations for abs-normal NLPs and MPECs. Then, equivalence of Abadie’s and Guignard’s constraint qualifications for all branch problems hold. Finally, we consider M-stationarity and B-stationarity concepts for abs-normal NLPs and prove corresponding first order optimality conditions.

1. Introduction

In [8] we have considered non-smooth nonlinear optimization problems of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) = 0, \quad h(x) \geq 0,$$

(NLP)

where $D^x \subset \mathbb{R}^n$ is open, the objective $f \in C^d(D^x, \mathbb{R})$ is a smooth function and the equality and inequality constraints $g \in C^d_{\text{abs}}(D^x, \mathbb{R}^{m_1})$ and $h \in C^d_{\text{abs}}(D^x, \mathbb{R}^{m_2})$ are non-smooth functions in abs-normal form [2]. That is, we find functions $c_{\mathcal{E}} \in C^d(D^x,[z],\mathbb{R}^{m_1})$, $c_{\mathcal{I}} \in C^d(D^x,[z],\mathbb{R}^{m_2})$ and $c_{\mathcal{Z}} \in C^d(D^x,[z],\mathbb{R}^s)$ with $D^x,[z] = D^x \times D^{[z]}$, $D^{[z]} \subset \mathbb{R}^s$ open and $0 \in D^{[z]}$ such that

$$g(x) = c_{\mathcal{E}}(x,|z|),$$

$$h(x) = c_{\mathcal{I}}(x,|z|),$$

$$z = c_{\mathcal{Z}}(x,|z|) \quad \text{with} \ \partial c_{\mathcal{Z}}(x,|z|) \text{ strictly lower triangular}.$$  

We introduce a single joint switching constraint $c_{\mathcal{Z}}$ for both $g$ and $h$, and reuse switching variables $z_i$ if the same argument repeats as an absolute value argument in $g$ and $h$. Due to the strictly lower triangular form of $\partial c_{\mathcal{Z}}(x,|z|)$, component $z_j$ of $z$ can be computed from $x$ and the components $z_i$, $i < j$. Hence, the variable $z$ is implicitly defined by $z = c_{\mathcal{Z}}(x,|z|)$, and to denote this dependence explicitly, we write $z(x)$ in the following. Whenever we address questions of solvability of this system, we make use of the reformulation $|z_i| = \text{sign}(z_i) z_i$.

Definition 1 (Signature of $z$). Let $x \in D^x$. We define the signature $\sigma(x)$ and the associated signature matrix $\Sigma(x)$ as

$$\sigma(x) := \text{sign}(z(x)) \in \{-1, 0, 1\}^s,$$

$$\Sigma(x) := \text{diag}(\sigma(x)).$$

A signature vector $\sigma(x) \in \{-1, 1\}^s$ is called definite, otherwise indefinite.

Date: August 14, 2019.
1991 Mathematics Subject Classification. 90C30, 90C33, 90C46.
Key words and phrases. Non-smooth NLPs, abs-normal form, MPECs, Abadie and Guignard type constraint qualifications, optimality conditions.

1
We may write \( |z(x)| = \Sigma(x)z(x) \) and may consider the system \( z = c_Z(x, \Sigma z) \) for fixed \( \Sigma = \Sigma(\hat{x}) \). By the implicit function theorem, the system has a locally unique solution \( z(x) \) with Jacobian
\[
\partial_{x} z(\hat{x}) = [I - \partial_{c} c_Z(\hat{x}, \Sigma z)]^{-1}\partial_{c} c_Z(\hat{x}, \Sigma z) \in \mathbb{R}^{s \times n}
\]

**Definition 2 (Active Switching Set).** We call the switching variable \( z_i \) active if \( z_i(x) = 0 \). The active switching set \( \alpha \) consists of all indices of active switching variables,
\[
\alpha(x) := \{1 \leq i \leq s: z_i(x) = 0\}.
\]
The numbers of active and inactive switching variables are \( |\alpha(x)| \) and \( |\sigma(x)| := s - |\alpha(x)| \).

### 1.1. Literature, Contributions, and Structure.

Griewank and Walther have developed a class of unconstrained abs-normal problems in [2, 3]. These problems offer particularly attractive theoretical features when generalizing KKT theory and stationarity concepts to non-smooth problems. Under certain regularity conditions, they are computationally tractable by active-set type algorithms with guaranteed convergence based on piecewise linearizations and using algorithmic differentiation techniques [4, 5].

In [6] we have shown that unconstrained abs-normal problems constitute a subclass of MPECs. In addition, we have studied regularity concepts of linear independence, Mangasarian-Fromovitz and Abadie type. As a direct generalization of unconstrained abs-normal problems we have considered NLPs with abs-normal constraints, which turned out to be equivalent to the class of MPECs. In [8] we have extended optimality conditions of unconstrained abs-normal problems to general abs-normal NLPs under the linear independence kink qualification using a reformulation of inequalities with absolute value slacks. We have compared these optimality conditions to concepts of MPECs in [7]. We have also shown that the above slack reformulation preserves constraint qualifications of linear independence type but not of Mangasarian-Fromovitz type.

In the present article we extend our detailed comparative study of general abs-normal NLPs and MPECs, considering constraint qualifications of Abadie and Guignard type both for the standard formulation and for the reformulation with absolute value slacks. In particular, we show that constraint qualifications of Abadie type are equivalent for abs-normal NLPs and MPECs and that they are preserved under the slack reformulation. For constraint qualifications of Guignard type we cannot prove equivalence but only certain implications. However, when considering branch problems of abs-normal NLPs and MPECs, we again obtain equivalence of constraint qualifications of Abadie and Guignard type, even under the slack reformulation. Finally we introduce Mordukhovich and Bouligand stationarity concepts for abs-normal NLPs and prove first order optimality conditions using the corresponding concepts for MPECs.

The remainder of this article is structured as follows. In Section 2 we state the general abs-normal NLP and its reformulation with absolute value slacks that permits to dispose of inequalities. We also present the branch structure of both formulations and introduce appropriate tangential and linearized cones. Using these tools, we introduce kink qualifications in the sense of Abadie and Guignard. In terms of these two kink qualifications, we then compare the regularity of the equality-constrained form of an abs-normal NLP to the inequality-constrained one. In Section 3 we introduce counterpart MPECs for the two formulations of abs-normal NLPs and discuss the associated MPEC-constraint qualifications, namely MPEC-ACQ and MPEC-GCQ. In Section 4 we investigate the interrelation of the regularity concepts for abs-normal NLPs and MPECs and find the situation to be more intricate than
shown in [8] for LICQ and MFCQ. Finally, in Section 5 we introduce abs-normal variants of M-stationarity and B-stationarity as first order necessary optimality conditions for abs-normal NLPs and prove equivalence relations for the respective MPEC stationarity conditions. We conclude in Section 6.

2. Inequality and Equality Constrained Formulations

In this section we consider two different treatments of inequality constraints for non-smooth NLPs in abs-normal form.

2.1. General Abs-Normal NLPs. Substituting the representation (ANF) of constraints in abs-normal form into the general problem (NLP), we obtain a general abs-normal NLP. Here, we use the variables \((t, z')\) instead of \((x, z)\).

**Definition 3** (Abs-Normal NLP). Let \(D^t\) be an open subset of \(\mathbb{R}^{m_t}\). A non-smooth NLP is called an abs-normal NLP if functions \(f \in C^d(D^t, \mathbb{R}), c_F \in C^d(D^t;|z'|, \mathbb{R}_{m_2}), c_Z \in C^d(D^t;|z'|, \mathbb{R}^{m_z})\) with \(d \geq 1\) exist such that it reads

\[
\min_{t,z'} f(t) \quad \text{s.t.} \quad c_F(t,|z'|) = 0, \quad c_Z(t,|z'|) \geq 0, \quad c_Z(t,|z'|) - z' = 0, \quad (\text{I-NLP})
\]

where \(0 \in D^t\) and \(\partial_Z c_Z(x,z')\) is strictly lower triangular. The feasible set of (I-NLP) is denoted by \(\mathcal{F}_{\text{abs}} := \{(t, z') : c_F(t,|z'|) = 0, c_Z(t,|z'|) \geq 0, c_Z(t,|z'|) = z'\}.

**Definition 4** (Active Set of Inequalities). Let \((t, z'(t)) \in \mathcal{F}_{\text{abs}}\). We call the inequality constraint \(i \in \mathcal{I}\) active if \(c_i(t,|z'(t)|) = 0\). The active set \(\mathcal{A}(t)\) consists of all indices of active inequality constraints, \(\mathcal{A}(t) = \{i \in \mathcal{I} : c_i(t,|z'(t)|) = 0\}\). We set \(c_A := [c_i]_{i \in \mathcal{A}(t)}\) and denote the number of active inequality constraints by \(|\mathcal{A}(t)|\).

With the goal of considering kink qualifications in the spirit of Abadie and Guignard, we define the tangential cone and the abs-normal-linearized cone.

**Definition 5** (Tangential Cone and Abs-Normal-Linearized Cone for (I-NLP)). Consider a feasible point \((\tilde{t}, \tilde{z}')\) of (I-NLP). The tangential cone to \(\mathcal{F}_{\text{abs}}\) at \((\tilde{t}, \tilde{z}')\) is

\[
\mathcal{T}_{\text{abs}}(\tilde{t}, \tilde{z}') := \left\{ (\delta t, \delta z') \, | \, \exists \tau_k, \forall k \geq 0, \mathcal{F}_{\text{abs}} \ni (t_k, z'_k) \rightarrow (\tilde{t}, \tilde{z}') : \right\}
\]

With \(\delta z'_i := |\delta z'_i|\) if \(i \in \alpha(\tilde{t})\) and \(\delta z'_i := \sigma_i(\tilde{t})\delta z'_i\) if \(i \notin \alpha(\tilde{t})\), the abs-normal-linearized cone is

\[
\mathcal{T}_{\text{lin}}(\tilde{t}, \tilde{z}') := \left\{ (\delta t, \delta z') \, | \, \begin{array}{l}
\partial c_F(\tilde{t},|\tilde{z}'|)\delta t + \partial c_A(\tilde{t},|\tilde{z}'|)\delta \zeta = 0, \\
\partial c_A(\tilde{t},|\tilde{z}'|)\delta t + \partial c_Z(\tilde{t},|\tilde{z}'|)\delta \zeta \geq 0, \\
\partial c_Z(\tilde{t},|\tilde{z}'|)\delta t + \partial_Z c_Z(\tilde{t},|\tilde{z}'|)\delta \zeta = \delta z'
\end{array} \right\}
\]

To prove that the tangential cone is a subset of the abs-normal-linearized cone, we follow an idea from [1], where an analogous result for MPECs was obtained. First, we need the definition of the smooth branch NLPs for (I-NLP) with their standard tangential cones and linearized cones.

**Definition 6** (Branch NLPs for (I-NLP)). Consider a feasible point \((\tilde{t}, \tilde{z}')\) of (I-NLP) and choose \(\Sigma^t = \text{diag}(\sigma^t)\) with \(\sigma^t_i \in \{-1, 1\}\) for \(i \in \alpha(\tilde{t})\) and \(\sigma^t_i = \sigma(\tilde{t})\) for \(i \notin \alpha(\tilde{t})\). The branch problem NLP (\(\Sigma^t\)) is defined as

\[
\min_{t,z'} f(t) \quad \text{s.t.} \quad c_F(t,\Sigma^t z') = 0, \quad c_Z(t,\Sigma^t z') \geq 0, \\
c_Z(t,\Sigma^t z') - z' = 0, \quad (\text{NLP}(\Sigma^t))
\]

The feasible set of (NLP(\(\Sigma^t\))), which always contains \((\tilde{t}, \tilde{z}')\), is denoted by \(\mathcal{F}_{\Sigma^t}\).
Remark 8. Observe that \( |\zeta| = \Sigma^t \zeta \) in Definitions 6 and 7, and for every \( \Sigma \) we have \( T_{\Sigma^t} \subseteq T_{\Sigma^t} \subseteq T_{\Sigma^t}(\hat{t}, \zeta^t) \) and \( T_{\Sigma^t} \hat{t}, \zeta^t \subseteq T_{\Sigma^t}(\hat{t}, \zeta^t) \).

Lemma 9. Let \((\hat{t}, \zeta^t)\) be feasible for \((I-NLP)\). Then, the following decompositions of the tangential cone and of the abs-normal-linearized cone of \((I-NLP)\) hold:

\[
T_{\Sigma^t}(\hat{t}, \zeta^t) = \bigcup_{\Sigma} T_{\Sigma^t}(\hat{t}, \zeta^t) \quad \text{and} \quad T_{\Sigma^t}(\hat{t}, \zeta^t) = \bigcup_{\Sigma} T_{\Sigma^t}(\hat{t}, \zeta^t).
\]

Proof. We first consider the tangential cones and show that a neighborhood \( N \) of \((\hat{t}, \zeta^t)\) exists such that

\[
F_{\text{abs}} \cap N = \bigcup_{\Sigma} (F_{\Sigma^t} \cap N).
\]

The inclusion \( \supseteq \) holds for every neighborhood \( N \) since \( F_{\Sigma^t} \subseteq F_{\text{abs}} \) for all \( \Sigma \). To show the inclusion \( \subseteq \), consider \( \Sigma \) containing \( \alpha \) with \( \epsilon > 0 \) exists with \( \sigma_i(t) = \sigma_i(\hat{t}) \in \{-1, 1\} \) for all \( t \in B_{\epsilon}(\hat{t}) \). Now set \( \epsilon := \min_{i \notin \alpha(\hat{t})} \epsilon_i, \quad N := B_{\epsilon} \times \mathbb{R}^n \), and consider \((t, \zeta^t) \in N \cap F_{\text{abs}} \). With the choice \( \sigma_i = \sigma_i(t) \) for \( i \notin \alpha(t) \) and \( \sigma_i = 1 \) for \( i \in \alpha(t) \) we find \( \Sigma^t = \text{diag}(\sigma^t) \) such that \((t, \zeta^t) \in N \cap F_{\Sigma^t} \) since \( \alpha(t) \subseteq \alpha(\hat{t}) \). Thus,

\[
F_{\text{abs}} \cap N = \bigcup_{\Sigma} (F_{\Sigma^t} \cap N).
\]

Now, let \( T(\hat{t}, \zeta^t; F) \) generically denote the tangential cone to \( F \) at \((\hat{t}, \zeta^t)\). Then,

\[
T_{\Sigma^t}(\hat{t}, \zeta^t) = T(\hat{t}, \zeta^t; F_{\Sigma^t}) = T(\hat{t}, \zeta^t; F_{\text{abs}} \cap N) = T(\hat{t}, \zeta^t; F_{\Sigma^t} \cap N) = \bigcup_{\Sigma} T(\hat{t}, \zeta^t; F_{\Sigma^t} \cap N) = \bigcup_{\Sigma} T_{\Sigma^t}(\hat{t}, \zeta^t).
\]

Here the fourth equality holds since the number of branch problems is finite. The decomposition of \( T_{\Sigma^t} \) follows directly by comparing definitions of \( T_{\Sigma^t} \) and \( T_{\Sigma^t} \).

Lemma 10. Let \((\hat{t}, \zeta^t)\) be feasible for \((I-NLP)\). Then,

\[
T_{\Sigma^t}(\hat{t}, \zeta^t) \subseteq T_{\Sigma^t}(\hat{t}, \zeta^t) \quad \text{and} \quad T_{\Sigma^t}(\hat{t}, \zeta^t)^* \supseteq T_{\Sigma^t}(\hat{t}, \zeta^t)^*.
\]

Proof. The branch NLPs are smooth, hence the inclusion \( T_{\Sigma^t}(\hat{t}, \zeta^t) \subseteq T_{\Sigma^t}(\hat{t}, \zeta^t) \) holds by standard NLP theory. Then, the first result follows directly from Lemma 9 and the second result follows by dualization.

In general, the reverse inclusions do not hold. This leads to the following definitions.

Definition 11 (Abadie’s Kink Qualification (AKQ) for \((I-NLP)\)). Consider a feasible point \( t \) of \((I-NLP)\). We say that Abadie’s Kink Qualification (AKQ) holds for \((I-NLP)\) at \( t \) if \( T_{\text{abs}}(\hat{t}, \zeta^t(t)) = T_{\Sigma^t}(\hat{t}, \zeta^t(t)) \).
Definition 12 (Guignard’s Kink Qualification (GKQ) for (I-NLP)). Consider a feasible point \( t \) of (I-NLP). We say that Guignard’s Kink Qualification (GKQ) holds for (I-NLP) at \( t \) if \( T_{abs}(t, \tilde{z}(t))^* = T_{abs}^{lin}(t, \tilde{z}(t))^* \).

The decomposition in Lemma 9 leads to the next results.

Theorem 13 (ACQ for all (NLP(\( \Sigma^I \)))) implies AKQ for (I-NLP)). If ACQ holds for all (NLP(\( \Sigma^I \)))) at a feasible point \( (\tilde{t}, \tilde{z}(\tilde{t})) \), then AKQ holds for (I-NLP) at \( \tilde{t} \).

Proof. This follows directly from Lemma 9.

Theorem 14 (GCQ for all (NLP(\( \Sigma^I \)))) implies GKQ for (I-NLP)). If GCQ holds for all (NLP(\( \Sigma^I \)))) at a feasible point \( (\tilde{t}, \tilde{z}(\tilde{t})) \), then GKQ holds for (I-NLP) at \( \tilde{t} \).

Proof. This follows directly from Lemma 9 by dualization.

2.2. Abs-Normal NLPs with Inequality Slacks. Here, we use absolute values of slack variables to get rid of the inequality constraints. This idea is due to Griewank. It has been introduced in [8] and has already been investigated in [7]. With slack variables \( w \in \mathbb{R}^{n_2} \), we reformulate (NLP) as follows:

\[
\min_{t,w} f(t) \quad \text{s.t.} \quad g(t) = 0, \quad h(t) - |w| = 0.
\]

Then, we express \( g \) and \( h \) in abs-normal form as in (ANF) and introduce additional switching variables \( z^w \) to handle \(|w|\). We obtain a class of purely equality-constrained abs-normal NLPs.

Definition 15 (Abs-Normal NLP with Inequality Slacks). An abs-normal NLP posed in the following form is called an abs-normal NLP with inequality slacks:

\[
\min_{t,w,z^t,z^w} f(t) \quad \text{s.t.} \quad c_E(t, |z^t|) = 0, \quad c_Z(t, |z^t|) - |z^w| = 0, \quad c_Z(t, |z^t|) = z^t, \quad w = z^w,
\]

(E-NLP)

where \( 0 \in D[|z^t|] \) and \( \partial_Z c_Z(x, |z^t|) \) is strictly lower triangular. The feasible set of (E-NLP) is denoted by \( F_{e-abs} \) and is a lifting of \( F_{abs} \).

Remark 16. Introducing \(|w|\) converts inequalities to pure equalities without a nonnegativity condition for the slack variables \( w \). In [8] we have used this formulation to simplify the presentation of first and second order conditions for the general abs-normal NLP under the linear independence kink qualification (LIKQ). Later we will see that constraint qualifications of Abadie type are preserved under reformulation. Nevertheless, this representation causes some problems. In [7] we have shown that, in contrast to LIKQ, constraint qualifications of Mangasarian-Fromovitz type are not preserved. Moreover, we cannot prove compatiblity of constraint qualifications of Guignard type. Also, note that the equation \( w - z^w = 0 \) (and hence \( w \)) cannot be eliminated as this would destroy the abs-normal form. Finally, the signs of nonzero components \( w_i \) can be chosen arbitrarily and thus the slack \( w \) is not uniquely determined. This needs to be taken into account when formulating kink qualifications (KQ) for (E-NLP).

We are now interested in defining Abadie’s and Guignard’s KQ for (E-NLP). To this end, we observe that the formulation (E-NLP) can be seen as a special case of (I-NLP): Let \( x = (t, w) \), \( z = (z^t, z^w) \), \( f(x) = f(t) \), \( c_E(x, |z|) = c_E(t, |z^t|) \), \( c_Z(t, |z^t|) - |z^w|, \) and \( c_Z(x, |z|) = c_Z(t, |z^t|), w \). Then, we can rewrite (E-NLP) as

\[
\min_{x,z} f(x) \quad \text{s.t.} \quad \tilde{c}_E(x, |z|) = 0, \quad \tilde{c}_Z(x, |z|) - z = 0.
\]

Hence, the following set of lemmas directly follows from results in the previous section.
Lemma 17 (Tangential Cone and Abs-Normal-Linearized Cone for (E-NLP)). The tangential cone to $\mathcal{F}_{\text{e-abs}}$ at $(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w)$ is

$$\mathcal{T}_{\text{e-abs}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w) = \left\{ \delta \mid \exists \tau_k \geq 0, \mathcal{F}_{\text{e-abs}} \ni (t_k, w_k, z_k', z_k^w) \rightarrow (\hat{t}, \hat{w}, \hat{z}', \hat{z}^w); \mathcal{T}_{k}^{-1}(t_k - \hat{t}, w_k - \hat{w}, z_k' - \hat{z}') \rightarrow (\delta t, \delta w, \delta z^w), \delta z^w = \delta w \right\},$$

with $\delta = (\delta t, \delta w, \delta z', \delta z^w)$ and the abs-normal-linearized cone is

$$\mathcal{T}_{\text{e-abs}}^{\text{lin}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w) = \left\{ \delta \mid \partial c_T(\hat{t}, z') |\delta t| + \partial c_T(\hat{t}, z') |\delta z'\rangle \delta z = \delta \omega, \delta z^w = \delta w \right\},$$

where $\omega = (\alpha', \alpha^w)$ and

$$\delta \zeta_i = \begin{cases} \sigma_i(\hat{t}) \delta z_i', & i \notin \alpha'(\hat{t}), \\ \delta \omega_i = \begin{cases} \sigma_i(\hat{w}) \delta z_i^w, & i \notin \alpha^w(\hat{w}), \\ \delta z_i^w, & i \in \alpha^w(\hat{w}). \end{cases} \end{cases}$$

Proof. This follows from Definition 5, the definition of $\mathcal{T}_{\text{e-abs}}^{\text{lin}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w)$. □

Lemma 18 (Branch NLPs for (E-NLP)). Consider a feasible point $(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w)$ of (E-NLP). Choose $\Sigma_i = \text{diag}(\sigma_i')$ with $\sigma_i' \in \{-1, 1\}$ for $i \in \alpha(\hat{t})$ and $\sigma_i' = \sigma(\hat{t})$ for $i \notin \alpha(\hat{t})$ as well as $\Sigma^w = \text{diag}(\sigma^w)$ with $\sigma^w \in \{-1, 1\}$ for $i \in \alpha(\hat{w})$ and $\sigma^w = \sigma(\hat{w})$ for $i \notin \alpha(\hat{w})$. The branch problem $\text{NLP}(\Sigma_i)$ for $\Sigma_i := \text{diag}(\Sigma_i', \Sigma^w)$ is defined as

$$\min_{t, w, z', z^w} f(t) \text{ s.t. } c_i(t, \Sigma_i' z') = 0, \quad c_T(t, \Sigma_i' z') - \Sigma^w z^w = 0, \quad c_T(t, \Sigma_i' z') - z' = 0, \quad w - z^w = 0, \quad (\text{NLP}(\Sigma_i))$$

$\Sigma_i' z' \geq 0, \quad \Sigma^w z^w \geq 0.$

The feasible set of $(\text{NLP}(\Sigma_i))$, which always contains $(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w)$, is denoted by $\mathcal{F}_{\Sigma_i}$ and is a lifting of $\mathcal{F}_{\Sigma_i}$. □

Lemma 19 (Tangential Cone and Abs-Normal-Linearized Cone for (NLP$(\Sigma_i)$)). Consider $(\text{NLP}(\Sigma_i))$ at a feasible point $(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w)$ of (I-NLP). The tangential cone to $\mathcal{F}_{\Sigma_i}$ at $(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w)$ is

$$\mathcal{T}_{\Sigma_i}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w) = \left\{ \delta \mid \exists \tau_k \geq 0, \mathcal{F}_{\Sigma_i} \ni (t_k, w_k, z_k', z_k^w) \rightarrow (\hat{t}, \hat{w}, \hat{z}', \hat{z}^w); \mathcal{T}_{k}^{-1}(t_k - \hat{t}, w_k - \hat{w}, z_k' - \hat{z}') \rightarrow (\delta t, \delta w, \delta z'), \delta z^w = \delta w \right\},$$

with $\delta = (\delta t, \delta w, \delta z', \delta z^w)$ and the linearized cone is

$$\mathcal{T}_{\Sigma_i}^{\text{lin}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w) = \left\{ \delta \mid \partial c_T \delta t + \partial c_T \Sigma_i' \delta z' - \Sigma^w \delta z^w = 0, \quad \delta z^w = \delta w, \quad (\delta t, \delta z') \in \mathcal{T}_{\Sigma_i}^{\text{lin}}(\hat{t}, \hat{z}'), \quad \alpha^w \delta w \geq 0, \quad i \in \alpha(\hat{w}). \right\}.$$ 

Here, all partial derivatives are evaluated at $(\hat{t}, \Sigma_i' \hat{z}')$. □

Proof. This follows from Definition 7. □

Moreover, we obtain the following decompositions by applying Lemma 9 to (E-NLP):

$$\mathcal{T}_{\text{e-abs}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w) = \bigcup_{\Sigma_i} \mathcal{T}_{\Sigma_i}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w), \quad \mathcal{T}_{\text{e-abs}}^{\text{lin}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w) = \bigcup_{\Sigma_i} \mathcal{T}_{\Sigma_i}^{\text{lin}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w).$$

As before, the tangential cone is a subset of the linearized cone and the reverse inclusion holds for the dual cones:

$$\mathcal{T}_{\text{e-abs}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w) \subseteq \mathcal{T}_{\text{e-abs}}^{\text{lin}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w), \quad \mathcal{T}_{\text{e-abs}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w)^* \supseteq \mathcal{T}_{\text{e-abs}}^{\text{lin}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w)^*.$$ 

This follows directly by applying Lemma 10 to (E-NLP). Again, equality does not hold in general, and we consider Abadie’s Kink Qualification (AKQ) and Guignard’s Kink Qualification (GKQ) for (E-NLP).
Lemma 20 (AKQ for (E-NLP)). Consider a feasible point \((\hat{t}, \hat{w})\) of (E-NLP). Then, AKQ for (E-NLP) at \((\hat{t}, \hat{w})\) is \(T_{e-abs}(\hat{t}, \hat{w}, z'(\hat{t})), z''(\hat{w})) = T_{e-abs}^{lin}(\hat{t}, \hat{w}, z'(\hat{t})), z''(\hat{w}))\).

Proof. This follows from Definition 11. \(\square\)

Lemma 21 (GKQ for (E-NLP)). Consider a feasible point \((\hat{t}, \hat{w})\) of (E-NLP). Then, GKQ for (E-NLP) at \((\hat{t}, \hat{w})\) is \(T_{e-abs}(\hat{t}, \hat{w}, z'(\hat{t})), z''(\hat{w})) = T_{e-abs}^{lin}(\hat{t}, \hat{w}, z'(\hat{t})), z''(\hat{w}))\).

Proof. This follows from Definition 12. \(\square\)

Remark 22. The possible slack values \(w \in W(t) := \{w: |w| = \sigma(t, |z'(t)|)\}\) just differ by the signs of components \(w_i\) for \(i \in A(t)\). Thus, neither AKQ nor GKQ depends on the particular choice of \(w\), and both conditions are well-defined for (E-NLP).

As before, AKQ or GKQ are implied if ACQ or GCQ hold for all branch problems.

Theorem 23 (ACQ for all (NLP(Σ e))). Consider a feasible point \((\hat{t}, \hat{w}, z'(\hat{t})), z''(\hat{w}))\) of (E-NLP). If ACQ holds for all \(\sigma(t, |z'(t)|)\) such that \(\hat{w} \in W(t)\), then AKQ holds for (E-NLP)

Proof. This follows from Theorem 13. \(\square\)

Theorem 24 (GCQ for all (NLP(Σ e))). Consider a feasible point \((\hat{t}, \hat{w}, z'(\hat{t})), z''(\hat{w}))\) of (E-NLP). If GCQ holds for all \(\sigma(t, |z'(t)|)\) such that \(\hat{w} \in W(t)\), then GKQ holds for (E-NLP)

Proof. This follows from Theorem 14. \(\square\)

2.3. Relations of Kink Qualifications for Abs-Normal NLPs. In this paragraph we discuss the relations of kink qualifications for the two different formulations introduced above. We consider the set \(W(t) = \{w: |w| = \sigma(t, |z'(t)|)\}\).

Theorem 25. AKQ for (I-NLP) holds at \(t \in D^i\) if and only if AKQ for (E-NLP) holds at \((\tilde{t}, \tilde{w})\) for \(t \in D^i \times \mathbb{R}^{m_2}\) for any \(w \in W(t)\) and hence for all \(w \in W(t)\) by Remark 22.

Proof. As \(T_{e-abs}(\tilde{t}, \tilde{z}') \subseteq T_{e-abs}(\tilde{t}, \tilde{z}')\) and \(T_{e-abs}(\tilde{t}, \tilde{z}') \subseteq T_{e-abs}^{lin}(\tilde{t}, \tilde{z}')\) always hold, we just need to prove

\[ T_{e-abs}(\tilde{t}, \tilde{z}') \subseteq T_{e-abs}^{lin}(\tilde{t}, \tilde{z}') \iff T_{e-abs}(\tilde{t}, \tilde{w}, \tilde{z}''(\tilde{w})) \subseteq T_{e-abs}^{lin}(\tilde{t}, \tilde{w}, \tilde{z}''(\tilde{w})). \]

We start with the implication “\(\subseteq\)”. Let \(\delta = (\delta t, \delta \omega, \delta z', \delta z'') \in T_{e-abs}(\tilde{t}, \tilde{w}, \tilde{z}', \tilde{z}'').\)

Then we have \(\delta = (\delta t, \delta \omega, \delta z', \delta z'') \in T_{e-abs}(\tilde{t}, \tilde{w}, \tilde{z}', \tilde{z}'').\)

Hence, there exist sequences \((t_k, z_k^w) \in T_{e-abs}\) and \(r_k \downarrow 0\)

Now, define

\[ \Sigma^w = \text{diag}(\sigma) \quad \text{with} \quad \sigma_i = \begin{cases} \sigma(\hat{w}_i), & i \notin \alpha^w(\hat{w}) \\ \text{sign}(\delta z^w_i), & i \in \alpha^w(\hat{w}) \end{cases} \]

and set \(z_k^w := w_k := \Sigma^w c_t(k, |z_k^t|)\).

Then, we have \(\hat{z}^w = \hat{w} = \Sigma^w c_t(\tilde{t}, |z_k^t|)\)

and obtain

\[ z_k^w - \hat{z}^w = \Sigma^w [c_t(k, |z_k^t|) - c_t(\tilde{t}, |z_k^t|)] = \Sigma^w [\partial_t c_t(\tilde{t}, |z_k^t|) (t_k - \hat{t}) + \partial_z c_t(\tilde{t}, |z_k^t|) (|z_k^t| - |\hat{z}^t|)] + o(\|t_k - \hat{t}, |z_k^t| - |\hat{z}^t|\|). \]

Further, for \(k\) large enough we have \(|z_k^t| - |\hat{z}^t| = \Sigma^w z_k^t - \Sigma^w \hat{z}^t\) using \(\Sigma^t = \text{diag}(\sigma^t)\) with \(\sigma^t = \sigma(z_k^t)\) and \(T_{e-abs}^{lin}(\tilde{t}, \tilde{w}, \tilde{z}'', \tilde{z}'').\)

Then, we obtain for \(z_k^t \neq 0\)

\[ T_{e-abs}(\tilde{t}, \tilde{z}') \subseteq T_{e-abs}^{lin}(\tilde{t}, \tilde{z}') \iff \tau_{\tilde{t}}^{-1} \sigma_{\tilde{t}}^t (z_k^t) \rightarrow \sigma_{\tilde{t}}^t (\hat{z}^t). \]

RELATIONS BETWEEN ABS-NORMAL NLPS AND MPECs 7
For $\tilde{z}_t^i = 0$ we have $\tau_k^{-1}(z_k^i) \rightarrow \delta z_t^i$ and hence
$$\tau_k^{-1}((z_k^i)_t) | (z_k^i)_t | | z_k^i | \rightarrow | \delta z_t^i |.$$ Thus, $\tau_k^{-1}((z_k^i)_t) | (z_k^i)_t | | z_k^i | \rightarrow \delta \zeta$ holds, and in total
$$\tau_k^{-1}(z_k^i-w_t) \rightarrow \Sigma^w\partial_1c_2(t,|\hat{z}_t^i|)dt + \partial_2c_2(t,|\hat{z}_t^i|)\delta \zeta = \Sigma^w \delta \zeta = \delta z^w.$$ Additionally, we obtain $\tau_k^{-1}(w_k - \hat{w}) \rightarrow \delta w$ and finally $d \in T_{\text{abs}}(i, w, z_t^i, z^w)$. To prove the implication $\Rightarrow$, consider $\delta = (\delta t, \delta z^i) \in T_{\text{lin}}^i(i, z_t^i)$. We define
$$\Sigma^w = \text{diag}(\sigma) \text{ with } \sigma_i = \begin{cases} \pm 1, & i \in A(i), \\ \text{sign}(\partial_1c_2(t,|\hat{z}_t^i|)dt + \partial_2c_2(t,|\hat{z}_t^i|)\delta \zeta), & i \notin A(i) \end{cases}$$
and set $\delta w = \delta z^w = \Sigma^w[\partial_1c_2(t,|\hat{z}_t^i|)dt + \partial_2c_2(t,|\hat{z}_t^i|)\delta \zeta]$. Then, we have $\delta = (\delta t, \delta w, \delta z^i, \delta z^w) \in T_{\text{abs}}(i, \hat{w}, z_t^i, z^w)$ for $\hat{w} = z^w = \Sigma^w c_2(t,|\hat{z}_t^i|)$. By assumption, $\delta \in T_{\text{abs}}(i, \hat{w}, z_t^i, z^w)$ holds, and this directly implies $\delta = (\delta t, \delta z^i) \in T_{\text{abs}}(i, z_t^i)$.

**Theorem 26.** GQK for (L-NLP) holds at $i \in D^i$ if GQK for (E-NLP) holds at $(i, \hat{w}) \in D^i \times R^{m_2}$ for any (and hence all) $\hat{w} \in W(t)$.

Proof. The inclusion $T_{\text{abs}}(i, z_t^i)^{\ast} \supseteq T_{\text{lin}}^i(i, z_t^i)^{\ast}$ is always satisfied. Thus, we just have to show
$$T_{\text{abs}}(i, z_t^i)^{\ast} \subseteq T_{\text{lin}}^i(i, z_t^i)^{\ast}.$$ Let $\omega = (\omega_t, \omega z^i) \in T_{\text{abs}}(i, z_t^i)^{\ast}$, i.e. $\omega^T \delta \geq 0$ for all $\delta = (\delta t, \delta z^i) \in T_{\text{abs}}(i, z_t^i)$. Then, set $\tilde{\omega} = (\omega_t, 0, 0, 0)$ and obtain $\tilde{\omega}^T \delta = \omega^T \delta \geq 0$ for all $\delta \in T_{\text{abs}}(i, \tilde{\omega}, z^i, z^w)$ where $\tilde{\omega} \in W(t)$ is arbitrary. By assumption, then $\tilde{\omega}^T \delta \geq 0$ for all $\hat{\delta} \in T_{\text{lin}}^i(i, \tilde{\omega}, z^i, z^w)$ holds. This implies $\omega^T \delta = \tilde{\omega}^T \delta \geq 0$ for all $\delta \in T_{\text{lin}}^i(i, z_t^i)^{\ast}$.

The converse is unlikely to hold, but we are, at the same time, not aware of a counterexample. Next, consider the branch problems and relations for ACQ and GCQ for all branch problems. Here, we can exploit sign informations and can also show equivalence for GCQ for all branch problems.

**Theorem 27.** ACQ for (NLP($\Sigma^i$)) holds at $(i, z^i(t)) \in D^i \times |z^i|$ if and only if ACQ for (NLP($\Sigma^i$)) holds at $(i, \tilde{w}, z^i(t), z^i(w)) \in D^i \times R^{m_2} \times D^i \times |z^i| \times |z^i|$ for any (and hence all) $w \in W(t)$.

Proof. The proof proceeds as shown in Theorem 25.

**Theorem 28.** GCQ for (NLP($\Sigma^i$)) holds at $(i, z^i(t)) \in D^i \times |z^i|$ if and only if GCQ for (NLP($\Sigma^i$)) holds at $(i, \tilde{w}, z^i(t), z^i(w)) \in D^i \times R^{m_2} \times D^i \times |z^i| \times |z^i|$ for any (and hence all) $w \in W(t)$.

Proof. The inclusions $T_{\Sigma^i}(i, z^i)^{\ast} \supseteq T_{\text{lin}}^i(i, z^i)^{\ast}$ and $T_{\Sigma^i}(i, z^i)^{\ast} \supseteq T_{\Sigma^i}^i(i, z^i)^{\ast}$ are always satisfied. Thus, we just need to prove
$$T_{\Sigma^i}(i, z^i)^{\ast} \subseteq T_{\text{lin}}^i(i, z^i)^{\ast} \iff T_{\Sigma^i}(i, \tilde{w}, z^i, z^w)^{\ast} \subseteq T_{\Sigma^i}^i(i, \tilde{w}, z^i, z^w)^{\ast}.$$ We start with the implication $\Rightarrow w$. Let $\omega = (\omega_t, \omega w, \omega z^i, \omega z^w) \in T_{\Sigma^i}(i, \tilde{w}, z^i, z^w)^{\ast}$, i.e. $\omega^T \delta \geq 0$ for all $\delta = (\delta_t, \delta w, \delta z^i, \delta z^w) \in T_{\Sigma^i}(i, \tilde{w}, z^i, z^w)$. Set
$$\tilde{\omega} = (\hat{\omega}_t, \hat{\omega} z^i) = (\omega_t, \omega z^i) + (\omega w + \omega z^w) \Sigma^w (\partial_1c_2(t,|\hat{z}_t^i|), \partial_2c_2(t,|\hat{z}_t^i|)).$$ Then, we have $\tilde{\omega}^T \delta = \omega^T \delta \geq 0$ for all $\delta = (\delta_t, \delta w, \delta z^i, \delta z^w) \in T_{\Sigma^i}^i(i, \tilde{w}, z^i, z^w)$ as $\omega^T \delta = \tilde{\omega}^T \delta$ holds. The reverse implication may be proved as shown in Theorem 26.
3. Counterpart MPECs

In this section we restate the MPEC counterpart problems for the two formulations (I-NLP) and (E-NLP) and we present the relations between them.

3.1. Counterpart MPEC for the General Abs-Normal NLP. In order to reformulate (I-NLP) as an MPEC, we partition $z^t$ into its nonnegative part and the modulus of its nonpositive part, $u^t := [z^t]^+ := \max(z^t, 0)$ and $v^t := [z^t]^- := \max(-z^t, 0)$. Then, we add complementarity of these two variables to replace $|z^t|$ by $u^t + v^t$ and $z^t$ itself by $u^t - v^t$.

**Definition 29** (Counterpart MPEC of (I-NLP)). The counterpart MPEC of the non-smooth NLP (I-NLP) reads

$$\min_{t, u^t, v^t} f(t) \quad \text{s.t.} \quad c_{\ell}(t, u^t + v^t) = 0, \quad c_Z(t, u^t + v^t) \geq 0,$$
$$c_Z(t, u^t + v^t) - (u^t - v^t) = 0,$$

(I-MPEC)

where $u^t, v^t \in \mathbb{R}^n$. The feasible set of (I-MPEC) is denoted by $F_{\text{mpec}}$.

To a given abs-normal NLP (I-NLP) and its counterpart MPEC (I-MPEC), the mapping $\phi: F_{\text{mpec}} \rightarrow F_{\text{abs}}$ defined as

$$\phi(t, u^t, v^t) = (t, u^t - v^t), \quad \phi^{-1}(t, z^t) = (t, [z^t]^+, [z^t]^-)$$

is a homeomorphism. This result was obtained in [7, Lemma 31].

Corresponding to the active switching set in the previous section, we introduce index sets for MPECs.

**Definition 30** (Index Sets). We denote by $U_0^t := \{ i \in \{1, \ldots, s_t \}: u^t_i = 0 \}$ the set of indices of active inequalities $u^t_i \geq 0$, and by $U_+^t := \{ i \in \{1, \ldots, s_t \}: u^t_i > 0 \}$ the set of indices of inactive inequalities $u^t_i > 0$. Analogous definitions hold of $V_0^t$ and $V_+^t$. By $D^t := U_0^t \cap V_0^t$ we denote the set of indices of non-strict (degenerate) complementarity pairs. Thus we have the partitioning $\{1, \ldots, s_t \} = U_0^t \cup V_0^t \cup D^t$.

In order to define MPEC-CQs in the spirit of Abadie and Guignard, we introduce the tangential cone, the complementarity cone, and the MPEC-linearized cone.

**Definition 31** (Tangential Cone and MPEC-Linearized Cone for (I-MPEC)). Consider a feasible point $(\hat{t}, \hat{u}^t, \hat{v}^t)$ of (I-MPEC). The tangential cone to $F_{\text{mpec}}$ at $(\hat{t}, \hat{u}^t, \hat{v}^t)$ is

$$T_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t) := \left\{ (\delta t, \delta u^t, \delta v^t) \mid \exists \gamma \geq 0, F_{\text{mpec}} \ni (t, u^t_k, v^t_k) \rightarrow (\hat{t}, \hat{u}^t, \hat{v}^t): \right. \left. \tau^{-1}_k(t_k - \hat{t}, u^t_k - \hat{u}^t, v^t_k - \hat{v}^t) \rightarrow (\delta t, \delta u^t, \delta v^t) \right\}.$$

The complementarity cone at $(\hat{u}^t, \hat{v}^t)$ is

$$T_{\perp}(\hat{u}^t, \hat{v}^t) := \left\{ (\delta u^t, \delta v^t) \mid \begin{array}{l} \delta u^t_i = 0, \ i \in V_0^t, \ \delta v^t_i = 0, \ i \in U_0^t, \\ 0 \leq \delta u^t_i \perp \delta v^t_i \geq 0, \ i \in D^t \end{array} \right\}.$$

The MPEC-linearized cone is

$$T_{\text{mpec}}^{\text{lin}}(\hat{t}, \hat{u}^t, \hat{v}^t) := \left\{ \left. \begin{array}{l} \delta t \\ \delta u^t \\ \delta v^t \end{array} \right| \begin{array}{l} \delta_1 c_{\ell}(\delta t, u^t, v^t) = 0, \\ \delta_2 c_{\ell}(\delta t, u^t, v^t) \geq 0, \\ \delta_1 c_Z(\delta t, u^t, v^t) = \delta u^t - \delta v^t, \\ (\delta u^t, \delta v^t) \in T_{\perp}(\hat{u}^t, \hat{v}^t) \end{array} \right\}.$$

Here all partial derivatives are evaluated at $(\hat{t}, \hat{u}^t + \hat{v}^t)$.

**Lemma 32.** The complementarity cone $T_{\perp}(\hat{u}, \hat{v})$ is the tangential cone and also the linearized cone to the complementarity set $\{(u, v) : 0 \leq u \perp v \geq 0\}$ at $(\hat{u}, \hat{v})$. 

Proof. Given a tangent vector \((\delta u, \delta v) = \lim \tau_k^{-1}(u_k - \hat{u}, v_k - \hat{v})\) where \(0 \leq u_k \perp v_k \geq 0\) and \(\tau_k \searrow 0\), we have for \(k\) large enough:

\[
\begin{align*}
    u_{ki} &> 0, \quad v_{ki} = 0, \quad i \in U_+ (\hat{u}_i > 0, \hat{v}_i = 0), \\
    u_{ki} = 0, \quad v_{ki} > 0, \quad i \in V_+ (\hat{u}_i = 0, \hat{v}_i > 0), \\
    0 \leq u_{ki} \perp v_{ki} \geq 0, \quad i \in D (\hat{u}_i = 0, \hat{v}_i = 0).
\end{align*}
\]

This implies \((\delta u, \delta v) \in T_{\hat{u}, \hat{v}}\). Conversely, every \((\delta u, \delta v) \in T_{\hat{u}, \hat{v}}\) is a tangent vector generated by the sequence \((u_k, v_k) = (\hat{u}, \hat{v}) + \tau_k (\delta u, \delta v)\) with \(\tau_k = 1/k\), \(k \in \mathbb{N}_{> 0}\). The linearized cone clearly coincides with the tangential cone. \(\square\)

Lemma 33. Given an abs-normal NLP (I-NLP) with counterpart MPEC (I-MPEC) and \((\hat{t}, \hat{z}^t) = \phi(\hat{t}, \hat{u}^t, \hat{v}^t) \in F_{\text{abs}},\) we have homeomorphisms \(\psi: T_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t) \to T_{\text{abs}}(\hat{t}, \hat{z}^t)\) and \(\psi: T_{\text{mpec}}^\text{lin}(\hat{t}, \hat{u}^t, \hat{v}^t) \to T_{\text{abs}}^\text{lin}(\hat{t}, \hat{z}^t)\), both defined as

\[
\psi(\delta t, \delta u^t, \delta v^t) = (\delta t, \delta u^t - \delta v^t), \quad \psi^{-1}(\delta t, \delta z^t) = (\delta t, \langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-)
\]

where \(\langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-\) map \(\delta z^t\) into the complementarity cone,

\[
\langle \delta z^t \rangle^+ = \begin{cases} \frac{\delta z^t}{\tau_k}, & \hat{v}^t = 0, \quad i \in U_+ (\hat{z}_i > 0) \\
\frac{\delta z^t}{\tau_k}, & \hat{u}^t = 0, \quad i \in V_+ (\hat{z}_i < 0) \\
\langle \delta z^t \rangle^+, & \hat{v}^t = 0, \hat{u}^t = 0, \quad i \in D (\hat{z}_i = 0) \end{cases}, \quad \langle \delta z^t \rangle^- = \begin{cases} 0, & \hat{v}^t = 0, \quad i \in U_+ (\hat{z}_i > 0) \\
-\delta z^t, & \hat{u}^t = 0, \quad i \in V_+ (\hat{z}_i < 0) \\
\langle \delta z^t \rangle^-, & \hat{v}^t = 0, \hat{u}^t = 0, \quad i \in D (\hat{z}_i = 0) \end{cases}.
\]

Proof. Given a vector \((\delta t, \delta u^t, \delta v^t) = \lim \tau_k^{-1}(t_k - \hat{t}, u_k^t - \hat{u}^t, v_k^t - \hat{v}^t)\) \(\in T_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t)\), set \((t_k, z_k^t) = \phi(t_k, u_k^t, v_k^t) = (t_k, u_k^t - v_k^t) \in F_{\text{abs}}\) to obtain

\[
\lim_{\tau_k} \frac{z_k^t - \hat{z}^t}{\tau_k} = \lim_{\tau_k} \frac{(u_k^t - \hat{u}^t) - (v_k^t - \hat{v}^t)}{\tau_k} = \delta u^t - \delta v^t \implies (\delta t, \delta u^t - \delta v^t) \in T_{\text{abs}}(\hat{t}, \hat{z}^t).
\]

Conversely, given a vector \((\delta t, \delta z^t) = \lim \tau_k^{-1}(t_k - \hat{t}, z_k^t - \hat{z}^t)\) \(\in T_{\text{abs}}(\hat{t}, \hat{z}^t)\), define \((t_k, u_k^t, v_k^t) = \phi^{-1}(t_k, z_k^t) = (t_k, z_k^t, \langle z_k^t \rangle^-) \in F_{\text{mpec}}\) to obtain

\[
\begin{align*}
    \hat{u}^t_i &> 0, \quad \hat{v}^t_i = 0, \quad u_{ki} > 0, \quad v_{ki} = 0 \text{ for } k \text{ large enough}, \\
    \hat{u}^t_i &> 0, \quad \hat{v}^t_i = 0, \quad u_{ki} = 0, \quad v_{ki} > 0 \text{ for } k \text{ large enough}, \\
    \hat{u}^t_i &> 0, \quad \hat{v}^t_i = 0, \quad 0 \leq u_{ki} \perp v_{ki} \geq 0 \text{ for } k \text{ large enough}, \\
    \hat{z}_i &> 0, \quad \hat{z}_i &< 0, \quad \hat{z}_i &< 0, \quad \hat{z}_i = 0.
\end{align*}
\]

By definition of \(\langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-\), this implies

\[
\lim_{\tau_k} \frac{t_k - \hat{t}, u_k^t - \hat{u}^t, v_k^t - \hat{v}^t}{\tau_k} = (\delta t, \langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-) \in T_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t).
\]

It is easily verified that \(\psi, \psi^{-1}\) are both continuous and inverse to each other.

Now, given \((\delta t, \delta u^t, \delta v^t) \in T_{\text{mpec}}^\text{lin}(\hat{t}, \hat{u}^t, \hat{v}^t)\), the vectors \(\delta z^t = \delta u^t - \delta v^t\) and \(\delta \zeta = \delta u^t + \delta v^t\) satisfy

\[
\begin{align*}
    \delta z^t_i &= \delta u^t_i - 0, \quad \delta \zeta_i = \delta u^t_i + 0 = \sigma_i(\hat{\ell})\delta z^t_i, \quad i \in U_+ (\hat{z}_i > 0), \\
    \delta z^t_i &= 0 - \delta v^t_i, \quad \delta \zeta_i = 0 + \delta v^t_i = \sigma_i(\hat{\ell})\delta z^t_i, \quad i \in V_+ (\hat{z}_i < 0), \\
    \delta z^t_i &= \delta u^t_i - \delta v^t_i, \quad \delta \zeta_i = \delta u^t_i + \delta v^t_i = \|\delta z^t_i\|, \quad i \in D (\hat{z}_i = 0).
\end{align*}
\]

Thus, \((\delta t, \delta z^t) = \psi(\delta t, \delta u^t, \delta v^t) \in T_{\text{abs}}^\text{lin}(\hat{t}, \hat{z}^t)\). Conversely, the same case distinction yields \(\psi^{-1}(\delta t, \delta z^t) \in T_{\text{mpec}}^\text{lin}(\hat{t}, \hat{u}^t, \hat{v}^t)\) for every \((\delta t, \delta z^t) \in T_{\text{abs}}^\text{lin}(\hat{t}, \hat{z}^t)\). \(\square\)
Definition 34 (Branch NLPs for (I-MPEC)). Consider a feasible point \((\hat{t}, \hat{u}^t, \hat{v}^t)\) of (I-NLP) and choose \(\mathcal{P}^t \subseteq D^t(\hat{t})\). The branch problem \(NLP(\mathcal{P}^t)\) is defined as

\[
\min_{t,u^t,v^t} f(t) \quad \text{s.t.} \quad \begin{align*}
&c_E(t, u^t + v^t) = 0, \\
&c_Z(t, u^t + v^t) \geq 0, \\
&c_Z(t, u^t + v^t) - (u^t - v^t) = 0, \\
&0 = u^t_i, \quad 0 \leq v^t_i, \quad i \in V^t(\hat{t}) \cup \mathcal{P}, \\
&0 \leq u^t_i, \quad 0 = v^t_i, \quad i \in U^t(\hat{t}) \cup \mathcal{P}.
\end{align*}
\]

\(NLP(\mathcal{P}^t)\)

Thus, the claim follows directly from Lemma 33.

Definition 35 (Tangential Cone and Abs-Normal-Linearized Cone for (NLP(\(\mathcal{P}^t\))). Consider corresponding branch problems \(NLP(\mathcal{P}^t)\) at a feasible point \((\hat{t}, \hat{u}^t, \hat{v}^t)\) of (I-MPEC). The tangential cone to \(\mathcal{F}_{\mathcal{P}^t}\) at \((\hat{t}, \hat{u}^t, \hat{v}^t)\) is

\[
\mathcal{T}_{\mathcal{P}^t}(\hat{t}, \hat{u}^t, \hat{v}^t) := \left\{ (\delta t, \delta u^t, \delta v^t) \mid \exists t_k \searrow 0, \mathcal{F}_{\mathcal{P}^t} \ni (t_k, u^t_k, v^t_k) \rightarrow (\hat{t}, \hat{u}^t, \hat{v}^t): \tau_k^{-1}(t_k - \hat{t}, u^t_k - \hat{u}^t, v^t_k - \hat{v}^t) \rightarrow (\delta t, \delta u^t, \delta v^t) \right\}.
\]

The linearized cone is

\[
\mathcal{T}^{\text{lin}}_{\mathcal{P}^t}(\hat{t}, \hat{u}^t, \hat{v}^t) := \left\{ \begin{array}{l}
\delta t \\
\delta u^t \\
\delta v^t
\end{array} \mid \begin{array}{l}
\partial_1 c_E \delta t + \partial_2 c_E (\delta u^t + \delta v^t) = 0, \\
\partial_1 c_A \delta t + \partial_2 c_A (\delta u^t + \delta v^t) \geq 0, \\
\partial_1 c_Z \delta t + \partial_2 c_Z (\delta u^t + \delta v^t) = \delta u^t - \delta v^t, \\
0 = \delta u^t_i \text{ for } i \in V^t(\hat{t}) \cup \mathcal{P}, \\
0 = \delta v^t_i \text{ for } i \in U^t(\hat{t}) \cup \mathcal{P}, \\
0 \leq \delta u^t_i \text{ for } i \in \mathcal{P}, \\
0 \leq \delta v^t_i \text{ for } i \in \mathcal{P}\end{array} \right\}.
\]

Here, all partial derivatives are evaluated at \((\hat{t}, \hat{u}^t + \hat{v}^t)\).

Lemma 36. Consider corresponding branch problems \((NLP(\Sigma^t))\) and \((NLP(\mathcal{P}^t))\) with \((\hat{t}, \hat{v}^t) = \phi(\hat{t}, \hat{u}^t, \hat{v}^t) \in \mathcal{F}_{\text{abs}}\) and \(\mathcal{P}^t = \{ i \in \alpha^t(\hat{t}) : \sigma_i = -1 \}\). We define \(\psi_{\mathcal{P}^t} := \psi|_{\mathcal{T}_{\mathcal{P}^t}}\), and \(\psi_{\Sigma^t} := \psi|_{\mathcal{T}^{\text{lin}}_{\Sigma^t}}\). Then,

\[
\psi_{\mathcal{P}^t}: \mathcal{T}_{\mathcal{P}^t}(\hat{t}, \hat{u}^t, \hat{v}^t) \rightarrow \mathcal{T}_{\Sigma^t}(\hat{t}, \hat{z}^t) \quad \text{and} \quad \psi_{\Sigma^t}: \mathcal{T}^{\text{lin}}_{\Sigma^t}(\hat{t}, \hat{u}^t, \hat{v}^t) \rightarrow \mathcal{T}^{\text{lin}}_{\Sigma^t}(\hat{t}, \hat{z}^t)
\]

are homeomorphisms.

Proof. Since \(\alpha^t(\hat{t}) = D^t(\hat{t})\), the following equalities of sets hold:

\[
\mathcal{P}^t = \{ i \in \alpha^t(\hat{t}) : \sigma_i = -1 \}, \quad V^t(\hat{t}) = \{ i \notin \alpha^t(\hat{t}) : \sigma_i = -1 \},
\]

\[
\mathcal{P}^t = \{ i \in \alpha^t(\hat{t}) : \sigma_i = +1 \}, \quad U^t(\hat{t}) = \{ i \notin \alpha^t(\hat{t}) : \sigma_i = +1 \}.
\]

Thus, the claim follows directly from Lemma 33. \(\square\)

Lemma 37. Let \((\hat{t}, \hat{u}^t, \hat{v}^t)\) be feasible for (I-MPEC). Then, the following decompositions of the tangential cone and of the abs-normal-linearized cone of (I-MPEC) hold:

\[
\mathcal{T}_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t) = \bigcup_{\mathcal{P}^t} \mathcal{T}_{\mathcal{P}^t}(\hat{t}, \hat{u}^t, \hat{v}^t) \quad \text{and} \quad \mathcal{T}^{\text{lin}}_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t) = \bigcup_{\mathcal{P}^t} \mathcal{T}^{\text{lin}}_{\mathcal{P}^t}(\hat{t}, \hat{u}^t, \hat{v}^t).
\]

Proof. A proof may be found in [1]. \(\square\)
Lemma 38. Let \((\hat{t}, \hat{u}^t, \hat{v}^t)\) be feasible for \((\text{I-MPEC})\). Then,
\[
\mathcal{T}_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t) \subseteq \mathcal{T}_{\text{mpec}}^\infty(\hat{t}, \hat{u}^t, \hat{v}^t) \quad \text{and} \quad \mathcal{T}_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t)^* \supseteq \mathcal{T}_{\text{mpec}}^\infty(\hat{t}, \hat{u}^t, \hat{v}^t)^*.
\]

Proof. A proof may be found in [1]. \(\square\)

In general, the converses do not hold. This motivates the definition of MPEC-ACQ and MPEC-GCQ.

Definition 39 (Abadie’s Constraint Qualification for MPEC (MPEC-ACQ) for \((\text{I-MPEC}))\). Consider a feasible point \((\hat{t}, \hat{u}^t, \hat{v}^t)\) of \((\text{I-MPEC})\). We say that Abadie’s Constraint Qualification for MPEC (MPEC-ACQ) holds for \((\text{I-MPEC})\) at \((\hat{t}, \hat{u}^t, \hat{v}^t)\) if \(\mathcal{T}_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t) = \mathcal{T}_{\text{mpec}}^\infty(\hat{t}, \hat{u}^t, \hat{v}^t)\).

Definition 40 (Guignard’s Constraint Qualification for MPEC (MPEC-GCQ) for \((\text{I-MPEC}))\). Consider a feasible point \((\hat{t}, \hat{u}^t, \hat{v}^t)\) of \((\text{I-MPEC})\). We say that Guignard’s Constraint Qualification for MPEC (MPEC-GCQ) holds for \((\text{I-MPEC})\) at \((\hat{t}, \hat{u}^t, \hat{v}^t)\) if \(\mathcal{T}_{\text{mpec}}(\hat{t}, \hat{u}^t, \hat{v}^t)^* = \mathcal{T}_{\text{mpec}}^\infty(\hat{t}, \hat{u}^t, \hat{v}^t)^*\).

Both MPEC-CQs are implied if the corresponding CQ holds for all branch problems.

Theorem 41 (ACQ for all \((\text{NLP}(\mathcal{P}^t))\) implies MPEC-ACQ for \((\text{I-MPEC}))\). If ACQ holds for all \((\text{NLP}(\mathcal{P}^t))\) at a feasible point \((\hat{t}, \hat{u}^t, \hat{v}^t)\), then MPEC-ACQ holds for \((\text{I-MPEC})\) at \((\hat{t}, \hat{u}^t, \hat{v}^t)\).

Proof. This follows directly from Lemma 37. \(\square\)

Theorem 42 (GCQ for all \((\text{NLP}(\mathcal{P}^t))\) implies MPEC-GCQ for \((\text{I-MPEC}))\). If GCQ holds for all \((\text{NLP}(\mathcal{P}^t))\) at a feasible point \((\hat{t}, \hat{u}^t, \hat{v}^t)\), then MPEC-GCQ holds for \((\text{I-MPEC})\) at \((\hat{t}, \hat{u}^t, \hat{v}^t)\).

Proof. This follows directly from Lemma 37 by dualization. \(\square\)

3.2. Counterpart MPEC for the Abs-Normal NLP with Inequality Slacks.
Using the same approach as in the preceding paragraph, we restate the counterpart MPEC of \((\text{E-NLP}))\).

Definition 43 (Counterpart MPEC of \((\text{E-NLP}))\). The counterpart MPEC of the non-smooth NLP \((\text{E-NLP}))\) reads:
\[
\min_{t,w,u^t,v^t,u^w,v^w} f(t) \quad \text{s.t.} \quad c_\mathcal{E}(t, u^t + v^t) = 0,
\]
\[
c_\mathcal{F}(t, u^t + v^t) - (u^w + v^w) = 0,
\]
\[
c_\mathcal{Z}(t, u^t + v^t) - (u^t - v^t) = 0,
\]
\[
\begin{align*}
& w - (u^w - v^w) = 0, \\
& 0 \leq u^t \perp v^t \geq 0, \\
& 0 \leq u^w \perp v^w \geq 0,
\end{align*}
\]
where \(u^t, v^t \in \mathbb{R}^n\) and \(u^w, v^w \in \mathbb{R}^{m_2}\). The feasible set is denoted by \(\mathcal{F}_{\text{e-mpec}}\) and is a lifting of \(\mathcal{F}_{\text{mpec}}\).

Clearly, the homeomorphism between \(\mathcal{F}_{\text{mpec}}\) and \(\mathcal{F}_{\text{abs}}\) extends to \(\mathcal{F}_{\text{e-mpec}}\) and \(\mathcal{F}_{\text{e-abs}}\). It is given by
\[
\tilde{\phi}(t, w, u^t, v^t, u^w, v^w) = (t, w, u^t - v^t, u^w - v^w),
\]
\[
\tilde{\phi}^{-1}(t, w, z^t, z^w) = (t, w, [z^t]^+, [z^t]^-, [z^w]^+, [z^w]^-).
\]

Just like in the abs-normal case, problem \((\text{E-MPEC})\) is a special case of \((\text{I-MPEC})\). Hence, we obtain the next lemmas from the corresponding definitions and lemmas for \((\text{I-MPEC}))\).
Lemma 44 (Tangential Cone and MPEC-Linearized Cone for (E-MPEC)). Consider a feasible point \( \hat{y} = (\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}') \) of (E-MPEC). The tangential cone to \( e\text{-mpec} \) at \( \hat{y} \) is

\[
\mathcal{T}_{e\text{-mpec}}(\hat{y}) = \left\{ \delta \mid \exists t_k \geq 0, F_{e\text{-mpec}} \ni y_k = (t_k, w_k, u_k', v_k', u_k^w, v_k^w) \to \hat{y} : \delta(t_k, \delta w, \delta u', \delta v', \delta u^w, \delta v^w) \rightarrow \mathcal{T}_{e\text{-mpec}}(\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}') \right\}.
\]

The MPEC-linearized cone is

\[
\mathcal{T}^\text{lin}_{e\text{-mpec}}(\hat{y}) = \left\{ \delta \mid \partial_1 c_T \delta t + \partial_2 c_T(\delta u' + \delta v') = \delta u^w + \delta v^w, \delta w = \delta u^w - \delta v^w \right\}.
\]

Here, all partial derivatives are evaluated at \((\hat{t}, \hat{u}', \hat{v}')\).

Proof. This follows from Definition 31. \(\square\)

Lemma 45. Given an abs-normal NLP (E-NLP) with its counterpart MPEC (E-MPEC) and \((\hat{t}, \hat{w}, \hat{z}', \hat{z}^w) = \hat{\phi}(\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}) \in F_{e\text{-abs}}\), we have homeomorphisms

\[
\hat{\psi}_e : \mathcal{T}_{e\text{-mpec}}(\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}) \to \mathcal{T}_{e\text{abs}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w),
\]

\[
\hat{\psi}_e : \mathcal{T}_{e\text{-mpec}}(\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}) \to \mathcal{T}_{e\text{abs}}(\hat{t}, \hat{w}, \hat{z}', \hat{z}^w),
\]

both defined as

\[
\hat{\psi}(\delta t, \delta w, \delta u', \delta v', \delta u^w, \delta v^w) = (\delta t, \delta w, \delta u^w - \delta v^w, \delta u^w - \delta v^w),
\]

\[
\hat{\psi}^{-1}(\delta t, \delta w, \delta z', \delta z^w) = (\delta t, \delta w, (\delta z')^+, (\delta z')^-, (\delta z^w)^+, (\delta z^w)^-).
\]

Proof. This follows immediately from Lemma 33. \(\square\)

Lemma 46 (Branch NLPs for (E-MPEC)). Consider a feasible point \( \hat{y} = (\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}) \) of (E-MPEC) and choose \( P^t \subseteq D^t(\hat{t}) \) as well as \( P^w \subseteq D^w(\hat{w}) \). The branch problem \( NLP(P_e) \) is defined as

\[
\min_{t, w, u', v', u^w, v^w} f(t) \quad \text{s.t.} \quad c_T(t, u' + v') = 0, \ c_T(t, u' + v') - (u^w + v^w) = 0, \ c_T(t, u' + v') - (u^w - v^w) = 0, \ c_T(t, u' + v') - (u^w - v^w) = 0, \ 0 = u', \ 0 \leq v', \ i \in V^t(\hat{t}) \cup P^t, \ (NLP(P_e))
\]

The feasible set of \( NLP(P_e) \), which always contains \( \hat{y} \), is a lifting of \( F_{P^t} \),

\[
F_{P_e} = \left\{ (t, w, u', v', u^w, v^w) \mid (t, u', v') \in F_{P^t}, w - (u^w - v^w) = 0, \ c_T(t, u' + v') - (u^w + v^w) = 0, \ 0 = u', \ 0 \leq v', \ i \in V^t(\hat{t}) \cup P^t, \ 0 = u^w, \ 0 \leq v^w, \ i \in V^w(\hat{w}) \cup P^w, \ 0 = u^w, \ 0 \leq v^w, \ i \in U^w(\hat{w}) \cup P^w \right\}.
\]

Proof. This follows from Definition 34. \(\square\)

Lemma 47 (Tangential Cone and MPEC-Linearized Cone for (NLP(P_e))). Consider \( (NLP(P_e)) \) at a feasible point \( \hat{y} = (\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}) \) of (E-MPEC). The tangential cone to \( e\text{-mpec} \) at \( \hat{y} \) is

\[
\mathcal{T}_{P_e}(\hat{y}) = \left\{ \delta \mid \exists t_k \geq 0, \mathcal{F}_{P_e} \ni (t_k, w_k, u_k', v_k', u_k^w, v_k^w) \to (\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}'; \hat{w}) : \delta(t_k, \delta w, \delta u', \delta v', \delta u^w, \delta v^w) \rightarrow \mathcal{T}_{e\text{-mpec}}(\hat{t}, \hat{w}, \hat{u}', \hat{v}', \hat{w}') \right\}.
\]
where $\delta = (\delta t, \delta w, \delta u^t, \delta v^t, \delta u^w, \delta v^w)$. The linearized cone is

$$\mathcal{T}_{\text{lin}}(\hat{y}) = \left\{ \delta \left| (\delta t, \delta u^t, \delta v^t) \in \mathcal{T}_{\text{lin}}^c, \partial_i c_2(\delta t + \partial_i c_2(\delta u^t + \delta v^t)) = (\delta u^w + \delta v^w), \delta w = \delta u^w + \delta v^w, 0 \leq \delta u^w_i \text{ for } i \in V^w \cup \mathcal{P}^w, 0 \leq \delta v^w_i \text{ for } i \in \mathcal{P}^w \right\}.$$  

Here, all partial derivatives are evaluated at $(\hat{t}, \hat{u}^t + \hat{v}^t)$. Moreover, the sets $U^t_+$ and $V^t_+$ are evaluated at $\hat{t}$ as well as $U^w_+$ and $V^w_+$ at $\hat{w}$.

\[\text{Proof.}\] This follows from Definition 35.

\textbf{Lemma 48.} Consider corresponding branch problems $(\text{NLP}(\Sigma_i))$ and $(\text{NLP}(\mathcal{P}^c))$ with $(\hat{t}, \hat{w}, \hat{z}^t, \hat{z}^w) = \overline{\delta}(\hat{t}, \hat{w}, \hat{u}^t, \hat{v}^t, \hat{u}^w, \hat{v}^w) \in \mathcal{F}_{\text{e-abs}}, \mathcal{P}^c = \{i \in \alpha' (\hat{t}) : \sigma^i_1 < 0\}$ and $\mathcal{P}^w = \{i \in \alpha^w(\hat{w}) : \sigma^i_w < 0\}$. We define $\psi_{\mathcal{P}_c} : = \psi|_{\mathcal{T}_{\mathcal{P}_c}}$ and $\bar{\psi}_{\mathcal{P}_c} : = \psi|_{\mathcal{T}_{\mathcal{P}_c}^*}$. Then, $\bar{\psi}_{\mathcal{P}_c} : \mathcal{T}_{\mathcal{P}_c}^c (\hat{t}, \hat{w}, \hat{u}^t, \hat{v}^t, \hat{u}^w, \hat{v}^w) \to \mathcal{T}_{\Sigma_i} (\hat{t}, \hat{w}, \hat{z}^t, \hat{z}^w)$, $\bar{\psi}_{\mathcal{P}_c} : \mathcal{T}_{\mathcal{P}_c}^c (\hat{t}, \hat{w}, \hat{u}^t, \hat{v}^t, \hat{u}^w, \hat{v}^w) \to \mathcal{T}_{\Sigma_i}^c (\hat{t}, \hat{w}, \hat{z}^t, \hat{z}^w)$ are homeomorphisms.

\[\text{Proof.}\] This follows directly from Lemma 45 by definition of the branch problems.

By applying Lemma 37 to (E-MPEC) we obtain the following decomposition of cones at $\hat{y} = (\hat{t}, \hat{w}, \hat{v}^t, \hat{v}^w, \hat{u}^w, \hat{v}^w)$:

$$\mathcal{T}_{\text{e-mpec}}(\hat{y}) = \bigcup_{\mathcal{P}^c} \mathcal{T}_{\mathcal{P}_c}(\hat{y}) \quad \text{and} \quad \mathcal{T}_{\text{e-mpec}}^c(\hat{y}) = \bigcup_{\mathcal{P}^w_c} \mathcal{T}_{\mathcal{P}_c}^c(\hat{y}).$$

Moreover, the tangential cone is included in the linearized cone and the converse hold for the dual cones:

$$\mathcal{T}_{\text{e-mpec}}(\hat{y}) \subseteq \mathcal{T}_{\text{e-mpec}}^c(\hat{y}) \quad \text{and} \quad \mathcal{T}_{\text{e-mpec}}(\hat{y})^* \supseteq \mathcal{T}_{\text{e-mpec}}^c(\hat{y})^*.$$  

Once again, the converses do not hold in general and we consider Abadie’s Constraint Qualification for (E-MPEC) and Guignard’s Constraint Qualification for (E-MPEC) at $\hat{y} = (\hat{t}, \hat{w}, \hat{v}^t, \hat{v}^w, \hat{u}^w, \hat{v}^w)$.

\textbf{Lemma 49 ((MPEC-ACQ) for (E-MPEC)).} Given a feasible point $\hat{y}$ of (E-MPEC), MPEC-ACQ at $\hat{y}$ reads $\mathcal{T}_{\text{e-mpec}}(\hat{y}) = \mathcal{T}_{\text{e-mpec}}^c(\hat{y})$.

\[\text{Proof.}\] This follows from Definition 39.

\textbf{Lemma 50 ((MPEC-GCQ) for (E-MPEC)).} Given a feasible point $\hat{y}$ of (E-MPEC), MPEC-GCQ at $\hat{y}$ reads $\mathcal{T}_{\text{e-mpec}}(\hat{y})^* = \mathcal{T}_{\text{e-mpec}}^c(\hat{y})^*.$

\[\text{Proof.}\] This follows from Definition 40.

\textbf{Remark 51.} Let

$$W(\hat{t}, \hat{u}^t, \hat{v}^t) = \{(w, u^w, v^w) : |w| = c_2(\hat{t}, \hat{v}^t + \hat{v}^t), u^w = |w|^+, v^w = |w|^-, \}.$$  

due to symmetry, the above equality of cones (respectively dual cones) holds for all elements $(\hat{w}, \hat{u}^w, \hat{v}^w) \in W(\hat{t}, \hat{v}^t, \hat{v}^t)$ if it holds for any element.

As with (I-MPEC), ACQ or GCQ for all $(\text{NLP} (\mathcal{P}^c))$ implies MPEC-ACQ or MPEC-GCQ for (E-MPEC).

\textbf{Theorem 52 (ACQ for all (NLP (\mathcal{P}^c)) implies MPEC-ACQ for (E-MPEC)).} If ACQ holds for all $(\text{NLP} (\mathcal{P}^c))$ at a feasible point $\hat{y} = (\hat{t}, \hat{w}, \hat{v}^t, \hat{v}^w, \hat{u}^w, \hat{v}^w)$, then MPEC-ACQ holds for (E-MPEC) at $\hat{y}$.

\[\text{Proof.}\] This follows from Theorem 41.
Theorem 53 (GCQ for all (NLP($P_c$)) implies MPEC-GCQ for (E-MPEC)). If GCQ holds for all (NLP($P_c$)) at a feasible point $\hat{y} = (\hat{t}, \hat{w}, \hat{v}, \hat{w}^w, \hat{v}^w$, then MPEC-GCQ holds for (E-MPEC) at $\hat{y}$.

Proof. This follows from Theorem 42. □

3.3. Relations of MPEC-CQs for Different Formulations. In this paragraph we prove relations between constraint qualifications for the two different formulations (I-MPEC) and (E-MPEC). Some relations follow from the results in the previous section and in the two following sections. For an illustration, see Fig. 1.

Theorem 54. MPEC-ACQ for (I-MPEC) holds at $(\hat{t}, \hat{u}, \hat{v}, \hat{t})$ if and only if MPEC-ACQ for (E-MPEC) holds at $(\hat{t}, \hat{w}, \hat{u}, \hat{v}, \hat{w}, \hat{v}^w)$ for any $(\hat{w}, \hat{u}, \hat{v}^w) \in W(\hat{t}, \hat{u}, \hat{v})$.

Proof. This follows immediately from Theorem 25, Theorem 58 and Theorem 62. □

Theorem 55. MPEC-GCQ for (I-MPEC) holds at $(\hat{t}, \hat{u}, \hat{v}, \hat{t})$ if MPEC-GCQ for (E-MPEC) holds at $(\hat{t}, \hat{w}, \hat{u}, \hat{v}, \hat{w}, \hat{v}^w)$ for any $(\hat{w}, \hat{u}, \hat{v}^w) \in W(\hat{t}, \hat{u}, \hat{v})$.

Proof. The inclusion $T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})^* \supseteq T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})^*$ always. Thus, it is left to show that $T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})^* \subseteq T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})^*$. Let $\omega = (\omega t, \omega w, \omega v) \in T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})^*$, i.e. $\omega^T \delta \geq 0$ for all $\delta = (\delta t, \delta u, \delta v) \in T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})$. Then, define $\hat{\omega} = (\omega t, 0, \omega w, \omega v, 0, 0)$ and obtain $\hat{\omega}^T \hat{\delta} = \omega^T \delta \geq 0$ for all $\hat{\delta} \in T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})$ where $\hat{w} \in W(\hat{t})$ is arbitrary. By assumption, we have $\hat{\omega}^T \hat{\delta} \geq 0$ for all $\hat{\delta} \in T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})$ which implies $\omega^T \delta = \hat{\omega}^T \hat{\delta} \geq 0$ for all $\delta \in T_{\text{mpec}}(\hat{t}, \hat{u}, \hat{v})$.

The converse of the previous theorem is unlikely to hold, but we do not know how to construct a counterexample. Moving to the branch problems equivalence for ACQ and GCQ holds.

Theorem 56. ACQ for (NLP($P^i$)) holds at $(\hat{t}, \hat{u}, \hat{v}, \hat{t})$ if and only if ACQ for (NLP($P_c$)) holds at $(\hat{t}, \hat{w}, \hat{u}, \hat{v}, \hat{w}, \hat{v}^w)$ for any $(\hat{w}, \hat{u}, \hat{v}^w) \in W(\hat{t}, \hat{u}, \hat{v})$.

Proof. This follows immediately from Theorem 27, Theorem 60 and Theorem 64. □

Theorem 57. GCQ for (NLP($P^i$)) holds at $(\hat{t}, \hat{u}, \hat{v}, \hat{t})$ if and only if GCQ for (NLP($P_c$)) holds at $(\hat{t}, \hat{w}, \hat{u}, \hat{v}, \hat{w}, \hat{v}^w)$ for any $(\hat{w}, \hat{u}, \hat{v}^w) \in W(\hat{t}, \hat{u}, \hat{v})$.

Proof. This follows immediately from Theorem 28, Theorem 61 and Theorem 65. □

4. Kink Qualifications and MPEC Constraint Qualifications

In this section we show relations between abs-normal NLPs and counterpart MPECs. Here, we consider both treatments of inequality constraints.

4.1. Relations of General Abs-Normal NLP and MPEC. In the following the variables $x$ and $z$ instead of $t$ and $z^t$ are used. Thus, the abs-normal NLP (I-NLP) reads:

$$\min_{x,z} f(x) \quad \text{s.t.} \quad c_E(x, |z|) = 0, \quad c_T(x, |z|) \geq 0, \quad c_Z(x, |z|) - z = 0.$$ 

The counterpart MPEC (I-MPEC) becomes:

$$\min_{x,u,v} f(x) \quad \text{s.t.} \quad c_E(x, u + v) = 0, \quad c_T(x, u + v) \geq 0,$$

$$c_Z(x, u + v) - (u - v) = 0, \quad 0 \leq u \perp v \geq 0.$$

Then, the subsequent relations of kink qualifications and MPEC constraint qualifications can be shown.
Theorem 58 (Equivalence of AKQ for (I-NLP) and MPEC-ACQ for (I-MPEC)). AKQ for (I-NLP) holds at \( x \in D^x \) if and only if MPEC-ACQ for (I-MPEC) holds at \( (x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^x \times \mathbb{R}^* \times \mathbb{R}^* \).

Proof. We need to show
\[
\mathcal{T}_{\text{abs}}(x, z) = \mathcal{T}_{\text{abs}}(x, z) \iff \mathcal{T}_{\text{mpec}}(x, u, v) = \mathcal{T}_{\text{mpec}}(x, u, v).
\]
This is obvious from the homeomorphisms \( \psi \) in Lemma 33.

Theorem 59 (MPEC-GCQ for (I-MPEC) implies GKQ for (I-NLP)). GKQ for (I-NLP) holds at \( x \in D^x \) if MPEC-GCQ for (I-MPEC) holds at \( (x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^x \times \mathbb{R}^* \times \mathbb{R}^* \).

Proof. The inclusion \( \mathcal{T}_{\text{abs}}(x, z)^* \subseteq \mathcal{T}_{\text{abs}}(x, z)^* \) holds always by Lemma 10. Thus, we just have to show
\[
\mathcal{T}_{\text{abs}}(x, z)^* \subseteq \mathcal{T}_{\text{abs}}(x, z)^*.
\]
Consider \( \omega = (\omega x, \omega z) \in \mathcal{T}_{\text{abs}}(x, z)^* \), i.e. \( \omega^T \delta \geq 0 \) for all \( \delta = (\delta x, \delta z) \in \mathcal{T}_{\text{abs}}(x, z) \). Set \( \tilde{\omega} = (\omega x, \omega z, -\omega z) \). For every \( \delta \in \mathcal{T}_{\text{abs}}(x, z) \) we then have
\[
\tilde{\omega}^T \psi^{-1}(\delta) = \omega^T x \delta x + \omega z^T (\delta z)^+ - \omega z^T (\delta z)^- = \omega^T x \delta x + \omega z^T (\delta z)^- = \omega^T \delta \geq 0.
\]
This means \( \tilde{\omega} \in \mathcal{T}_{\text{mpec}}(x, u, v)^* \) and hence, by assumption, \( \tilde{\omega} \in \mathcal{T}_{\text{mpec}}(x, u, v)^* \). We thus have \( \omega^T \delta = \tilde{\omega}^T \psi^{-1}(\delta) \geq 0 \) for every \( \delta \in \mathcal{T}_{\text{abs}}(x, z)^* \), which means \( \omega \in \mathcal{T}_{\text{abs}}(x, z)^* \).

The converse is unlikely to hold, although we are not, at this time, aware of a counterexample. Once again, moving to branch problems allows to exploit additional sign information.

Theorem 60 (Equivalence of ACQ for (NLP(\( \Sigma' \))) and ACQ for (NLP(\( P^\mu \)))), ACQ for (NLP(\( \Sigma' \))) holds at \( (x, z(x)) \) if and only if ACQ for the corresponding (NLP(\( P^\mu \))) holds at \( (x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^x \times \mathbb{R}^* \times \mathbb{R}^* \).

Proof. We need to show
\[
\mathcal{T}_{\Sigma'}(x, z) = \mathcal{T}_{\Sigma'}^\text{lin}(x, z) \iff \mathcal{T}_{P^\mu}(x, u, v) = \mathcal{T}_{P^\mu}^\text{lin}(x, u, v).
\]
This is obvious from the homeomorphisms \( \psi_P \) in Lemma 36.

Theorem 61 (Equivalence of GCQ for (NLP(\( \Sigma' \))) and GCQ for (NLP(\( P^\mu \)))) GCQ for (NLP(\( \Sigma' \))) holds at \( (x, z(x)) \) if and only if GCQ for the corresponding (NLP(\( P^\mu \))) holds at \( (x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^x \times \mathbb{R}^* \times \mathbb{R}^* \).

Proof. The inclusions \( \mathcal{T}_{\Sigma'}^\text{lin}(x, u, v)^* \subseteq \mathcal{T}_{P^\mu}(x, u, v)^* \) and \( \mathcal{T}_{\Sigma'}^\text{lin}(x, z)^* \subseteq \mathcal{T}_{\Sigma'}(x, z)^* \) hold always. Thus, we just have to show
\[
\mathcal{T}_{\Sigma'}(x, z)^* \supseteq \mathcal{T}_{P^\mu}^\text{lin}(x, z)^* \iff \mathcal{T}_{P^\mu}(x, u, v)^* \supseteq \mathcal{T}_{P^\mu}^\text{lin}(x, u, v)^*.
\]
First, we show the implication \( \Rightarrow \). Consider \( \omega = (\omega x, \omega u, \omega v) \in \mathcal{T}_{P^\mu}(x, u, v)^* \), i.e. \( \omega^T \delta \geq 0 \) for all \( \delta = (\delta x, \delta u, \delta v) \in \mathcal{T}_{P^\mu}(x, u, v) \). Set \( \tilde{\omega} = (\omega x, \omega z) \) with
\[
\omega_{z_i} = \begin{cases} +\omega u_i, & i \in \mathcal{U}_u \cup \mathcal{P}, \\ -\omega v_i, & i \in \mathcal{V}_u \cup \mathcal{P}. \end{cases}
\]
This leads to
\[
\tilde{\omega}^T \psi_P(\delta) = \omega^T x \delta x + \omega u^T \delta u + \omega v^T \delta v = \omega^T \delta \geq 0
\]
for every \( \delta \in \mathcal{T}_{P^\mu}(x, u, v) \), i.e. \( \tilde{\omega} \in \mathcal{T}_{\Sigma'}(x, z)^* \). Then, the assumption yields \( \tilde{\omega} \in \mathcal{T}_{\Sigma'}(x, z)^* \). As we have \( \omega^T \delta = \tilde{\omega}^T \psi_P(\delta) \geq 0 \) for every \( \delta \in \mathcal{T}_{P^\mu}^\text{lin}(x, u, v) \), we obtain \( \omega \in \mathcal{T}_{P^\mu}^\text{lin}(x, u, v)^* \). The reverse implication follows as in Theorem 59.
4.2. Relations of Abs-Normal NLP and MPEC with Inequality Slacks.

Now, the relations for the slack reformulations are stated. These are special cases of the general problem formulations, hence we simply cite the previous proofs.

**Theorem 62** (Equivalence of AKQ for (E-NLP) and MPEC-ACQ for (E-MPEC)). AKQ for (E-NLP) holds at \( x \in D^\pi \) if and only if MPEC-ACQ for (E-MPEC) holds at \( (x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^\pi \times \mathbb{R}^{s_1+m_2} \times \mathbb{R}^{s_2+m_2} \).

*Proof.* This follows as in the proof of Theorem 58. \(\square\)

**Theorem 63** (MPEC-GCQ for (E-MPEC) implies GKQ for (E-NLP)). GKQ for (E-NLP) holds at \( x \in D^\pi \) if MPEC-GCQ for (E-MPEC) holds at \( (x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^\pi \times \mathbb{R}^{s_1+m_2} \times \mathbb{R}^{s_2+m_2} \).

The converse is unlikely to hold, but to date we are not aware of a counterexample.

*Proof.* This follows as in the proof of Theorem 59. \(\square\)

**Theorem 64** (Equivalence of ACQ for (NLP(\(\Sigma\)) and ACQ for (NLP(\(\mathcal{P}\))). ACQ for (NLP(\(\Sigma\))) holds at \( (x, z(x)) \) if and only if ACQ for the corresponding (NLP(\(\mathcal{P}\))) holds at \( (x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^\pi \times \mathbb{R}^s \times \mathbb{R}^s \).

*Proof.* This follows as in the proof of Theorem 60. \(\square\)

**Theorem 65** (Equivalence of GCQ for (NLP(\(\Sigma\))) and GCQ for (NLP(\(\mathcal{P}\))). GCQ for (NLP(\(\Sigma\))) holds at \( (x, z(x)) \) if and only if GCQ for the corresponding (NLP(\(\mathcal{P}\))) holds at \( (x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in D^\pi \times \mathbb{R}^s \times \mathbb{R}^s \).

*Proof.* This follows as in the proof of Theorem 61. \(\square\)

5. First Order Stationarity Concepts

In this section, we introduce definitions of Mordukhovich stationarity and Bouligand stationarity for abs-normal NLPs and compare these definitions to M-stationarity and B-stationarity for MPECs. We give proofs based on the general formulation.

5.1. Mordukhovich Stationarity. In this paragraph we have a closer look at M-stationarity which is a necessary optimality condition for MPECs under MPEC-ACQ.

**Definition 66** (M-stationarity for (I-MPEC)). A feasible point \((x^\ast, u^\ast, v^\ast)\) of (I-MPEC) is an M-stationary point if there exist multipliers \(\lambda = (\lambda_\mathcal{E}, \lambda_\mathcal{I}, \lambda_\mathcal{Z})\) and \(\mu = (\mu_u, \mu_v)\) such that the following conditions are satisfied:

\[
\partial_{x,u,v}\mathcal{L}_\mathcal{I}(x^\ast, u^\ast, v^\ast, \lambda, \mu) = 0, \\
((\mu_u)_i > 0, (\mu_v)_i > 0) \lor (\mu_u)_i(\mu_v)_i = 0, \quad i \in \mathcal{D}(x^\ast) \\
(\mu_u)_i = 0, \quad i \in \mathcal{U}_u(x^\ast), \\
(\mu_v)_i = 0, \quad i \in \mathcal{V}_v(x^\ast), \\
\lambda_\mathcal{Z} \geq 0, \\
\lambda_\mathcal{E}^T c_\mathcal{E}(x^\ast, u^\ast, v^\ast) = 0.
\]

Herein, \(\mathcal{L}_\mathcal{I}\) is the MPEC-Lagrangian function

\[
\mathcal{L}_\mathcal{I}(x, u, v, \lambda, \mu) := f(x) + \lambda_\mathcal{E}^T c_\mathcal{E}(x, u + v) - \lambda_\mathcal{Z}^T c_\mathcal{Z}(x, u + v) + \lambda_\mathcal{E}^T c_\mathcal{E}(x, u + v) - (u - v)] - \mu_u^T u' - \mu_v^T v'.
\]

**Theorem 67.** Under MPEC-ACQ all local minimizers of (I-MPEC) are M-stat. points.
whether the reverse implications hold.

Note that in Theorem 26, Theorem 55, Theorem 59 and Theorem 63 we have only proved one-sided implications and it is open whether the reverse implications hold.

Figure 1. Solid arrows: Relations between AKQ and MPEC-ACQ; dashed arrows: Relations between GKQ and MPEC-GCQ.
Proof. A proof may be found in [1]. □

Definition 68 (M-Stationarity for (I-NLP)). A feasible point \((x^*, z^*)\) of (I-NLP) is an M-stationary point if there exist multipliers \(\lambda = (\lambda_x, \lambda_I, \lambda_Z)\) such that the following conditions are satisfied:

\[
\begin{align*}
    f'(x^*) + \lambda^T_x \partial_1 c_E - \lambda^T_I \partial_1 c_I + \lambda^T_Z \partial_1 c_Z &= 0, \\
\left[\lambda^T_x \partial_2 c_E - \lambda^T_I \partial_2 c_I + \lambda^T_Z \partial_2 c_Z\right]_i &= (\lambda_z)_i, i \notin \alpha(x^*), \\
(\mu^+_i)(\mu^-_i) &= 0 \quad \vee \quad \left[\lambda^T_x \partial_2 c_E - \lambda^T_I \partial_2 c_I + \lambda^T_Z \partial_2 c_Z\right]_i > |(\lambda_z)_i|, i \in \alpha(x^*), \\
\lambda_I &\geq 0, \\
\lambda^T_Z c_Z &= 0. 
\end{align*}
\]

Here we use the notation

\[
\begin{align*}
    \mu^+_i &:= \left[\lambda^T_x \partial_2 c_E - \lambda^T_I \partial_2 c_I + \lambda^T_Z \partial_2 c_Z - I\right]_i, \\
    \mu^-_i &:= \left[\lambda^T_x \partial_2 c_E - \lambda^T_I \partial_2 c_I + \lambda^T_Z \partial_2 c_Z + I\right]_i, 
\end{align*}
\]

and the constraints and the partial derivatives are evaluated at \((x^*, z^*)\).

Theorem 69 (M-Stationarity for (I-MPEC) is M-Stationarity for (I-NLP)). A feasible point \((x^*, z^*)\) of (I-NLP) is M-stationary if and only if \((x^*, u^*, v^*) = (x^*, [z^+]^+, [z^-]^-)\) of (I-MPEC) is M-stationary.

Proof. For indices that satisfy the first condition in (1b), the equivalence with the second condition in (2c) was shown in [7, Theorem 33]. Thus, we just need to consider the alternative conditions. For (I-MPEC) we have the relations

\[
\begin{align*}
    \left[\lambda^T_x \partial_2 c_E - \lambda^T_I \partial_2 c_I + \lambda^T_Z \partial_2 c_Z - I\right]_i &= (\mu_u)_i, i \in \mathcal{D}(x^*), \\
    \left[\lambda^T_x \partial_2 c_E - \lambda^T_I \partial_2 c_I + \lambda^T_Z \partial_2 c_Z + I\right]_i &= (\mu_v)_i, i \in \mathcal{D}(x^*), 
\end{align*}
\]

which was also shown in [7, Theorem 33]. These are exactly the definitions of \(\mu^+_i\) and \(\mu^-_i\) in the definition of M-Stationarity for (I-NLP). □

Theorem 70 (Minimizers and M-Stationarity for (I-NLP)). Assume that \((x^*, z^*)\) is a local minimizer of (I-NLP) and that AKQ holds at \(x^*\). Then, \((x^*, z^*)\) is M-stationary for (I-NLP).

Proof. First, we note that \((x^*, z^*)\) is a local minimizer of (I-NLP) if and only if \((x^*, u^*, v^*) = (x^*, [z^+]^+, [z^-]^-)\) is a local minimizer of (I-MPEC), then the point \((x^*, u^*, v^*)\) is a local minimizer of the counterpart MPEC, and MPEC-ACQ holds by Theorem 58. Now, Theorem 67 implies that \((x^*, u^*, v^*)\) is M-stationary for (I-MPEC) and thus Theorem 69 implies that \((x^*, z^*)\) is M-stationary for (I-NLP). □

5.2. MPEC-linearized Bouligand Stationarity. Finally, we introduce MPEC-linearized Bouligand stationarity, which is defined via smooth subproblems.

Definition 71 (B-Stationarity for (I-MPEC)). A feasible point \((x^*, u^*, v^*)\) of (I-MPEC) is a B-stationary point if it is a stationary point of all branch problems \((NLP(P^i))\) for \(P^t = P \subseteq D(x^*)\).

Theorem 72. Under GCQ for all \((NLP(P^i))\) all local minimizers of (I-MPEC) are MPEC-linearized B-stationary points.

Proof. This follows directly by KKT theory for smooth optimization problems. □

Definition 73 (Abs-Normal-Linearized B-Stationarity for (I-NLP)). A feasible point \((x^*, z^*)\) of (I-NLP) is an abs-normal-linearized B-stationary point if it is a stationary point of all branch problems \((NLP(\Sigma_i^j))\) for \(\Sigma^t = \Sigma = \text{diag}(\sigma)\) with \(\sigma_i = \sigma(x_i), i \notin \alpha(x^*)\) and \(\sigma_i \in \{-1, 1\}, i \in \alpha(x^*)\).
Theorem 74 (MPEC-linearized B-stationarity for (I-MPEC) is abs-normal-linearized B-stationarity for (I-NLP)). A feasible point \((x^*, z^*)\) of (I-NLP) is abs-normal-linearized B-stationary if and only if \((x^*, u^*, v^*) = (x^*, [z^+]^+, [z^+]^-)\) of (I-MPEC) is MPEC-linearized B-stationary.

Proof. This follows directly from Lemma 36.

Theorem 75 (Minimizers and abs-normal-linearized B-Stationarity for (I-NLP)). Assume that \((x^*, z^*)\) is a local minimizer of (I-NLP) and that GCQ holds at \(x^*\) for all \((\text{NLP}(\Sigma_t))\). Then, \((x^*, z^*)\) is abs-normal-linearized B-stationary for (I-NLP).

Proof. The point \((x^*, z^*)\) is a local minimizer of (I-NLP) if and only if \((x^*, u^*, v^*) = (x^*, [z^+]^+, [z^+]^-)\) is a local minimizer of (I-MPEC). Moreover, GCQ for all \((\text{NLP}(\Sigma_t'))\) and GCQ for all \((\text{NLP}(P_t'))\) are equivalent by Theorem 61. Thus, \((x^*, u^*, v^*)\) is a local minimizer of the counterpart MPEC and GCQ holds for all \((\text{NLP}(P_t'))\). Then, it is MPEC-linearized B-stationary by Lemma 72 and finally \((x^*, z^*)\) is abs-normal-linearized B-stationary by Theorem 74.

Remark 76. In [5], Griewank and Walther have presented a stationarity concept that holds without any kink qualification for minimizers of the unconstrained abs-normal NLP

\[
\min_x f(x), \quad f \in C^{cd}_{\text{abs}}(D^x, \mathbb{R}).
\]

Indeed, this concept is precisely abs-normal-linearized Bouligand stationarity: it requires the conditions of Definition 73 specialized to (3). Now, the question arises why no regularity assumption is needed. The answer is that the abs-normal form provides a certain degree of built-in regularity: we have shown in [6] that MPEC-ACQ is always satisfied (and thus every local minimizer is an M-stationary point). Analogously one can show that ACQ for all branch problems \((\text{NLP}(P_t'))\) is always satisfied for (3). Now, ACQ for all branch problems \((\text{NLP}(P_t'))\) is equivalent to ACQ for all branch problems \((\text{NLP}(\Sigma_t'))\) by Theorem 60, which in turn implies GCQ for all branch problems \((\text{NLP}(\Sigma_t'))\). Thus, GCQ for all branch problems \((\text{NLP}(\Sigma_t'))\) is always satisfied for (3) and Theorem 75 holds.

6. Conclusions

We have shown that general abs-normal NLPs are essentially the same problem class as MPECs. The two problem classes permit the definition of corresponding constraint qualifications, and optimality conditions of first order under weak constraint qualifications. We have also shown that the slack reformulation from [8], preserves constraint qualifications of Abadie type, whereas for Gugnland type we could only prove some implications. Here, one subtle drawback is the non-uniqueness of slack variables. Thus, we have introduced branch formulations of general abs-normal NLPs and counterpart MPECs. Then, constraint qualifications of Abadie and Guignard type are preserved.

Acknowledgement

C. Kirches was supported by the German Federal Ministry of Education and Research through grants no. 05M17MBA-MoPhaPro, 05M18MBA-MORNet, and 01/S17089C-ODINE, and by Deutsche Forschungsgemeinschaft (DFG) through Priority Programme 1962, grants Ki1839/1-1 and Ki1839/1-2.
References


L. C. Hegerhorst-Schultchen, Leibniz Universität Hannover, Institut für Angewandte Mathematik, Welfengarten 1, 30167 Hannover, Germany

Email address: hegerhorst@ifam.uni-hannover.de

Technische Universität Carolo-Wilhelmina zu Braunschweig, Institut für Mathematische Optimierung, Universitätsplatz 2, 38106 Braunschweig, Germany.

Email address: c.kirches@tu-bs.de

M.C. Steinbach, Leibniz Universität Hannover, Institut für Angewandte Mathematik, Welfengarten 1, 30167 Hannover, Germany

Email address: mcs@ifam.uni-hannover.de