Mathematical programs with multiobjective generalized Nash equilibrium problems in the constraints

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Abstract. This paper considers a class of mathematical programs that includes multiobjective generalized Nash equilibrium problems in the constraints. For the lower level, we deal with weakly efficient generalized Nash equilibria. Although this kind of problems has some interesting applications, there is no research focusing on it due to the difficulty resulting from its hierarchical structure and the multiplicity of objectives at the lower level. In this paper, we present a single level reformulation for this kind of problems and show the equivalence in terms of global and local minimizers. We find that the reformulation is a special case of a mathematical program with equilibrium constraints which is extensively studied in the literature.
Key Words. Multiobjective optimization, bilevel program, generalized Nash equilibrium

1 Introduction

Mathematical programs with optimization problems in the constraints was first given by Bracken and McGill in [6]. This kind of problems has a hierarchical structure and is highly nonconvex, resulting in that it is difficult to deal with. Its history can date back to von Stackelberg who (in 1934 in monograph [24]) formulated a hierarchical game of two players now called Stackelberg game. Nowadays this kind of problems is widely called the bilevel (or two level) programming problem where the upper level is called "leader" and the low level is called "follower". They have many applications in many fields such as economics and management, transportation, engineering, and others. Although bilevel programs are quite difficult to deal with due to their hierarchical structure, there have been great progresses made on theoretical and numerical issues. See the monographs of Dempe [10] and Bard [3] and the bibliography reviews by Vicente and Calamai [25], Dempe [9], and Colson et al. [8].

Motivated by the fact that decision makers often have several conflict objectives that should be optimized simultaneously when facing a real-life problem, recently bilevel programs

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with multiple objectives are received attention in the literature; see, e.g., [1, 4, 7, 12, 14, 18, 22] for optimality conditions, numerical algorithms, and applications.

In practice, at the lower level of a two-level hierarchical program there may be multiple followers who compete in a noncooperative generalized Nash game (see Section 3). In this paper, we call such a kind of problems "the mathematical program with multiobjective generalized Nash equilibrium problems in the constraints". We illustrate this case with an example. Let us consider a regional pollution control problem. This region consists of several areas each of which determines the pollution quantities retained by herself/himself to remove and transferred to other areas, aiming to maximize the health benefit of this area and minimize the area's pollution removal cost simultaneously. The strategy of each area is dependent on the strategies of other rivals. The regional administrator can decide about the pollution transfer tax rate that will influence all the areas in the removal quantities of pollution. Hence the pollution removal problem among the areas in the region can be modeled on the lower level with the transfer tax rate as a parameter (competing in a generalized Nash game), and with the solutions influencing the objective of the region on the upper level. It is not hard to see that the regional pollution problem can be modeled as a mathematical program with a generalized Nash equilibrium problem in the constraints.

One main approach to deal with problems with hierarchical structure is to transform them into single-level optimization problems by replacing the lower-level problem with its KKT conditions (see, e.g., [11, 19]). The transformed reformulation is a special case of a mathematical program with equilibrium constraints (MPEC) whose theoretical results such as necessary optimality and stability, and numerical algorithms such as relaxation methods and augmented Lagrangian methods have been extensively studied in the literature.

In this paper, we propose the mathematical program with generalized Nash equilibrium problems in the constraints and we show that it can be equivalently reformulated as an MPEC in terms of global minimizers and local minimizers under some convexity conditions and constraint regularity conditions. This suggests that theoretical results and numerical algorithms for MPECs can apply to this kind of problems directly.

The rest of the paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we introduce the equilibrium concept for the generalized Nash equilibrium problem with multiple objectives and give its scalarization method. In Section 4, we give a single level MPEC reformulation for our problem and show the equivalence from the views of global minimizers and local minimizers.

2 Preliminaries

In this section, we review some background materials on multiobjective optimization to keep the paper self-contained; see, e.g., [13] and the references therein for more details.

Let $C \subseteq \Re^m$ be a pointed, closed, and convex cone with nonempty interior. A vector-valued function $f : \Re^n \to \Re^m$ is said to be C-convex if we have

$$f(\lambda x_1 + (1-\lambda)x_2) \preceq_C \lambda f(x_1) + (1-\lambda) f(x_2) \quad \forall x_1, x_2 \in \Re^n, \lambda \in (0,1).$$

where $a \preceq_C b$ means that $b - a \in C$. When $C = \Re^m_+ := \{x \in \Re^m : x \geq 0\}$, the C-convexity function f is simply said to be convex for simplicity.

Let us consider the following multiobjective optimization problem with respect to a partial order induced by a pointed, closed, and convex cone $C \subseteq \Re^m$:

$$\min\nolimits_C \ \varphi(x) \quad \text{s.t. } x \in D, \tag{2.1}$$

where "$\min_C$" means that the objective function values are compared using the partial order induced by C, $\varphi : \Re^n \to \Re^m$ represents a vector-valued function, and D is a nonempty feasible region.

A point $x^* \in D$ is said to be a weakly efficient solution of problem (2.1) if there is no $x \in D$ such that

$$\varphi(x^*) - \varphi(x) \in \operatorname{int} C, \tag{2.2}$$

where "int" denotes the topological interior of the set considered. The set of weakly efficient solutions is denoted by $\mathcal{S}_{\text{wef}}$. A point $x^* \in D$ is said to be a locally weakly efficient solution of problem (2.1) if for some neighborhood V of x^*, there is no $x \in D \cap V$ such that

$$\varphi(x^*) - \varphi(x) \in \operatorname{int} C.$$

If $C = \Re^m_+$, then (2.2) can be simply rewritten as

$$\varphi(x) < \varphi(x^*).$$

For the sake of simplicity, throughout this paper we use the partial order induced by $C = \Re^m_+$ to make comparisons between vectors and let "min" stand for a minimization problem where the objective function values are compared by using the partial order induced by $C = \Re^m_+$.

There are many approaches including the scalarization and nonscalarization techniques to deal with multiobjective optimization problems. One traditional scalarization approach is the so-called weighted sum scalarization technique, which consists of solving the following problem

$$\min \ \langle z, \varphi(x) \rangle \quad \text{s.t. } x \in D, \tag{2.3}$$

where z is a nonnegative parameter in a unit sphere, i.e., $z \in Z$ where

$$Z := \{z \in \Re^m : z \geq 0, \sum_{i=1}^m z_i = 1\}. \tag{2.4}$$

We let $\mathcal{S}(z)$ denote the optimal solution set of problem (2.3). It is not hard to verify that when φ is a convex function and D is a convex set, the function value set $\varphi(D) := \{\varphi(x) : x \in D\}$ is

$\Re^m_+$-convex, i.e., $\varphi(D)+\Re^m_+$ is convex. Thus, by e.g., [13, Theorem 3.5], we have the following result that relates the single objective problem (2.3) with the multiobjective problem (2.1).

Proposition 2.1 *Assume that the objective function φ is convex and the constraint region D is convex. Then we have*

$$\mathcal{S}_{\text{wef}} = \mathcal{S}(Z) := \bigcup\{\mathcal{S}(z) : z \in Z\}.$$

3 Multiobjective generalized Nash equilibrium problem

The standard Nash equilibrium problem requires that each player has a feasible region that is independent of the rivals' strategies [21]. However, in many cases the interaction among the players can take place at the feasible set level [2, 20, 23]. When a player's feasible region is dependent of other players' strategies, this game is called the generalized Nash equilibrium problem (GNEP) by OR researchers in recent years [15]. Formally, the GNEP consists of K players, each player controlling the variables $y^k \in \Re^{m_k}$. We denote by y the vector formed by all decision variables: $y := (y^1, y^2, \dots, y^K)^T$ which has dimension $m = \sum_{k=1}^K m_k$, and by x^{-k} the decision variables from the k-th player's rivals. To emphasize the k-th player's variables within y, we sometimes write $y = (y^k, y^{-k})$ instead of y that is still the vector $(y^1, y^2, \dots, y^K)$. Each player has an objective $\theta^k : \Re^{m_k} \to \Re$, called utility function or payoff function, which depends on his own variables y^k as well as the variables y^{-k} of the other players. Moreover, each player's strategies belong to a set $Y^k(y^{-k})$ that depends on the rival players' strategies. The GNEP is such that for any given $k = 1, \dots, K$ and y^{-k}, the k-th player chooses y^k that solves

$$\min\ \theta^k(y^k, y^{-k}) \quad \text{s.t. } y^k \in Y(y^{-k}). \tag{3.1}$$

The solution set of problem (3.1) is denoted by $S^k(y^{-k})$ for any y^{-k}. We point out that if $Y(y^{-k})$ is independent of the rivals' strategies for any k, the GNEP reduces to the standard Nash equilibrium problem.

A generalized Nash equilibrium y^* is such that

$$y^k_* \in S^k(y^{-k}_*) \quad \forall k = 1, \dots, K.$$

In other words, an equilibrium is such that no player can decrease his objective by changing unilaterally y^k_* to any other feasible point.

In practice, all players may have multiple criterions when making decisions. Naturally, the GNEP can be easily extended to that for all $k = 1, \dots, K$ and y^{-k}, the k-th player solves the following problem

$$\text{“min”}\ \ f^k(y^k, y^{-k}) \quad \text{s.t. } y^k \in Y(y^{-k}), \tag{3.2}$$

where $f^k : \Re^{m_k} \to \Re^{p_k}$ is a vector-valued function instead of a scalar function. We denote by $\mathcal{S}^k_{\text{wef}}(y^{-k})$ the weakly efficient solution set of problem (3.2) for any y^{-k}. For notational simplicity, we let M-GNEP stand for the generalized Nash equilibrium problem with multiobjectives. The notion of equilibria of a Nash game in [21] can be easily extended to the M-GNEP as follows.

Definition 3.1 *A point $y_* = (y_*^1, \ldots, y_*^K)$ is called a weakly efficient generalized Nash equilibrium of the M-GNEP if*

$$y_*^k \in \mathcal{S}_{\mathrm{wef}}^k(y_*^{-k}) \quad \forall k = 1, \ldots, K. \tag{3.3}$$

We let $\mathcal{E}_{\mathrm{wef}}$ denote the set of the weakly efficient generalized Nash equilibria.

We now investigate how to transform the M-GNEP into a GNEP with a single objective. As done in Section 2, we transform problem (3.2) into the following scalarization problem:

$$\min \, \langle z^k, f^k(y^k, y^{-k}) \rangle \quad \text{s.t. } y^k \in Y(y^{-k}). \tag{3.4}$$

Here $z^k \in Z^k$ is a nonnegative parameter where Z^k is defined as in (2.4). Denote by $\mathcal{S}^k(y^{-k}, z^k)$ the solution set of problem (3.4) for any y^{-k} and z^k. Denote $z := (z^1, \ldots, z^K)$, $Z := Z^1 \times \cdots \times Z^K$, and

$$\mathcal{E}(z) := \{y^* : y_*^k \in \mathcal{S}^k(y_*^{-k}, z^k) \quad \forall k = 1, \ldots, K\}.$$

The following result relates the M-GNEP with a GNEP with a single objective.

Theorem 3.1 *Assume that for all $k = 1, \ldots, K$, $f^k(\cdot, y^{-k})$ is convex and $Y(y^{-k})$ is convex for any y^{-k}. Then we have*

$$\mathcal{E}_{\mathrm{wef}} = \bigcup \{\mathcal{E}(z) : z \in Z\}.$$

Proof. Let $y_* \in \mathcal{E}_{\mathrm{wef}}$. Then by the definition, we have $y_*^k \in \mathcal{S}_{\mathrm{wef}}^k(y_*^{-k})$ for all $k = 1, \ldots, K$. By Proposition 2.1, it follows that for all $k = 1, \ldots, K$, there exists $z^k \in Z^k$ such that $y_*^k \in \mathcal{S}^k(y_*^{-k}, z^k)$, which implies that $y_* \in \mathcal{E}(z)$.

Fix an arbitrary z satisfying $z^k \in Z^k$ for all $k = 1, \ldots, K$. Let $y_* \in \mathcal{E}(z)$. Then by the definition, we have $y_*^k \in \mathcal{S}^k(y_*^{-k}, z^k)$ for all $k = 1, \ldots, K$. By Proposition 2.1, it follows that $y_*^k \in \mathcal{S}_{\mathrm{wef}}^k(y_*^{-k})$ for all $k = 1, \ldots, K$. This implies that $y_* \in \mathcal{E}_{\mathrm{wef}}$. ∎

It is well known that a GNEP is associated with a so-called quasi variational inequality problem which is difficult to solve [17]. In 1965, Rosen considered a class of shared constraints games which is a special case of GNEPs. In Rosen's setting, a convex compact set $Y \subseteq \Re^m$ is given and, given y^{-k}, the k-th player's feasible region is given by

$$Y(y^{-k}) := \{y^k : (y^k, y^{-k}) \in Y\}. \tag{3.5}$$

This situation often occurs in practice such as when all the players share some resources that are available in limited amount.

Proposition 3.1 *Assume that for all $k = 1, \ldots, K$, f^k is convex in y^k for any given y^{-k}. Assume further that the set $Y(y^{-k})$ are defined by (3.5) with Y being convex and compact. Then a generalized Nash equilibrium exists for the scalarization of the M-GNEP for each $z \in Z$, i.e., $\mathcal{E}(z) \neq \emptyset \quad \forall z \in Z$.*

Proof. The convexity of f^k in y^k and the nonnegativeness of z^k imply that the product function $\langle z^k, f^k \rangle$ is also convex in y^k. Then the desired result follows immediately from [23, Theorem 1]. ■

4 Mathematical program with GNEPs in the constraints

Given a leader's decision variables $x \in X \subseteq \Re^n$, consider a K-person GNEP parameterized by x. In the setting of hierarchical problems, we call each player a follower. We assume that every follower has a vector-valued objective function $f^k : \Re^n \times \Re^m \to \Re^{p_k}$. For all $k = 1, \ldots, K$ and y^{-k}, the k-th follower solves an optimization problem parameterized by x and y^{-k}:

$$P(x, y^{-k}) \qquad \underset{y^k}{\text{“min”}} \quad f^k(x, y^k, y^{-k}) \qquad \text{s.t.} \quad y^k \in Y^k(x, y^{-k}) \tag{4.1}$$

where $Y^k(x, y^{-k})$ is a closed subset in $\Re^{m_k}$. This K-person GNEP is an M-GNEP parameterized by x. We let $\mathcal{E}_{\text{wef}}(x)$ the set of weakly efficient Nash equilibria at x. Then the mathematical program with GNEPs in the constraints considered in the paper is of the form

$$\begin{aligned} \underset{x}{\text{“min”}} \quad & F(x, y) \\ \text{s.t.} \quad & x \in X, \\ & y \in \mathcal{E}_{\text{wef}}(x), \end{aligned} \tag{4.2}$$

where x stands for the leader's decision variables, y stands for the followers' decision variables, and the vector-valued function $F : \Re^n \times \Re^m \to \Re^l$ denotes the leader's objective function. Let Ω denote the feasible region of problem (4.2), i.e.,

$$\Omega := \{(x, y) \in \Re^n \times \Re^m : x \in X, y \in \mathcal{E}_{\text{wef}}(x)\}.$$

This formulation (4.2) is a little ambiguous when the weakly efficient solution set $\mathcal{E}_{\text{wef}}(x)$ at the lower level is not unique. At least two ways out of this unpleasing situation can be proposed as what has done for bilevel programming [10]. The first is the so-called optimistic reformulation of problem (4.2)

$$\begin{aligned} \underset{x,y}{\text{“min”}} \quad & F(x, y) \\ \text{s.t.} \quad & x \in X, \\ & y \in \mathcal{E}_{\text{wef}}(x), \end{aligned} \tag{4.3}$$

and the second is the so-called pessimistic reformulation

$$\underset{x \in X}{\text{“min”}} \; \underset{y \in \mathcal{E}_{\text{wef}}(x)}{\text{“max”}} \; F(x, y). \tag{4.4}$$

For the optimistic formulation, the followers are assumed to choose in favor of the leader while the pessimistic formulation refers to the case where the leader protects himself/herself against the worst possible situation. In this paper, we focus on the optimistic formulation (4.3) since in this case it is possible to reformulate it as a multiobjective optimization problem with complementarity constraints as done in Subsection 4.1. We leave the investigation of the pessimistic formulation (4.4) for further research.

The notion of weakly efficient solutions for multiobjective optimization as introduced in Subsection 2 can be easily extended to problem (4.3).

Definition 4.1 *We say that* (x_*, y_*) *is a weakly efficient Stackelberg-Nash equilibrium (SNE) if* $x_* \in X,\ y_* \in \mathcal{E}_{\rm wef}(x_*)$ *and* (x_*, y_*) *is a weakly efficient solution of problem* (4.3)*, that is, there is no* $(x, y) \in \Omega$ *such that* $F(x, y) < F(x_*, y_*)$*.*

We say that (x_*, y_*) *is a locally weakly efficient SNE if* $x_* \in X, y_* \in \mathcal{E}_{\rm wef}(x_*)$*, and* (x_*, y_*) *is a locally weakly efficient solution of problem* (4.3)*, that is, there exists a neighborhood* V *of* (x_*, y_*) *such that there is no* $(x, y) \in \Omega \cap V$ *such that* $F(x, y) < F(x_*, y_*)$*.*

4.1 A single level reformulation

One efficient way to deal with hierarchical problems is to provide a single level reformulation. In the rest of this subsection, we investigate how to reformulate problem (4.3) as a single level problem. We need to describe the k-th player's feasible region more explicitly. Given any x and y^{-k}, we shall describe $Y^k(x, y^{-k})$ by means of the mapping $g^k(x, \cdot, y^{-k}) : \Re^{m_k} \to \Re^{q_k}$, i.e.,

$$Y^k(x, y^{-k}) = \{y^k : g^k(x, y^k, y^{-k}) \leq 0\}.$$

By Theorem 3.1, we have $\mathcal{E}_{\rm wef}(x) = \bigcup\{\mathcal{E}(x, z) : z \in Z\}$. Here

$$\mathcal{E}(x, z) := \{y : y^k \in \mathcal{S}^k(x, y^{-k}, z^k) \quad \forall k = 1, \ldots, K\}, \tag{4.5}$$

where $\mathcal{S}^k(x, y^{-k}, z^k)$ is the optimal solution set of the following scalarization problem

$$\begin{array}{ll} \min\limits_{y^k} & \langle z^k, f^k(x, y^k, y^{-k})\rangle \\ \text{s.t.} & y^k \in Y^k(x, y^{-k}). \end{array} \tag{4.6}$$

Hence we can replace problem (4.2) by the following multiobjective problem where each follower has a single objective:

$$\begin{array}{ll} \text{``}\min\limits_{x,y,z}\text{''} & F(x, y) \\ \text{s.t.} & x \in X,\ z \in Z, \\ & y \in \mathcal{E}(x, z). \end{array} \tag{4.7}$$

We denote by $\Omega^{\mathbf{s}}$ the feasible region of problem (4.7). Observe that the objective function $F(x, y)$ is independent of variables z. Thus, if (x_*, y_*, z_*) is a weakly efficient solution of problem (4.7) in the usual sense, then any feasible point $(x_*, y_*, z_0) \in \Omega^{\mathbf{s}}$ is also a weakly efficient solution of problem (4.7).

We next show the equivalence of problem (4.3) and problem (4.7) in the sense of global and local minimizers respectively. We suppose that the following assumption holds in the rest of this paper.

Assumption 4.1 *For all $k = 1, \ldots, K$ and $(x, y^{-k}) \in X \times \Re^{m-m_k}$, each component of $f^k(x, \cdot, y^{-k})$ and each component of $g^k(x, \cdot, y^{-k})$ are convex.*

We let $\Lambda^k(x, y_*^k, y^{-k}, z^k)$ denote the optimal multiplier set of problem (4.6) at y_*^k, i.e.,

$$\Lambda^k(x, y_*^k, y^{-k}, z^k) := \left\{ \lambda^k \in \Re_+^{p_k} : \begin{array}{c} \nabla_{y^k} f^k(x, y_*^k, y^{-k}) z^k + \nabla_{y^k} g^k(x, y_*^k, y^{-k}) \lambda^k = 0 \\ \langle g^k(x, y_*^k, y^{-k}), \lambda^k \rangle = 0, \ g^k(x, y_*^k, y^{-k}) \le 0 \end{array} \right\}.$$

Under Assumption 4.1, the fact $\Lambda^k(x, y_*^k, y^{-k}, z^k) \neq \emptyset$ implies that $y_*^k \in \mathcal{S}^k(x, y^{-k}, z^k)$. If the following Slater's condition holds as well, the converse is also true

Definition 4.2 *We say that Slater's condition hold at $x \in X$ if for each $k = 1, \ldots, K$, there exists $y^k(x, y^{-k}) \in \Re^{m_k}$ such that $g^k(x, y^k(x, y^{-k}), y^{-k}) < 0$.*

Lemma 4.1 *Let Slater's condition hold at x_*. Then the equilibrium solution mapping $\mathcal{E}$ defined in* (4.5) *is closed at (x_*, z_*, y_*) in the sense that if $(x_\nu, z_\nu) \to (x_*, z_*)$, $y_\nu \to y_*$ with $y_\nu \in \mathcal{E}(x_\nu, z_\nu)$, then $y^* \in \mathcal{E}(x_*, z_*)$. Moreover, if $y^* \in \mathcal{E}(x_*, z_*)$, then for each $k = 1, \ldots, K$, we have $\Lambda^k(x_*, y_*^k, y_*^{-k}, z_*^k) \neq \emptyset$.*

Proof. The relation $y_\nu \in \mathcal{E}(x_\nu, z_\nu)$ implies that $y_\nu^k \in \mathcal{S}^k(x_\nu, y_\nu^{-k}, z_\nu^k)$ for each $k = 1, \ldots, K$. Since Slater's condition holds at x_*, it then follows that the so-called MFCQ holds at y_*^k for the system $\{y^k : g^k(x, y^k, y^{-k}) \le 0\}$ when $x = x_*$ and $y^{-k} = y_*^{-k}$. Thus it follows from [16, Theorem 3.3] that $y_*^k \in \mathcal{S}^k(x_*, y_*^{-k}, z_*^k)$ for each $k = 1, \ldots, K$. This indicates that $y^* \in \mathcal{E}(x_*, z_*)$. We now show the second part. One can easily have that for each $k = 1, \ldots, K$, $y_*^k \in \mathcal{S}^k(x_*, y_*^{-k}, z_*^k)$ by the relation $y^* \in \mathcal{E}(x_*, z_*)$, and Slater's condition holds for the system $\{y^k : g^k(x_*, y^k, y_*^{-k}) \le 0\}$ by Slater's condition at x_*. Then by [5, Theorem 3.2.8], we have $\Lambda^k(x_*, y_*^k, y_*^{-k}, z_*^k) \neq \emptyset$ for each $k = 1, \ldots, K$. ■

The following results show the equivalence between problem (4.3) and problem (4.7). Denote $\mathcal{Z}(x, y) := \{z \in Z : y \in \mathcal{E}(x, z)\}$ for any $(x, y) \in X \times \Re^m$.

Theorem 4.1 *(i) The point (x_*, y_*) is a weakly efficient SNE of problem* (4.3) *if and only if $\mathcal{Z}(x_*, y_*) \neq \emptyset$ and for any $z_* \in \mathcal{Z}(x_*, y_*)$, the point (x_*, y_*, z_*) is a weakly efficient solution of problem* (4.7) *in the usual sense.*

(ii) The point (x_*, y_*) is a locally weakly efficient SNE of problem (4.3) *if and only if $\mathcal{Z}(x_*, y_*) \neq \emptyset$ and for any $z_* \in \mathcal{Z}(x_*, y_*)$, the point (x_*, y_*, z_*) is a locally weakly efficient solution of problem* (4.7) *in the usual sense.*

Proof. We only give the proof for case (ii). The proof for case (i) can be obtained similarly. We first show the necessity part. Let (x_*, y_*) be a locally weakly efficient SNE of problem (4.3). It follows that $x_* \in X$ and $y_* \in \mathcal{E}_{\rm wef}(x_*)$. By Theorem 3.1, there exists $z_* \in Z$ such that $y_* \in \mathcal{E}(x_*, z_*)$. Then we have $\mathcal{Z}(x_*, y_*) \neq \emptyset$. Moreover, it is easy to see that (x_*, y_*, z_*) is a feasible solution of problem (4.7), i.e., $(x_*, y_*, z_*) \in \Omega^{\mathbf{s}}$. Assume that for some $z_0 \in Z$ satisfying $y_* \in \mathcal{E}(x_*, z_0)$, the point (x_*, y_*, z_0) is not a locally weakly efficient solution of problem (4.7). Then there must exist a sequence $(x_\nu, y_\nu, z_\nu) \in \Omega^{\mathbf{s}}$ converging to (x_*, y_*, z_0) such that

$$F(x_\nu, y_\nu) < F(x_*, y_*). \tag{4.8}$$

We claim that $(x_\nu, y_\nu) \in \Omega$ for all ν. Since $(x_\nu, y_\nu, z_\nu) \in \Omega^{\mathbf{s}}$, it follows that $x_\nu \in X$, $z_\nu \in Z$ and $y_\nu \in \mathcal{E}(x_\nu, z_\nu)$. By Theorem 3.1, the relations $z_\nu \in Z$ and $y_\nu \in \mathcal{E}(x_\nu, z_\nu)$ imply that $y_\nu \in \mathcal{E}_{\rm wef}(x_\nu)$, suggesting that $(x_\nu, y_\nu) \in \Omega$ for all ν. This together with (4.8) contradicts the assumption that (x_*, y_*) is a locally weakly efficient SNE of problem (4.3).

We now show the sufficiency part. Let (x_*, y_*, z_*) be a locally weakly efficient solution of problem (4.7) for any given $z_* \in \mathcal{Z}(x_*, y_*)$. It follows that $x_* \in X, z_* \in Z$, and $y_* \in \mathcal{E}(x_*, z_*)$. By Theorem 3.1, one can easily have $y_* \in \mathcal{E}_{\rm wef}(x_*)$. Hence (x_*, y_*) is feasible to problem (4.3). Assume for contradiction that (x_*, y_*) is not a locally weakly efficient SNE of problem (4.3). Then there must exist a sequence $(x_\nu, y_\nu) \in \Omega$ converging to (x_*, y_*) such that (4.8) holds true. The relation $(x_\nu, y_\nu) \in \Omega$ implies that $x_\nu \in X$ and $y_\nu \in \mathcal{E}_{\rm wef}(x_\nu)$. Then by Theorem 3.1, there exists $z_\nu \in Z$ such that $y_\nu \in \mathcal{E}(x_\nu, z_\nu)$. Since Z is a bounded set, we may choose a subsequence such that $z_\nu \in Z$ converges to some $z_0 \in Z$ on the subsequence. Without loss of generality, we may assume that $z_\nu \to z_0$ as $k \to \infty$. By Lemma 4.1, it follows that the equilibrium solution mapping $\mathcal{E}$ is closed. This means that $y_* \in \mathcal{E}(x_*, z_0)$. This together with the fact that $x_* \in X$ by the closedness of X implies that $(x_*, y_*, z_0) \in \Omega^{\mathbf{s}}$. Combing all these facts, we may find a sequence $(x_\nu, y_\nu, z_\nu) \in \Omega^{\mathbf{s}}$ converging to $(x_*, y_*, z_0) \in \Omega^{\mathbf{s}}$ such that $F(x_\nu, y_\nu) < F(x_*, y_*)$, suggesting that (x_*, y_*, z_0) is not a locally weakly efficient SNE of problem (4.7). This is a contradiction. ∎

In the constraints of problem (4.7), there is still a GNEP and hence it is also intractable. By using the first-order conditions of problem (4.6), we may transform problem (4.7) into a multiobjective optimization problem with more tractable constraints:

$$\begin{array}{ll} \underset{x,y,z,\lambda}{\text{“min”}} & F(x,y) \\ \text{s.t.} & x \in X,\ z^k \in Z^k \quad \forall k = 1, \ldots, K, \\ & \nabla_{y^k} f^k(x,y) z^k + \nabla_{y^k} g^k(x,y) \lambda^k = 0 \quad \forall k = 1, \ldots, K, \\ & \lambda^k \geq 0,\ g^k(x,y) \leq 0,\ \langle \lambda^k, g^k(x,y) \rangle = 0 \quad \forall k = 1, \ldots, K. \end{array} \tag{4.9}$$

Problem (4.9) is a special case of the multiobjective optimization with equilibrium constraints considered in e.g, [26]. We denote by $\Omega^{\rm kkt}$ the feasible region of the above problem and

$$\Lambda(x,y,z) := \{(\lambda^1, \ldots, \lambda^K) : \lambda^k \in \Lambda^k(x, y^k, y^{-k}, z^k) \quad \forall k = 1, \ldots, K\}$$

The following theorem shows that problems (4.7) and (4.9) are equivalent in the sense of global solutions.

Theorem 4.2 *(i) If (x_*, y_*, z_*) is a weakly efficient solution of problem (4.7) and Slater's condition holds at x_*, then $\Lambda(x_*, y_*, z_*) \neq \emptyset$ and for any $\lambda_* \in \Lambda(x_*, y_*, z_*)$, the point $(x_*, y_*, z_*, \lambda_*)$ is a weakly efficient solution of problem (4.9).*

(ii) Let Slater's condition hold at each $x \in X$. If $(x_*, y_*, z_*, \lambda_*)$ is a weakly efficient solution of problem (4.9), then (x_*, y_*, z_*) is a weakly efficient solution of problem (4.7).

Proof. (i) Let (x_*, y_*, z_*) be a weakly efficient solution of problem (4.7). It follows that $y^* \in \mathcal{E}(x_*, z_*)$. This together with Lemma 4.1 implies that $\Lambda^k(x_*, y_*^k, y_*^{-k}, z_*^k) \neq \emptyset$ for each $k = 1, \ldots, K$. Equivalently, we have $\Lambda(x_*, y_*, z_*) \neq \emptyset$. Let $\lambda_* \in \Lambda(x_*, y_*, z_*)$. It is easy to see that $(x_*, y_*, z_*, \lambda_*) \in \Omega^{\mathrm{kkt}}$. Assume for contradiction that $(x_*, y_*, z_*, \lambda_*)$ is not a weakly efficient solution of problem (4.9). Then there is a point $(x_0, y_0, z_0, \lambda_0) \in \Omega^{\mathrm{kkt}}$ such that $F(x_0, y_0) < F(x_*, y_*)$. By Assumption 4.1, the relation $(x_0, y_0, z_0, \lambda_0) \in \Omega^{\mathrm{kkt}}$ implies that $y_0^k \in \mathcal{S}^k(x_0, y_0^{-k}, z_0^k)$ for each $k = 1, \ldots, K$, suggesting that $y_0 \in \mathcal{E}(x_0, z_0)$. Thus, we have $(x_0, y_0, z_0) \in \Omega^{\mathbf{s}}$. This together with the relation $F(x_0, y_0) < F(x_*, y_*)$ contradicts the assumption that (x_*, y_*, z_*) is a weakly efficient solution of problem (4.7).

(ii) Let $(x_*, y_*, z_*, \lambda_*)$ be a weakly efficient solution of problem (4.9). By Assumption 4.1, it is easy to verify that $(x_*, y_*, z_*) \in \Omega^{\mathbf{s}}$ is a feasible solution of problem (4.7). Assume for contradiction that (x_*, y_*, z_*) is not a weakly efficient solution of problem (4.7). Then there exists a point $(x_0, y_0, z_0) \in \Omega^s$ such that $F(x_0, y_0) < F(x_*, y_*)$. The relation $(x_0, y_0, z_0) \in \Omega^s$ means that $y_0 \in \mathcal{E}(x_0, z_0)$. Then by Lemma 4.1, there exists $\lambda_0^k \in \Lambda^k(x_0, y_0^k, y_0^{-k}, z_0^k)$ for each $k = 1, \ldots, K$. Thus we have $(x_0, y_0, z_0, \lambda_0) \in \Omega^{\mathrm{kkt}}$. This together with the relation $F(x_0, y_0) < F(x_*, y_*)$ contradicts the assumption that $(x_*, y_*, z_*, \lambda_*)$ is a weakly efficient solution of problem (4.9). ■

For nonconvex programming problems including problems with hierarchical structure as special cases, it is very difficult to find a global solution and in many practical cases one needs to be happy with just obtaining a local solution. Thus for our problem considered, it is important to investigate the relation between locally weakly efficient solutions of problem (4.7) and problem (4.9).

Theorem 4.3 *Let Slater's condition hold at x_*. The point (x_*, y_*, z_*) is a locally weakly efficient solution of problem (4.7) if and only if $\Lambda(x_*, y_*, z_*) \neq \emptyset$ and for any $\lambda_* \in \Lambda(x_*, y_*, z_*)$, the point $(x_*, y_*, z_*, \lambda_*)$ is a locally weakly efficient solution of problem (4.9).*

Proof. We first show the necessity part. As done in the first part of the proof in Theorem 4.2, we have $\Lambda(x_*, y_*, z_*) \neq \emptyset$ and $(x_*, y_*, z_*, \lambda_*) \in \Omega^{\mathrm{kkt}}$ is feasible to problem (4.9) for any $\lambda_* \in \Lambda(x_*, y_*, z_*)$. Assume for contradiction that $(x_*, y_*, z_*, \lambda_*)$ is not a locally weakly efficient solution of problem (4.9). Then there must exist a sequence $(x_\nu, y_\nu, z_\nu, \lambda_\nu) \in \Omega^{\rm kkt}$ converging to $(x_*, y_*, z_*, \lambda_*)$ such that $F(x_\nu, y_\nu) < F(x_*, y_*)$. By Assumption 4.1, the relation $(x_\nu, y_\nu, z_\nu, \lambda_\nu) \in \Omega^{\rm kkt}$ implies that $(x_\nu, y_\nu, z_\nu) \in \Omega^{\bf s}$. This together with the relation $F(x_\nu, y_\nu) < F(x_*, y_*)$ contradicts the assumption that (x_*, y_*, z_*) is a locally weakly efficient solution of problem (4.7).

We now show the sufficiency part. As done in the second part of the proof in Theorem 4.2, we can show that $(x_*, y_*, z_*) \in \Omega^{\bf s}$. Assume for contradiction that (x_*, y_*, z_*) is not a locally weakly efficient solution of problem (4.7). Then there must exist a sequence $(x_\nu, y_\nu, z_\nu) \in \Omega^{\bf s}$ converging to (x_*, y_*, z_*) such that $F(x_\nu, y_\nu) < F(x_*, y_*)$. The relation $(x_\nu, y_\nu, z_\nu) \in \Omega^{\bf s}$ means that $y_\nu^k \in \mathcal{S}^k(x_\nu, y_\nu^{-k}, z_\nu^k)$ for each $k = 1, \ldots, K$. By Slater's condition at x_*, it is easy to see that the MFCQ holds at y_*^k for the system $\{y^k : g^k(x, y^k, y^{-k}) \le 0\}$ when $x = x_*$ and $y^{-k} = y_*^{-k}$. Then by [16, Theorem 3.4], there exists a sequence $\lambda_\nu^k \in \Lambda^k(x_\nu, y^k, y_\nu^{-k}, z_\nu^k)$ that is convergent on some subsequence and the accumulation point λ_0^k is in $\Lambda^k(x_*, y_*^k, y_*^{-k}, z_*^k)$. Combing all these facts, we find a subsequence $(x_\nu, y_\nu, z_\nu, \lambda_\nu) \in \Omega^{\rm kkt}$ converging to $(x_*, y_*, z_*, \lambda_0) \in \Omega^{\rm kkt}$ such that $F(x_\nu, y_\nu) < F(x_*, y_*)$. This contradicts the assumption that $(x_*, y_*, z_*, \lambda_0)$ is a locally weakly efficient solution of problem (4.9). ■

Combing Theorem 4.1 with Theorems 4.2–4.3, we derive our main results of this paper which show that problem (4.3) and problem (4.9) are equivalent in the sense of global and local minimizers respectively.

Theorem 4.4 *(i) Let (x_*, y_*) be a weakly efficient SNE of problem (4.3) and Slater's condition hold at x_*. Then for any $z_* \in \mathcal{Z}(x_*, y_*)$ and for any $\lambda_* \in \Lambda(x_*, y_*, z_*)$, the point $(x_*, y_*, z_*, \lambda_*)$ is a weakly efficient solution of problem (4.9). Conversely, assume that Slater's condition holds at each $x \in X$ and $(x_*, y_*, z_*, \lambda_*)$ is a weakly efficient solution of problem (4.9), then (x_*, y_*) is a weakly efficient SNE of problem (4.3).*

(ii) Let Slater's condition hold at x_*. The point (x_*, y_*) is a locally weakly efficient SNE of problem (4.3) if and only if $\mathcal{Z}(x_*, y_*) \neq \emptyset$ and for any $z^* \in \mathcal{Z}(x_*, y_*)$ and any $\lambda_* \in \Lambda(x_*, y_*, z_*)$, the point $(x_*, y_*, z_*, \lambda_*)$ is a locally weakly efficient solution of problem (4.9).

Proof. (i) Since the objective function F of problem (4.9) is independent of (z, λ), we observe that if $(x_*, y_*, z_*, \lambda_*)$ is a weakly efficient solution of problem (4.9), then $\Lambda(x_*, y_*, z_*) \neq \emptyset$, $\mathcal{Z}(x_*, y_*) \neq \emptyset$ by Assumption 4.1, and for any $\bar{z} \in \mathcal{Z}(x_*, y_*)$ and $\bar{\lambda} \in \Lambda(x_*, y_*, z_*)$, the point $(x_*, y_*, \bar{z}, \bar{\lambda})$ is a weakly efficient solution of problem (4.9). Then the result follows from Theorem 4.1(i) and Theorem (4.2) immediately.

(ii) We note that when $\mathcal{Z}(x_*, y_*) \neq \emptyset$, by Slater's condition at x_* and Lemma 4.1, we have $\Lambda(x_*, y_*, z_*) \neq \emptyset$. Then this result follows from Theorem 4.1(ii) and Theorem (4.3) immediately. ■

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