

Proximal splitting algorithms: Relax them all!

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Abstract

Convex optimization problems, whose solutions live in very high dimensional spaces, have become ubiquitous. To solve them, proximal splitting algorithms are particularly adequate: they consist of simple operations, by handling the terms in the objective function separately. We present several existing proximal splitting algorithms and we derive new ones, within a unified framework, which consists in applying splitting methods for monotone inclusions, like the forward–backward algorithm, in primal–dual product spaces with well-chosen metric. This allows us to derive new convergence theorems with larger parameter ranges. In particular, when the smooth term in the objective function is quadratic, e.g. for least-squares problems, convergence is guaranteed with larger values of the relaxation parameter than previously known. Indeed, it is often the case in practice that the larger the relaxation parameter, the faster the convergence.

1 Introduction

Many problems in statistics, machine learning, signal and image processing, or control can be formulated as convex optimization problems [1–7]. In the age of ‘big data’, with the explosion in size and complexity of the data to process, it is increasingly important to be able to solve convex optimization problems, whose solutions live in very high dimensional spaces [8–12]. There is an extensive literature about *proximal splitting* algorithms for solving convex optimization problems, with applications in various fields [9, 13–20]. They consist of simple, easy to compute, steps that can deal with the terms in the objective function separately. In this paper, we present several existing proximal splitting algorithms, which are more or less known, and we derive new ones, within a unified framework, which consists in applying splitting methods for monotone inclusions, like the forward–backward algorithm, in well-chosen primal–dual product spaces, with preconditioning.

In addition, we show that, if the optimization problem involves a smooth term, which is quadratic, convergence is guaranteed with larger values of the relaxation parameter than previously known. By relaxation, we mean the following: let us consider an iterative algorithm of the form $z^{(i+1)} = T(z^{(i)})$, for some operator T , which converges to some fixed point and solution z^* . Then $z^{(i+1)}$ tends to

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be closer than $z^{(i)}$ to z^\star , on average. Hence, we may want, starting at $z^{(i)}$, to move further in the direction $z^{(i+1)} - z^{(i)}$, which improves the estimate. This yields the *relaxed* iteration:

$$\begin{cases} z^{(i+\frac{1}{2})} = T(z^{(i)}) \\ z^{(i+1)} = z^{(i)} + \rho^{(i)}(z^{(i+\frac{1}{2})} - z^{(i)}). \end{cases} \quad (1)$$

While there is no interest in doing underrelaxation with $\rho^{(i)}$ less than 1, overrelaxation with $\rho^{(i)}$ larger than 1 may be beneficial to the convergence speed, as often observed in practice. Thus, in this paper, we emphasize the possibility to overrelax most, if not all, proximal splitting algorithms.

The paper is organized as follows: in Section 2, we present relaxed versions of the forward–backward splitting algorithm to solve monotone inclusions. This analysis is subsequently applied in Section 3 to the Loris–Verhoeven algorithm, which is a primal–dual forward–backward algorithm. In Section 4, we analyse the Chambolle–Pock algorithm and its particular case the Douglas–Rachford algorithm, which is shown to be equivalent to the Alternating Direction Method of Multipliers (ADMM). For them too, we derive convergence results for relaxed versions, which extend previously known results. In Section 5, we study an algorithm, which we call the Generalized Chambolle–Pock algorithm, since it is applicable to the minimization of two functions composed with two linear operators, and reverts back to the Chambolle–Pock algorithm when one the linear operator is the identity. The algorithm is not new and has been presented in the literature as a linearized or preconditionned version of the ADMM, but without relaxation. In Section 6, we study an algorithm proposed independently by the first author and by B. C. Vũ, which is also a primal–dual forward–backward algorithm. Section 7 is devoted to recently proposed algorithms based on a three-operator splitting scheme, which can be viewed as the fusion of the forward–backward and Douglas–Rachford two-operator splitting scheme. Finally, in Section 8, we propose parallel versions of all these algorithms, by applying them in product spaces. Along the way, in Sections 6.1 and 7.1, we capitalize on the gained insights to propose new algorithms.

1.1 Notions of convex analysis

Let us introduce some notions and notations, which will be used throughout the paper. For more notions of convex analysis and operator theory, we refer the reader to the textbooks [21, 22] or to the papers [17, 23]. For the rest of this section, let \mathcal{X} be a real Hilbert space, with its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$. Let $f : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function. The domain of f is the convex set $\text{dom } f = \{x \in \mathcal{X} : f(x) \neq +\infty\}$. f is said proper if its domain is nonempty and lower semicontinuous if the convex set $\{x \in \mathcal{X} : f(x) \leq \alpha\}$ is closed, for every $\alpha \in \mathbb{R}$. We denote by $\Gamma_0(\mathcal{X})$ the set of convex, proper, lower semicontinuous functions from \mathcal{X} to $\mathbb{R} \cup \{+\infty\}$ [22]. We define the subdifferential of f as the set-valued operator $\partial f : x \in \mathcal{X} \mapsto \{u \in \mathcal{X} : (\forall y \in \mathcal{X}) f(x) + \langle y - x, u \rangle \leq f(y)\}$. If f is differentiable at $x \in \mathcal{X}$, $\partial f(x) = \{\nabla f(x)\}$, where $\nabla f(x)$ denotes the gradient of f at x . We will consider several instances of the generic convex optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \sum_{m=1}^M g_m(L_m x), \quad (2)$$

where $M \geq 1$, each L_m is a bounded linear operator from \mathcal{X} to some real Hilbert space \mathcal{U}_m , possibly the identity Id if $\mathcal{U}_m = \mathcal{X}$, and each $g_m \in \Gamma_0(\mathcal{U}_m)$. From the well known Fermat rule [22, Theorem 16.3], $x \in \mathcal{X}$ is a solution to the optimization problem (2) if and only if¹

$$0 \in \partial \left(\sum_{m=1}^M g_m \circ L_m \right) (x), \quad (3)$$

¹Fermat’s rule is easy to demonstrate: let $f \in \Gamma_0(\mathcal{X})$ and $x^\star \in \mathcal{X}$. Then $x^\star \in \arg \min_{x \in \mathcal{X}} f(x) \Leftrightarrow \forall x \in \mathcal{X}, f(x^\star) \leq f(x) \Leftrightarrow \forall x \in \mathcal{X}, f(x^\star) + \langle x - x^\star, 0 \rangle \leq f(x) \Leftrightarrow 0 \in \partial f(x^\star)$.

where \circ denotes the composition of functions or operators. For every $x \in \mathcal{X}$, we have

$$\left(\sum_{m=1}^M L_m^* \circ \partial g_m \circ L_m \right)(x) \subset \partial \left(\sum_{m=1}^M g_m \circ L_m \right)(x), \quad (4)$$

but the inclusion may be strict. All the algorithms considered in the paper will actually find a solution to (2) by finding $x \in \mathcal{X}$ such that

$$0 \in \sum_{m=1}^M L_m^* \partial g_m(L_m x). \quad (5)$$

Therefore, throughout the paper, we assume that for every optimization problem considered, **the solution set of the corresponding monotone inclusion (5) is nonempty**. Given (4), this assumption implies that the solution set of the optimization problem (2) is nonempty. If the functions satisfy a so-called *qualification constraint*, we have an equality instead of an inclusion in (4). In that case, it is sufficient to show that a solution to (2) exists, since the solution sets to (2) and (5) are the same. A qualification constraint is [24, Proposition 4.3]:

$$\bigcap_{m=1}^M L_m^{-1}(\text{sri dom } g_m) \neq \emptyset, \quad (6)$$

where, for any linear operator $L : \mathcal{X} \rightarrow \mathcal{U}$ and $\Omega \subset \mathcal{U}$, $L^{-1}(\Omega) = \{x \in \mathcal{X} : Lx \in \Omega\}$, and $\text{sri } \Omega$ denotes the strong relative interior of a convex set Ω , which is the set of points $u \in \Omega$ such that the cone generated by $-u + \Omega$ is a closed linear subspace. Other qualification constraints can be found in [24, Proposition 4.3]. Note that for $\sum_{m=1}^M g_m \circ L_m$ to be proper, it is necessary to have $\bigcap_{m=1}^M L_m^{-1}(\text{dom } g_m) \neq \emptyset$. Since the strong relative interior is even larger than the interior of a set, the condition (6) is really not a lot to ask.

We denote by 2^Ω the set of all subsets of a set Ω . Let $M : \mathcal{X} \rightarrow 2^\mathcal{X}$ be a set-valued operator. The graph of M is the set $\{(x, v) \in \mathcal{X}^2 : v \in Mx\}$. The inverse operator of M , denoted by M^{-1} , is the set-valued operator with graph $\{(v, x) \in \mathcal{X}^2 : v \in Mx\}$. M is said to be monotone if for every $(x, x') \in \mathcal{X}^2$ and every $v \in Mx$ and $v' \in Mx'$, $\langle x - x', v - v' \rangle \geq 0$. A monotone operator M is said to be maximally monotone if there exists no monotone operator whose graph strictly contains the graph of M . Let \mathcal{U} be a real Hilbert space. The adjoint operator $L^* : \mathcal{U} \rightarrow \mathcal{X}$ of a linear operator $L : \mathcal{X} \rightarrow \mathcal{U}$ is the only linear operator such that $\langle Lx, u \rangle_{\mathcal{U}} = \langle x, L^*u \rangle_{\mathcal{X}}$, for every $x \in \mathcal{X}$ and $u \in \mathcal{U}$, where $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ denote the inner products in \mathcal{X} and \mathcal{U} , respectively. L is said to be self-adjoint if $L^* = L$. A self-adjoint linear operator P on \mathcal{X} is said to be positive if $\langle x, Px \rangle \geq 0$, for every $x \in \mathcal{X}$, and strongly positive if $P - a\text{Id}$ is positive, for some real $a > 0$.

For any function $f \in \Gamma_0(\mathcal{X})$, we define its proximity operator [25] as

$$\text{prox}_f : \mathcal{X} \rightarrow \mathcal{X}, x \mapsto \arg \min_{x' \in \mathcal{X}} \left\{ f(x') + \frac{1}{2} \|x - x'\|^2 \right\}. \quad (7)$$

Then $\text{prox} = J_{\partial f} = (\partial f + \text{Id})^{-1}$, where $J_N = (N + \text{Id})^{-1}$ denotes the resolvent of an operator N . There are fast and exact methods to compute the proximity operators for a large class of functions [13, 26–30], see also the website <http://proximity-operator.net>. In this paper, we consider *proximal splitting algorithms*, which are iterative algorithms producing an estimate of a solution to a convex optimization problem of the form (2), by calling at each iteration the proximity operator or the gradient of each function g_m , as well as the operators L_m or their adjoint operators L_m^* . For every $f \in \Gamma_0(\mathcal{X})$, we denote by f^* the conjugate of f , defined by $f^* : x \mapsto \sup_{x' \in \mathcal{X}} \{\langle x, x' \rangle - f(x')\}$, which belongs to $\Gamma_0(\mathcal{X})$. The following hold: $f^{**} = f$ and $\partial f^* = (\partial f)^{-1}$. We also recall the Moreau identity, which allows us to compute the proximity operator of f^* from the one of f , and conversely: for every $\tau > 0$,

$$\text{prox}_{\tau f}(x) = x - \tau \text{prox}_{f^*/\tau}(x/\tau). \quad (8)$$

Let $T : \mathcal{X} \rightarrow \mathcal{X}$ be a single-valued operator. T is said to be β -Lipschitz continuous, for some real $\beta \geq 0$, if

$$\|Tx - Tx'\| \leq \beta \|x - x'\|, \text{ for every } (x, x') \in \mathcal{X}^2 \quad (9)$$

(we may use the shortened notation Tx for $T(x)$, even if T is a nonlinear operator). T is said to be nonexpansive if it is 1-Lipschitz continuous. T is said to be α -averaged, for some $\alpha \in (0, 1)$, if there exists a nonexpansive operator $T' : \mathcal{X} \rightarrow \mathcal{X}$ such that $T = (1 - \alpha)\text{Id} + \alpha T'$. Firmly nonexpansive means $(1/2)$ -averaged. T is said to be ξ -cocoercive, for some real $\xi > 0$, if ξT is firmly nonexpansive; or, equivalently, if

$$\xi \|Tx - Tx'\|^2 \leq \langle x - x', Tx - Tx' \rangle, \quad (10)$$

for every $(x, x') \in \mathcal{X}^2$ [22, Proposition 4.4]. The resolvent J_M of a maximally monotone operator $M : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ is defined and single-valued on \mathcal{X} and firmly nonexpansive. The subdifferential ∂f of a function $f \in \Gamma_0(\mathcal{X})$ is maximally monotone, so that $J_{\partial f} = \text{prox}_f$ is firmly nonexpansive.

All algorithms presented in the paper are iterative algorithms having the simple form (1). Note that the variable z in (1) may not be the variable converging to a solution of the optimization problem at hand: it can be a concatenation of variables, like $z^{(i)} = (x^{(i)}, u^{(i)})$, with $x^{(i)}$ converging to a solution and $u^{(i)}$ an auxiliary variable, the value of which is not of primary concern. Or it can be that there is an internal variable $x^{(i)}$, among the set of operations modeled by the abstract operator T , which converges to a solution, whereas $z^{(i)}$ is not interesting in itself. In any case, in this paper, we rely on the following theorem to establish convergence of the algorithms:

Krasnosel'skiĭ–Mann theorem for (1) [22, Proposition 5.16] Let \mathcal{X} be a real Hilbert space and let T be an α -averaged operator on \mathcal{X} , for some $\alpha \in (0, 1)$, with a nonempty set of fixed points. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 1/\alpha]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(1 - \alpha\rho^{(i)}) = +\infty$. Let $z^{(0)} \in \mathcal{X}$. Then the sequence $(z^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (1) converges weakly to a fixed point of T .

2 The forward–backward algorithm

Let \mathcal{X} be a real Hilbert space. Let $M : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a set-valued maximally monotone operator and let $C : \mathcal{X} \rightarrow \mathcal{X}$ be a ξ -cocoercive operator, for some real $\xi > 0$. Let us consider the monotone inclusion

$$0 \in Mz + Cz, \quad (11)$$

whose solution set is supposed nonempty. Let $z^{(0)} \in \mathcal{X}$ be some initial estimate of a solution and let $\gamma > 0$ be some real parameter. The classical forward–backward iteration, to find a solution of (11), consists in iterating

$$z^{(i+1)} = J_{\gamma M}(z^{(i)} - \gamma Cz^{(i)}). \quad (12)$$

This method, proposed by Mercier [31], was further developed by many authors [32–38].

A not-so-well known extension of the forward–backward iteration consists in *relaxing* it. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of relaxation parameters. The iteration becomes:

Forward–backward iteration for (11): for $i = 0, 1, \dots$

$$\begin{cases} z^{(i+\frac{1}{2})} = J_{\gamma M}(z^{(i)} - \gamma Cz^{(i)}) \\ z^{(i+1)} = z^{(i)} + \rho^{(i)}(z^{(i+\frac{1}{2})} - z^{(i)}). \end{cases} \quad (13)$$

To our knowledge, Theorem 25.8 of [39] is the first convergence result about the relaxed forward–backward algorithm, with a smaller relaxation range than in Theorem 2.1 below, though.

We can remark that the explicit mapping from $z^{(i)}$ to $z^{(i+\frac{1}{2})}$ in (13) can be equivalently written under the implicit form:

$$0 \in Mz^{(i+\frac{1}{2})} + Cz^{(i)} + \frac{1}{\gamma}(z^{(i+\frac{1}{2})} - z^{(i)}). \quad (14)$$

The now standard convergence result for the forward–backward iteration is the following:

Theorem 2.1 (forward–backward algorithm (13)) Let $z^{(0)} \in \mathcal{X}$. Let $\gamma \in (0, 2\xi)$. Set $\delta = 2 - \gamma/(2\xi)$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequence $(z^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (13) converges weakly to a solution of (11).

This result is Lemma 4.4 in [40], see also Theorem 26.14 in [22]. It follows from the Krasnosel’skiĭ–Mann theorem and the fact that the operator $T = (\gamma M + \text{Id})^{-1} \circ (\text{Id} - \gamma C)$, which maps $z^{(i)}$ to $z^{(i+\frac{1}{2})}$, is $(1/\delta)$ -averaged. Moreover, the fixed points of T are the solutions to (11), as is clearly visible in (14).

We can allow $\gamma = 2\xi$ in Theorem 2.1, but in this case $\delta = 1$ and we cannot set $\rho^{(i)} = 1$. Since we are interested in overrelaxation, we write all theorems such that the default choice $\rho^{(i)} = 1$ is allowed.

We can note that in this algorithm, like in every algorithm of the paper, the sequence $(z^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ converges to the same solution as $(z^{(i)})_{i \in \mathbb{N}}$. For any i , $z^{(i+\frac{1}{2})}$ may be a better estimate of the solution than $z^{(i)}$, since it is in the domain of M ; that is, $Mz^{(i+\frac{1}{2})} \neq \emptyset$.

We can also remark that if γ is close to 2ξ , δ is close to 1, so that overrelaxation cannot be used. This explains why the relaxed forward–backward iteration is not so well known.

Now, let P be a self-adjoint, strongly positive, bounded linear operator on \mathcal{X} . Clearly, solving (11) is equivalent to solving

$$0 \in P^{-1}Mz + P^{-1}Cz. \quad (15)$$

Let \mathcal{X}_P be the Hilbert space obtained by endowing \mathcal{X} with the inner product $(x, x') \mapsto \langle x, x' \rangle_P = \langle x, Px' \rangle$. Then $P^{-1}M$ is maximally monotone in \mathcal{X}_P [22, Proposition 20.24]. However, the cocoercivity of $P^{-1}C$ in \mathcal{X}_P has to be checked on a case-by-case basis. The *preconditioned* forward–backward iteration to solve (15) is

$$\begin{aligned} & \text{Preconditioned forward–backward iteration for (11): for } i = 0, 1, \dots \\ & \left[\begin{array}{l} z^{(i+\frac{1}{2})} = (P^{-1}M + \text{Id})^{-1}(z^{(i)} - P^{-1}Cz^{(i)}) \\ z^{(i+1)} = z^{(i)} + \rho^{(i)}(z^{(i+\frac{1}{2})} - z^{(i)}). \end{array} \right. \end{aligned} \quad (16)$$

The corresponding convergence result follows:

Theorem 2.2 (preconditioned forward–backward algorithm (16)) Suppose that $P^{-1}C$ is χ -cocoercive in \mathcal{X}_P , for some $\chi > \frac{1}{2}$. Set $\delta = 2 - 1/(2\chi)$. Let $z^{(0)} \in \mathcal{X}$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequence $(z^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (16) converges weakly to a solution of (11).

Proof This is a straightforward application of Theorem 2.1 in \mathcal{X}_P instead of \mathcal{X} , with $\gamma = 1$. Weak convergence in \mathcal{X}_P is equivalent to weak convergence in \mathcal{X} , and the solution sets of (11) and (15) are the same. \square

We can remark that the explicit mapping from $z^{(i)}$ to $z^{(i+\frac{1}{2})}$ in (16) can be equivalently written under the implicit form:

$$0 \in Mz^{(i+\frac{1}{2})} + Cz^{(i)} + P(z^{(i+\frac{1}{2})} - z^{(i)}). \quad (17)$$

If $C = 0$, the forward–backward iteration reduces to the proximal point algorithm [22, 41]. Let us give a formal convergence statement for the preconditioned proximal point algorithm. Let $M : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ be a maximally monotone operator. The problem is to solve

$$0 \in Mz, \quad (18)$$

whose solution set is supposed nonempty. Let $z^{(0)} \in \mathcal{X}$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of relaxation parameters. The relaxed and preconditioned proximal point algorithm consists in the iteration:

Proximal point iteration for (18): for $i = 0, 1, \dots$

$$\begin{cases} z^{(i+\frac{1}{2})} = (P^{-1}M + \text{Id})^{-1}z^{(i)} \\ z^{(i+1)} = z^{(i)} + \rho^{(i)}(z^{(i+\frac{1}{2})} - z^{(i)}). \end{cases} \quad (19)$$

The mapping from $z^{(i)}$ to $z^{(i+\frac{1}{2})}$ in (19) can be equivalently written as

$$0 \in Mz^{(i+\frac{1}{2})} + P(z^{(i+\frac{1}{2})} - z^{(i)}). \quad (20)$$

The convergence of the proximal point algorithm can be stated as follows:

Theorem 2.3 (proximal point algorithm (19)) Let $z^{(0)} \in \mathcal{X}$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequence $(z^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (19) converges weakly to a solution of (18).

Proof $P^{-1}M$ is maximally monotone in \mathcal{X}_P [22, Proposition 20.24], so its resolvent $(P^{-1}M + \text{Id})^{-1}$ is firmly nonexpansive in \mathcal{X}_P . By virtue of the Krasnosel'skiĭ–Mann theorem, $(z^{(i)})_{i \in \mathbb{N}}$ converges weakly in \mathcal{X}_P to some element $z^* \in \mathcal{X}$ with $0 \in P^{-1}Mz^*$, so that $0 \in Mz^*$. \square

2.1 The case where C is affine

Let us continue with our analysis of the forward–backward iteration to solve (11). In this section, we suppose that, in addition to being ξ -cocoercive, C is affine; that is,

$$C : z \in \mathcal{X} \mapsto Qz + c, \quad (21)$$

for some self-adjoint, positive, nonzero, bounded linear operator Q on \mathcal{X} and some element $c \in \mathcal{X}$. We have $\xi = 1/\|Q\|$, where the operator norm of Q is $\|Q\| = \sup \{\|Qz\| : z \in \mathcal{X}, \|z\| \leq 1\}$. Now, we can write (14) as

$$0 \in (M + C)z^{(i+\frac{1}{2})} + P(z^{(i+\frac{1}{2})} - z^{(i)}), \quad (22)$$

where

$$P = \frac{1}{\gamma} \text{Id} - Q. \quad (23)$$

Therefore, the forward–backward iteration (13) can be interpreted as a preconditioned proximal point iteration (19), applied to find a zero of $M + C$! Since P must be strongly positive, we must have $\gamma \in (0, \xi)$, so that the admissible range for γ is halved. But in return, we get the larger range $(0, 2)$ for relaxation. Hence, we have the following new convergence result:

Theorem 2.4 (forward–backward algorithm (13), affine case) Let $z^{(0)} \in \mathcal{X}$, let $\gamma \in (0, \xi)$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequence $(z^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (13) converges weakly to a solution of (11).

Proof In view of (20) and (22), this is Theorem 2.3 applied to the problem (18) with $M + C$ as the maximally monotone operator. \square

Now, let us improve the range $(0, \xi)$ for γ in Theorem 2.4 into $(0, \xi]$. Let $\gamma \in (0, \xi]$. We first recall that for every self-adjoint, positive, bounded linear operator P on \mathcal{X} , there exists a unique self-adjoint, positive, bounded linear operator on \mathcal{X} , denoted by \sqrt{P} and called the positive square root of P , such that $\sqrt{P}\sqrt{P} = P$. So, let A be a bounded linear operator from \mathcal{X} to some real Hilbert space \mathcal{A} , such that $A^*A = Q$. $A = \sqrt{Q}$ is a valid choice, but in some cases, $Q = A^*A$ for some A in the first place, for instance in least-squares problems, see eq. (37). In any case, we do not need to exhibit A , the fact that it exists is sufficient here. Furthermore, let B be a bounded linear operator from some

real Hilbert space \mathcal{B} to \mathcal{A} , such that $AA^* + BB^* = (1/\gamma)\text{Id}$. Since $\|AA^*\| = \|A^*A\| = \|\mathcal{Q}\| = 1/\xi \leq 1/\gamma$, $(1/\gamma)\text{Id} - AA^*$ is positive and we can choose $B = \sqrt{(1/\gamma)\text{Id} - AA^*}$. Again, we do not have to exhibit B , the fact that it exists is sufficient. Then we can rewrite the problem (11) as finding a pair $w = (z, b) \in \mathcal{W} = \mathcal{X} \times \mathcal{B}$ solution to the monotone inclusion

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} A^*(Az + Bb) \\ B^*(Az + Bb) \end{pmatrix}}_{S w} + \underbrace{\begin{pmatrix} Mz + c \\ (\mathcal{B} \text{ if } b = 0, \emptyset \text{ else}) \end{pmatrix}}_{N w}. \quad (24)$$

The operators S and N defined in (24) are maximally monotone in \mathcal{W} . To solve a monotone inclusion involving two monotone operators, the Douglas–Rachford splitting algorithm [32, 37, 42, 43] is a natural choice. So, let us consider a general monotone inclusion in some arbitrary real Hilbert space \mathcal{W} , with nonempty solution set:

$$0 \in S w + N w, \quad (25)$$

for some maximally monotone operators S and N on \mathcal{W} . Let $w^{(0)} \in \mathcal{W}$ and $y^{(0)} \in \mathcal{W}$, let $\tau > 0$ be a real parameter and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of relaxation parameters. The Douglas–Rachford iteration can be written in several ways; one of them is the following:

Douglas–Rachford iteration for (25): for $i = 0, 1, \dots$

$$\begin{cases} y^{(i+\frac{1}{2})} = J_{S^{-1}/\tau}(y^{(i)} + w^{(i)}/\tau) \\ w^{(i+\frac{1}{2})} = J_{\tau N}(w^{(i)} - \tau(2y^{(i+\frac{1}{2})} - y^{(i)})) \\ y^{(i+1)} = y^{(i)} + \rho^{(i)}(y^{(i+\frac{1}{2})} - y^{(i)}) \\ w^{(i+1)} = w^{(i)} + \rho^{(i)}(w^{(i+\frac{1}{2})} - w^{(i)}). \end{cases} \quad (26)$$

We have the following convergence theorem [22, Theorem 26.11]:

Theorem 2.5 (Douglas–Rachford algorithm (26)) Let $w^{(0)} \in \mathcal{W}$ and $y^{(0)} \in \mathcal{W}$, let $\tau > 0$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequence $(w^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (26) converges weakly to a solution of (25).

Thus, let us apply the Douglas–Rachford algorithm to (24), in the lifted space $\mathcal{W} = \mathcal{X} \times \mathcal{B}$, with $\tau = \gamma$. For this, we need to compute the resolvents; we have [44, eq. 15]:

$$J_{S^{-1}/\gamma} : (z, b) \in \mathcal{W} \mapsto (A^*a, B^*a) \text{ with } a = \frac{\gamma}{2}(Az + Bb) \quad (27)$$

$$J_{\gamma N} : (z, b) \in \mathcal{W} \mapsto (J_{\gamma M}(z - \gamma c), 0) \quad (28)$$

Hence, assuming that $w^{(0)} = (z^{(0)}, 0)$ for some $z^{(0)} \in \mathcal{X}$ and subsequently keeping only the first part of the variable w as the variable z , the Douglas–Rachford iteration in this setting is

Douglas–Rachford iteration for (24): for $i = 0, 1, \dots$

$$\begin{cases} a^{(i+\frac{1}{2})} = \frac{\gamma}{2}(A(y_z^{(i)} + z^{(i)}/\gamma) + B y_b^{(i)}) \\ y_z^{(i+\frac{1}{2})} = A^*a^{(i+\frac{1}{2})} \\ y_b^{(i+\frac{1}{2})} = B^*a^{(i+\frac{1}{2})} \\ z^{(i+\frac{1}{2})} = J_{\gamma M}(z^{(i)} - \gamma(2y_z^{(i+\frac{1}{2})} - y_z^{(i)}) - \gamma c) \\ y_z^{(i+1)} = y_z^{(i)} + \rho^{(i)}(y_z^{(i+\frac{1}{2})} - y_z^{(i)}) \\ y_b^{(i+1)} = y_b^{(i)} + \rho^{(i)}(y_b^{(i+\frac{1}{2})} - y_b^{(i)}) \\ z^{(i+1)} = z^{(i)} + \rho^{(i)}(z^{(i+\frac{1}{2})} - z^{(i)}). \end{cases} \quad (29)$$

Suppose that $(y_z^{(0)}, y_z^{(b)}) = (A^*a^{(0)}, B^*a^{(0)})$, for some $a^{(0)} \in \mathcal{A}$ and set $a^{(i+1)} = a^{(i)} + \rho^{(i)}(a^{(i+\frac{1}{2})} - a^{(i)})$, for every $i \in \mathbb{N}$. Since for every $i \in \mathbb{N}$, $y_z^{(i)} = A^*a^{(i)}$ and $y_b^{(i)} = B^*a^{(i)}$, then $Ay_z^{(i)} + By_b^{(i)} = (AA^* + BB^*)a^{(i)} = a^{(i)}/\gamma$. Therefore, we have

$$z^{(i+\frac{1}{2})} = J_{\gamma M}(z^{(i)} - \gamma(2y_z^{(i+\frac{1}{2})} - y_z^{(i)}) - \gamma c) \quad (30)$$

$$= J_{\gamma M}(z^{(i)} - \gamma^2 A^*(Ay_z^{(i)} + Az^{(i)})/\gamma + By_b^{(i)} + \gamma y_z^{(i)} - \gamma c) \quad (31)$$

$$= J_{\gamma M}(z^{(i)} - \gamma A^*Az^{(i)} - \gamma A^*a^{(i)} + \gamma y_z^{(i)} - \gamma c) \quad (32)$$

$$= J_{\gamma M}(z^{(i)} - \gamma(Qz^{(i)} + c)). \quad (33)$$

So, we can remove all variables but z and we recover the forward–backward iteration (13)! It is remarkable that the forward–backward algorithm can be viewed as an instance of the Douglas–Rachford algorithm, since they are both fundamental algorithms to find a zero of a sum of two monotone operators. This is the case only because the cocoercive operator C is supposed here to be affine, though.

Hence, as an application of Theorem 2.5, we have the following convergence result, which extends Theorem 2.4 with the possibility of setting $\gamma = \xi$:

Theorem 2.6 (forward–backward algorithm (13), affine case) Let $z^{(0)} \in \mathcal{X}$, let $\gamma \in (0, \xi]$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequence $(z^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (13) converges to a solution of (11).

Thus, if $\gamma = \xi$, we can set $\rho^{(i)} = 1.49$ according to Theorem 2.1; with Theorem 2.6, we can do better and set $\rho^{(i)} = 1.99$.

So, which value of γ should be used in practice? If Q is badly conditioned and we do not use overrelaxation ($\rho^{(i)} = 1$), γ close to 2ξ is probably the best choice. If Q is well conditioned and we still do not use overrelaxation, $\gamma = \xi$ may be a better choice. Indeed, if $Q = (1/\xi)\text{Id}$ and $\gamma = \xi$, then $z^{(1)} = (\gamma M + \text{Id})^{-1}(z^{(0)} - \gamma C z^{(0)}) = (\gamma M + \text{Id})^{-1}(\gamma c)$ and $z^{(1)}$ is a solution to (11), so that the algorithm converges in one iteration! In any case, a value of γ less than ξ is not interesting. Thus, the authors' recommendation for a given practical problem is to try the three settings:

- $\gamma = (2 - \epsilon)\xi$, for a small $\epsilon > 0$, and $\rho^{(i)} = 1$,
- $\gamma = \xi$ and $\rho^{(i)} = 1$,
- $\gamma = \xi$ and $\rho^{(i)} = 2 - \epsilon$, for a small $\epsilon > 0$.

2.2 Applications to convex optimization

Let \mathcal{X} be a real Hilbert space. Let $f \in \Gamma_0(\mathcal{X})$ and let $h : \mathcal{X} \rightarrow \mathbb{R}$ be a convex and Fréchet differentiable function, whose gradient ∇h is β -Lipschitz continuous, for some real $\beta > 0$. We consider the convex optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + h(x). \quad (34)$$

From Fermat's rule [22, Theorem 16.3], the problem (34) is equivalent to (11) with $M = \partial f$, which is maximally monotone, and $C = \nabla h$, which is ξ -cocoercive, with $\xi = 1/\beta$ [22, Corollary 18.17] (and as mentioned in the introduction, a zero of $\partial f + \nabla h$ is supposed to exist). So, it is natural to use the forward–backward iteration to solve (34). Let $\gamma > 0$, let $x^{(0)} \in \mathcal{X}$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of relaxation parameters. The forward–backward iteration is

Forward–backward iteration for (34): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\gamma f}(x^{(i)} - \gamma \nabla h(x^{(i)})) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}). \end{cases} \quad (35)$$

As a direct consequence of Theorem 2.1, we have:

Theorem 2.7 (forward–backward algorithm (35)) Let $x^{(0)} \in \mathcal{X}$ and let $\gamma \in (0, 2/\beta)$. Set $\delta = 2 - \gamma\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequence $(x^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (35) converges weakly to a solution of (34).

Now, let us focus on the case where h is quadratic:

$$h : x \mapsto \frac{1}{2} \langle x, Qx \rangle + \langle x, c \rangle + t, \quad (36)$$

for some self-adjoint, positive, nonzero, bounded linear operator Q on \mathcal{X} , some $c \in \mathcal{X}$, and some $t \in \mathbb{R}$. We thereafter omit the constant t in the notations of quadratic functions, since it does not play any role, so we can assume that $t = 0$.

A very common example is a least-squares penalty, in particular to solve inverse problems [5]; that is,

$$h : x \mapsto \frac{1}{2} \|Ax - y\|^2, \quad (37)$$

for some bounded linear operator A from \mathcal{X} to some real Hilbert space \mathcal{Y} and some $y \in \mathcal{Y}$. Clearly, (37) is an instance of (36), with $Q = A^*A$ and $c = -A^*y$.

We have, for every $x \in \mathcal{X}$,

$$\nabla h(x) = Qx + c. \quad (38)$$

So, $\beta = \|Q\|$. Suppose that $\gamma \in (0, 1/\beta)$ and set $P = \frac{1}{\gamma} \text{Id} - Q$, which is strongly positive. Then the update step in (35) can be written as

$$x^{(i+\frac{1}{2})} = \arg \min_{x \in \mathcal{X}} \left\{ f(x) + h(x) + \frac{1}{2} \|x - x^{(i)}\|_P^2 \right\}, \quad (39)$$

where we introduce the norm $\|\cdot\|_P : x \mapsto \sqrt{\langle x, Px \rangle}$. So, $x^{(i+\frac{1}{2})}$ can be viewed as the result of applying the proximity operator of $f + h$ with the preconditioned norm $\|\cdot\|_P$. Thus, convergence follows from Theorem 2.4. Furthermore, as a direct consequence of Theorem 2.6, we have:

Theorem 2.8 (forward–backward algorithm (35), quadratic case) Suppose that h is quadratic. Let $x^{(0)} \in \mathcal{X}$, let $\gamma \in (0, 1/\beta]$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequence $(x^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (35) converges weakly to a solution of (34).

3 The Loris–Verhoeven algorithm

Let \mathcal{X} and \mathcal{U} be two real Hilbert spaces. Let $g \in \Gamma_0(\mathcal{U})$ and let $h : \mathcal{X} \rightarrow \mathbb{R}$ be a convex and Fréchet differentiable function with β -Lipschitz continuous gradient, for some real $\beta > 0$. Let $L : \mathcal{X} \rightarrow \mathcal{U}$ be a bounded linear operator.

Often, the template problem (34) of minimizing the sum of two functions is too simple and we would like, instead, to

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad g(Lx) + h(x). \quad (40)$$

We assume that there is no simple way to compute the proximity operator of $g \circ L$. The corresponding monotone inclusion, which we will actually solve, is

$$0 \in L^* \partial g(Lx) + \nabla h(x). \quad (41)$$

To get rid of the annoying operator L , we introduce an auxiliary variable $u \in \partial g(Lx)$, which shall be called the *dual* variable, so that the problem now consists in finding $x \in \mathcal{X}$ and $u \in \mathcal{U}$ such that

$$\begin{cases} u \in \partial g(Lx) \\ 0 = L^*u + \nabla h(x) \end{cases} \quad (42)$$

The interest in increasing the dimension of the problem is that we obtain a system of two monotone inclusions, which are decoupled: ∂g and ∇h appear separately in the two inclusions. So, equivalently, the problem is to find a pair of objects $z = (x, u)$ in $\mathcal{X} = \mathcal{X} \times \mathcal{U}$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} L^*u \\ -Lx + (\partial g)^{-1}u \end{pmatrix}}_{Mz} + \underbrace{\begin{pmatrix} \nabla h(x) \\ 0 \end{pmatrix}}_{Cz}. \quad (43)$$

The operator $M : \mathcal{X} \rightarrow 2^{\mathcal{X}}$, $(x, u) \mapsto (L^*u, -Lx + (\partial g)^{-1}u)$ is maximally monotone [22, Proposition 26.32 (iii)] and $C : \mathcal{X} \rightarrow \mathcal{X}$, $(x, u) \mapsto (\nabla h(x), 0)$ is ξ -cocoercive, with $\xi = 1/\beta$. Thus, it is natural to think of the forward–backward iteration to solve the problem (43). However, to make the resolvent of M computable with the proximity operator of g , we need preconditioning. The solution consists in the iteration, that we first write in implicit form:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} L^*u^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix}}_{Mz^{(i+\frac{1}{2})}} + \underbrace{\begin{pmatrix} \nabla h(x^{(i)}) \\ 0 \end{pmatrix}}_{Cz^{(i)}} + \underbrace{\begin{pmatrix} \frac{1}{\tau}\text{Id} & 0 \\ 0 & \frac{1}{\sigma}\text{Id} - \tau LL^* \end{pmatrix}}_P \underbrace{\begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}}_{z^{(i+\frac{1}{2})} - z^{(i)}}, \quad (44)$$

where $\tau > 0$ and $\sigma > 0$ are two real parameters, $z^{(i)} = (x^{(i)}, u^{(i)})$ and $z^{(i+\frac{1}{2})} = (x^{(i+\frac{1}{2})}, u^{(i+\frac{1}{2})})$. It is not straightforward to see that this yields an explicit iteration. The key is to remark that we have

$$x^{(i+\frac{1}{2})} = x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^* u^{(i+\frac{1}{2})}, \quad (45)$$

so that we can update the primal variable $x^{(i+\frac{1}{2})}$, once the dual variable $u^{(i+\frac{1}{2})}$ is available. So, the first step of the algorithm is to construct $u^{(i+\frac{1}{2})}$. It depends on $Lx^{(i+\frac{1}{2})}$, which is not yet available, but using (45), we can express it using $x^{(i)}$ and $LL^*u^{(i+\frac{1}{2})}$. This last term is canceled in the preconditioner P to make the update explicit. That is,

$$0 \in -Lx^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} + (\frac{1}{\sigma}\text{Id} - \tau LL^*)(u^{(i+\frac{1}{2})} - u^{(i)}) \quad (46)$$

$$\Leftrightarrow 0 \in -Lx^{(i)} + \tau L \nabla h(x^{(i)}) + \tau LL^* u^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} + (\frac{1}{\sigma}\text{Id} - \tau LL^*)(u^{(i+\frac{1}{2})} - u^{(i)}) \quad (47)$$

$$\Leftrightarrow 0 \in -Lx^{(i)} + \tau L \nabla h(x^{(i)}) + \tau LL^* u^{(i)} + (\partial g)^{-1}u^{(i+\frac{1}{2})} + \frac{1}{\sigma}(u^{(i+\frac{1}{2})} - u^{(i)}) \quad (48)$$

$$\Leftrightarrow (\sigma(\partial g)^{-1} + \text{Id})u^{(i+\frac{1}{2})} \ni \sigma Lx^{(i)} - \tau \sigma L \nabla h(x^{(i)}) - \tau \sigma LL^* u^{(i)} + u^{(i)} \quad (49)$$

$$\Leftrightarrow u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} \left(\sigma L(x^{(i)} - \tau \nabla h(x^{(i)})) + u^{(i)} - \tau \sigma LL^* u^{(i)} \right). \quad (50)$$

We can note that (42) is equivalent to

$$\begin{cases} Lx \in (\partial g)^{-1}u \\ x \in (\nabla h)^{-1}(-L^*u) \end{cases}, \quad (51)$$

which implies that $0 \in (\partial g)^{-1}u - L(\nabla h)^{-1}(-L^*u)$; this is the first-order characterization of the convex optimization problem

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad g^*(u) + h^*(-L^*u), \quad (52)$$

which is called the *dual* problem to the *primal* problem (40). So, if a pair $(x, u) \in \mathcal{X} \times \mathcal{U}$ is a solution to (43), then x is a solution to (40) and u is a solution to (52).

Thus, let $\tau > 0$ and $\sigma > 0$, let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of relaxation parameters. The primal–dual forward–backward iteration, which we call the Loris–Verhoeven iteration is

Loris–Verhoeven iteration for (40) and (52): for $i = 0, 1, \dots$

$$\begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} \left(u^{(i)} + \sigma L(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^* u^{(i)}) \right) \\ x^{(i+1)} = x^{(i)} - \rho^{(i)} \tau (\nabla h(x^{(i)}) + L^* u^{(i+\frac{1}{2})}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)} (u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (53)$$

This algorithm was first proposed by Loris and Verhoeven, in the case where h is a least-squares term [45]. It was then rediscovered several times and named *Primal–Dual Fixed-Point algorithm based on the Proximity Operator* (PDFP2O) [46] or *Proximal Alternating Predictor–Corrector* (PAPC) algorithm [47]. The above interpretation of the algorithm as a primal–dual forward–backward iteration has been presented in [48].

We can note that the Loris–Verhoeven iteration can be rewritten by adding auxiliary variables $b^{(i)} = \nabla h(x^{(i)})$ and $l^{(i)} = L^* u^{(i)}$, for $i \in \mathbb{N}$, so that there is only one call to ∇h and L^* per iteration:

Loris–Verhoeven iteration for (40) and (52): for $i = 0, 1, \dots$

$$\begin{cases} b^{(i)} = \nabla h(x^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} \left(u^{(i)} + \sigma L(x^{(i)} - \tau(b^{(i)} + l^{(i)})) \right) \\ l^{(i+\frac{1}{2})} = L^* u^{(i+\frac{1}{2})} \\ x^{(i+1)} = x^{(i)} - \rho^{(i)} \tau (b^{(i)} + l^{(i+\frac{1}{2})}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)} (u^{(i+\frac{1}{2})} - u^{(i)}) \\ l^{(i+1)} = l^{(i)} + \rho^{(i)} (l^{(i+\frac{1}{2})} - l^{(i)}) \end{cases} \quad (54)$$

(see (165) for an alternative formulation).

As an application of Theorem 2.2, we obtain the following convergence result:

Theorem 3.1 (Loris–Verhoeven algorithm (53)) Let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau \in (0, 2/\beta)$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 < 1$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (53) converge weakly to a solution of (40) and to a solution of (52), respectively.

Proof In view of (44) and (17), this is Theorem 2.2 applied to the problem (43). For this, P must be strongly positive, which is the case if and only if $\sigma\tau\|L\|^2 < 1$. Moreover, $P^{-1}C$ is $1/(\tau\beta)$ -cocoercive in \mathcal{X}_P . \square

The following result makes it possible to have $\sigma\tau\|L\|^2 = 1$; it is a consequence of the analysis by O’Connor and Vandenberghe [44] of the PD3O algorithm [49], discussed in Section 7, of which the Loris–Verhoeven algorithm is a particular case. See also Theorems 3.4 and 3.5 in [46] for the same result, but without relaxation.

Theorem 3.2 (Loris–Verhoeven algorithm (53)) Suppose that \mathcal{X} and \mathcal{U} are of finite dimension. Let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau \in (0, 2/\beta)$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 \leq 1$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (53) converge to a solution of (40) and to a solution of (52), respectively.

Note that the finite dimension assumption is not necessary in the proof technique [44], so it could be removed.

Thus, in practice, one can keep τ as the only parameter to tune and set $\sigma = 1/(\tau\|L\|^2)$. For primal–dual algorithms, whose iteration satisfies an inclusion like (44), we want P to be as close to 0 as possible, broadly speaking. We also want the proximal parameters, here σ and τ , to be as large as possible. But these objectives are antagonistic: if τ is large, σ must be small. Nevertheless, for a

given value of τ , setting σ to the largest possible value $1/(\tau\|L\|^2)$ is the best we can do.

If $\mathfrak{X} = \mathfrak{U}$, $L = \text{Id}$, and we set $\sigma = 1/\tau$, the Loris–Verhoeven iteration becomes:

$$\begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{g^*/\tau}(x^{(i)}/\tau - \nabla h(x^{(i)})) \\ x^{(i+\frac{1}{2})} = x^{(i)} - \tau \nabla h(x^{(i)}) - \tau u^{(i+\frac{1}{2})} \\ \quad = \text{prox}_{\tau g}(x^{(i)} - \tau \nabla h(x^{(i)})) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (55)$$

So, we can discard the dual variable and we recover the forward–backward iteration (35). It is interesting that in this primal algorithm, there is an implicit dual variable $u^{(i+1)} = -\nabla h(x^{(i)}) + (x^{(i)} - x^{(i+\frac{1}{2})})/\tau$, which converges to a solution of the dual problem; that is, to a minimizer of $h^*(-u) + g^*(u)$.

Again, let us focus on the case where h is quadratic; that is, $h : x \mapsto \frac{1}{2}\langle x, Qx \rangle + \langle x, c \rangle$, for some self-adjoint, positive, nonzero, bounded linear operator Q on \mathfrak{X} , and some $c \in \mathfrak{X}$. We have $\beta = \|Q\|$. We can rewrite the primal–dual inclusion (44), which characterizes the Loris–Verhoeven iteration (53), as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} \nabla h(x^{(i+\frac{1}{2})}) + L^*u^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix} + \begin{pmatrix} \frac{1}{\tau}\text{Id} - Q & 0 \\ 0 & \frac{1}{\sigma}\text{Id} - \tau LL^* \end{pmatrix} \begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}. \quad (56)$$

We recognize the structure of a primal–dual proximal point algorithm, so we can apply Theorem 2.3. We can do better and apply Theorem 2.6 to obtain convergence under the conditions $\tau \in (0, 1/\beta]$ and $\sigma\tau\|L\|^2 < 1$. Indeed, we want to solve $0 \in P^{-1}Mz + P^{-1}Cz$ in \mathfrak{X}_P , with P defined in (44). We have $P^{-1}C : (x, u) \mapsto (\tau Qx + \tau c, 0)$, which is affine and $1/(\tau\beta)$ -cocoercive in \mathfrak{X}_P . Hence, Theorem 2.6 can be applied with $\gamma = 1$.

We can do even better and apply Theorem 5.2 below, since according to (120)–(121), the Loris–Verhoeven algorithm with h quadratic can be viewed as a primal–dual Douglas–Rachford algorithm. Hence, we have:

Theorem 3.3 (Loris–Verhoeven algorithm (53), quadratic case) Suppose that h is quadratic. Let $x^{(0)} \in \mathfrak{X}$ and $u^{(0)} \in \mathfrak{U}$. Let $\tau \in (0, 1/\beta]$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (53) converge weakly to a solution of (40) and to a solution of (52), respectively.

3.1 The Primal–Dual Fixed Point (PDFP) algorithm

Let us extend the problem (40) to

$$\underset{x \in \mathfrak{X}}{\text{minimize}} \quad f(x) + g(Lx) + h(x), \quad (57)$$

using the same assumptions as before in this Section, with an additional function $f \in \Gamma_0(\mathfrak{X})$. The dual problem is

$$\underset{u \in \mathfrak{U}}{\text{minimize}} \quad (f + h)^*(-L^*u) + g^*(u). \quad (58)$$

Can we modify the Loris–Verhoeven algorithm to handle this extended problem? We will see one way to do this in Section 7, with the PD3O algorithm. Meanwhile, let us extend the Loris–Verhoeven algorithm, by keeping its primal–dual forward–backward structure. For this, we suppose that, for any $\gamma > 0$, the proximity operator of γf is affine; that is, there exist some self-adjoint, positive, bounded linear operator R_γ on \mathfrak{X} , with $\|R_\gamma\| \leq 1$, and some element $a_\gamma \in \mathfrak{X}$, such that

$$\text{prox}_{\gamma f} : x \in \mathfrak{X} \mapsto R_\gamma x + a_\gamma. \quad (59)$$

We can mention two cases of practical interest, where the proximity operator is affine. First, this is the case if f is quadratic: $f : x \in \mathcal{X} \mapsto \frac{1}{2}\langle x, Q'x \rangle + \langle x, c' \rangle + t'$, for some self-adjoint, positive, bounded linear operator Q' , some $c' \in \mathcal{X}$, and some $t' \in \mathbb{R}$, with $R_\gamma = (\gamma Q' + \text{Id})^{-1}$ and $a_\gamma = -(\gamma Q' + \text{Id})^{-1}c'$. Second, this is the case if f is the indicator function of an affine subspace: $f : x \mapsto (0 \text{ if } x \in \mathcal{A}, +\infty \text{ else})$, where \mathcal{A} is a closed affine subspace of \mathcal{X} ; for instance, $\mathcal{A} = \{x \in \mathcal{X} : Ax = y\}$, for some bounded linear operator A and some element y in the range of A ; then $\text{prox}_{\gamma f}$ is the projector onto \mathcal{A} , which does not depend on γ .

Let $\tau > 0$ and $\sigma > 0$. The extended Loris–Verhoeven iteration, written in implicit form, is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} \partial f(x^{(i+\frac{1}{2})}) + L^*u^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix}}_{Mz^{(i+\frac{1}{2})}} + \underbrace{\begin{pmatrix} \nabla h(x^{(i)}) \\ 0 \end{pmatrix}}_{Cz^{(i)}} + \underbrace{\begin{pmatrix} \frac{1}{\tau}\text{Id} & 0 \\ 0 & \frac{1}{\sigma}\text{Id} - \tau LR_\tau L^* \end{pmatrix}}_P \underbrace{\begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}}_{z^{(i+\frac{1}{2})} - z^{(i)}}. \quad (60)$$

This corresponds to the iteration:

PDFP iteration for (57) and (58): for $i = 0, 1, \dots$

$$\begin{cases} \tilde{x}^{(i)} = \text{prox}_{\tau f}(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^*u^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma L\tilde{x}^{(i)}) \\ x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^*u^{(i+\frac{1}{2})}) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (61)$$

This algorithm has been analyzed by Chen et al. [50] and called the Primal–Dual Fixed Point (PDFP) algorithm, so we give it this name.

Since the algorithm has the structure of a primal–dual forward–backward algorithm, we can, again, invoke Theorem 2.2, and we obtain:

Theorem 3.4 (PDFP algorithm (61), affine prox_f case) Suppose that prox_f is affine. Let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau \in (0, 2/\beta)$ and $\sigma > 0$ be such that $\sigma\tau\|LR_\tau L^*\| < 1$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (61) converge weakly to a solution of (57) and to a solution of (58), respectively.

We can note that $\|LR_\tau L^*\| \leq \|L\|^2$, so if $\sigma\tau\|L\|^2 < 1$, then $\sigma\tau\|LR_\tau L^*\| < 1$.

Chen et al. have proved convergence of the PDFP algorithm with any function f , not only in the affine proximity operator case, as follows [50, Theorem 3.1]:

Theorem 3.5 (PDFP algorithm (61), general case) Suppose that \mathcal{X} and \mathcal{U} are of finite dimension. Let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau \in (0, 2/\beta)$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 < 1$. Set $\rho^{(i)} = 1$, for every $i \in \mathbb{N}$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (61) converge to a solution of (57) and to a solution of (58), respectively.

It remains an open question whether the PDFP algorithm can be relaxed like in Theorem 3.4, for an arbitrary f .

We can check that when $h = \frac{\beta}{2}\|\cdot\|^2$, it is equivalent to apply the Loris–Verhoeven algorithm to minimize $g \circ L + h$ or to apply the PDFP to minimize $f + g \circ L$, with $f = \frac{\beta}{2}\|\cdot\|^2$. Indeed, in the later case, let $\tau' > 0$ and $\sigma > 0$ be the parameters of the PDFP algorithm. Set $\tau = \frac{\tau'}{1+\tau'\beta}$. We have $\text{prox}_{\tau' f} : x \in \mathcal{X} \mapsto \frac{1}{1+\tau'\beta}x$, which is linear. Then the PDFP algorithm (61) becomes:

PDFP iteration: for $i = 0, 1, \dots$

$$\begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} \left(u^{(i)} + \frac{1}{1+\tau'\beta} L(x^{(i)} - \tau L^* u^{(i)}) \right) \\ \quad = \text{prox}_{\sigma g^*} \left(u^{(i)} + L(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^* u^{(i)}) \right) \\ x^{(i+\frac{1}{2})} = \frac{1}{1+\tau'\beta} (x^{(i)} - \tau L^* u^{(i+\frac{1}{2})}) \\ \quad = x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^* u^{(i+\frac{1}{2})} \\ x^{(i+1)} = x^{(i)} + \rho^{(i)} (x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)} (u^{(i+\frac{1}{2})} - u^{(i)}), \end{cases} \quad (62)$$

and we recover the Loris–Verhoeven algorithm (53) applied to minimize $g \circ L + h$, with parameters τ and σ . The conditions for convergence from the PDFP interpretation are $\tau < 1/\beta$, $\sigma\tau\|L\|^2 < 1$, and $\delta = 2$, so that we do not gain anything in comparison with Theorem 3.3.

4 The Chambolle–Pock and the Douglas–Rachford algorithms

We now consider a similar problem to (40), but this time we want to make use of the proximity operators of the two functions. So, let \mathcal{X} and \mathcal{U} be two real Hilbert spaces. Let $f \in \Gamma_0(\mathcal{X})$ and $g \in \Gamma_0(\mathcal{U})$. Let $L : \mathcal{X} \rightarrow \mathcal{U}$ be a bounded linear operator. We want to

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + g(Lx). \quad (63)$$

The corresponding monotone inclusion is

$$0 \in \partial f(x) + L^* \partial g(Lx). \quad (64)$$

Again, to get rid of the annoying operator L , we introduce an auxiliary variable $u \in \partial g(Lx)$, which will be called the dual variable, so that the problem now consists in finding $x \in \mathcal{X}$ and $u \in \mathcal{U}$ such that

$$\begin{cases} u \in \partial g(Lx) \\ 0 \in L^* u + \partial f(x) \end{cases} \quad (65)$$

Let us define the dual convex optimization problem associated to the primal problem (63):

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad f^*(-L^* u) + g^*(u). \quad (66)$$

If a pair $(x, u) \in \mathcal{X} \times \mathcal{U}$ is a solution to (65), then x is a solution to (63) and u is a solution to (66).

To solve the primal and dual problems (63) and (66) jointly, Chambolle and Pock [51] proposed the following algorithms (without relaxation), see also Esser et al. [52] and Zhang et al. [53]:

Chambolle–Pock iteration, form I, for (63) and (66): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f} (x^{(i)} - \tau L^* u^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} (u^{(i)} + \sigma L(2x^{(i+\frac{1}{2})} - x^{(i)})) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)} (x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)} (u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (67)$$

Chambolle–Pock iteration, form II, for (63) and (66): for $i = 0, 1, \dots$

$$\begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} (u^{(i)} + \sigma Lx^{(i)}) \\ x^{(i+\frac{1}{2})} = \text{prox}_{\tau f} (x^{(i)} - \tau L^* (2u^{(i+\frac{1}{2})} - u^{(i)})) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)} (u^{(i+\frac{1}{2})} - u^{(i)}) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)} (x^{(i+\frac{1}{2})} - x^{(i)}). \end{cases} \quad (68)$$

We can note that the Chambolle–Pock algorithm is self-dual: if we apply the Chambolle–Pock iteration form I to the problem (66), to minimize $\tilde{f} + \tilde{g} \circ \tilde{L}$ with $\tilde{f} = g^*$, $\tilde{g} = f^*$, $\tilde{L} = -L^*$, and the roles of x and u are switched, as well as the roles of τ and σ , we obtain exactly the Chambolle–Pock iteration form II.

Chambolle and Pock proved the convergence in the finite-dimensional case, assuming that $\tau\sigma\|L\|^2 < 1$ and $\rho^{(i)} = 1$ [51]. The convergence was proved in a different way by He and Yuan, with a constant relaxation parameter $\rho^{(i)} = \rho \in (0, 2)$ and the same other hypotheses [54]; indeed, they observed that the Chambolle–Pock algorithm is a primal–dual proximal point algorithm to find a primal–dual pair $z = (x, u)$ in $\mathcal{X} = \mathcal{X} \times \mathcal{U}$, solution to the monotone inclusion

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} \partial f(x) + L^*u \\ -Lx + (\partial g)^{-1}u \end{pmatrix}}_{Mz}. \quad (69)$$

The operator $M : \mathcal{X} \rightarrow 2^{\mathcal{X}}, (x, u) \mapsto (\partial f(x) + L^*u, -Lx + (\partial g)^{-1}u)$ is maximally monotone [22, Proposition 26.32 (iii)]. Then, one can observe that the Chambolle–Pock iteration form I satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} \partial f(x^{(i+\frac{1}{2})}) + L^*u^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix}}_{Mz^{(i+\frac{1}{2})}} + \underbrace{\begin{pmatrix} \frac{1}{\tau}\text{Id} & -L^* \\ -L & \frac{1}{\sigma}\text{Id} \end{pmatrix}}_P \underbrace{\begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}}_{z^{(i+\frac{1}{2})} - z^{(i)}}. \quad (70)$$

The Chambolle–Pock iteration form II satisfies the same primal–dual inclusion, but with $-L$ replaced by L in P .

Thus, the Chambolle–Pock iteration is a preconditioned primal–dual proximal point algorithm and, as a consequence of Theorem 2.3, we have [40, Theorem 3.2]:

Theorem 4.1 (Chambolle–Pock algorithm (67) or (68)) Let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau > 0$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 < 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined either by the iteration (67) or by the iteration (68) converge weakly to a solution of (63) and to a solution of (66), respectively.

In addition, the first author proved that in the finite-dimensional setting, one can set $\sigma\tau\|L\|^2 = 1$ [40, Theorem 3.3], see also O’Connor and Vandenberghe [44] for another proof. If we apply Theorem 5.2 below, with $K = \text{Id}$ and $\eta = 1/\tau$, we obtain:

Theorem 4.2 (Chambolle–Pock algorithm (67) or (68)) Let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau > 0$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined either by the iteration (67) or by the iteration (68) converge weakly to a solution of (63) and to a solution of (66), respectively.

The difference between Theorems 4.1 and 4.2 is that $\sigma\tau\|L\|^2 = 1$ is allowed in the latter. This is a significant improvement: in practice, one can set $\sigma = 1/(\tau\|L\|^2)$ in the algorithms and have only one parameter left, namely τ , to tune.

4.1 The Proximal Method of Multipliers

Let \mathcal{X} and \mathcal{U} be two real Hilbert spaces. Let $g \in \Gamma_0(\mathcal{U})$, $c \in \mathcal{X}$ and $L : \mathcal{X} \rightarrow \mathcal{U}$ be a bounded linear operator. We consider the problem

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad g(Lx) + \langle x, c \rangle \quad (71)$$

and the dual problem

$$\underset{u \in \mathcal{U}}{\text{minimize}} \ g^*(u) \quad \text{s.t.} \quad L^*u + c = 0. \quad (72)$$

We can view the term $\langle x, c \rangle$ as a smooth function h with constant gradient $\nabla h = c$ and apply the Loris–Verhoeven algorithm. Thus, let $\tau > 0$ and $\sigma > 0$, let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of relaxation parameters. The iteration is (we change the name of the variable x into s):

$$\begin{aligned} & \textbf{Loris–Verhoeven iteration for (71) and (72): for } i = 0, 1, \dots \\ & \begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma L(s^{(i)} - \tau L^*u^{(i)} - \tau c)) \\ s^{(i+1)} = s^{(i)} - \rho^{(i)}\tau(L^*u^{(i+\frac{1}{2})} + c) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \end{aligned} \quad (73)$$

Each iteration of the algorithm satisfies the inclusion

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} c + L^*u^{(i+\frac{1}{2})} \\ -Ls^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix} + \begin{pmatrix} \frac{1}{\tau}\text{Id} & 0 \\ 0 & \frac{1}{\sigma}\text{Id} - \tau LL^* \end{pmatrix} \begin{pmatrix} s^{(i+\frac{1}{2})} - s^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}. \quad (74)$$

Since the algorithm has the structure of a proximal point algorithm, we can apply Theorem 2.3 and obtain convergence under the condition $\sigma\tau\|L\|^2 < 1$.

An alternative is to see the term $\langle x, c \rangle$ as a function f , with proximity operator $\text{prox}_{\tau f} : x \mapsto x - \tau c$. So, we can solve (71) and (72) by applying the Chambolle–Pock algorithm (we change the name of its variable $x^{(i+\frac{1}{2})}$ into $s^{(i)}$):

$$\begin{aligned} & \textbf{Chambolle–Pock iteration, form I, for (71) and (72): for } i = 0, 1, \dots \\ & \begin{cases} s^{(i)} = x^{(i)} - \tau L^*u^{(i)} - \tau c \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma L(2s^{(i)} - x^{(i)})) \\ \quad = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma L(s^{(i)} - \tau L^*u^{(i)} - \tau c)) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(s^{(i)} - x^{(i)}) \\ \quad = x^{(i)} - \rho^{(i)}\tau(L^*u^{(i)} + c) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \end{aligned} \quad (75)$$

We can discard the variable x and the iteration becomes:

$$\begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma L(s^{(i)} - \tau L^*u^{(i)} - \tau c)) \\ s^{(i+1)} = s^{(i)} - \rho^{(i)}\tau(L^*u^{(i+\frac{1}{2})} + c) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}), \end{cases} \quad (76)$$

and we exactly recover the iteration (73). So, the Loris–Verhoeven algorithm and the Chambolle–Pock algorithm are the same for the specific problems (71) and (72); we call this algorithm the Proximal Method of Multipliers [55].

Furthermore, the form (75) provides us with a compact form of the algorithm:

$$\begin{aligned} & \textbf{Proximal Method of Multipliers for (71) and (72): for } i = 0, 1, \dots \\ & \begin{cases} a^{(i)} = L^*u^{(i)} + c \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}\left(\text{prox}_{\sigma g^*}(u^{(i)} + \sigma L(x^{(i)} - 2\tau a^{(i)})) - u^{(i)}\right) \\ x^{(i+1)} = x^{(i)} - \rho^{(i)}\tau a^{(i)}, \end{cases} \end{aligned} \quad (77)$$

and since the two updates are independent, they can be performed in parallel.

Theorem 4.2 yields:

Theorem 4.3 (Proximal Method of Multipliers (77)) Let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau > 0$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (77) converge weakly to a solution of (71) and to a solution of (72), respectively.

We can apply the form II of the Chambolle–Pock algorithm as well, with same convergence properties as in Theorem 4.3:

Chambolle–Pock iteration, form II, for (71) and (72): for $i = 0, 1, \dots$

$$\begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma Lx^{(i)}) \\ x^{(i+1)} = x^{(i)} - \rho^{(i)}\tau(L^*(2u^{(i+\frac{1}{2})} - u^{(i)}) + c) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (78)$$

4.2 The Douglas–Rachford algorithm

Let us consider the particular case of the problem (63) when $\mathcal{X} = \mathcal{U}$ and $L = \text{Id}$. The problem becomes

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + g(x). \quad (79)$$

By setting $\sigma = 1/\tau$ in the Chambolle–Pock algorithm, we recover the Douglas–Rachford algorithm [32, 37, 42, 43] as a particular case:

Douglas–Rachford iteration for (79): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(x^{(i)} - \tau u^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{g^*/\tau}(u^{(i)} + (2x^{(i+\frac{1}{2})} - x^{(i)})/\tau) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (80)$$

Note that the Douglas–Rachford algorithm is not a primal–dual proximal point algorithm, since the operator P in (70) is not strongly positive any more. So, its weak convergence is difficult to establish; it was shown only in 2011 by Svaiter [43]. Let us have a quick look at his structure. First, we define the auxiliary variable

$$s^{(i)} = x^{(i)} - \tau u^{(i)}. \quad (81)$$

The Douglas–Rachford iteration only depends on this concatenated variable, not on the full pair $(x^{(i)}, u^{(i)})$, and we can rewrite it as:

Douglas–Rachford iteration for (79): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(s^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{g^*/\tau}((2x^{(i+\frac{1}{2})} - s^{(i)})/\tau) \\ s^{(i+1)} = s^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - \tau u^{(i+\frac{1}{2})} - s^{(i)}), \end{cases} \quad (82)$$

or, equivalently,

Douglas–Rachford iteration for (79): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(s^{(i)}) \\ s^{(i+1)} = s^{(i)} + \rho^{(i)}(\text{prox}_{\tau g}(2x^{(i+\frac{1}{2})} - s^{(i)}) - x^{(i+\frac{1}{2})}). \end{cases} \quad (83)$$

Or, keeping only the variable s :

$$\begin{cases} s^{(i+\frac{1}{2})} = (\frac{1}{2}(2\text{prox}_{\tau g} - \text{Id}) \circ (2\text{prox}_{\tau f} - \text{Id}) + \frac{1}{2}\text{Id})s^{(i)} \\ s^{(i+1)} = s^{(i)} + \rho^{(i)}(s^{(i+\frac{1}{2})} - s^{(i)}). \end{cases} \quad (84)$$

Since the operator mapping $s^{(i)}$ to $s^{(i+\frac{1}{2})}$ is firmly nonexpansive, $(s^{(i)})_{i \in \mathbb{N}}$ converges weakly, after the Krasnosel’skiĭ–Mann theorem, for every $\tau > 0$ and sequence $(\rho^{(i)})_{i \in \mathbb{N}}$ in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 -$

$\rho^{(i)} = +\infty$. But in infinite dimension, one cannot deduce that $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ converges. In finite dimension, however, the proximity operator is continuous, so that one can indeed conclude that $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ converges to some minimizer of $f + g$.

As an application of Corollary 28.3 in [22], which proves weak convergence using fine properties of the operators, we have:

Theorem 4.4 (Douglas–Rachford algorithm (83)) Let $s^{(0)} \in \mathfrak{X}$. Let $\tau > 0$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequence $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ defined by the iteration (83) converges weakly to a solution of (79).

Note that it would be misleading to state Theorem 4.4 by invoking Theorem 4.2, since Theorem 4.2 is itself based on the convergence result of the Douglas–Rachford algorithm [22, Corollary 28.3]. In other words, weak convergence of the Douglas–Rachford algorithm is the fundamental result, from which we deduce weak convergence of the Chambolle–Pock algorithm, which generalizes it.

Let us show that, like the Chambolle–Pock algorithm, the Douglas–Rachford algorithm is self-dual [56]: it is equivalent to apply it to the primal problem (79) or to the dual problem:

$$\underset{u \in \mathfrak{X}}{\text{minimize}} \quad f^*(-u) + g^*(u). \quad (85)$$

Using the Moreau identity (8) and starting from the form (80), we can write the Douglas–Rachford iteration as:

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{f^*(-\cdot)/\tau}(u^{(i)} - \frac{1}{\tau}x^{(i)}) \\ x^{(i+\frac{1}{2})} = x^{(i)} - \tau(u^{(i)} - v^{(i+\frac{1}{2})}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{g^*/\tau}(v^{(i+\frac{1}{2})} + \frac{1}{\tau}x^{(i+\frac{1}{2})}) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (86)$$

If we introduce the variable $\tilde{s}^{(i)} = \frac{-1}{\tau}s^{(i)} = u^{(i)} - \frac{1}{\tau}x^{(i)}$, we can rewrite the iteration as:

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{f^*(-\cdot)/\tau}(\tilde{s}^{(i)}) \\ x^{(i+\frac{1}{2})} = x^{(i)} - \tau(u^{(i)} - v^{(i+\frac{1}{2})}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{g^*/\tau}(2v^{(i+\frac{1}{2})} - \tilde{s}^{(i)}) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}) \\ \tilde{s}^{(i+1)} = u^{(i+1)} - \frac{1}{\tau}x^{(i+1)} \\ \quad = \tilde{s}^{(i)} + \rho^{(i)}(\text{prox}_{g^*/\tau}(2v^{(i+\frac{1}{2})} - \tilde{s}^{(i)}) - v^{(i+\frac{1}{2})}), \end{cases} \quad (87)$$

which can be simplified as:

Douglas–Rachford iteration for (85): for $i = 0, 1, \dots$

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{f^*(-\cdot)/\tau}(\tilde{s}^{(i)}) \\ \tilde{s}^{(i+1)} = \tilde{s}^{(i)} + \rho^{(i)}(\text{prox}_{g^*/\tau}(2v^{(i+\frac{1}{2})} - \tilde{s}^{(i)}) - v^{(i+\frac{1}{2})}). \end{cases} \quad (88)$$

Thus, we recognize the form (83) of the Douglas–Rachford algorithm applied to the dual problem (85), with parameter $1/\tau$. All in all, there are only two different instances of the Douglas–Rachford algorithm: the one given here, and the one obtained by switching the roles of f and g (which we could obtain as a particular case of the Chambolle–Pock algorithm, form II).

Douglas–Rachford splitting has been shown to be the only way, in some sense, to minimize a sum of two functions, by calling their proximity operators [57].

4.2.1 A slightly modified version of the Douglas–Rachford algorithm

Let us consider a slightly modified version of the Douglas–Rachford algorithm, which we will use in the next section. In addition to f and g as before, let $c \in \mathfrak{X}$. We consider the problem:

$$\underset{x \in \mathfrak{X}}{\text{minimize}} \quad f(x) + g(x) + \langle x, c \rangle. \quad (89)$$

The dual problem is

$$\underset{u \in \mathfrak{X}}{\text{minimize}} \quad f^*(-u - c) + g^*(u). \quad (90)$$

The linear term $\langle x, c \rangle$ does not add any difficulty: one can apply the Douglas–Rachford algorithm to minimize $\tilde{f} + g$, where $\tilde{f} = f + \langle \cdot, c \rangle$, using the fact that $\text{prox}_{\tau \tilde{f}}(x) = \text{prox}_{\tau f}(x - \tau c)$. So, the modified Douglas–Rachford algorithm is

Douglas–Rachford iteration for (89): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(s^{(i)} - \tau c) \\ s^{(i+1)} = s^{(i)} + \rho^{(i)}(\text{prox}_{\tau g}(2x^{(i+\frac{1}{2})} - s^{(i)}) - x^{(i+\frac{1}{2})}). \end{cases} \quad (91)$$

Using the Moreau identity (8) and starting from the form (80), we can write the algorithm as

Douglas–Rachford iteration for (89): for $i = 0, 1, \dots$

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{f^*/\tau}(\frac{1}{\tau}x^{(i)} - u^{(i)} - c) \\ x^{(i+\frac{1}{2})} = x^{(i)} - \tau(u^{(i)} + v^{(i+\frac{1}{2})} + c) \\ u^{(i+\frac{1}{2})} = \text{prox}_{g^*/\tau}(\frac{1}{\tau}x^{(i+\frac{1}{2})} - v^{(i+\frac{1}{2})} - c) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}) \end{cases} \quad (92)$$

(note that the variable v here is not the same as the one in (88) or (93); it is its opposite).

As an application of Corollary 28.3 in [22], we have:

Theorem 4.5 (Douglas–Rachford algorithm (92)) Let $x^{(0)} \in \mathfrak{X}$ and $u^{(0)} \in \mathfrak{X}$. Let $\tau > 0$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (92) converge weakly to a solution of (89) and to a solution of (90), respectively. Moreover, $(u^{(i+\frac{1}{2})} + v^{(i+\frac{1}{2})} + c)_{i \in \mathbb{N}}$ converges strongly to 0.

4.3 The Alternating Direction Method of Multipliers (ADMM)

The Alternating Direction Method of Multipliers (ADMM) goes back to Glowinski and Marocco [58], and Gabay and Mercier [59]. This algorithm has been studied extensively, see e.g. [42, 60–66]. The ADMM was rediscovered by Osher et al. and called *Split Bregman Algorithm* [67, 68]; this method has received significant attention in image processing [61, 69–72]. The equivalence between the ADMM and the Split Bregman Algorithm is now well established [52, 68, 73]. Also, the ADMM has been popularized in image processing by a series of papers of Figueiredo, Bioucas-Dias et al., e.g. [74–76].

It is well known that the ADMM is equivalent to the Douglas–Rachford algorithm [34, 42, 77]. Let us show it: starting from the form (80) of the Douglas–Rachford algorithm and using the Moreau identity (8), we can rewrite it as

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(x^{(i)} - \tau u^{(i)}) \\ v^{(i+\frac{1}{2})} = u^{(i)} + \frac{1}{\tau}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ w^{(i+1)} = \text{prox}_{\tau g}(\tau v^{(i+\frac{1}{2})} + x^{(i+\frac{1}{2})}) \\ u^{(i+\frac{1}{2})} = v^{(i+\frac{1}{2})} + \frac{1}{\tau}(x^{(i+\frac{1}{2})} - w^{(i+1)}) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (93)$$

Let us introduce the variable $v^{(i+1)} = v^{(i+\frac{1}{2})} + \frac{\rho^{(i)}-1}{\tau}(x^{(i+\frac{1}{2})} - w^{(i+1)})$, for every $i \in \mathbb{N}$. Let $w^{(0)} \in \mathcal{X}$ and $v^{(0)} \in \mathcal{X}$ be such that $x^{(0)} - \tau u^{(0)} = w^{(0)} - \tau v^{(0)}$. Then for every $i \in \mathbb{N}$, $x^{(i+1)} - \tau u^{(i+1)} = (1 - \rho^{(i)})(x^{(i)} - \tau u^{(i)}) + \rho^{(i)}(x^{(i+\frac{1}{2})} - \tau u^{(i+\frac{1}{2})}) = (1 - \rho^{(i)})(x^{(i+\frac{1}{2})} - \tau v^{(i+\frac{1}{2})}) + \rho^{(i)}(w^{(i+1)} - \tau v^{(i+\frac{1}{2})}) = w^{(i+1)} - \tau v^{(i+1)}$. Moreover, we introduce the scaled variables $\tilde{v}^{(i)} = \tau v^{(i)}$ and $\tilde{v}^{(i+\frac{1}{2})} = \tau v^{(i+\frac{1}{2})}$. Hence, we can remove the variables x and u and rewrite the iteration as

ADMM iteration for (79) and (85): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(w^{(i)} - \tilde{v}^{(i)}) \\ \tilde{v}^{(i+\frac{1}{2})} = \tilde{v}^{(i)} + x^{(i+\frac{1}{2})} - w^{(i)} \\ w^{(i+1)} = \text{prox}_{\tau g}(\tilde{v}^{(i+\frac{1}{2})} + x^{(i+\frac{1}{2})}) \\ \tilde{v}^{(i+1)} = \tilde{v}^{(i+\frac{1}{2})} + (\rho^{(i)} - 1)(x^{(i+\frac{1}{2})} - w^{(i+1)}). \end{cases} \quad (94)$$

As an application of Corollary 28.3 in [22], we have:

Theorem 4.6 (ADMM (94)) Let $w^{(0)} \in \mathcal{X}$ and $\tilde{v}^{(0)} \in \mathcal{X}$. Let $\tau > 0$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ and $(w^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (94) both converge weakly to some element $x^* \in \mathcal{X}$ solution to (79). Moreover, the sequences $(\tilde{v}^{(i+\frac{1}{2})}/\tau)_{i \in \mathbb{N}}$ and $(\tilde{v}^{(i)}/\tau)_{i \in \mathbb{N}}$ defined by the iteration (94) both converge weakly to some element $u^* \in \mathcal{X}$ solution to (85). In addition, $(x^{(i+\frac{1}{2})} - w^{(i+1)})_{i \in \mathbb{N}}$ converges strongly to 0.

Thus, in the Douglas–Rachford algorithm, or equivalently the ADMM, there are two primal variables x and w and two dual variables u and v , as can be seen in (93). But depending on how the algorithm is written, only some of these variables appear. Also, two monotone inclusions are satisfied in parallel at every iteration:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} \partial f(x^{(i+\frac{1}{2})}) + u^{(i+\frac{1}{2})} \\ -x^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix} + \begin{pmatrix} \frac{1}{\tau}\text{Id} & -\text{Id} \\ -\text{Id} & \tau\text{Id} \end{pmatrix} \begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}, \quad (95)$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} -(\partial f)^{-1}(-v^{(i+\frac{1}{2})}) + w^{(i+1)} \\ -v^{(i+\frac{1}{2})} + \partial g(w^{(i+1)}) \end{pmatrix} + \begin{pmatrix} \tau\text{Id} & -\text{Id} \\ -\text{Id} & \frac{1}{\tau}\text{Id} \end{pmatrix} \begin{pmatrix} v^{(i+\frac{1}{2})} - v^{(i)} \\ w^{(i+1)} - w^{(i)} \end{pmatrix}. \quad (96)$$

Note that in the ADMM form (94), the final extrapolation step from $v^{(i+\frac{1}{2})}$ to $v^{(i+1)}$ accounts for the absence of relaxation on the variable w , in such a way that $w^{(i+1)} - \tau v^{(i+1)}$ takes the appropriate value. That is, $w^{(i+1)} - \tau v^{(i+1)} = w^{(i)} - \tau v^{(i)} + \rho^{(i)}((w^{(i+1)} - \tau v^{(i+\frac{1}{2})}) - (w^{(i)} - \tau v^{(i)}))$, as if the two variables v and w had been relaxed as usually.

We can remark that τ should not be chosen as large as possible in the Douglas–Rachford algorithm or the ADMM: we see in the primal–dual inclusions (95) and (96) that the antagonistic values τ and $1/\tau$ control the primal and dual updates. So there is a tradeoff to achieve in the choice of τ . We insist on the fact that the Douglas–Rachford algorithm and the ADMM are primal–dual algorithms: it is equivalent to apply them on the primal or on the dual problem. There are only two different algorithms to minimize $f + g$: the one given here, with prox_f or prox_{f^*} applied first, and the one obtained by exchanging f and g .

Finally, one can find the following relaxed version of the ADMM in the literature:

ADMM iteration for (79) and (85): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(w^{(i)} - \tilde{v}^{(i)}) \\ \tilde{v}^{(i+1)} = \tilde{v}^{(i)} + \rho(x^{(i+\frac{1}{2})} - w^{(i)}) \\ w^{(i+1)} = \text{prox}_{\tau g}(\tilde{v}^{(i+1)} + x^{(i+\frac{1}{2})}), \end{cases} \quad (97)$$

which is proved to converge for $\rho \in (0, (\sqrt{5} + 1)/2)$ [78]. This is different from the relaxation considered here. It would be interesting to compare the two relaxed forms in practice.

4.4 Infimal postcompositions

Suppose that we want to solve an optimization problem, which involves a term $g(Lx)$. Throughout the paper, we are studying splitting algorithms, which allow us to solve optimization problems, by calling the proximity operator of g , L , and L^* separately. Here we look at an alternative, which consists in a *change of variables*: we introduce the variable $r = Lx$ and we express the optimization problem with respect to r instead of x . Then an algorithm to solve the reformulated problem will construct a sequence $(r^{(i)})_{i \in \mathbb{N}}$ converging to $r^* = Lx^*$, where x^* is a solution to the initial problem, that will be recovered as a byproduct. For instance, suppose that we want to minimize $f(x) + g(Lx)$ over $x \in \mathfrak{X}$, where f , g , L are as before. This is the same as minimizing $f(x) + g(r)$, subject to $r = Lx$. We can rewrite the problem with respect to r only, by ‘hiding’ x as a subvariable, as follows: minimize $\tilde{f}(r) + g(r)$, where $\tilde{f} : r \in \mathcal{U} \mapsto \inf_{x \in \mathfrak{X}} f(x) + \iota_0(Lx - r)$ and we introduce the indicator function $\iota_0 : s \mapsto (0 \text{ if } s = 0, +\infty \text{ else})$. $\tilde{f} \in \Gamma_0(\mathcal{U})$ under some mild mathematical safeguards; for instance, we will require that the infimum is attained in the definition of \tilde{f} . Clearly, minimizing $\tilde{f}(r) + g(r)$ will force r to be in the range of L , else $\tilde{f}(r) = +\infty$. That is, a solution will be $r^* = Lx^*$ for some $x^* \in \arg \min_{\{x \in \mathfrak{X} : Lx = r^*\}} f(x)$, which is the actual element we are interested in. To apply a proximal splitting algorithm to this problem, we do not need to evaluate \tilde{f} , but we must be able to apply its proximity operator $\text{prox}_{\tilde{f}} : r \in \mathcal{U} \mapsto Lx'$, where $x' \in \arg \min_{x \in \mathfrak{X}} \{f(x) + \frac{1}{2}\|Lx - r\|^2\}$. That is, $\text{prox}_{\tilde{f}} = L(\partial f + L^*L)^{-1}L^*$, which is indeed single-valued: even if x' in its definition is not unique, Lx' is unique. We have $\tilde{f}^* : u \in \mathcal{U} \mapsto \sup_{x \in \mathfrak{X}} \langle u, Lx \rangle - f(x) = f^*(L^*u)$ [22, Proposition 13.24]. Thus, the dual problem to $\min_x f(x) + g(Lx) = \min_r \tilde{f}(r) + g(r)$ is $\min_u \tilde{f}^*(-u) + g^*(u) = \min_u f^*(-L^*u) + g^*(u)$, which is consistent with our formulations of dual problems so far. The function \tilde{f} is called The *infimal postcomposition* of f by L , which is thereafter denoted by $L \triangleright f$ [22, Definition 12.34]. As we have just explained, behind this technical notion there is simply a change of variables, which is helpful if $\text{prox}_{L \triangleright f}$ is easy to compute.

Let us apply this notion. Let \mathfrak{X} , \mathfrak{Y} , \mathcal{U} be real Hilbert spaces. Let $f \in \Gamma_0(\mathfrak{X})$, $g \in \Gamma_0(\mathfrak{Y})$, $c \in \mathcal{U}$. Let $K : \mathfrak{Y} \rightarrow \mathcal{U}$ and $L : \mathfrak{X} \rightarrow \mathcal{U}$ be nonzero bounded linear operators. We consider the problem:

$$\underset{(x,y) \in \mathfrak{X} \times \mathfrak{Y}}{\text{minimize}} \quad f(x) + g(y) \quad \text{s.t.} \quad Lx + Ky = c. \quad (98)$$

The dual problem is

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad f^*(-L^*u) + g^*(-K^*u) + \langle u, c \rangle. \quad (99)$$

The corresponding primal–dual monotone inclusion is: find $(x, y, u) \in \mathfrak{X} \times \mathfrak{Y} \times \mathcal{U}$, such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} \partial f(x) + L^*u \\ \partial g(y) + K^*u \\ -Lx - Ky + c \end{pmatrix}. \quad (100)$$

For instance, the minimization of $f(x) + g(Lx)$ is a particular case of this problem, with $c = 0$, $\mathcal{U} = \mathfrak{Y}$, $K = -\text{Id}$. We want to solve (98) by applying two infimal postcompositions, so that the problem amounts to find an element $r^* = Lx^* = -Ky^* + c$, where (x^*, y^*) is a solution to (98). That is, we rewrite (98) as

$$\underset{a \in \mathcal{U}}{\text{minimize}} \quad L \triangleright f(r) + (-K) \triangleright g(r - c), \quad (101)$$

where

$$L \triangleright f : r \in \mathcal{U} \mapsto \inf_{x \in \mathfrak{X}} f(x) + \iota_0(Lx - r) \quad (102)$$

$$(-K) \triangleright g : r \in \mathcal{U} \mapsto \inf_{y \in \mathfrak{Y}} g(y) + \iota_0(Ky + r). \quad (103)$$

For the problem to be well posed, we suppose that the following assumptions hold:

- (i) The solution set of (100) is nonempty.

- (ii) The infimal postcompositions are exact [22, Definition 12.34]; that is, for every $r \in \mathcal{U}$, the infimum in (102) and (103) is attained, so it is a minimum (its value can be $+\infty$). In other words, for every r in the range of L , denoted by $\text{ran } L$, the set of minimizers of f over the set $L^{-1}\{r\}$ is nonempty. Likewise, for every $r \in \text{ran } K$, the set of minimizers of g over $K^{-1}\{-r\}$ is nonempty.
- (iii) $L \triangleright f$ is lower semicontinuous. A sufficient condition for that is $0 \in \text{sri}(\text{ran } L^* - \text{dom } f^*)$ [22, Corollary 25.44]. Likewise, $(-K) \triangleright g$ is lower semicontinuous.

Note that $L \triangleright f$ and $(-K) \triangleright g$ are convex and proper [22, Proposition 12.36]. So (iii) implies that $L \triangleright f \in \Gamma_0(\mathcal{U})$ and $(-K) \triangleright g \in \Gamma_0(\mathcal{U})$. Moreover, $(L \triangleright f)^* = f^* \circ L^*$ and $((-K) \triangleright g)^* = g^* \circ -K^*$ [22, Proposition 13.24].

Then, let us apply the ADMM (94) to the problem (101):

$$\begin{aligned} & \text{ADMM iteration for (98) and (99): for } i = 0, 1, \dots \\ & \begin{cases} x^{(i+\frac{1}{2})} \in \arg \min_{x \in \mathcal{X}} \left\{ \tau f(x) + \frac{1}{2} \|Lx + Ky^{(i)} - c + \tilde{v}^{(i)}\|^2 \right\} \\ \tilde{v}^{(i+\frac{1}{2})} = \tilde{v}^{(i)} + Lx^{(i+\frac{1}{2})} + Ky^{(i)} - c \\ y^{(i+1)} \in \arg \min_{y \in \mathcal{Y}} \left\{ \tau g(y) + \frac{1}{2} \|Lx^{(i+\frac{1}{2})} + Ky - c + \tilde{v}^{(i+\frac{1}{2})}\|^2 \right\} \\ \tilde{v}^{(i+1)} = \tilde{v}^{(i+\frac{1}{2})} + (\rho^{(i)} - 1)(Lx^{(i+\frac{1}{2})} + Ky^{(i+1)} - c) \end{cases} \end{aligned} \quad (104)$$

($x^{(i+\frac{1}{2})}$ and $w^{(i)}$ in (94) become $Lx^{(i+\frac{1}{2})}$ and $-Ky^{(i)} + c$ in (101), respectively).

Since the ADMM and the Douglas–Rachford algorithm are equivalent, we can write the iteration like in (83) instead:

$$\begin{aligned} & \text{Douglas–Rachford iteration for (98): for } i = 0, 1, \dots \\ & \begin{cases} x^{(i+\frac{1}{2})} \in \arg \min_{x \in \mathcal{X}} \left\{ \tau f(x) + \frac{1}{2} \|Lx - s^{(i)}\|^2 \right\} \\ y^{(i+1)} \in \arg \min_{y \in \mathcal{Y}} \left\{ \tau g(y) + \frac{1}{2} \|Ky - c + 2Lx^{(i+\frac{1}{2})} - s^{(i)}\|^2 \right\} \\ s^{(i+1)} = s^{(i)} - \rho^{(i)}(Lx^{(i+\frac{1}{2})} + Ky^{(i+1)} - c). \end{cases} \end{aligned} \quad (105)$$

In case $\rho^{(i)} > 1$, the Douglas–Rachford form (105) is to be preferred to the ADMM form (104), since it involves less operations.

As an application of Theorem 4.6, we have:

Theorem 4.7 (ADMM (104) or Douglas–Rachford algorithm (105)) Let $y^{(0)} \in \mathcal{Y}$, $\tilde{v}^{(0)} \in \mathcal{U}$, $s^{(0)} \in \mathcal{U}$ be such that $s^{(0)} = -Ky^{(0)} + c - \tilde{v}^{(0)}$. Let $\tau > 0$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(Lx^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ and $(-Ky^{(i)} + c)_{i \in \mathbb{N}}$ defined by the iteration (104), or equivalently by the iteration (105), both converge weakly to some element $r^* \in \mathcal{X}$ solution to (101). Moreover, the sequences $(\tilde{v}^{(i+\frac{1}{2})}/\tau)_{i \in \mathbb{N}}$ and $(\tilde{v}^{(i)}/\tau)_{i \in \mathbb{N}}$ defined by the iteration (104) both converge weakly to some element $u^* \in \mathcal{X}$ solution to (99). In addition, $(Lx^{(i+\frac{1}{2})} + Ky^{(i+1)} - c)_{i \in \mathbb{N}}$ converges strongly to 0.

Furthermore, suppose that \mathcal{X} , \mathcal{Y} , \mathcal{U} are of finite dimension, and that, for every $r \in \mathcal{U}$, the minimizers of $\tau f + \frac{1}{2} \|L \cdot -r\|^2$ and $\tau g + \frac{1}{2} \|K \cdot +r\|^2$ are unique. Then the sequence $(x^{(i+\frac{1}{2})}, y^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (104), or equivalently by the iteration (105), converges to a solution of (98).

Proof Let us prove the second part of the Theorem. The operators $(\tau \partial f + L^*L)^{-1}L^*$ and $(\tau \partial g + K^*K)^{-1}K^*$ are supposed single-valued on \mathcal{U} . Therefore, $(\tau \partial f + L^*L)^{-1}$ is single-valued on $\text{ran } L^*$ and, since it is maximally monotone, it is continuous on $\text{ran } L^*$. Likewise, $(\tau \partial g + K^*K)^{-1}$ is continuous on $\text{ran } K^*$. Hence, $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ converges to $x^* = (\tau \partial f + L^*L)^{-1}L^*(r^* - \tau u^*)$ and $(y^{(i)})_{i \in \mathbb{N}}$ converges to $y^* = (\tau \partial g + K^*K)^{-1}K^*(-r^* + c - \tau u^*)$. Since $Lx^* = r^*$ and $-Ky^* + c = r^*$, we have $0 \in \tau \partial f(x^*) + \tau L^*u^*$ and $0 \in \tau \partial g(y^*) + \tau K^*u^*$. In addition, $Lx^* + Ky^* = c$. Therefore, (x^*, y^*, u^*) is a solution to (100), which implies that (x^*, y^*) is a solution to (98). \square

Here we used infimal postcompositions in the Douglas–Rachford algorithm, but they could be used in any other splitting algorithm. One should study, on a case-by-case basis, if the subproblems corresponding to the proximity operators of the infimal postcompositions are easy to solve, and if this is the case, if it is better to do so or to split the problem with separate calls to the linear operators.

5 The Generalized Chambolle–Pock algorithm

Let \mathcal{X} , \mathcal{U} , \mathcal{V} be real Hilbert spaces. Let $f \in \Gamma_0(\mathcal{V})$, $g \in \Gamma_0(\mathcal{U})$, $c \in \mathcal{X}$. Let $K : \mathcal{X} \rightarrow \mathcal{V}$ and $L : \mathcal{X} \rightarrow \mathcal{U}$ be nonzero bounded linear operators. We consider the following problem, generalizing (63):

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(Kx) + g(Lx) + \langle x, c \rangle. \quad (106)$$

The corresponding monotone inclusion, which we will actually solve, is

$$0 \in K^* \partial f(Kx) + L^* \partial g(Lx) + c. \quad (107)$$

We introduce two dual variables $u \in \mathcal{U}$ and $v \in \mathcal{V}$, so that the problem is to find (x, u, v) such that

$$\begin{cases} v \in \partial f(Kx) \\ u \in \partial g(Lx) \\ 0 \in K^*v + L^*u + c \end{cases}. \quad (108)$$

Accordingly, the dual problem is

$$\underset{(u, v) \in \mathcal{U} \times \mathcal{V}}{\text{minimize}} \quad f^*(v) + g^*(u) \quad \text{s.t.} \quad K^*v + L^*u + c = 0. \quad (109)$$

Clearly, if $K = \text{Id}$ and $L = \text{Id}$, (106) and (109) revert to (89) and (90), and we can use the Douglas–Rachford algorithm (92) to solve these problems. On the other hand, if $K = \text{Id}$ and $c = 0$, (106) and (109) revert to (63) and (66), and we can use the Chambolle–Pock algorithm (67) or (68).

Let $\tau > 0$, $\sigma > 0$, $\eta > 0$, let $x^{(0)} \in \mathcal{X}$, $u^{(0)} \in \mathcal{U}$, $v^{(0)} \in \mathcal{V}$, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of relaxation parameters. We consider the algorithm:

Generalized Chambolle–Pock iteration for (106) and (109): for $i = 0, 1, \dots$

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{\eta f^*} \left(v^{(i)} + \eta K \left(x^{(i)} - \tau(L^*u^{(i)} + K^*v^{(i)} + c) \right) \right) \\ x^{(i+\frac{1}{2})} = x^{(i)} - \tau(L^*u^{(i)} + K^*v^{(i+\frac{1}{2})} + c) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} \left(u^{(i)} + \sigma L \left(2x^{(i+\frac{1}{2})} - x^{(i)} \right) \right) \\ \quad = \text{prox}_{\sigma g^*} \left(u^{(i)} + \sigma L \left(x^{(i+\frac{1}{2})} - \tau(L^*u^{(i)} + K^*v^{(i+\frac{1}{2})} + c) \right) \right) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}) \\ v^{(i+1)} = v^{(i)} + \rho^{(i)}(v^{(i+\frac{1}{2})} - v^{(i)}) \end{cases} \quad (110)$$

The primal–dual inclusion satisfied at every iteration is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} c + L^*u^{(i+\frac{1}{2})} + K^*v^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + \partial g^*(u^{(i+\frac{1}{2})}) \\ -Kx^{(i+\frac{1}{2})} + \partial f^*(v^{(i+\frac{1}{2})}) \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{\tau} \text{Id} & -L^* & 0 \\ -L & \frac{1}{\sigma} \text{Id} & 0 \\ 0 & 0 & \frac{1}{\eta} \text{Id} - \tau KK^* \end{pmatrix}}_P \begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \\ v^{(i+\frac{1}{2})} - v^{(i)} \end{pmatrix}. \quad (111)$$

P is strongly positive if and only if

$$\tau\sigma\|L\|^2 < 1 \quad \text{and} \quad \tau\eta\|K\|^2 < 1. \quad (112)$$

Therefore, since the algorithm is a preconditioned primal–dual proximal point algorithm, we can apply Theorem 2.3 and we get:

Theorem 5.1 (Generalized Chambolle–Pock algorithm (110)) Let $x^{(0)} \in \mathcal{X}$, $u^{(0)} \in \mathcal{U}$, $v^{(0)} \in \mathcal{V}$. Let $\tau > 0$, $\sigma > 0$, $\eta > 0$ be such that $\tau\sigma\|L\|^2 < 1$ and $\tau\eta\|K\|^2 < 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)}, v^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (110) converge weakly to a solution of (106) and to a solution of (109), respectively.

The Generalized Chambolle–Pock algorithm (110) has appeared in the literature under different names, like Alternating Proximal Gradient Method [79], Generalized Alternating Direction Method of Multipliers [80], or Preconditioned ADMM [81]. The convergence results derived in this section generalize previously known results.

If $K = \text{Id}$ and we set $\eta = 1/\tau$, we recover the Chambolle–Pock algorithm form I, to minimize $\tilde{f} + g \circ L$, with $\tilde{f} = f + \langle \cdot, c \rangle$. Indeed, in that case, the first two updates become:

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{f^*/\tau}(x^{(i)}/\tau - L^*u^{(i)} - c) \\ x^{(i+\frac{1}{2})} = x^{(i)} - \tau L^*u^{(i)} - \tau v^{(i+\frac{1}{2})} - \tau c \\ \quad = \text{prox}_{\tau f}(x^{(i)} - \tau L^*u^{(i)} - \tau c) \\ \quad = \text{prox}_{\tau \tilde{f}}(x^{(i)} - \tau L^*u^{(i)}). \end{cases} \quad (113)$$

Therefore, the Generalized Chambolle–Pock indeed generalizes the Chambolle–Pock algorithm to any linear operator K . But we can remark that setting $\eta = 1/\tau$ is not allowed in Theorem 5.1. So, to extend the range of parameters to $\tau\sigma\|L\|^2 \leq 1$ and $\tau\eta\|K\|^2 \leq 1$, we now analyze the algorithm from another point of view: we show that it is a Douglas–Rachford algorithm in a primal–dual space. This analysis is inspired by the one of O’Connor and Vandenberghe [44], showing that the Chambolle–Pock algorithm is a primal–dual Douglas–Rachford algorithm (in a different way than in Section 7.1 with $Q = 0$, mentioned by Yan [49]).

Let $\tau > 0$, $\sigma > 0$, $\eta > 0$ be such that $\tau\sigma\|L\|^2 \leq 1$ and $\tau\eta\|K\|^2 \leq 1$. Let A be a linear operator from \mathcal{A} to \mathcal{V} , for some real Hilbert space \mathcal{A} , such that $KK^* + AA^* = (\tau\eta)^{-1}\text{Id}$; for instance $A = \sqrt{(\tau\eta)^{-1}\text{Id} - KK^*}$ is a valid choice. We do not need to exhibit A , the fact that it exists is sufficient here. Similarly, Let B be a linear operator from \mathcal{B} to \mathcal{U} , for some real Hilbert space \mathcal{B} , such that $LL^* + BB^* = (\tau\sigma)^{-1}\text{Id}$. We introduce the real Hilbert space $\mathcal{X} = \mathcal{X} \times \mathcal{A} \times \mathcal{B}$ and the functions of $\Gamma_0(\mathcal{X})$:

$$F : (x, a, b) \in \mathcal{X} \mapsto f(Kx + Aa) + \iota_0(b), \quad (114)$$

$$G : (x, a, b) \in \mathcal{X} \mapsto g(Lx + Bb) + \iota_0(a), \quad (115)$$

where $\iota_0(s) = (0$ if $s = 0$, $+\infty$ else). Then we can rewrite the problem (106) as:

$$\underset{z \in \mathcal{X}}{\text{minimize}} F(z) + G(z) + \langle z, (c, 0, 0) \rangle. \quad (116)$$

We can now apply the Douglas–Rachford algorithm (92) in the augmented space \mathcal{X} . For this, we need to observe that the proximity operators of F^* and G^* are easy to compute: for any $\tau > 0$, we have [44, eq. 15]

$$\text{prox}_{F^*/\tau} : (x, a, b) \in \mathcal{X} \mapsto (K^*v, A^*v, b) \text{ with } v = \text{prox}_{\eta f^*}(\tau\eta Kx + \tau\eta Aa) \quad (117)$$

$$\text{prox}_{G^*/\tau} : (x, a, b) \in \mathcal{X} \mapsto (L^*u, a, B^*u) \text{ with } u = \text{prox}_{\sigma g^*}(\tau\sigma Lx + \tau\sigma Bb) \quad (118)$$

After some substitutions, notably replacing AA^* by $(\tau\eta)^{-1}\text{Id} - K^*K$ and BB^* by $(\tau\sigma)^{-1}\text{Id} - L^*L$, we recover exactly the algorithm in (110).

Hence, as an application of Theorem 4.5, we get:

Theorem 5.2 (Generalized Chambolle–Pock algorithm (110)) Let $x^{(0)} \in \mathcal{X}$, $u^{(0)} \in \mathcal{U}$, $v^{(0)} \in \mathcal{V}$. Let $\tau > 0$, $\sigma > 0$, $\eta > 0$ be such that $\tau\sigma\|L\|^2 \leq 1$ and $\tau\eta\|K\|^2 \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)}, v^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (110) converge to a solution of (106) and to a solution of (109), respectively. Moreover, $(L^*u^{(i+\frac{1}{2})} + K^*v^{(i+\frac{1}{2})} + c)_{i \in \mathbb{N}}$ converges strongly to 0.

Thus, in practice, one can keep τ as the single parameter to tune and set $\sigma = 1/(\tau\|L\|^2)$ and $\eta = 1/(\tau\|K\|^2)$.

We can write the Generalized Chambolle–Pock algorithm in a different form, with only one call to K, K^*, L, L^* per iteration. For this, we introduce the scaled variable $\tilde{x} = x/\tau$ and auxiliary variables $b = K^*v$ and $r = L^*u + K^*v + c$. Set $b^{(0)} = K^*v^{(0)}$ and $l^{(0)} = L^*u^{(0)} + b^{(0)} + c$. Then we have:

Generalized Chambolle–Pock iteration for (106) and (109): for $i = 0, 1, \dots$

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{\eta f^*}(v^{(i)} + \eta\tau K(\tilde{x}^{(i)} - l^{(i)})) \\ l^{(i+\frac{1}{2})} = l^{(i)} + K^*v^{(i+\frac{1}{2})} - b^{(i)} \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma\tau L(\tilde{x}^{(i)} - 2l^{(i+\frac{1}{2})})) \\ \tilde{x}^{(i+1)} = \tilde{x}^{(i)} - \rho^{(i)}l^{(i+\frac{1}{2})} \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}) \\ v^{(i+1)} = v^{(i)} + \rho^{(i)}(v^{(i+\frac{1}{2})} - v^{(i)}) \\ b^{(i+1)} = b^{(i)} + \rho^{(i)}(l^{(i+\frac{1}{2})} - l^{(i)}) \\ l^{(i+1)} = L^*u^{(i+1)} + b^{(i+1)} + c. \end{cases} \quad (119)$$

Furthermore, let us show that we recover the Loris–Verhoeven algorithm as a particular case when $h : x \mapsto \frac{1}{2}\langle x, Qx \rangle + \langle x, c \rangle$ is quadratic, with $Q = L^*L$ (again, given Q , such an operator L exists). Thus, the problem is to minimize $f \circ K + h$, or equivalently $f \circ K + g \circ L + \langle \cdot, c \rangle$, with $g = g^* = \frac{1}{2}\|\cdot\|^2$. Let us apply the generalized Chambolle–Pock algorithm (110) on this latter problem, with $\sigma = 1$. Since $\text{prox}_{\sigma g^*} = \frac{1}{2}\text{Id}$, we can write the update of u as:

$$u^{(i+\frac{1}{2})} = \frac{1}{2}u^{(i)} + Lx^{(i+\frac{1}{2})} - \frac{1}{2}Lx^{(i)}, \quad (120)$$

so that, if $u^{(0)} = Lx^{(0)}$, we have $u^{(i)} = Lx^{(i)}$, for every $i \in \mathbb{N}$. Hence, we can remove the variable u and rewrite the iteration as:

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{\eta f^*}(v^{(i)} + \eta K(x^{(i)} - \tau(L^*Lx^{(i)} + K^*v^{(i)} + c))) \\ x^{(i+\frac{1}{2})} = x^{(i)} - \tau(L^*Lx^{(i)} + K^*v^{(i+\frac{1}{2})} + c) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ v^{(i+1)} = v^{(i)} + \rho^{(i)}(v^{(i+\frac{1}{2})} - v^{(i)}), \end{cases} \quad (121)$$

which is exactly the Loris–Verhoeven iteration (53). Thus, the Loris–Verhoeven algorithm can be viewed as a primal–dual forward–backward algorithm, but also as a primal–dual Douglas–Rachford algorithm, when h is quadratic.

6 The Condat–Vũ algorithm

Let us consider the primal optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + g(Lx) + h(x), \quad (122)$$

where $f \in \Gamma_0(\mathcal{X})$, $g \in \Gamma_0(\mathcal{U})$, $h : \mathcal{X} \rightarrow \mathbb{R}$ is a convex and Fréchet differentiable function with β -Lipschitz continuous gradient ∇h , for some real $\beta > 0$, and $L : \mathcal{X} \rightarrow \mathcal{U}$ is a bounded linear operator. The corresponding monotone inclusion is

$$0 \in \partial f(x) + L^* \partial g(Lx) + \nabla h(x). \quad (123)$$

Again, we introduce a dual variable u , so that we can rewrite the problem (123) with respect to a pair of objects $z = (x, u)$ in $\mathcal{F} = \mathcal{X} \times \mathcal{U}$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} \partial f(x) + L^* u \\ -Lx + (\partial g)^{-1}(u) \end{pmatrix}}_{Mz} + \underbrace{\begin{pmatrix} \nabla h(x) \\ 0 \end{pmatrix}}_{Cz}. \quad (124)$$

If $(x, u) \in \mathcal{X} \times \mathcal{U}$ is a solution to (124), then x is a solution to (122) and $u \in \partial g(Lx)$ is a solution to the dual problem

$$\underset{u \in \mathcal{U}}{\text{minimize}} (f + h)^*(-L^*u) + g^*(u). \quad (125)$$

The operator $M : \mathcal{F} \rightarrow 2^{\mathcal{F}}$, $(x, u) \mapsto (\partial f(x) + L^*u, -Lx + (\partial g)^{-1}u)$ is maximally monotone [22, Proposition 26.32 (iii)] and $C : \mathcal{F} \rightarrow \mathcal{F}$, $(x, u) \mapsto (\nabla h(x), 0)$ is ξ -cocoercive, with $\xi = 1/\beta$. Thus, it is again natural to think of the forward–backward iteration, with preconditioning. The difference with the construction in Section 3 is the presence of the nonlinear operator ∂f , which prevents us to express $x^{(i+\frac{1}{2})}$ in terms of $x^{(i)}$ and $u^{(i+\frac{1}{2})}$. Instead, the iteration is made explicit by canceling the dependence of $x^{(i+\frac{1}{2})}$ from $u^{(i+\frac{1}{2})}$ in P . That is, the iteration, written in implicit form, is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} \partial f(x^{(i+\frac{1}{2})}) + L^*u^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix}}_{Mz^{(i+\frac{1}{2})}} + \underbrace{\begin{pmatrix} \nabla h(x^{(i)}) \\ 0 \end{pmatrix}}_{Cz^{(i)}} + \underbrace{\begin{pmatrix} \frac{1}{\tau}\text{Id} & -L^* \\ -L & \frac{1}{\sigma}\text{Id} \end{pmatrix}}_P \underbrace{\begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}}_{z^{(i+\frac{1}{2})} - z^{(i)}}, \quad (126)$$

where $\tau > 0$ and $\sigma > 0$ are two real parameters, $z^{(i)} = (x^{(i)}, u^{(i)})$ and $z^{(i+\frac{1}{2})} = (x^{(i+\frac{1}{2})}, u^{(i+\frac{1}{2})})$. Thus, the primal–dual forward–backward iteration is

Condat–Vũ iteration, form I, for (122) and (125): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^*u^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma L(2x^{(i+\frac{1}{2})} - x^{(i)})) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (127)$$

This algorithm was proposed independently by the first author [40] and by B. C. Vũ [82].

An alternative is to update u before x , instead of the converse. This yields a different algorithm, characterized by the primal–dual inclusion

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} \partial f(x^{(i+\frac{1}{2})}) + L^*u^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix}}_{Mz^{(i+\frac{1}{2})}} + \underbrace{\begin{pmatrix} \nabla h(x^{(i)}) \\ 0 \end{pmatrix}}_{Cz^{(i)}} + \underbrace{\begin{pmatrix} \frac{1}{\tau}\text{Id} & L^* \\ L & \frac{1}{\sigma}\text{Id} \end{pmatrix}}_P \underbrace{\begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}}_{z^{(i+\frac{1}{2})} - z^{(i)}}. \quad (128)$$

This corresponds to the primal–dual forward–backward iteration:

Condat–Vũ iteration, form II, for (122) and (125): for $i = 0, 1, \dots$

$$\begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma Lx^{(i)}) \\ x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^*(2u^{(i+\frac{1}{2})} - u^{(i)})) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}). \end{cases} \quad (129)$$

As an application of Theorem 2.2, we obtain the following convergence result [40, Theorem 3.1]:

Theorem 6.1 (Condat–Vũ algorithm (127) or (129)) Let $x^{(0)} \in \mathfrak{X}$ and $u^{(0)} \in \mathfrak{U}$. Let $\tau > 0$ and $\sigma > 0$ be such that $\tau(\sigma\|L\|^2 + \frac{\beta}{2}) < 1$. Set $\delta = 2 - \frac{\beta}{2}(\frac{1}{\tau} - \sigma\|L\|^2)^{-1} > 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined either by the iteration (127) or by the iteration (129) converge weakly to a solution of (122) and to a solution of (125), respectively.

Proof In view of (126) and (128), this is Theorem 2.2 applied to the problem (124). The condition on τ and σ implies that $\sigma\tau\|L\|^2 < 1$, so that P is strongly positive, by virtue of the properties of the Schur complement. Let us establish the cocoercivity of $P^{-1}C$ in \mathfrak{X}_P . In both cases (126) and (128), we have, for every $z = (x, u)$ and $z' = (x', u')$ in \mathfrak{X} ,

$$\|P^{-1}Cz - P^{-1}Cz'\|_P^2 = \langle P^{-1}Cz - P^{-1}Cz', Cz - Cz' \rangle \quad (130)$$

$$= \left\langle \frac{1}{\sigma} \left(\frac{1}{\sigma\tau} \text{Id} - L^*L \right)^{-1} (\nabla h(x) - \nabla h(x')), \nabla h(x) - \nabla h(x') \right\rangle \quad (131)$$

$$\leq \left(\frac{1}{\tau} - \sigma\|L\|^2 \right)^{-1} \langle \nabla h(x) - \nabla h(x'), \nabla h(x) - \nabla h(x') \rangle \quad (132)$$

$$\leq \beta \left(\frac{1}{\tau} - \sigma\|L\|^2 \right)^{-1} \langle x - x', \nabla h(x) - \nabla h(x') \rangle \quad (133)$$

$$= \beta \left(\frac{1}{\tau} - \sigma\|L\|^2 \right)^{-1} \langle z - z', Cz - Cz' \rangle \quad (134)$$

$$= \beta \left(\frac{1}{\tau} - \sigma\|L\|^2 \right)^{-1} \langle z - z', P^{-1}Cz - P^{-1}Cz' \rangle_P. \quad (135)$$

Thus, $P^{-1}C$ is χ -cocoercive in \mathfrak{X}_P , with $\chi = \frac{1}{\beta} \left(\frac{1}{\tau} - \sigma\|L\|^2 \right)$. Moreover, $\chi > 1/2$ if and only if $\tau(\sigma\|L\|^2 + \beta/2) < 1$. Finally, $\delta = 2 - 1/(2\chi)$. \square

We can observe that if $h = 0$, the Condat–Vũ iteration reverts to the Chambolle–Pock iteration, so the former can be viewed as a generalization of the latter. Accordingly, if we set $\beta = 0$ in Theorem 6.1, we recover Theorem 4.1.

The Condat–Vũ algorithm and the Loris–Verhoeven algorithms are both primal–dual forward–backward algorithms, but they are different. When $f = 0$, larger values of τ and σ are allowed in the latter than in the former; this may be beneficial to the convergence speed in practice.

We can mention a different proximal splitting algorithm to solve (122) and (125), proposed by Combettes and Pesquet [24] before the Condat–Vũ algorithm. It is based on the forward–backward–forward splitting technique [22, section 26.6].

For the Condat–Vũ algorithm too, let us focus on the case where h is quadratic; that is,

$$h : x \mapsto \frac{1}{2} \langle x, Qx \rangle + \langle x, c \rangle, \quad (136)$$

for some self-adjoint, positive, nonzero, bounded linear operator Q on \mathfrak{X} and some $c \in \mathfrak{X}$. We have $\beta = \|Q\|$. We can rewrite the primal–dual inclusion (126), which characterizes the Condat–Vũ iteration (127), as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} (\partial f + \nabla h)(x^{(i+\frac{1}{2})}) + L^*u^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + (\partial g)^{-1}u^{(i+\frac{1}{2})} \end{pmatrix}}_{Mz^{(i+\frac{1}{2})}} + \underbrace{\begin{pmatrix} \frac{1}{\tau} \text{Id} - Q & -L^* \\ -L & \frac{1}{\sigma} \text{Id} \end{pmatrix}}_P \underbrace{\begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \end{pmatrix}}_{z^{(i+\frac{1}{2})} - z^{(i)}}. \quad (137)$$

Similarly, we can rewrite the primal–dual inclusion (128), which characterizes the second form of the Condat–Vũ iteration (129), as (137), with $-L$ replaced by L in P .

In both cases, using the properties of the Schur complement, P is strongly positive if and only if

$$\tau\|Q + \sigma L^*L\| < 1 \quad (138)$$

(which implies that $\tau < 1/\beta$). A sufficient condition for this inequality to hold is $\tau(\sigma\|L\|^2 + \beta) < 1$. However, in some applications, $\|Q + \sigma L^*L\|$ may be smaller than $\sigma\|L\|^2 + \beta$, so that larger stepsizes τ and σ may be used when h is quadratic, for the benefit of convergence speed.

Thus, when h is quadratic, the Condat–Vũ iteration can be viewed as a preconditioned Chambolle–Pock iteration. Accordingly, we can apply Theorem 2.3 and obtain convergence, under the condition $\tau\|Q + \sigma L^*L\| < 1$. Instead, let us apply the stronger Theorem 2.6 to allow $\tau\|Q + \sigma L^*L\| = 1$. For this, let us go back to the forward–backward analysis in (126) and (128). We suppose that $\tau\sigma\|L\|^2 < 1$. Then we can strengthen the analysis in (130)–(135): $P^{-1}C$ is χ -cocoercive in \mathcal{E}_P , with $\chi = \|(\frac{1}{\tau}\text{Id} - \sigma L^*L)^{-1}Q\|^{-1}$. Then $\chi \geq 1$ if $\tau\|Q + \sigma L^*L\| \leq 1$. Hence, we have:

Theorem 6.2 (Condat–Vũ algorithm (127) or (129), quadratic case) Suppose that h is quadratic. Let $x^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau > 0$ and $\sigma > 0$ be such that $\tau\sigma\|L\|^2 < 1$ and $\tau\|Q + \sigma L^*L\| \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined either by the iteration (127) or by the iteration (129) converge weakly to a solution of (122) and to a solution of (125), respectively.

We can note that $\tau(\beta + \sigma\|L\|^2) \leq 1$ implies $\tau\sigma\|L\|^2 < 1$ and $\tau\|Q + \sigma L^*L\| \leq 1$.

6.1 An Extended Generalized Chambolle–Pock algorithm

Now that we have seen the different ways to construct primal–dual algorithms based on the forward–backward or proximal point algorithms, we can imagine different variations. For instance, we can extend the Generalized Chambolle–Pock algorithm in Section 5 to deal with additional smooth terms on u and v in the dual problem. Let us look at one particular case of this extension: we add a smooth quadratic term on v . The problem is:

$$\underset{(u,v) \in \mathcal{U} \times \mathcal{V}}{\text{minimize}} \quad f^*(v) + \frac{1}{2}\langle v, Qv \rangle + \langle v, t \rangle + g^*(u) \quad \text{s.t.} \quad K^*v + L^*u + c = 0, \quad (139)$$

for some self-adjoint, positive, nonzero, bounded linear operator Q on \mathcal{V} and some element $t \in \mathcal{V}$. The problem (139) has applications, for instance, to solve inverse problems in imaging regularized with the total variation, as defined by the first author [7]. Again, we design a primal–dual forward–backward algorithm, whose iteration satisfies:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \underbrace{\begin{pmatrix} c + L^*u^{(i+\frac{1}{2})} + K^*v^{(i+\frac{1}{2})} \\ -Lx^{(i+\frac{1}{2})} + \partial g^*(u^{(i+\frac{1}{2})}) \\ -Kx^{(i+\frac{1}{2})} + \partial f^*(v^{(i+\frac{1}{2})}) + t \end{pmatrix}}_{Mz^{(i+\frac{1}{2})}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ Qv^{(i)} \end{pmatrix}}_{Cz^{(i)}} + \underbrace{\begin{pmatrix} \frac{1}{\tau}\text{Id} & -L^* & 0 \\ -L & \frac{1}{\sigma}\text{Id} & 0 \\ 0 & 0 & \frac{1}{\eta}\text{Id} - \tau KK^* \end{pmatrix}}_P \underbrace{\begin{pmatrix} x^{(i+\frac{1}{2})} - x^{(i)} \\ u^{(i+\frac{1}{2})} - u^{(i)} \\ v^{(i+\frac{1}{2})} - v^{(i)} \end{pmatrix}}_{z^{(i+\frac{1}{2})} - z^{(i)}}. \quad (140)$$

Accordingly, the algorithm is

Primal–dual iteration for (139): for $i = 0, 1, \dots$

$$\begin{cases} v^{(i+\frac{1}{2})} = \text{prox}_{\eta f^*} \left(v^{(i)} + \eta K(x^{(i)} - \tau(L^*u^{(i)} + K^*v^{(i)} + c)) - \eta(Qv^{(i)} + t) \right) \\ x^{(i+\frac{1}{2})} = x^{(i)} - \tau(L^*u^{(i)} + K^*v^{(i+\frac{1}{2})} + c) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} \left(u^{(i)} + \sigma L(2x^{(i+\frac{1}{2})} - x^{(i)}) \right) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}) \\ v^{(i+1)} = v^{(i)} + \rho^{(i)}(v^{(i+\frac{1}{2})} - v^{(i)}). \end{cases} \quad (141)$$

The analysis is the same as for the Condat–Vũ algorithm: suppose that $\tau\sigma\|L\|^2 < 1$ and $\tau\eta\|K\|^2 < 1$. Then P is strongly positive and $P^{-1}C$ is χ -cocoercive in \mathcal{E}_P , with $\chi = \|(\frac{1}{\eta}\text{Id} - \tau KK^*)^{-1}Q\|^{-1}$. Moreover, $\chi \geq 1$ if $\eta\| \tau KK^* + Q \| \leq 1$. Hence, we can apply Theorem 2.6 with $\gamma = 1$ and we obtain:

Theorem 6.3 (primal–dual algorithm (141)) Let $x^{(0)} \in \mathcal{X}$, $u^{(0)} \in \mathcal{U}$, $v^{(0)} \in \mathcal{V}$. Let $\tau > 0$, $\sigma > 0$, $\eta > 0$ be such that $\tau\sigma\|L\|^2 < 1$, $\tau\eta\|K\|^2 < 1$, and $\eta\|\tau KK^* + Q\| \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then $(u^{(i)}, v^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (141) converges weakly to a solution of (139).

7 The Primal–Dual Three-Operator Splitting (PD3O) and the Davis–Yin algorithms

We consider the same problem as in the previous section, which involves three functions and a linear operator:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + g(Lx) + h(x). \quad (142)$$

The dual problem is, again:

$$\underset{u \in \mathcal{U}}{\text{minimize}} \quad (f + h)^*(-L^*u) + g^*(u). \quad (143)$$

Let $\tau > 0$ and $\sigma > 0$, let $s^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of relaxation parameters. The Primal–Dual Three-Operator Splitting (PD3O) algorithm proposed by Yan [49] is

PD3O iteration for (142): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(s^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}\left(u^{(i)} + \sigma L(2x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})}) - \tau L^*u^{(i)})\right) \\ s^{(i+1)} = s^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})}) - \tau L^*u^{(i+\frac{1}{2})}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}) \end{cases} \quad (144)$$

(see (175) for a more compact form with one call to ∇h and L^* per iteration).

If h is quadratic (∇h is affine) or if $\rho^{(i)} = 1$, we have $s^{(i)} = x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^*u^{(i)}$ and the PD3O iteration can be rewritten as

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau L^*u^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}\left(u^{(i)} + \sigma L(2x^{(i+\frac{1}{2})} - x^{(i)} + \tau \nabla h(x^{(i)}) - \tau \nabla h(x^{(i+\frac{1}{2})}))\right) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}) \end{cases} \quad (145)$$

Thus, we recover the Chambolle–Pock algorithm when $h = 0$. When $f = 0$, the PD3O algorithm reverts to the Loris–Verhoeven algorithm (with $s^{(i)} = x^{(i+\frac{1}{2})}$ in the former playing the role of $x^{(i)}$ in the latter).

From the analysis by O’Connor and Vandenberghe [44], we have:

Theorem 7.1 (PD3O algorithm (144)) Let $s^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau \in (0, 2/\beta)$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 < 1$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (144) converge weakly to a solution of (142) and to a solution of (143), respectively.

The following result makes it possible to have $\sigma\tau\|L\|^2 = 1$ [44]:

Theorem 7.2 (PD3O algorithm (144)) Suppose that \mathcal{X} and \mathcal{U} are of finite dimension. Let $s^{(0)} \in \mathcal{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau \in (0, 2/\beta)$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 \leq 1$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined by the iteration (144) converge to a solution of (142) and to a solution of (143), respectively.

Note that the finite dimension assumption is not necessary in the proof [44], so it could be removed.

When $\mathcal{X} = \mathcal{U}$ and $L = \text{Id}$, the problem (142) becomes:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + g(x) + h(x), \quad (146)$$

and if we set $\sigma = 1/\tau$, the PD3O algorithm reverts to the Davis–Yin algorithm [83]:

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(s^{(i)}) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*} \left((2x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})})) / \tau \right) \\ \quad = (\text{Id} - \text{prox}_{\tau g})(2x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})})) / \tau \\ s^{(i+1)} = s^{(i)} + \rho^{(i)} (x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})}) - \tau u^{(i+\frac{1}{2})}) \\ \quad = s^{(i)} + \rho^{(i)} \left(\text{prox}_{\tau g}(2x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})})) - x^{(i+\frac{1}{2})} \right), \end{cases} \quad (147)$$

which can be simplified as:

$$\begin{cases} \text{Davis–Yin iteration for (146): for } i = 0, 1, \dots \\ x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(s^{(i)}) \\ s^{(i+1)} = s^{(i)} + \rho^{(i)} \left(\text{prox}_{\tau g}(2x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})})) - x^{(i+\frac{1}{2})} \right). \end{cases} \quad (148)$$

If $f = 0$, the Davis–Yin algorithm reverts to the forward–backward algorithm. If $h = 0$, it reverts to the Douglas–Rachford algorithm. The Davis–Yin algorithm generalizes the forward–Douglas–Rachford algorithm and the Generalized forward–backward algorithm [84–86].

Its main convergence result is: [83]:

Theorem 7.3 (Davis–Yin algorithm (148)) Let $s^{(0)} \in \mathcal{X}$. Let $\tau \in (0, 2/\beta)$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequence $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ defined by the iteration (148) converges weakly to a solution of (146).

Contrary to the Condat–Vũ algorithm, the PD3O algorithm is not a primal–dual forward–backward algorithm; if $\sigma\tau\|L\|^2 < 1$, it can be viewed as a primal–dual Davis–Yin algorithm [44, 49], the same way the Loris–Verhoeven is a primal–dual forward–backward algorithm. The conditions for convergence are the same as for the Loris–Verhoeven algorithm, so that we get the additional function f for free.

7.1 A new algorithm for the quadratic case

Let us continue with the minimization of $f + g \circ L + h$, in the quadratic case where $h(x) = \frac{1}{2}\langle x, Qx \rangle + \langle x, c \rangle$, for some self-adjoint, positive, nonzero, bounded linear operator Q on \mathcal{X} and some $c \in \mathcal{X}$. Set $\beta = \|Q\|$. There seems to be no way to interpret the PD3O algorithm in a different way, to enlarge the relaxation range to $(0, 2)$. However, we can propose a new algorithm, which has the widest range of parameters seen so far.

Let $\tau \in (0, 2/\beta)$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 < 1$. We introduce the self-adjoint, strongly positive, bounded linear operator P in \mathcal{X} :

$$P = \begin{pmatrix} \frac{1}{\tau}\text{Id} - \frac{1}{2}Q & 0 \\ 0 & \frac{1}{\sigma}\text{Id} - \tau LL^* \end{pmatrix}. \quad (149)$$

We define the real Hilbert space \mathcal{E}_P as \mathcal{E} endowed with the inner product $\langle \cdot, \cdot \rangle_P : (z, z') \mapsto \langle z, Pz' \rangle$. We will solve the following monotone inclusion in \mathcal{E}_P :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \underbrace{P^{-1} \begin{pmatrix} \partial f(x) + \frac{1}{2}Qx + c \\ 0 \end{pmatrix}}_{Mz} + \underbrace{P^{-1} \begin{pmatrix} \frac{1}{2}Qx + L^*u \\ -Lx + (\partial g)^{-1}u \end{pmatrix}}_{Nz} \quad (150)$$

Let $s^{(0)} \in \mathfrak{X}$. The operators M and N are maximally monotone in \mathfrak{X}_P , so we will use the Douglas–Rachford algorithm to solve (150) (with stepsize $\gamma = 1$); it can be written as

Douglas–Rachford iteration for (150): for $i = 0, 1, \dots$

$$\begin{cases} z^{(i+\frac{1}{2})} = J_M(s^{(i)}) \\ w^{(i+\frac{1}{2})} = J_N(2z^{(i+\frac{1}{2})} - s^{(i)}) \\ s^{(i+1)} = s^{(i)} + \rho^{(i)}(w^{(i+\frac{1}{2})} - z^{(i+\frac{1}{2})}). \end{cases} \quad (151)$$

We have $J_M : (x, u) \mapsto (\text{prox}_{\tau f}(x - \frac{\tau}{2}Qx - \tau c), u)$, and the resolvent J_N amounts to one iteration of the Loris–Verhoeven algorithm (56), so it maps (x, u) to (x', u') , with:

$$\begin{cases} u' = \text{prox}_{\sigma g^*}(u + \sigma L(x - \frac{\tau}{2}Qx - \tau L^*u)) \\ x' = x - \frac{\tau}{2}Qx - \tau L^*u'. \end{cases} \quad (152)$$

Substituting these two expressions of the resolvents in (151), we obtain the new primal–dual Douglas–Rachford iteration:

Primal–dual Douglas–Rachford iteration, form I, for (142): for $i = 0, 1, \dots$

$$\begin{cases} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(s^{(i)} - \frac{\tau}{2}Qs^{(i)} - \tau c) \\ u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma L(2x^{(i+\frac{1}{2})} - s^{(i)} - \frac{\tau}{2}Q(2x^{(i+\frac{1}{2})} - s^{(i)}) - \tau L^*u^{(i)})) \\ s^{(i+\frac{1}{2})} = x^{(i+\frac{1}{2})} - \frac{\tau}{2}Q(2x^{(i+\frac{1}{2})} - s^{(i)}) - \tau L^*u^{(i+\frac{1}{2})} \\ s^{(i+1)} = s^{(i)} + \rho^{(i)}(s^{(i+\frac{1}{2})} - s^{(i)}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (153)$$

If we switch the roles of M and N in (151), we obtain the iteration:

Primal–dual Douglas–Rachford iteration, form II, for (142): for $i = 0, 1, \dots$

$$\begin{cases} u^{(i+\frac{1}{2})} = \text{prox}_{\sigma g^*}(u^{(i)} + \sigma L(s^{(i)} - \frac{\tau}{2}Qs^{(i)} - \tau L^*u^{(i)})) \\ y^{(i+\frac{1}{2})} = s^{(i)} - \frac{\tau}{2}Qs^{(i)} - \tau L^*u^{(i+\frac{1}{2})} \\ x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(2y^{(i+\frac{1}{2})} - s^{(i)} - \frac{\tau}{2}Q(2y^{(i+\frac{1}{2})} - s^{(i)}) - \tau c) \\ s^{(i+1)} = s^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - y^{(i+\frac{1}{2})}) \\ u^{(i+1)} = u^{(i)} + \rho^{(i)}(u^{(i+\frac{1}{2})} - u^{(i)}). \end{cases} \quad (154)$$

From the convergence of the Douglas–Rachford algorithm (151) [22, Theorem 26.11], we obtain:

Theorem 7.4 (Primal–dual Douglas–Rachford algorithm (153) or (154), quadratic case) Let $s^{(0)} \in \mathfrak{X}$ and $u^{(0)} \in \mathcal{U}$. Let $\tau \in (0, 2/\beta)$ and $\sigma > 0$ be such that $\sigma\tau\|L\|^2 < 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ and $(u^{(i)})_{i \in \mathbb{N}}$ defined either by the iteration (153) or by the iteration (154) converge weakly to a solution of (142) and to a solution of (143), respectively.

In comparison with the Condat–Vũ and the PD3O algorithms, with the new algorithm, τ is allowed to be in $(0, 2/\beta)$, the condition on τ and σ does not depend on Q , and the relaxation range is $(0, 2)$. Like them, the new algorithm reverts to the Chambolle–Pock algorithm when $Q = 0$. The price to pay is two evaluations of Q per iteration.

8 Parallel versions of the algorithms

In this section, we generalize some of the algorithms seen so far, to make it possible to solve optimization problems involving an arbitrary number of functions g_m , $m = 1, \dots, M$, instead of only

one function g . The derived algorithm are parallel, in the sense that the proximity operators of the functions g_m or g_m^* are independent of each other and can be applied in parallel. We propose two techniques for this generalization, which consist in reformulating the problems in product spaces. When $M = 1$, we recover the initial algorithms. But we should keep in mind that the variables, which are updated in parallel with respect to antagonistic functions, are then essentially averaged to form the update of the variable x of interest. This means that the larger M , the slower the convergence to a solution. For instance, it should be much faster to use the Douglas–Rachford algorithm to minimize $f + g$ than to apply any parallel algorithm to minimize $g_1 + g_2$. Similarly, the Generalized Chambolle–Pock algorithm in Section 5 to minimize $f \circ K + g \circ L$ should be faster than the parallel version of the Proximal Method of Multipliers in Section 4.1 to minimize $g_1 \circ L_1 + g_2 \circ L_2$.

Technique 1: Let $M \geq 1$ be an integer. Let \mathcal{X} be a real Hilbert space. We consider the minimization of $f(x) + \sum_{m=1}^M g_m(x) + h(x)$ over $x \in \mathcal{X}$, for some $f \in \Gamma_0(\mathcal{X})$, $g_m \in \Gamma_0(\mathcal{X})$, differentiable and convex function h with β -Lipschitz continuous gradient, for some $\beta \geq 0$. Let $(\omega_m)_{m=1}^M$ be a sequence of positive weights, whose sum is 1. We introduce the real Hilbert space $\mathfrak{X} = \mathcal{X} \times \cdots \times \mathcal{X}$ (M times), endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{X}} : (\mathbf{x}, \mathbf{x}') \mapsto \sum_{m=1}^M \omega_m \langle x_m, x'_m \rangle$. We introduce the function $l_{=} : \mathbf{x} = (x_m)_{m=1}^M \in \mathfrak{X} \mapsto (0 \text{ if } x_1 = \cdots = x_M, +\infty \text{ else})$. Then we can rewrite the function to minimize as $f(\mathbf{x}) + g(\mathbf{x}) + h(\mathbf{x})$, with $f : \mathbf{x} \in \mathfrak{X} \mapsto f(x_1) + l_{=}(x)$, $g : \mathbf{x} \in \mathfrak{X} \mapsto \sum_{m=1}^M g_m(x_m)$, and $h : \mathbf{x} \in \mathfrak{X} \mapsto \sum_{m=1}^M \omega_m \nabla h(x_m)$. Then we have the gradient $\nabla h : \mathbf{x} \mapsto (\nabla h(x_1), \dots, \nabla h(x_M))$, which is β -Lipschitz continuous, and the proximity operators: for any $\gamma > 0$, $\text{prox}_{\gamma f} : \mathbf{x} \mapsto \text{prox}_{\gamma f}(\sum_{m=1}^M \omega_m x_m)$ and $\text{prox}_{\gamma g} : \mathbf{x} \mapsto (\text{prox}_{(\gamma/\omega_1)g_1}(x_1), \dots, \text{prox}_{(\gamma/\omega_M)g_M}(x_M))$. Note that one must be careful when changing the metric; for instance, we have $\partial g : \mathbf{x} \mapsto (\frac{1}{\omega_1} \partial g_1(x_1), \dots, \frac{1}{\omega_M} \partial g_M(x_M))$, $g^*(\mathbf{x}) = (g_1^*(\omega_1 x_1), \dots, g_M^*(\omega_M x_M))$, and $\text{prox}_{\gamma g^*} : \mathbf{x} \mapsto (\frac{1}{\omega_1} \text{prox}_{\gamma \omega_1 g_1^*}(\omega_1 x_1), \dots, \frac{1}{\omega_M} \text{prox}_{\gamma \omega_M g_M^*}(\omega_M x_M))$.

Technique 2: Let $M \geq 1$ be an integer. Let \mathcal{X} and \mathcal{U}_m be real Hilbert spaces. We consider an optimization problem, which involves a term $\sum_{m=1}^M g_m(L_m x)$, for some $g_m \in \Gamma_0(\mathcal{U}_m)$ and bounded linear operators $L_m : \mathcal{X} \rightarrow \mathcal{U}_m$. Let $(\omega_m)_{m=1}^M$ be a sequence of positive weights. We introduce the real Hilbert space $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_M$, endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{U}} : (\mathbf{u}, \mathbf{u}') \mapsto \sum_{m=1}^M \omega_m \langle u_m, u'_m \rangle$. Then $\sum_{m=1}^M g_m \circ L_m = g \circ L$, with $g : \mathbf{u} \in \mathcal{U} \mapsto \sum_{m=1}^M g_m(u_m)$ and $L : x \in \mathcal{X} \mapsto (L_1 x, \dots, L_M x) \in \mathcal{U}$. Then we have, for any $\gamma > 0$, $\text{prox}_{\gamma g} : \mathbf{u} \mapsto (\text{prox}_{(\gamma/\omega_1)g_1}(u_1), \dots, \text{prox}_{(\gamma/\omega_M)g_M}(u_M))$. One must be again careful when changing the metric; for instance, we have $L^* : \mathbf{u} \in \mathcal{U} \mapsto \sum_{m=1}^M \omega_m L_m^* u_m$, $\|L^* \mathbf{u}\| = \|\sum_{m=1}^M \omega_m L_m^* u_m\|$, and $\text{prox}_{\gamma g^*} : \mathbf{u} \mapsto (\frac{1}{\omega_1} \text{prox}_{\gamma \omega_1 g_1^*}(\omega_1 u_1), \dots, \frac{1}{\omega_M} \text{prox}_{\gamma \omega_M g_M^*}(\omega_M u_M))$.

With both techniques, the dual variables must be multiplied by ω_m , so that they converge to solutions to the original dual problems, not the ones with respect to \mathcal{U} with distorted metric.

8.1 The Douglas–Rachford algorithm

Let $M \geq 1$ be an integer. Let \mathcal{X} be a real Hilbert space. Let $f \in \Gamma_0(\mathcal{X})$ and $g_m \in \Gamma_0(\mathcal{X})$, for $m = 1, \dots, M$. We consider the convex optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + \sum_{m=1}^M g_m(x). \quad (155)$$

Let $(\omega_m)_{m=1}^M$ be a sequence of positive weights, whose sum is 1. Let $\tau > 0$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of nonnegative reals. Let $s_m^{(0)} \in \mathcal{X}$, for $m = 1, \dots, M$. By applying Technique 1 to the

Douglas–Rachford algorithm (83), we obtain the iteration:

$$\begin{array}{l}
\text{Douglas–Rachford iteration, form I, for (155): for } i = 0, 1, \dots \\
\left[\begin{array}{l}
x^{(i+\frac{1}{2})} = \text{prox}_{\tau f} \left(\sum_{m=1}^M \omega_m s_m^{(i)} \right) \\
\text{For } m = 1, \dots, M, \\
s_m^{(i+1)} = s_m^{(i)} + \rho^{(i)} \left(\text{prox}_{\frac{\tau}{\omega_m} g_m} (2x^{(i+\frac{1}{2})} - s_m^{(i)}) - x^{(i+\frac{1}{2})} \right).
\end{array} \right.
\end{array} \tag{156}$$

If we switch the roles of f and g , we obtain the iteration:

$$\begin{array}{l}
\text{Douglas–Rachford iteration, form II, for (155): for } i = 0, 1, \dots \\
\left[\begin{array}{l}
\text{For } m = 1, \dots, M, \\
x_m^{(i+\frac{1}{2})} = \text{prox}_{\frac{\tau}{\omega_m} g_m} (s_m^{(i)}) \\
x^{(i+\frac{1}{2})} = \text{prox}_{\tau f} \left(\sum_{m=1}^M \omega_m (2x_m^{(i+\frac{1}{2})} - s_m^{(i)}) \right) \\
\text{For } m = 1, \dots, M, \\
s_m^{(i+1)} = s_m^{(i)} + \rho^{(i)} (x^{(i+\frac{1}{2})} - x_m^{(i+\frac{1}{2})})
\end{array} \right.
\end{array} \tag{157}$$

As an application of Theorem 4.4, we have:

Theorem 8.1 (Douglas–Rachford algorithm (156)) Let $s_m^{(0)} \in \mathcal{X}$, for $m = 1, \dots, M$. Let $\tau > 0$, let $\omega_1 > 0, \dots, \omega_M > 0$ be reals, whose sum is 1, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequence $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ defined either by the iteration (156) or by the iteration (157) converges weakly to a solution of (155).

8.2 The Chambolle–Pock algorithm

Let $M \geq 1$ be an integer. Let \mathcal{X} and \mathcal{U}_m , $m = 1, \dots, M$, be real Hilbert spaces. Let $f \in \Gamma_0(\mathcal{X})$ and let $g_m \in \Gamma_0(\mathcal{U}_m)$, $m = 1, \dots, M$. Let $L_m : \mathcal{X} \rightarrow \mathcal{U}_m$, $m = 1, \dots, M$, be bounded linear operators. We consider the convex optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + \sum_{m=1}^M g_m(L_m x) \tag{158}$$

The dual problem is:

$$\underset{(u_1, \dots, u_M) \in \mathcal{U}_1 \times \dots \times \mathcal{U}_M}{\text{minimize}} \quad f^* \left(-\sum_{m=1}^M L_m^* u_m \right) + \sum_{m=1}^M g_m^*(u_m). \tag{159}$$

Let $(\omega_m)_{m=1}^M$ be a sequence of positive weights. Let $\tau > 0$, $\sigma > 0$, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of nonnegative reals. Let $x^{(0)} \in \mathcal{X}$ and $u_m^{(0)} \in \mathcal{U}_m$, for $m = 1, \dots, M$. Applying Technique 2 to the Chambolle–Pock algorithm (67) or (68) and setting $\sigma_m = \sigma \omega_m$, we obtain the iterations:

$$\begin{array}{l}
\text{Chambolle–Pock iteration, form I, for (158) and (159): for } i = 0, 1, \dots \\
\left[\begin{array}{l}
x^{(i+\frac{1}{2})} = \text{prox}_{\tau f} \left(x^{(i)} - \tau \sum_{m=1}^M L_m^* u_m^{(i)} \right) \\
x^{(i+1)} = x^{(i)} + \rho^{(i)} (x^{(i+\frac{1}{2})} - x^{(i)}) \\
\text{For } m = 1, \dots, M, \\
u_m^{(i+\frac{1}{2})} = \text{prox}_{\sigma_m g_m^*} \left(u_m^{(i)} + \sigma_m L_m (2x^{(i+\frac{1}{2})} - x^{(i)}) \right) \\
u_m^{(i+1)} = u_m^{(i)} + \rho^{(i)} (u_m^{(i+\frac{1}{2})} - u_m^{(i)}).
\end{array} \right.
\end{array} \tag{160}$$

Chambolle–Pock iteration, form II, for (158) and (159): for $i = 0, 1, \dots$

$$\left\{ \begin{array}{l} \text{For } m = 1, \dots, M, \\ \left[\begin{array}{l} u_m^{(i+\frac{1}{2})} = \text{prox}_{\sigma_m g_m^*} (u_m^{(i)} + \sigma_m L_m x^{(i)}) \\ u_m^{(i+1)} = u_m^{(i)} + \rho^{(i)} (u_m^{(i+\frac{1}{2})} - u_m^{(i)}) \end{array} \right. \\ x^{(i+\frac{1}{2})} = \text{prox}_{\tau f} (x^{(i)} - \tau \sum_{m=1}^M L_m^* (2u_m^{(i+\frac{1}{2})} - u_m^{(i)})) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)} (x^{(i+\frac{1}{2})} - x^{(i)}). \end{array} \right. \quad (161)$$

We can remark that when both Technique 1 and Technique 2 are applicable, they give the same result: if $L_m = \text{Id}$, the Chambolle–Pock algorithm (160), with $\sum_{m=1}^M \omega_m = 1$ and $\sigma = 1/\tau$, gives the Douglas–Rachford algorithm (156).

As an application of Theorem 4.2, we have:

Theorem 8.2 (Chambolle–Pock algorithm (160) or (161)) Let $x^{(0)} \in \mathcal{X}$, $u_1^{(0)} \in \mathcal{U}_1, \dots, u_M^{(0)} \in \mathcal{U}_M$. Let $\tau > 0$ and $\sigma_1 > 0, \dots, \sigma_M > 0$ be such that $\tau \|\sum_{m=1}^M \sigma_m L_m^* L_m\| \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)} (2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $((u_1^{(i)}, \dots, u_M^{(i)})_{i \in \mathbb{N}})$ defined either by the iteration (160) or by the iteration (161) converge weakly to a solution of (158) and to a solution of (159), respectively.

8.3 The Loris–Verhoeven algorithm

Let $M \geq 1$ be an integer. Let \mathcal{X} and \mathcal{U}_m , $m = 1, \dots, M$, be real Hilbert spaces. Let $g_m \in \Gamma_0(\mathcal{U}_m)$, $m = 1, \dots, M$, and let $h : \mathcal{X} \rightarrow \mathbb{R}$ be a convex and Fréchet differentiable function with β -Lipschitz continuous gradient, for some real $\beta > 0$. Let $L_m : \mathcal{X} \rightarrow \mathcal{U}_m$, $m = 1, \dots, M$, be bounded linear operators. We consider the convex optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \sum_{m=1}^M g_m(L_m x) + h(x). \quad (162)$$

The dual problem is:

$$\underset{(u_1, \dots, u_M) \in \mathcal{U}_1 \times \dots \times \mathcal{U}_M}{\text{minimize}} \sum_{m=1}^M g_m^*(u_m) + h^*(-\sum_{m=1}^M L_m^* u_m), \quad (163)$$

Let $(\omega_m)_{m=1}^M$ be a sequence of positive weights. Let $\tau > 0$, $\sigma > 0$, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of nonnegative reals. Let $x^{(0)} \in \mathcal{X}$ and $u_m^{(0)} \in \mathcal{U}_m$, for $m = 1, \dots, M$. Applying Technique 2 to the Loris–Verhoeven algorithm (53) and setting $\sigma_m = \sigma \omega_m$, we obtain the iteration:

Loris–Verhoeven iteration for (162) and (163): for $i = 0, 1, \dots$

$$\left\{ \begin{array}{l} \text{For } m = 1, \dots, M, \\ \left[\begin{array}{l} u_m^{(i+\frac{1}{2})} = \text{prox}_{\sigma_m g_m^*} (u_m^{(i)} + \sigma_m L_m (x^{(i)} - \tau \nabla h(x^{(i)}) - \tau \sum_{m=1}^M L_m^* u_m^{(i)})) \\ u_m^{(i+1)} = u_m^{(i)} + \rho^{(i)} (u_m^{(i+\frac{1}{2})} - u_m^{(i)}) \end{array} \right. \\ x^{(i+1)} = x^{(i)} - \rho^{(i)} \tau (\nabla h(x^{(i)}) + \sum_{m=1}^M L_m^* u_m^{(i+\frac{1}{2})}). \end{array} \right. \quad (164)$$

If we introduce the variables $\tilde{s}^{(i)} = x^{(i)} + \tau \sum_{m=1}^M L_m^* u_m^{(i)}$ and $a^{(i)} = x^{(i)} - \tau \nabla h(x^{(i)}) - \tilde{s}^{(i)} =$

$-\tau \nabla h(x^{(i)}) - \tau \sum_{m=1}^M L_m^* u_m^{(i)}$, for every $i \in \mathbb{N}$, we can rewrite the iteration as:

$$\begin{aligned} & \text{Loris-Verhoeven iteration for (162) and (163): for } i = 0, 1, \dots \\ & \begin{cases} a^{(i)} = x^{(i)} - \tau \nabla h(x^{(i)}) - \tilde{s}^{(i)} \\ \tilde{s}^{(i+1)} = \tilde{s}^{(i)} + \rho^{(i)} a^{(i)} \\ \text{For } m = 1, \dots, M, \\ \left[\begin{array}{l} u_m^{(i+1)} = u_m^{(i)} + \rho^{(i)} \left(\text{prox}_{\sigma_m g_m^*} (u_m^{(i)} + \sigma_m L_m (x^{(i)} + a^{(i)})) - u_m^{(i)} \right) \\ x^{(i+1)} = \tilde{s}^{(i+1)} - \tau \sum_{m=1}^M L_m^* u_m^{(i+1)}. \end{array} \right. \end{cases} \end{aligned} \quad (165)$$

As an application of Theorem 3.1 and 3.2, we have:

Theorem 8.3 (Loris-Verhoeven algorithm (165)) Let $x^{(0)} \in \mathcal{X}$, $u_1^{(0)} \in \mathcal{U}_1, \dots, u_M^{(0)} \in \mathcal{U}_M$. Set $\tilde{s}^{(0)} = x^{(0)} + \tau \sum_{m=1}^M L_m^* u_m^{(0)}$. Let $\tau \in (0, 2/\beta)$ and $\sigma_1 > 0, \dots, \sigma_M > 0$ be such that $\tau \|\sum_{m=1}^M \sigma_m L_m^* L_m\| < 1$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $((u_1^{(i)}, \dots, u_M^{(i)}))_{i \in \mathbb{N}}$ defined by the iteration (165) converge weakly to a solution of (162) and to a solution of (163), respectively.

Theorem 8.4 (Loris-Verhoeven algorithm (165)) Suppose that \mathcal{X} and the \mathcal{U}_m are all of finite dimension. Let $x^{(0)} \in \mathcal{X}$, $u_1^{(0)} \in \mathcal{U}_1, \dots, u_M^{(0)} \in \mathcal{U}_M$. Set $\tilde{s}^{(0)} = x^{(0)} + \tau \sum_{m=1}^M L_m^* u_m^{(0)}$. Let $\tau \in (0, 2/\beta)$ and $\sigma_1 > 0, \dots, \sigma_M > 0$ be such that $\tau \|\sum_{m=1}^M \sigma_m L_m^* L_m\| \leq 1$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $((u_1^{(i)}, \dots, u_M^{(i)}))_{i \in \mathbb{N}}$ defined by the iteration (165) converge to a solution of (162) and to a solution of (163), respectively.

When h is quadratic, as an application of Theorem 3.3, we have:

Theorem 8.5 (Loris-Verhoeven algorithm (165), quadratic case) Suppose that h is quadratic. Let $x^{(0)} \in \mathcal{X}$, $u_1^{(0)} \in \mathcal{U}_1, \dots, u_M^{(0)} \in \mathcal{U}_M$. Set $\tilde{s}^{(0)} = x^{(0)} + \tau \sum_{m=1}^M L_m^* u_m^{(0)}$. Let $\tau \in (0, 1/\beta]$ and $\sigma_1 > 0, \dots, \sigma_M > 0$ be such that $\tau \|\sum_{m=1}^M \sigma_m L_m^* L_m\| \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $((u_1^{(i)}, \dots, u_M^{(i)}))_{i \in \mathbb{N}}$ defined by the iteration (165) converge weakly to a solution of (162) and to a solution of (163), respectively.

8.4 The Condat-Vũ algorithm

Let $M \geq 1$ be an integer. Let \mathcal{X} and \mathcal{U}_m , $m = 1, \dots, M$, be real Hilbert spaces. Let $f \in \Gamma_0(\mathcal{X})$, let $g_m \in \Gamma_0(\mathcal{U}_m)$, $m = 1, \dots, M$, and let $h : \mathcal{X} \rightarrow \mathbb{R}$ be a convex and Fréchet differentiable function with β -Lipschitz continuous gradient, for some real $\beta > 0$. Let $L_m : \mathcal{X} \rightarrow \mathcal{U}_m$, $m = 1, \dots, M$, be bounded linear operators. We consider the convex optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + \sum_{m=1}^M g_m(L_m x) + h(x). \quad (166)$$

The dual problem is:

$$\underset{(u_1, \dots, u_M) \in \mathcal{U}_1 \times \dots \times \mathcal{U}_M}{\text{minimize}} \quad (f + h)^* \left(-\sum_{m=1}^M L_m^* u_m \right) + \sum_{m=1}^M g_m^*(u_m). \quad (167)$$

Let $(\omega_m)_{m=1}^M$ be a sequence of positive weights. Let $\tau > 0$, $\sigma > 0$, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of nonnegative reals. Let $x^{(0)} \in \mathcal{X}$ and let $u_m^{(0)} \in \mathcal{U}_m$, for $m = 1, \dots, M$. Applying Technique 2 to the

Condat–Vũ algorithm (127) or (129) and setting $\sigma_m = \sigma \omega_m$, we obtain the iterations:

$$\begin{aligned}
& \text{Condat–Vũ iteration, form I, for (166) and (167): for } i = 0, 1, \dots \\
& \left[\begin{array}{l} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau \sum_{m=1}^M L_m^* u_m^{(i)}) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}) \\ \text{For } m = 1, \dots, M, \\ \left[\begin{array}{l} u_m^{(i+\frac{1}{2})} = \text{prox}_{\sigma_m g_m^*}(u_m^{(i)} + \sigma_m L_m(2x^{(i+\frac{1}{2})} - x^{(i)})) \\ u_m^{(i+1)} = u_m^{(i)} + \rho^{(i)}(u_m^{(i+\frac{1}{2})} - u_m^{(i)}). \end{array} \right. \end{array} \right. \quad (168)
\end{aligned}$$

$$\begin{aligned}
& \text{Condat–Vũ iteration, form II, for (166) and (167): for } i = 0, 1, \dots \\
& \left[\begin{array}{l} \text{For } m = 1, \dots, M, \\ \left[\begin{array}{l} u_m^{(i+\frac{1}{2})} = \text{prox}_{\sigma_m g_m^*}(u_m^{(i)} + \sigma_m L_m x^{(i)}) \\ u_m^{(i+1)} = u_m^{(i)} + \rho^{(i)}(u_m^{(i+\frac{1}{2})} - u_m^{(i)}) \end{array} \right. \\ x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(x^{(i)} - \tau \nabla h(x^{(i)}) - \tau \sum_{m=1}^M L_m^*(2u_m^{(i+\frac{1}{2})} - u_m^{(i)})) \\ x^{(i+1)} = x^{(i)} + \rho^{(i)}(x^{(i+\frac{1}{2})} - x^{(i)}). \end{array} \right. \quad (169)
\end{aligned}$$

If $h = 0$, the Condat–Vũ algorithm reverts to the Chambolle–Pock algorithm (160), (161).
As an application of Theorem 6.1, we have:

Theorem 8.6 (Condat–Vũ algorithm (168) or (169)) Let $x^{(0)} \in \mathcal{X}$, $u_1^{(0)} \in \mathcal{U}_1, \dots, u_M^{(0)} \in \mathcal{U}_M$. Let $\tau > 0$ and $\sigma_1 > 0, \dots, \sigma_M > 0$ be such that $\tau(\|\sum_{m=1}^M \sigma_m L_m^* L_m\| + \frac{\beta}{2}) < 1$. Set $\delta = 2 - \frac{\beta}{2}(\frac{1}{\tau} - \|\sum_{m=1}^M \sigma_m L_m^* L_m\|)^{-1} > 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $((u_1^{(i)}, \dots, u_M^{(i)}))_{i \in \mathbb{N}}$ defined either by the iteration (168) or by the iteration (169) converge weakly to a solution of (166) and to a solution of (167), respectively.

When h is quadratic, as an application of Theorem 6.2, we have:

Theorem 8.7 (Condat–Vũ algorithm (168) or (169), quadratic case) Suppose that h is quadratic. Let $x^{(0)} \in \mathcal{X}$, $u_1^{(0)} \in \mathcal{U}_1, \dots, u_M^{(0)} \in \mathcal{U}_M$. Let $\tau > 0$ and $\sigma_1 > 0, \dots, \sigma_M > 0$ be such that $\tau\|\sum_{m=1}^M \sigma_m L_m^* L_m\| < 1$ and $\tau\|Q + \sum_{m=1}^M \sigma_m L_m^* L_m\| \leq 1$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, 2]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(2 - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i)})_{i \in \mathbb{N}}$ and $((u_1^{(i)}, \dots, u_M^{(i)}))_{i \in \mathbb{N}}$ defined either by the iteration (168) or by the iteration (169) converge weakly to a solution of (166) and to a solution of (167), respectively.

8.5 The Davis–Yin algorithm

Let $M \geq 1$ be an integer. Let \mathcal{X} be a real Hilbert space. Let $f \in \Gamma_0(\mathcal{X})$, let $g_m \in \Gamma_0(\mathcal{X})$, $m = 1, \dots, M$, and let $h : \mathcal{X} \rightarrow \mathbb{R}$ be a convex and Fréchet differentiable function with β -Lipschitz continuous gradient, for some real $\beta > 0$. We consider the convex optimization problem:

$$\text{minimize}_{x \in \mathcal{X}} f(x) + \sum_{m=1}^M g_m(x) + h(x) \quad (170)$$

Let $(\omega_m)_{m=1}^M$ be a sequence of positive weights, whose sum is 1. Let $\tau > 0$ and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of nonnegative reals. Let $s_m^{(0)} \in \mathcal{X}$, for $m = 1, \dots, M$. By applying Technique 1 to the

Davis–Yin algorithm (148), we obtain the iteration:

$$\begin{array}{l} \text{Davis–Yin iteration for (170): for } i = 0, 1, \dots \\ \left[\begin{array}{l} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f} \left(\sum_{m=1}^M \omega_m s_m^{(i)} \right) \\ \text{For } m = 1, \dots, M, \\ s_m^{(i+1)} = s_m^{(i)} + \rho^{(i)} \left(\text{prox}_{\frac{\tau}{\omega_m} g_m} \left(2x^{(i+\frac{1}{2})} - s_m^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})}) \right) - x^{(i+\frac{1}{2})} \right). \end{array} \right. \end{array} \quad (171)$$

As an application of Theorem 7.3, we have:

Theorem 8.8 (Davis–Yin algorithm (171)) Let $s_m^{(0)} \in \mathcal{X}$, for $m = 1, \dots, M$. Let $\tau \in (0, 2/\beta)$ and let $\omega_1 > 0, \dots, \omega_M > 0$ be reals, whose sum is 1. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequence $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ defined by the iteration (171) converges weakly to a solution of (170).

If $h = 0$, we recover the Douglas–Rachford algorithm (156). On the other hand, if $f = 0$, we recover the Generalized forward–backward algorithm [84, 86].

8.6 The PD3O algorithm

Let $M \geq 1$ be an integer. Let \mathcal{X} and \mathcal{U}_m , $m = 1, \dots, M$, be real Hilbert spaces. Let $f \in \Gamma_0(\mathcal{X})$, let $g_m \in \Gamma_0(\mathcal{U}_m)$, $m = 1, \dots, M$, and let $h : \mathcal{X} \rightarrow \mathbb{R}$ be a convex and Fréchet differentiable function with β -Lipschitz continuous gradient, for some real $\beta > 0$. Let $L_m : \mathcal{X} \rightarrow \mathcal{U}_m$, $m = 1, \dots, M$, be bounded linear operators. We consider the convex optimization problem:

$$\underset{x \in \mathcal{X}}{\text{minimize}} \quad f(x) + \sum_{m=1}^M g_m(L_m x) + h(x). \quad (172)$$

The dual problem is:

$$\underset{(u_1, \dots, u_M) \in \mathcal{U}_1 \times \dots \times \mathcal{U}_M}{\text{minimize}} \quad (f + h)^* \left(-\sum_{m=1}^M L_m^* u_m \right) + \sum_{m=1}^M g_m^*(u_m). \quad (173)$$

Let $(\omega_m)_{m=1}^M$ be a sequence of positive weights. Let $\tau > 0$, $\sigma > 0$, and let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence of nonnegative reals. Let $s^{(0)} \in \mathcal{X}$ and let $u_m^{(0)} \in \mathcal{U}_m$, for $m = 1, \dots, M$. Applying Technique 2 to the PD3O algorithm (144) and setting $\sigma_m = \sigma \omega_m$, we obtain the iteration:

$$\begin{array}{l} \text{PD3O iteration for (172) and (173): for } i = 0, 1, \dots \\ \left[\begin{array}{l} x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(s^{(i)}) \\ \text{For } m = 1, \dots, M, \\ \left[\begin{array}{l} u_m^{(i+\frac{1}{2})} = \text{prox}_{\sigma_m g_m^*} \left(u_m^{(i)} + \sigma_m L_m \left(2x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})}) \right) - \tau \sum_{m=1}^M L_m^* u_m^{(i)} \right) \\ u_m^{(i+1)} = u_m^{(i)} + \rho^{(i)} \left(u_m^{(i+\frac{1}{2})} - u_m^{(i)} \right). \end{array} \right. \\ s^{(i+1)} = s^{(i)} + \rho^{(i)} \left(x^{(i+\frac{1}{2})} - s^{(i)} - \tau \nabla h(x^{(i+\frac{1}{2})}) - \tau \sum_{m=1}^M L_m^* u_m^{(i+\frac{1}{2})} \right) \end{array} \right. \end{array} \quad (174)$$

If we introduce the variables $\tilde{s}^{(i)} = s^{(i)} + \tau \sum_{m=1}^M L_m^* u_m^{(i)}$ and $a^{(i)} = x^{(i+\frac{1}{2})} - \tau \nabla h(x^{(i+\frac{1}{2})}) - \tilde{s}^{(i)}$, for

every $i \in \mathbb{N}$, we can rewrite the iteration as:

$$\begin{aligned}
& \text{PD3O iteration for (172) and (173): for } i = 0, 1, \dots \\
& \left[\begin{array}{l}
x^{(i+\frac{1}{2})} = \text{prox}_{\tau f}(\tilde{s}^{(i)} - \tau \sum_{m=1}^M L_m^* u_m^{(i)}) \\
a^{(i)} = x^{(i+\frac{1}{2})} - \tau \nabla h(x^{(i+\frac{1}{2})}) - \tilde{s}^{(i)} \\
\tilde{s}^{(i+1)} = \tilde{s}^{(i)} + \rho^{(i)} a^{(i)} \\
\text{For } m = 1, \dots, M, \\
\left[u_m^{(i+1)} = u_m^{(i)} + \rho^{(i)} \left(\text{prox}_{\sigma_m g_m^*}(u_m^{(i)} + \sigma_m L_m(x^{(i+\frac{1}{2})} + a^{(i)})) - u_m^{(i)} \right). \right.
\end{array} \right. \tag{175}
\end{aligned}$$

If $f = 0$, the PD3O algorithm (175) reverts to the Loris–Verhoeven algorithm (165). If $h = 0$, it reverts to the Chambolle–Pock algorithm (160).

We can check that if $L_m = \text{Id}$ for every m and $\omega = 1$, by setting $\sigma_m = \omega_m \sigma$, $\sigma = 1/\tau$, $s_m^{(i)} = s^{(i)} - \tau u_m^{(i)} + \tau \sum_{m=1}^M \omega_m u_m^{(i)}$ in the PD3O algorithm (174), we recover the Davis–Yin algorithm (171).

As an application of Theorem 7.2, we have:

Theorem 8.9 (PD3O algorithm (175)) Suppose that \mathcal{X} and \mathcal{U} are of finite dimension. Let $s^{(0)} \in \mathcal{X}$ and $u_1^{(0)} \in \mathcal{U}_1, \dots, u_M^{(0)} \in \mathcal{U}_M$. Set $\tilde{s}^{(0)} = s^{(0)} + \tau \sum_{m=1}^M L_m^* u_m^{(0)}$. Let $\tau \in (0, 2/\beta)$, $\sigma_1 > 0, \dots, \sigma_M > 0$ be such that $\tau \|\sum_{m=1}^M \sigma_m L_m^* L_m\| \leq 1$. Set $\delta = 2 - \tau\beta/2$. Let $(\rho^{(i)})_{i \in \mathbb{N}}$ be a sequence in $[0, \delta]$ such that $\sum_{i \in \mathbb{N}} \rho^{(i)}(\delta - \rho^{(i)}) = +\infty$. Then the sequences $(x^{(i+\frac{1}{2})})_{i \in \mathbb{N}}$ and $((u_1^{(i)}, \dots, u_M^{(i)}))_{i \in \mathbb{N}}$ defined by the iteration (175) converge to a solution of (172) and to a solution of (173), respectively.

9 Conclusion

We have performed a small tour of proximal splitting algorithms and we have shown how a principled analysis in terms of primal–dual monotone inclusions allows us to derive existing and new algorithms, with general convergence guarantees. Several connections between apparently distinct algorithms have been established, and parallel versions for an arbitrary number of functions have been derived. Thus, we have unleashed the relaxation potential of a large class of proximal splitting algorithms. We recommend everyone to try overrelaxation as follows: if for instance convergence is guaranteed for $\rho^{(i)} \in (0, 2)$, first find the appropriate parameter values, like τ , with $\rho^{(i)} = 1$, and only then try $\rho^{(i)} = 1.9$. Overrelaxation will accelerate convergence in most cases. The paper is by no way exhaustive and covers only a subset of proximal splitting methods. For instance, we did not cover the forward–backward–forward splitting method and its applications [24, 87, 88] [22, section 26.6], or Dykstra-like algorithms [89, 90] [22, section 30.2]. Moreover, research around proximal splitting techniques is still an active topic, with recent contributions like the asymmetric forward-backward-adjoint algorithm [91], the forward-backward-half forward algorithm [92], the forward-reflected-backward algorithm [93], or the Forward-Douglas–Rachford-Forward algorithm [94]. There are many possible extensions of proximal splitting algorithms, including the use of variable stepsizes or metrics [48, 95], the study of convergence rates and development of accelerated variants [96–100], or the design of randomized versions [101–105]. It will be interesting to combine these notions with the new gained insights on proximal splitting.

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