Multiobjective Optimization Under Uncertainty: A Multiobjective Robust (Relative) Regret Approach

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Abstract

Consider a multiobjective decision problem with uncertainty given as a set of scenarios. In the single criteria case, robust optimization methodology helps to identify solutions which remain feasible and of good quality for all possible scenarios. An alternative method is to compare the possible decisions under uncertainty against the optimal decision with the benefit of hindsight, i.e. to minimize the (possibly scaled) regret of not having chosen the optimal decision. In this exposition, we extend the concept of regret to the multiobjective setting and introduce a proper definition of multivariate (relative) regret. All early attempts in such a setting mix scalarization and optimization, whereas we first model regret and then solve the resulting problem separately. Moreover, in contrast to the existing approaches, we are not limited to a finite uncertainty sets or interval uncertainty and further, computations remain tractable in most common special cases.

Keywords: multiobjective optimization, robust optimization, minmax regret, scalarization, semi-infinite optimization

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1 Introduction

1.1 Motivation

The task of decision making under uncertainty appears in various fields. Quite often, the considered decision problem cannot be expressed by standard formulation as an optimization problem with a single objective. Instead, multiple conflicting criteria have to be considered and thus a formulation as an uncertain (or parametric) multiobjective problem is required.\footnote{Although we very briefly recall the main concepts necessary for following our exposition, we assume a certain familiarity of the reader with the concepts of robust optimization and multiobjective optimization.} In recent past, several promising approaches have emerged which allow to generalize the idea of a robust counterpart of an uncertain single-objective problem to the multicriteria setup, see for instance [7, 8, 16]. All these approaches have in common that they rely on the simple and intuitive (albeit quite conservative) strategy to maximize the worst possible outcome among the possible scenarios.

Although the concept of a robust counterpart to an uncertain single objective problem in the gist of [11] is probably most popular, there also exists a more credible alternative based on the notion of regret, see e.g. [12, 14] for the first treatment of regret and [11] for a recent similar analysis in the single-objective setting. In summary, it can be noted that robust regret leads to computationally harder problems than classical robust optimization – however, the most common cases still remain computationally tractable. In Figure 1 we have illustrated the difference of the two approaches in the single objective case.

In contrast to the single-objective case, no corresponding concept of regret is available for the case of an uncertain multiobjective problem. The only existing approaches in this direction so far were given by [19] and [23]. Still, none of these approaches actually considers a proper concept of regret in the multiobjective case. Instead, it is suggested to first scalarize the multiobjective problem which is then tackled within the known single-objective regret setting. In the following, we will close this gap and introduce a proper concept of multivariate (relative) regret. We will motivate the choice of this regret formulation and show that this indeed represents a generalization of the single-objective setting to the multicriteria case. We further analyze the structure of the corresponding multiobjective maximum regret formulation. We especially show that this problem is almost as easily solvable as a standard multiobjective problem in a variety of common special cases. This analysis is in complete analogy to the analysis in the single-objective case, cf. [11]. Finally, we compare our approach to the related approaches by [5, 19, 23] in more detail. This especially allows to put these approaches into broader context and discuss similarities and differences on a novel basis.

1.2 Problem formulation

Before we introduce the general multiobjective setup, let us first briefly recall the main idea of (relative) regret in the single-objective case.
Figure 1: Illustration of worst-case optimization versus regret optimization under the assumption $F(x^*_{\text{RC}}) = \max_{u \in U} f^*(u)$.

1.2.1 Uncertain single-objective problems

Let us consider the following (family of) uncertain optimization problem(s)

$$\min_{x \in X} f(x, u)$$

(P(u))

where $f : X \times U \to \mathbb{R}$ is some continuous function, $x \in X \subset \mathbb{R}^n$ represents the decision variables and $u \in U \subset \mathbb{R}^n$ represents uncertain parameters. For simplicity of presentation, let us assume that both $X$ and $U$ are compact. The robust counterpart to (P(u)) in the gist of [1] is given as

$$\min_{x \in X} F(x)$$

(RC(U))

with $F(x) := \max_{u \in U} f(x, u)$. It is well-known, see e.g. [1], that (RC(U)) can be solved efficiently, if the original uncertain problem satisfies certain structural requirements, e.g. $f$ has to be convex in $x$ and $F$ needs to be easily computable. However, it is also well-known that (RC(U)) leads to rather conservative solutions due to the fact that its solution $x^*_{\text{RC}}$ is focused on the worst case instance only. As a remedy to this problem, the more credible concept of (relative) regret can be applied, see e.g. [14] for a more detailed discussion. In the single-objective setting, the regret of a decision $x$ in a scenario $u \in U$ is defined as

$$r(x, u) := f(x, u) - f^*(u), \quad \text{with} \quad f^*(u) := \min_{x \in X} f(x, u),$$

while the relative or scaled regret of a decision $x$ in a scenario $u \in U$ can be represented as

$$s(x, u) := \frac{f(x, u) - f^*(u)}{f^*(u)} = \frac{f(x, u)}{f^*(u)} - 1 = \frac{1}{f^*(u)} r(x, u),$$

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if \( f^*(u) > 0 \) holds for all \( u \in U \). It usually depends on the given application, whether regret or relative regret is considered by the decision maker. Based on the (relative) regret, the decision maker can now optimize with respect to regret. In uncertain environments, similar to the robust counterpart, a worst case (relative) regret is often deemed appropriate; hence we introduce

\[
R(x) := \max_{u \in U} r(x, u),
\]

and, analogously,

\[
S(x) := \max_{u \in U} s(x, u).
\]

This leads to the corresponding robust (relative) regret counterparts

\[
\min_{x \in X} R(x), \quad (RR(U))
\]

and

\[
\min_{x \in X} S(x), \quad (RS(U))
\]

The differences of the robust counterpart to the robust regret counterpart are illustrated in Figure 1. As it can be observed, the worst case performance of the robust counterpart is (by definition) better than the worst case performance of the minimum regret decision. The latter, however, usually comes with a much lower worst case opportunity loss. Accordingly, in [11], solutions to these problems are called robust (relative) regret solutions and in [14], they are called robust (relative) deviation decisions. For interval uncertainty, this approach can also already be found in [12].

**Remark 1.1.** It needs to be mentioned that stochastic programming represents an important (quite related) approach to optimization under uncertainty. There, the uncertain parameter \( u \) is treated as a random object and probabilistic criteria are applied instead of a worst case criterion. The same distinction can of course be made here, leading to a framework of stochastic regret, where instead the expected regret or some risk measure of regret could be considered. As this is beyond the scope of this exposition, we prefer to leave this for future research.

We have already mentioned that the robust counterpart can be solved efficiently in a variety of common setups; let us refer to [1] for an overview and more details. We will demonstrate in the following that although the robust regret counterpart represents a somewhat more involved concept (due to the fact that the optimal value \( f^*(u) \) appears in the formulation), it can still be solved reasonably fast in common specific situations, see e.g. [11] for a detailed analysis of regret in the single-objective case. As we will see later in Section 4, this remains to be true for our generalization of robust (relative) regret to the multicriteria setup.

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2If the goal is not to compare the decision to the optimal solution, but to some given benchmark, one can consider the so-called benchmark regret as introduced in [20]. In this case we only need to consider some benchmark performance \( b(u) \) replacing the individual optimal solutions \( f^*(u) \) in the regret formulation plus requiring the same structural properties for \( b(u) \) as for \( f^*(u) \).
1.2.2 Uncertain multiobjective problems

If instead of an uncertain single-objective problem an uncertain multicriteria problem is given, the above considerations get somewhat more involved. As already mentioned, there have been successful approaches how to formulate a robust counterpart for an uncertain multiobjective problem, while the corresponding concept for (relative) regret is still missing. For this purpose, let now \( f : X \times U \to \mathbb{R}^m \) be an \( m \)-variate continuous objective function, representing potentially conflicting aims. The uncertain multiobjective problem then reads as

\[
\min_{x \in X} \mathbb{R}^m \geq f(x, u).
\]

Multiobjective problems constitute a special case of vector optimization problems (see for example [13]), where the optimization is carried out with respect to a general ordering cone \( K \). In the case of multiobjective problems, the corresponding ordering cone is always given by the specific cone \( \mathbb{R}^m_{\geq} \). In contrast to the single-objective setup, minima of the image set \( \phi(u) := \{ f(x, u) \mid x \in X \} \) will not exist in general. Therefore, usually minimal elements of \( \phi(u) \), or, respectively, so-called efficient (or Pareto optimal) solutions in \( \phi^{-1}(u) \) are sought for (see e.g. [6, 13]). In our setup, a solution \( \bar{x} \in X \) is called efficient (for fixed parameter \( u \)) iff there does not exist \( x \in X \) with \( f_i(x, u) \leq f_i(\bar{x}, u) \) for every \( i = 1, \ldots, m \) and \( f_j(x, u) < f_j(\bar{x}, u) \) for at least one \( j \in \{1, \ldots, m\} \). A slightly weaker concept is the following: a solution \( \bar{x} \in X \) is called weakly efficient iff there does not exist \( x \in X \) with \( f_i(x, u) < f_i(\bar{x}, u) \) for every \( i = 1, \ldots, m \).

As recently suggested by [7, 8, 16], a reasonable, albeit rather conservative formulation for a multiobjective robust counterpart, taking into account all uncertainty, looks as follows:

\[
\min_{x \in X} \mathbb{R}^m \geq F(x),
\]

where, in complete analogy to the single-objective case, \( F_i(x) := \max_{u \in U} f_i(x, u) \) for \( i = 1, \ldots, m \).

**Remark 1.2.** Let us point out that although formulation \( \min_{x \in X} \mathbb{R}^m \geq F(x) \) seems quite straightforward, a thorough discussion of the pros and cons of such a formulation had been appropriate and necessary, see especially [7] and [8]. As pointed out there, it is not instantaneously clear how to interpret the term ”\( \max_{u \in U} f(x, u) \)" for an \( m \)-variate \( f \). Several interpretations, ranging from the straightforward idea of a kind of anti-efficient frontier with respect to the negative ordering cone to a set-valued interpretation have appeared in the literature. For the standard cone, the above formulation has shown to be quite successful in terms of interpretation, application and ease of computation.

Subsequently, we will show that we can again obtain a multiobjective robust (relative) regret formulation along the same lines as the multiobjective robust counterpart has been obtained, thus generalizing the concept of regret from the univariate to the multivariate setup. For this purpose, the remainder of the paper is organized as follows: In Section 2 we introduce and motivate the generalization of robust regret to the multivariate case. To solve these multiobjective problems, scalarization methods are usually employed, for which reason we provide a brief survey in Section 3. We pay special attention to the order of robustification and scalarization. We specifically prove
that weighted Chebyshev scalarization actually commutes with robustification in our context. We then move to the numerical analysis of the multiobjective robust regret formulations in Section 4 and we show that these can be solved efficiently (or at least be well-approximated) in the most common setups. We close our exposition by a detailed comparison to existing approaches in Section 5.

2 Multiobjective robust (relative) regret

As already discussed, there is currently no extension of the robust (relative) regret approach for uncertain problems to the multiobjective setting. The main question in this context is how to replace the scalar term $f^*(u)$ in a multivariate regret formulation. It has to be noted that in the multiobjective setup, the optimal value is no longer an unique scalar value, but a whole efficient frontier. From this observation, as outlined in Remark 1.2, it is not straightforward, what quantity should be used to compare to $f(x, u)$ to obtain a meaningful notion of regret.

2.1 Extension of robust regret to the multiobjective setting

In the following, we will argue that a meaningful choice will be obtained, if the scalar value $f^*(u)$ is replaced by the corresponding ideal point $\pi(u)$ in the multiobjective setting, defined by

$$
\pi_i(u) := f^*_i(u) := \min_{x \in X} f_i(x, u), \quad \text{for } i = 1, \ldots, m.
$$

For this argumentation, let us have a closer look at the single-objective setup. In this setup it holds:

$$
\min_{x \in X} R(x) = \min_{x \in X} \max_{u \in U} r(x, u).
$$

Introducing a slack variable $\alpha$ for the objective function leads to

$$
\min_{x \in X} R(x) = \min_{x \in X} \alpha \quad \alpha \in \mathbb{R}
$$

s.t. $r(x, u) \leq \alpha \quad \forall u \in U.$

The latter constitutes a semi-infinite problem, as we have to consider an infinite number of constraints (for general uncertainty $U$). We can continue the reformulation and obtain

$$
\min_{x \in X} R(x) = \min_{x \in X} \alpha \quad \alpha \in \mathbb{R}
$$

s.t. $f(x, u) \leq \alpha + f(y, u) \quad \forall u \in U, \forall y \in X.$

Now we make the important observation that this final semi-infinite reformulation can be easily generalized to multivariate functions $f$ (together with a corresponding slack $\alpha \in \mathbb{R}^m$) and general
ordering cones $K \subset \mathbb{R}^m$ by replacing the ordinary inequality $\leq$ with the cone inequality $\leq_K$:

$$\min_{x \in X} \alpha \quad \text{s.t.} \quad f(x, u) \leq_K \alpha + f(y, u) \quad \forall u \in U, \forall y \in X.$$ 

Quite importantly, as we only consider the specific setup $K = \mathbb{R}^m$, it is sufficient to consider each row of the cone inequality separately and thus we obtain

$$\min_{x \in X} \alpha \quad \text{s.t.} \quad f_i(x, u) \leq \alpha_i + f_i^*(u) \quad \forall u \in U, \forall y \in X, \forall i = 1, \ldots, m.$$ 

Due to the special structure of the constraint, which can be interpreted as rowwise uncertainty, we can get rid of the semi-infinite constraint in $y$, which yields

$$\min_{x \in X} \alpha \quad \text{s.t.} \quad f_i(x, u) \leq \alpha_i + f_i^*(u) \quad \forall u \in U, \forall i = 1, \ldots, m.$$ 

This can be further simplified to

$$\min_{x \in X} \alpha \quad \text{s.t.} \quad \max_{u \in U} f_i(x, u) - f_i^*(u) \leq \alpha_i \quad \forall i = 1, \ldots, m.$$ 

Eliminating the slack variable $\alpha$, we finally arrive at the multiobjective robust regret problem

$$\min_{x \in X} R(x) \quad (RR^{(m)}(U))$$

with $R_i(x) := \max_{u \in U} r_i(u, x) := \max_{u \in U} f_i(x, u) - f_i^*(u)$. We immediately see that this indeed represents a generalization of the single-objective regret formulation to a multivariate setting. Quite obviously, the same arguments can be repeated to obtain the multiobjective robust relative (or scaled) regret problem

$$\min_{x \in X} S(x) \quad (RS^{(m)}(U))$$

Let us point out that both $(RR^{(m)}(U))$ and $(RS^{(m)}(U))$, like the robust counterpart $(RC(U))$ constitute classical (potentially non-convex) multiobjective problems. Nevertheless, the subsequent proposition shows that in case of convex uncertain problems, convexity transfers to the robust regret problems as well.

**Proposition 2.1.** Let $f$ be convex in $x$ for all $u$ in $U$. Then $R$ and $S$ are convex as well.

**Proof.** This follows directly from the well-known fact that the supremum of convex functions is
again convex and that $r(x, u)$ and $s(x, u)$ are convex in $x$ for all $u$.

### 2.2 Alternative motivation

By a closer inspection of the robust (relative) regret, it can be observed that there is a close relationship to the robust counterpart. Indeed, starting with the family of uncertain problems $(P(u))$, we can shift (or scale and shift, respectively) each objective function to obtain the uncertain families

$$
\min_{x \in X} f(x, u) - f^*(u), \quad (P'(u))
$$

and

$$
\min_{x \in X} \frac{f(x, u)}{f^*(u)} - 1, \quad (P''(u))
$$

respectively. Let us note that these transformations do not change the optimal solution in each scenario, and hence can be seen as equivalent to $(P(u))$. Now, it becomes obvious that the robust counterpart to $(P'(u))$ is exactly $(RR(U))$ and the robust counterpart to $(P''(u))$ coincides with $(RS(U))$. The same is of course true for the multiobjective case. Let us emphasize that – in alternative to our motivation presented above – we could have introduced the robust multiobjective regret concept via the above considerations instead.

These observations especially imply that all discussions on robust counterparts to multiobjective problems directly transfer to the case of multiobjective regret. We specifically would like to emphasize that experience over recent years has shown that it is more beneficial to first formulate a version of a multiobjective robust problem, and then to scalarize, instead of the other way round, cf. [8].

### 3 Scalarization techniques to solve certain multiobjective problems

For the discussion of scalarization techniques, let us drop the uncertain parameter $u$ for the moment and let us consider the (certain) multiobjective problem

$$
\min_{x \in X} \begin{pmatrix} g_1(x) \\
\vdots \\
g_m(x) \end{pmatrix} =: \min_{x \in X} g(x) \quad (MOP)
$$

for some $g : X \to \mathbb{R}^m$. In the following, we focus on the most important properties of the three most common scalarization techniques to solve $(MOP)$ and especially problems as in $(RR^{(m)}(U))$ or $(RS^{(m)}(U))$. All these methods have in common that a solution of the scalarized problem is at least weakly efficient for $(MOP)$. Under additional assumptions, solutions of scalarized problems are even known to be efficient. Further, under stronger assumptions like convexity, these methods actually yield all (weakly) efficient solutions.
3.1 The $\varepsilon$-constraint scalarization method

As a first, rather general method, let us consider the $\varepsilon$-constraint scalarization method, cf. [6, Section 4.1]. For some predefined $\varepsilon \in \mathbb{R}^m$ and a fixed index $j \in \{1, \ldots, m\}$, we consider the problem

$$\min_{x \in X} \ g_j(x) \quad (ECM_j(\varepsilon))$$

$$\text{s.t. } g_k(x) \leq \varepsilon_k \quad \forall k = 1, \ldots, m, \ k \neq j.$$ 

Then each optimal solution of $(ECM_j(\varepsilon))$ is already a weakly efficient solution of $(MOP)$, cf. [6, Proposition 4.3]. Moreover, as the following result states, all efficient solutions of $(MOP)$ can be found, cf. [6, Theorem 4.5].

**Proposition 3.1.** A feasible solution $x \in X$ is an efficient solution of $(MOP)$ if and only if there exists $\varepsilon \in \mathbb{R}^m$ such that $x$ is an optimal solution of all $(ECM_j(\varepsilon))$ for $j = 1, \ldots, m$.

However, it is not possible to recover all weakly efficient solutions by this scalarization method. We further note that the $\varepsilon$-constraint scalarization method works without any additional assumptions like convexity, etc.

3.2 The weighted sum method

The weighted sum scalarization, probably the most common approach to solve $(MOP)$, considers a weight vector $\lambda \in \mathbb{R}^m_\geq$ and the corresponding single criteria problem

$$\min_{x \in X} \sum_{i=1}^{m} \lambda_i g_i(x). \quad (WSS(\lambda))$$

If $\lambda > 0$ ($\lambda \geq 0$), solving $(WSS(\lambda))$ returns a (weakly) efficient solution, c.f. [6], Proposition 3.9. In a convex setting, a stronger result can be obtained, cf. [6], Proposition 3.10, which yields that all weakly efficient solutions can be obtained by weighted sum scalarization.

**Proposition 3.2.** If $X$ is convex and $g_1, \ldots, g_m$ are convex functions, for any weakly efficient solution $\bar{x}$ of $(MOP)$ there exists $\lambda \geq 0$ such that $\bar{x}$ is as an optimal solution of $(WSS(\lambda))$.

3.3 Weighted Chebyshev scalarization

The third approach we would like to mention is called weighted Chebyshev scalarization, c.f. [2], also called compromise solution, cf. [6], Section 4.5, and [9]. Unfortunately, the literature is not completely consistent when considering the definition of the weighted Chebyshev scalarization: in some references the ideal point $\pi$ is used, whereas other references consider some arbitrary utopian point $\mu$ instead (i.e. $\mu < \pi$). Further, some references work with a weight vector $w \in \mathbb{R}^m_\geq$, whereas other work with the reciprocal weight vector instead. We here follow [6], Section 4.5, who works with the ideal point and a weight vector $w \in \mathbb{R}^m$ for the weighted Chebyshev scalarization:

$$\min_{x \in X} \max_{i=1, \ldots, m} \ w_i (g_i(x) - \pi_i). \quad (WCS(w))$$
According to [6], Proposition 4.22, each solution of (WCS(w)) yields a weakly efficient solution of (MOP). If we instead use an utopian point $\mu$, some stronger result can be obtained, cf. [6, Theorem 4.24].

**Proposition 3.3.** If an utopian point $\mu$ is used instead of $\pi$, a feasible solution $x \in X$ is weakly efficient if and only if some weight vector $w \in \mathbb{R}_+^m$ exists such that $x$ solves (WCS(w)).

This means that all weakly efficient solutions can be obtained in such a way. Let us note that these results are valid without any additional convexity assumption.

In the setup of multiobjective regret, we are in a rather special situation: In our case, 0 can often be assumed to be an utopian point both for $R$ and $S$, i.e. $R_i^* > 0$ and $S_i^* > 0$, while clearly 0 is just an ideal point (and not utopian) for $r(x, u)$ and $s(x, u)$ for any fixed $u$. If 0 is not utopian to $R$ or $S$ it would mean that there is an $\hat{x} \in X$ which is optimal for each uncertain parameter $u$ – a situation which indicates that uncertainty does actually not play a role for this component of the objective function. Thus, according to the above, the weighted Chebyshev scalarizations with 0 as utopian point yield all weakly efficient solutions both for $R$ and $S$, while it remains open if all weakly efficient solutions can be found in case $R^*$ or $S^*$ has a 0 component. Nevertheless, this can be easily avoided by scalarizing based on a reference point with all entries slightly negative.

### 3.4 Remarks on the order of regret and scalarization

Here, it is in order to say a few words about the question, whether one should first scalarize an uncertain multiobjective problem and then consider regret, or, if first a multiobjective regret formulation is established and then scalarized. As the regret formulation is closely linked to a robust counterpart formulation (as argued in Section 2.2), let us note here the following: Applying the weighted sum scalarization (or the $\varepsilon$-constraint method) first and then robustifying is not the same as robustifying in the multicriteria setting and then applying the mentioned scalarization. A more detailed discussion of this issue can for instance be found in [8]. Hence, robustification and scalarization will not commute in general. We believe that as scalarization is a computational tool to solve a given problem and not a modeling paradigm as such, our setting seems to be the more natural one. Furthermore, in our opinion, sticking to the multicriteria setting as we do, is more intuitive from a modelers perspective. Finally, our approach now allows to put all existing approaches, cf. Section 5, into more context and also allows for better interpretation.

### 3.5 Chebyshev scalarization commutes with robustification

Interestingly, to the best of our knowledge, until now, no scalarization method has been identified which commutes with robustification. In the following, we will show that in the given situation it holds that weighted Chebyshev scalarization indeed commutes with robustification.

As can be easily seen, 0 represents the ideal point of $r(x, u)$ as well as for $s(x, u)$ for each fixed $u$. Thus, it holds for the robustified scalarized regret:

$$\max_{u \in U} \max_{1 \leq i \leq m} w_i r_i(x, u) = \max_{u \in U} \max_{1 \leq i \leq m} w_i r_i(x, u) = \max_{u \in U} \max_{1 \leq i \leq m} w_i R_i(x).$$
The right hand side of this equation coincides with the scalarized robust regret using 0 as reference point for \( R \), thus Chebyshev scalarization commutes with robustification. The same arguments hold of course for \( S \) instead of \( R \). We further note that this argumentation remains true for a general uncertain \( f \) instead of \( r \) or \( s \), as long as 0 represents a reasonable reference point – thus extending the analysis on commutation given in [8]. The reason lying behind this surprising result is the connection of the weighted Chebyshev scalarization to the \( \varepsilon \)-constraint scalarization method. As shown in [8], the \( \varepsilon \)-constraint scalarization method commutes with robustification of generalized instances of uncertain problems, a property which has been used above when switching the order of maximization in \( u \) and \( i \).

4 Numerical analysis

Subsequently, we will analyse the numerical tractability of problems \( RR^{(m)}(U) \) and \( RS^{(m)}(U) \) in specific settings. As it will turn out, the multiobjective situation is in complete analogy to the situation in the single-objective case: In accordance with [11] we will demonstrate that multiobjective robust regret can indeed be solved efficiently in a variety of setups.

Since we are interested in cases in which the above approach yields numerically tractable formulations, we only consider convex instances \( P(u) \) from here on, i.e. \( f \) is assumed to be convex in \( x \) for all \( u \in U \), and further \( X \) is also assumed to be convex. If this is not the case, already the original instances \( P(u) \) constitute problems which are, in general, not easily solvable. Please note that compactness of \( X \) and \( U \) has already been assumed right from the beginning.

We now investigate under which further conditions on \( U \) and \( f \) the multiobjective regret formulations remain computationally tractable. We will focus our investigations on the most important practical cases, which are

- \( U \) is finite,
- \( U \) is a convex polytope, or,
- \( U \) is convex.

In the following, we will show that we can actually solve problems \( RR^{(m)}(U) \) and \( RS^{(m)}(U) \) rather efficiently in the first two cases or at least approximate it to a sufficient degree in the most general third case.

4.1 Finite uncertainty set

Here, let \( |U| < \infty \) such that \( U \) is given as a finite set of scenarios, i.e. \( U = \{u_1, \ldots, u_p\} \). In this case, it is possible to precompute all optimal values \( f_i^*(u) \) for all \( i = 1, \ldots, m \) and for all \( u \in U \). Then

\[
R_i(x) = \max_{u \in U} f_i(x, u) - f_i^*(u) = \max_{j \in \{1, \ldots, p\}} f_i(x, u_j) - f_i^*(u_j)
\]
for every $i = 1, \ldots, m$. Thus, problem $\text{(RR}^m(U))$ can be rewritten, again by introducing a slack variable $\alpha \in \mathbb{R}^m$, as

$$\min_{\alpha \in \mathbb{R}^m} \alpha \quad \text{s.t.} \quad f_i(x, u_j) - f_i^*(u_j) \leq \alpha_i \quad \forall i = 1, \ldots, m, \forall j = 1, \ldots, p.$$ 

This immediately proves the following proposition.

**Proposition 4.1.** Let $U = \{u_1, \ldots, u_p\}$. Then $\text{(RR}^m(U))$ is

1. a linear multiobjective problem, if $X$ is polyhedral and $f$ is affine-linear in $x$ for all $u \in U$, and
2. a standard convex multiobjective problem, if $X$ is convex and $f$ is convex in $x$ for all $u \in U$.

The same is true for $\text{(RS}^m(U))$ under the additional assumption that $f_i^*(u) > 0$ for all $u \in U$ and for all $i = 1, \ldots, m$.

Please note that due to the convex structure, the complete efficient frontier of both $\text{(RR}^m(U))$ and $\text{(RS}^m(U))$ can be computed by a suitable scalarization technique, see Section 3. The main difference in the computational effort between the solution of an arbitrary instance of $\text{(P}^m(U))$ and $\text{(RR}^m(U))$ or $\text{(RS}^m(U))$ is due to the consideration of $mp$ additional constraints (or dealing with non-smooth objective functions) and the precomputation of all $mp$ individual optima.

### 4.2 Polytopial uncertainty set

In contrast to the previous subsection, where finiteness of $U$ was assumed, we now instead assume that the uncertainty set $U$ is given as a convex polytope. As we will show in the following, this can be used to reduce the polytopial case to the previous setup of finite $U$, given that $f$ has some additional structure in $u$. Let us start with the following observation.

**Proposition 4.2.** Let $U$ be a convex polytope and let $V(U)$ denote the finite set of its vertices. Further, let $f$ be affine-linear in $u$ for all $x \in X$. Then

$$\max_{u \in U} f_i(x, u) - f_i^*(u) = \max_{u \in V(U)} f_i(x, u) - f_i^*(u)$$

holds for every $i \in \{1, \ldots, m\}$. In this case, if $f_i^*(u) > 0$ for all $u \in U$, it also holds that

$$\max_{u \in U} \frac{f_i(x, u) - f_i^*(u)}{f_i^*(u)} = \max_{u \in V(U)} \frac{f_i(x, u) - f_i^*(u)}{f_i^*(u)}.$$ 

**Proof.** Let us first consider the first statement: As $u \mapsto f_i(x, u)$ is linear for each $x \in X$, the mapping $u \mapsto f_i^*(u) = \min_{x \in X} f_i(x, u)$ is concave. Thus, the function $g_i(u) = f_i(x, u) - f_i^*(u)$ is convex in $u$. Since convex functions that attain their maximum also attain it in one of the extreme points of the feasible domain (in our case in one of the vertices), the first statement of the proposition follows.
For the second statement, note that it is no longer true that the function \( g_i(u) = \frac{f_i(x,u) - f^*_i(u)}{f^*_i(u)} \) is convex in \( u \). Still, as the nominator is convex in \( u \) and the denominator is concave in \( u \) (and positive), it is straightforward to see that \( g_i \) is quasiconvex in \( u \). Since quasiconvex functions that attain their maximum also attain it in one of the vertices of the feasible domain (see e.g. [17]), the second statement of the proposition follows.

Thanks to the above proposition we can replace the uncertainty set \( U \) by its finite set of vertices \( V(U) = \{u_1, \ldots, u_p\} \) in case of affine-linear uncertainty, which again yields

\[ R_i(x) = \max_{u \in U} f_i(x,u) - f^*_i(u) = \max_{u \in V(U)} f_i(x,u) - f^*_i(u) = \max_{j \in \{1,\ldots,p\}} f_i(x,u_j) - f^*_i(u_j) \]

for every \( i = 1, \ldots, m \), and similarly for \( S_i \). We then proceed as in the finite setting described in the previous subsection and obtain the complete analogous formulation of Proposition 4.1 for the polytopial case.

**Proposition 4.3.** Let \( U \) be a convex polytope with vertex set \( V(U) = \{u_1, \ldots, u_p\} \) and let \( f \) be affine-linear in \( u \) for all \( x \in X \). Then \( (\text{RR}^{(m)}(U)) \) is

1. a linear multiobjective problem, if \( X \) is polyhedral and \( f \) is affine-linear in \( x \) for all \( u \in U \), and
2. a standard convex multiobjective problem, if \( X \) is convex and \( f \) is convex in \( x \) for all \( u \in U \).

The same is true for \( (\text{RS}^{(m)}(U)) \) under the additional assumption that \( f^*_i(u) > 0 \) for all \( u \in U \) and for all \( i = 1, \ldots, m \).

As in the finite case, the computational burden to solve \( (\text{RR}^{(m)}(U)) \) or \( (\text{RS}^{(m)}(U)) \) is comparable to that for solving \( (P^{(m)}(u)) \). The additional effort is due to the consideration of \( mp \) additional constraints (or dealing with non-smooth objective functions) and the precomputation of all \( mp \) individual optima. It needs to be noted that \( p \) might become large, i.e. exponential in the dimension of \( u \) for the classical box uncertainty, whereas it remains moderate if some simplicial uncertainty is considered.

### 4.3 Convex uncertainty set

Let us now consider the most general situation where the uncertainty set \( U \) is assumed to be convex. This especially includes the important case of ellipsoidal uncertainty sets, as often used in the optimization under uncertainty framework. To tackle problems \( (\text{RR}^{(m)}(U)) \) or \( (\text{RS}^{(m)}(U)) \) in this setting, we can either consider some inner and outer polytopial approximations to the uncertainty set \( U \), or, alternatively, borrow results from semi-infinite programming.

#### 4.3.1 Approximation by inner and outer polytopial approximation

Let us start with the consideration of inner and outer polytopial approximations. For this purpose, let \((U^I_k)_{k \in \mathbb{N}}\) and \((U^O_k)_{k \in \mathbb{N}}\) denote sequences of convex polytopes such that

\[ U^I_1 \subseteq U^I_2 \subseteq \cdots \subseteq U \subseteq \cdots \subseteq U^O_2 \subseteq U^O_1 \quad \text{(A1)} \]
gives an inner and an outer approximation of $U$, refined in every step. We further require that both the inner as well as the outer approximations become more and more accurate, i.e. we require
\[ d_H(U_k^I, U) \to 0 \quad \text{and} \quad d_H(U_k^O, U) \to 0 \quad \text{for} \ k \to \infty \quad (A2) \]
where $d_H$ denotes the usual Hausdorff distance between sets. To make the dependence of the (scaled) regret more explicit, we write in the following $R(x, U)$ and $S(x, U)$ instead of simply $R(x)$ and $S(x)$. By assumption (A1), it holds for all $k \in \mathbb{N}$:
\[ \forall x \in X : \ R(x, U_k^I) \leq R(x, U) \leq R(x, U_k^O), \quad (1) \]
and analogously for $S$. Under some Lipschitz condition, we then have the following theorem. This result relies on the technical Lemma A.1 which we have moved to the appendix for ease of presentation.

**Theorem 4.4.** Let $(U_k^I)_{k \in \mathbb{N}}$ and $(U_k^O)_{k \in \mathbb{N}}$ denote sequences of sets that satisfy (A1) and (A2). Further, let $f$ be locally Lipschitz with respect to $(x, u)$ jointly. Then there exists a constant $K > 0$ such that uniformly for all $x \in X$
\[ ||R(x, U) - R(x, U_k^I)||_1 \leq Kd_H(U, U_k^I) \quad \text{and} \quad ||R(x, U) - R(x, U_k^O)||_1 \leq Kd_H(U, U_k^O). \]
as well as
\[ ||S(x, U) - S(x, U_k^I)||_1 \leq Kd_H(U, U_k^I) \quad \text{and} \quad ||S(x, U) - S(x, U_k^O)||_1 \leq Kd_H(U, U_k^O). \]

**Proof.** We only prove the statement for the case $m = 1$ and the inner approximation, the proof for $S$, the generalization to arbitrary dimensions and to the outer approximation is then straightforward. To show the statement, let us note that $R(x, U) = r(x, u^*(x))$ for some $u^*(x)$ which maximizes $r(x, u)$ in $u \in U$. We can now replace $u^*(x)$ by some $v_k^*(x) \in U_k^I$ with $||u^*(x) - v_k^*(x)||_1 \leq \sqrt{m}d_H(U, U_k^I)$ for each $k$. Since $R(x, U_k^I) \geq r(x, v_k^*(x))$ and since $r$ is globally Lipschitz continuous with Lipschitz constant $2mL$ thanks to Lemma A.1 we get
\[ R(x, U) - R(x, U_k^I) \leq r(x, u^*(x)) - r(x, v_k^*(x)) \leq 2mL||u^*(x) - v_k^*(x)||_1 \leq Kd_H(U, U_k^I), \]
which shows the claim with $K = 2mn'L$. \[ \square \]

Combining this theorem with inequality (1) yields that the objective function of the original problem is bracketed between the inner and outer approximation with known approximation quality, i.e. for all $x \in X$:
\[ R(x, U) - Kd_H(U, U_k^I) \leq R(x, U_k^I) \leq R(x, U) \leq R(x, U_k^O) \leq R(x, U) + Kd_H(U, U_k^O) \quad (2) \]
and analogously for $S$. Using the results from the previous subsection, we see that the inner and outer approximation can be computed with reasonable effort if we assume that $f$ is affine-linear in $u$ and convex in $x$. Of course, the computational burden is closely linked to the (increasing) number of vertices of the inner and outer approximation, as at each vertex, $f^*(u)$ has to be computed.
by the solution of a convex optimization problem. It is well-known, see for example [3], that an
$\varepsilon$-approximation (with respect to the Hausdorff metric in $d$ dimensions) can be obtained by (inner
and outer) polytopes with $O\left(\frac{1}{\varepsilon^d} \right)$ vertices.

4.3.2 Approximation by semi-infinite programming

As the number of vertices for the above inner and outer approximations might become prohibitively
large, alternative approaches are sought for. Of course, the computation of the scalarized multi-
objective robust (relative) regret can be formulated as a standard semi-infinite problem, if the
$\varepsilon$-constraint scalarization method or the weighted Chebyshev scalarization method is employed.
For example, in case of the weighted Chebyshev method, the scalarization with reference point 0
yields:

$$\min_{x \in X} \min_{\alpha \in \mathbb{R}} \quad \text{s.t. } \alpha \geq w_i R_i(x) \quad \text{for } i = 1, \ldots, m,$$

which constitutes an ordinary convex semi-infinite program:

$$\min_{x \in X} \min_{\alpha \in \mathbb{R}} \quad \text{s.t. } \alpha \geq w_i r_i(x, u) \quad \text{for } i = 1, \ldots, m, \forall u \in U.$$

Therefore, any suitable method from convex semi-infinite programming can be used to approximately
solve the corresponding scalarized problems. Among the large variety of available algorithms we especially suggest cutting plane methods. Discretization methods cannot be suggested as method of choice as they are closely linked to the inner polytopial approximation which should be preferred in our specific setup (as this usually avoids evaluating "inner" $u$). For an introduction
and overview of these methods let us refer to the recent overview article [22] or the more detailed
books [18] or [10].

As $r_i(x, u) = f_i(x, u) - f^*_i(u)$ still contains the computation of $f^*(u)$, let us give a few more
remarks on the example of a cutting plane method:

1. For $k = 0$, the cutting plane method is initialized with some finite set $U_0 := \{u, \ldots, u_p\} \subset U$ such that $(SIP(U_0))$ possesses an optimal solution $(x_0^*, \alpha_0^*)$.
2. We solve $(SIP(U_k))$ and obtain an optimal solution $(x_k^*, \alpha_k^*)$.
3. Now the most violating $u \in U$ has to be computed, i.e. we need to solve

$$\max_{u \in U} \max_{1 \leq i \leq m} w_i \left( f_i(x_k^*, u) - f^*_i(u) \right) - \alpha_k^*$$

which, even for $f$ affine-linear in $u$, represents a hard-to-solve (concave) problems (as shown
before).
4. Then, set $U_{k+1} := U_k \cup \{u_{k+1}\}$, $k = k + 1$, and go to Step 2.
Although typically representing an infeasible method, let us point out that for cutting plane methods, it is possible to obtain performance guarantees such that an early stopping with predefined accuracy and feasible iterate is possible, see for example [4] for more details.

As major obstacle in the computation of an optimal solution we have thus identified the computation of the most violating \( u \) in Step 3. Unfortunately, this generally remains a hard-to-solve problem, which can only be solved to reasonable accuracy (e.g. by discretization) for low-dimensional \( U \), i.e. for \( n' \) small. Of course, re-using precomputed values of \( f^*_i(u) \) will still allow for a certain speedup.

5 Connection to existing approaches

As already mentioned, other authors [5, 19, 23] have already studied robust regret approaches to solve uncertain multiobjective optimization problems. Albeit, all these approaches are based on the main idea to first scalarize the uncertain multiobjective problem by some suitable scalarization method and then to apply the single-objective robust regret approach. The main drawback in this approach, as we see it, is that it mixes the computational tool scalarization with the modelling paradigm multiobjective optimization. Instead, we favor a clear separation between problem modelling and problem solution, including a transparent definition of what we understand as a solution. In the following, let us remark more details on these individual previous approaches.

5.1 Connection to Drezner et al. [5]

The approach by Drezner et al in [5] can be seen as a first step into multiobjective regret, introducing (certain) relative multiobjective regret for the first time. To be more precise,

- Drezner et al. do not consider uncertainty, or, to fit within our setting, they assume \( U = \{ \bar{u} \} \) to be a singleton; and,

- similar to our approach, the authors work with the ideal point based relative regret as

\[
\begin{align*}
\bar{s}_i(x, \bar{u}) &= \frac{f_i(x, \bar{u}) - f^*_i(\bar{u})}{f^*_i(\bar{u})},
\end{align*}
\]

for each objective function \( f_1(x, \bar{u}), \ldots, f_m(x, \bar{u}) \) individually.

- By applying a specific Chebyshev scalarization with weight vector \( w \) as the all-ones vector and using the ideal point \( 0 \) (which is indeed the ideal point for \( s \) in the given setup, whereas \( 0 \) becomes an utopian point in the uncertain setup), the authors finally suggest to consider the single-objective robust optimization problem

\[
\min_{x \in X} \max_{i=1, \ldots, m} \bar{s}_i(x, \bar{u}).
\]

- The authors do not consider different weight vectors and thus obtain a single weakly efficient point only\(^3\)

\(^3\)Here, it needs to be mentioned that one specific solution was sufficient for the specific application in [5].
The insight that this approach indeed yields a weakly efficient solution within our setting is of course only possible within our setup and thus not discussed by the authors.

5.2 Connection to Xidonas et al. [23]

An approach probably closer to ours than the previous one is due to Xidonas et al. in [23]. It can be seen as a second step, adding uncertainty to the approach by Drezner et al. However, instead of considering general uncertain setups, in [23] the authors focus on the special case where the uncertainty set is a finite set of scenarios $U = \{u_1, \ldots, u_p\}$. Further, our understanding of the authors’ framework is that they also focus solely on linear multiobjective problems. The authors apply the weighted sum scalarization method prior to the regret formulation in order to work with the single-criteria regret. Again, to be more precise,

- the authors consider the same approach as in Drezner et al., except for the chosen scalarization technique. Let $s_i(x, u)$ again denote the individual regret\(^4\) for every $i$. Then the corresponding scalarized single-objective function reads as

$$\sum_{i=1}^{m} \lambda_i s_i(x, u).$$

Applying the classical robust counterpart to this objective function finally introduces the robust scalarized relative regret optimization problem:

$$\min_{x \in X} \max_{u \in U} \sum_{i=1}^{m} \lambda_i s_i(x, u).$$

- To obtain different solutions of the single-objective formulation, they suggest to vary the weights accordingly to the weighted sum scalarization method.

- As the authors do not discuss any (robust regret) multiobjective formulation or solution concept thereof, they simply take the single-objective solutions as solutions to the original question. This is in strong contrast to our approach which starts by introducing a corresponding multiobjective formulation of robust regret together with the usual concepts of (weakly) efficient solutions.

Without further strong assumptions on the uncertainty (e.g. like separability) it cannot be expected that any solution obtained by the above approach is (weakly) efficient in our setting.

5.3 Connection to Rivaz et al. [19]

Finally, the following approach by Rivaz et al. in [19] is again motivated by a priori scalarization. We have translated their maximization approach (with corresponding adjusted definition of regret) to our notation for convenience.

\(^4\)Instead of classically normalizing as we do and as suggested by [14], the authors prefer to apply a different normalization. As their normalization seems justifiable for linear problems as considered by authors, but hardly extendable to nonlinear settings or infinite uncertainty sets, we have adopted the classical normalization instead for presentation here.
• The method is similar to the model by Drezner et al., but focuses on linear objective functions with interval uncertainty for the coefficients of the objective functions.

• Translated to our notation, the authors consider the same definitions as we do, but instead of first setting up a multiobjective problem, the authors directly start with a scalarized formulation:

\[
\min_{x \in X} \max_{u \in U} \max_{1 \leq i \leq m} r_i(x, u).
\]

This represents the minimization of the robustified scalarized regret, where a Chebyshev scalarization with ideal point 0 (for \(r(x, u)\)) has been used together with the weight vector of all-ones.

• Since for the Chebyshev scalarization we can swap robustification and scalarization (see Section 3.5), this can be equivalently reformulated as

\[
\min_{x \in X} \max_{1 \leq i \leq m} \max_{u \in U} r_i(x, u).
\]

(3)

The same observation was made by the authors in [19], however, without explicitly noting that this indeed allows to swap the order of robustification and scalarization in general, as observed in Section 3.5.

In (3) the authors first robustify then scalarize by Chebyshev scalarization, however, without referring to a corresponding multiobjective setup. Nevertheless, Problem (3) can be seen as another step towards introducing a kind of scalarized robust regret for the first time, of course limited to the special setting considered in [19].

6 Conclusion

Motivated by the comparison to the approaches in [5], [23] and [19], our contribution can be seen as the completion of the concept of (relative) regret in a multicriteria framework. In contrast to the mentioned early approaches, we are not limited to linear objective functions and/or finite uncertainty sets, or linear interval uncertainty, resp. Furthermore, we purely work in a multicriteria setting and introduce a multivariate (relative) regret based on a clear separation between problem modelling and problem solution, whereas the mentioned methods first scalarize the problem to introduced a scalarized robust regret. Moreover, we show that most common cases remain computationally tractable. In the most general case of \(U\) being a compact convex set, we introduce two approaches to approximate the multivariate relative regret for the first time.
References


Appendix

A Proofs

Lemma A.1. Let $f : X \times U \to \mathbb{R}^m$ be locally Lipschitz in $(x, u)$ jointly, and let $X$ and $U$ be compact. Then there exists a constant $L > 0$ such that

$$\forall x, y \in X, u, v \in U : \quad ||f(x, u) - f(y, v)||_1 \leq L(||x - y||_1 + ||u - v||_1)$$

and

$$\forall u, v \in U : \quad ||f^*(u) - f^*(v)||_1 \leq mL||u - v||_1.$$ 

Further, $r$ is also globally Lipschitz continuous with constant $2mL$. Finally, $s$ is also globally Lipschitz continuous with some (larger) constant $\tilde{L} > 0$.

Proof. For the proof, let us assume that $m = 1$; the generalization to arbitrary $m$ is straightforward. As $f$ is locally Lipschitz on the compact set $X \times U$, it is already globally Lipschitz on $X \times U$. The first statement then follows from the equivalence of norms in finite dimensional spaces, and the additivity of the $1$-norm.

For the second claim, consider the following inequality for $f^*$:

$$f^*(u) - f^*(v) = \min_{x \in X} f(x, u) - \min_{x \in X} f(x, v) \leq \min_{x \in X} f(x, v) + L||u - v||_1 - \min_{x \in X} f(x, v) = L||u - v||_1.$$ 

Swapping the roles of $u$ and $v$ and generalising to $m$ dimensions shows the second statement. The third claim follows directly from the definition of $r$ as the sum of two globally Lipschitz continuous functions.

The analogous statement for $s$ is a bit more involved. For this purpose, let us define the following constants

$$A := \max_{u \in U} \frac{1}{f^*(u)}, \quad \text{and} \quad B := \max_{u \in U} f^*(u).$$

Then

$$s(x, u) - s(y, v) = \frac{f(x, u)}{f^*(u)} - \frac{f(y, v)}{f^*(v)} = \frac{1}{f^*(u)f^*(v)} \left( f(x, u)f^*(v) - f(y, v)f^*(u) \right).$$

As for the term in brackets we have

$$f(x, u)f^*(v) - f(y, v)f^*(u) = f(x, u)f^*(v) - f(x, u)f^*(u) + f(x, u)f^*(u) - f(y, v)f^*(u),$$

setting $\tilde{L} = mABL$ shows the claim for $s$. □