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Primal Necessary Characterizations of Transversality Properties

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Abstract This paper continues the study of general nonlinear transversality properties of collections of sets and focuses on primal necessary (in some cases also sufficient) characterizations of the properties. We formulate geometric, metric and slope characterizations, particularly in the convex setting. The Hölder case is given a special attention. Quantitative relations between the nonlinear transversality properties of collections of sets and the corresponding regularity properties of set-valued mappings as well as two nonlinear transversality properties of a convex set-valued mapping to a convex set in the range space are discussed.

Keywords Transversality · Subtransversality · Semitransversality · Regularity · Subregularity · Semiregularity · Slope

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1 Introduction and Prelimiaries

Transversality properties of collections of sets are fundamental for many applications in optimization and variational analysis. They are involved, e.g., in constraint qualifications, qualification conditions in subdifferential, normal cone and coderivative calculus, and convergence analysis of computational algorithms. The properties have been intensively studied during the last 25 years in various settings (convex and nonconvex, linear and nonlinear, finite and infinite dimensional, and finite and infinite collections of sets); cf. [1–3, 5, 6, 18, 19, 21, 26, 27, 32–34, 37, 38, 40, 41, 43, 44].

This paper continues a series of recent publications dedicated to general nonlinear transversality properties of collections of sets [11–13] and focuses on primal necessary (in some cases also sufficient) characterizations of the properties. We formulate geometric, metric and slope characterizations, particularly in the convex setting. The Hölder case is given a special attention. Our aim is not formally extending the earlier results from the linear and Hölder cases to a more general nonlinear setting, but rather

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developing a comprehensive theory of transversality. The nonlinearity is just a simple setting, which allows us to unify the existing results on the topic.

As an application, we provide in the last section quantitative characterizations in the convex setting of two nonlinear transversality properties of a set-valued mapping to a set in the range space, one of which was studied recently by Ioffe [24].

Our basic notation is standard, see, e.g., [17,36,42]. Throughout the paper, X and Y are either metric or, more often, normed spaces. The open unit ball in any space is denoted by $\mathbb{B}$, and $B_\delta(x)$ stands for the open ball with center x and radius $\delta>0$. If not explicitly stated otherwise, products of normed spaces are assumed to be equipped with the maximum norm $\|(x,y)\|:=\max\{\|x\|,\|y\|\}$, $(x,y)\in X\times Y$. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ denote the real line (with the usual norm) and the set of all nonnegative real numbers, respectively.

Given a set Ω, its interior and boundary are denoted by $\operatorname{int}\Omega$ and $\operatorname{bd}\Omega$, respectively. The distance from a point x to Ω is defined by $d(x,\Omega):=\inf_{u\in\Omega}\|u-x\|$, and we use the convention $d(x,\emptyset)=+\infty$. The indicator function of Ω is defined as follows: $i_\Omega(x)=0$ if $x\in\Omega$ and $i_\Omega(x)=+\infty$ if $x\notin\Omega$.

A set-valued mapping $F:X\rightrightarrows Y$ between two sets X and Y is a mapping, which assigns to every $x\in X$ a subset (possibly empty) $F(x)$ of Y. We use the notations $\operatorname{gph}F:=\{(x,y)\in X\times Y\mid y\in F(x)\}$ and $\operatorname{dom}F:=\{x\in X\mid F(x)\neq\emptyset\}$ for the graph and the domain of F, respectively, and $F^{-1}:Y\rightrightarrows X$ for the inverse of F. This inverse (which always exists with possibly empty values at some y) is defined by $F^{-1}(y):=\{x\in X\mid y\in F(x)\}$, $y\in Y$. Obviously $\operatorname{dom}F^{-1}=F(X)$.

For an extended-real-valued function $f:X\to\mathbb{R}\cup\{+\infty\}$ on a metric space X, its domain and epigraph are defined, respectively, by $\operatorname{dom}f:=\{x\in X\mid f(x)<+\infty\}$ and $\operatorname{epi}f:=\{(x,\alpha)\in X\times\mathbb{R}\mid f(x)\le\alpha\}$. The inverse of f (if it exists) is denoted by f^{-1}. The *slope* [16] and *nonlocal slope* [28, 39] of f at $x\in\operatorname{dom}f$ are defined, respectively, by

$$|\nabla f|(x):=\limsup_{u\to x,\,u\neq x}\frac{[f(x)-f(u)]_+}{d(x,u)}\quad\text{and}\quad|\nabla f|^{\diamond}(x):=\sup_{u\neq x}\frac{[f(x)-f_+(u)]_+}{d(x,u)},$$

where $\alpha_+:=\max\{0,\alpha\}$ for any $\alpha\in\mathbb{R}$. The limit $|\nabla f|(x)$ provides the rate of steepest descent of f at x. If X is a normed space, and f is Fréchet differentiable at x, then $|\nabla f|(x)=\|f'(x)\|$. When $x\notin\operatorname{dom}f$, we set $|\nabla f|(x)=|\nabla f|^{\diamond}(x):=+\infty$.

Proposition 1.1 *Let X be a metric space, $f:X\to\mathbb{R}\cup\{+\infty\}$, and $x\in X$ with $f(x)>0$.*

(i) $|\nabla f|(x)\le|\nabla f|^{\diamond}(x)$.

(ii) *If X is a normed space and f is convex, then $|\nabla f|(x)=|\nabla f|^{\diamond}(x)$.*

Proof (i) follows from the definitions of the slopes.

(ii) (Cf. the proof of [20, Theorem 5(ii)].) Suppose X is a normed space, and f is convex. If $f(x)=+\infty$, the equality holds by convention. If f attains its (finite) minimum at x, then $|\nabla f|(x)=|\nabla f|^{\diamond}(x)=0$. Let $u\in X$ and $f(u)<f(x)<+\infty$. Set $u_t:=(1-t)x+tu$, $t>0$. Observe that f_+ is convex and $f_+(x)=f(x)$. Thus, for any $t\in]0,1]$, we have $\|u_t-x\|=t\|u-x\|$ and $f(u_t)-f(x)\le f_+(u_t)-f(x)\le t(f_+(u)-f(x))$, and consequently,

$$\frac{[f(x)-f_+(u)]_+}{\|x-u\|}=\frac{f(x)-f_+(u)}{\|x-u\|}\le\lim_{t\downarrow0}\frac{f(x)-f(u_t)}{\|u_t-x\|}\le\limsup_{u\to x,\,u\neq x}\frac{[f(x)-f(u)]_+}{\|u-x\|}.$$

Hence, $|\nabla f|^{\diamond}(x)\le|\nabla f|(x)$. In view of (i), the proof is complete. □

The next statement provides a chain rule for slopes; cf. [13, Lemma 1.2].

Lemma 1.1 *Let X be a metric space, $f:X\to\mathbb{R}\cup\{+\infty\}$, $\varphi:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$, $x\in\operatorname{dom}f$ and $f(x)\in\operatorname{dom}\varphi$. Suppose φ is nondecreasing on $\mathbb{R}$ and differentiable at $f(x)$ with $\varphi'(f(x))>0$. Then $|\nabla(\varphi\circ f)|(x)=\varphi'(f(x))|\nabla f|(x)$.*

The rest of the paper is organized as follows. The next Section 2 sets the scene for the rest of the paper. It contains the definitions of the three nonlinear transversality properties studied in the paper and collects several basic facts about these properties including their geometric and metric characterizations, as well as simplified versions of the definitions and characterizations of the nonlinear semitransversality and transversality properties in the convex setting. Section 3 is dedicated to slope necessary conditions for the properties. Besides being of interest on their own, these conditions make the foundation for the necessary dual characterizations of the respective properties in [12]. In Section 4, we provide quantitative relations between the nonlinear transversality properties of collections of sets and the corresponding regularity properties of set-valued mappings, and discuss in the convex setting two nonlinear transversality properties of a set-valued mapping to a set in the range space. The final Section 5 contains some concluding remarks and a few items identified for future work.

2 Definitions and Basic Relations

The working model in this paper is a collection of $n \geq 2$ arbitrary subsets $\Omega_1,\ldots,\Omega_n$ of a normed space X, having a common point $\bar{x} \in \cap_{i=1}^n \Omega_i$. The nonlinearity in the definitions of the properties is determined by a continuous strictly increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(0) = 0$ and $\lim_{t\to+\infty} \varphi(t) = +\infty$. The collection of all such functions is denoted by $\mathscr{C}$. We denote by $\mathscr{C}^1$ the subfamily of functions from $\mathscr{C}$ which are differentiable on $]0,+\infty[$ with $\varphi'(t) > 0$ for all $t > 0$. Obviously, if $\varphi \in \mathscr{C}$ ($\varphi \in \mathscr{C}^1$), then $\varphi^{-1} \in \mathscr{C}$ ($\varphi^{-1} \in \mathscr{C}^1$). If not explicitly stated otherwise, we assume from now on that $\varphi \in \mathscr{C}$.

Remark 2.1 For the purposes of this paper, it suffices to assume that functions $\varphi \in \mathscr{C}$ are defined and invertible near 0.

Definition 2.1 The collection $\{\Omega_1,\ldots,\Omega_n\}$ is

(i) $\varphi-$semitransversal at $\bar{x}$ if there exists a $\delta > 0$ such that

$$\bigcap_{i=1}^{n} (\Omega_i - x_i) \cap B_\rho(\bar{x}) \neq \emptyset \tag{1}$$

for all $\rho \in]0,\delta[$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\leq i\leq n} \|x_i\|) < \rho$;

(ii) $\varphi-$subtransversal at $\bar{x}$ if there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\bigcap_{i=1}^{n} \Omega_i \cap B_\rho(x) \neq \emptyset$$

for all $\rho \in]0,\delta_1[$ and $x \in B_{\delta_2}(\bar{x})$ with $\varphi(\max_{1\leq i\leq n} d(x,\Omega_i)) < \rho$;

(iii) $\varphi-$transversal at $\bar{x}$ if there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\bigcap_{i=1}^{n} (\Omega_i - \omega_i - x_i) \cap (\rho\mathbb{B}) \neq \emptyset \tag{2}$$

for all $\rho \in]0,\delta_1[$, $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\leq i\leq n} \|x_i\|) < \rho$.

Each of the properties in Definition 2.1 is determined by a function $\varphi \in \mathscr{C}$, and a number $\delta > 0$ in item (i) or numbers $\delta_1 > 0$ and $\delta_2 > 0$ in items (ii) and (iii). The function plays the role of a kind of rate or modulus of the respective property, while the role of the δ's is more technical: they control the size of the interval for the values of ρ and, in the case of $\varphi-$subtransversality and $\varphi-$transversality in parts (ii) and (iii), the size of the neighbourhoods of $\bar{x}$ involved in the respective definitions. Of course, if a property is satisfied with some $\delta_1 > 0$ and $\delta_2 > 0$, it is satisfied also with the single $\delta := \min\{\delta_1, \delta_2\}$ in place of both δ_1 and δ_2. We use two different parameters to emphasise their different roles in the definitions and the corresponding characterizations. Moreover, we are going to provide quantitative estimates for the values of these parameters.

The most important realization of the three properties in Definition 2.1 corresponds to the Hölder setting, i.e. φ being a power function, given for all $t \geq 0$ by $\varphi(t) := \alpha^{-1}t^q$ with some $\alpha > 0$ and $q > 0$. In this case, Definition 2.1 reduces to a (slight modification of) [32, Definition 1], and we refer to the respective properties as *$\alpha-$semitransversality*, *$\alpha-$subtransversality* and *$\alpha-$transversality* of order q at $\bar{x}$. With $q = 1$ (linear case), the properties were studied in [7,26,27,33]. For more discussions of the Hölder transversality properties, readers are referred to [13, 32]. Another important for applications class of functions from $\mathscr{C}$ is given by the so called *Hölder-type* [4, 35] ones, i.e. functions of the form $t \mapsto \alpha^{-1}(t^q + t)$, frequently used in the error bound theory.

The next statement collects several basic facts about the properties; cf. [13, Propositions 2.2 & 2.3].

Proposition 2.1 (i) *If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then it is $\varphi-$semitransversal at $\bar{x}$ with δ_1 and $\varphi-$subtransversal at $\bar{x}$ with any $\delta_1' \in]0, \delta_1]$ and $\delta_2' > 0$ such that $\varphi^{-1}(\delta_1') + \delta_2' \leq \delta_2$.*

(ii) *If $\bar{x} \in \operatorname{int} \cap_{i=1}^n \Omega_i$, then all three properties in Definition 2.1 are satisfied.*

(iii) *Suppose $\cap_{i=1}^n \Omega_i$ is closed, and $\bar{x} \in \operatorname{bd} \cap_{i=1}^n \Omega_i$. If $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$subtransversal (in particular, if it is $\varphi-$transversal) at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then there exists a $\bar{t} \in]0, \min\{\varphi^{-1}(\delta_1), \delta_2\}[$ such that $\varphi(t) \geq t$ for all $t \in]0, \bar{t}]$.*

The next proposition provides alternative geometric representations of $\varphi-$transversality. They differ from those in Definition 2.1(iii) by values of the parameters δ_1 and δ_2. Note also the relations between the values of the parameters in the two groups of representations and observe the similarity with those in Proposition 2.1(iii). One of the advantages of the alternative representations of $\varphi-$transversality given below is their direct relations with those in the definition of $\varphi-$subtransversality. Some other advantages will be exposed later.

Proposition 2.2 *Let $\delta_1 > 0$ and $\delta_2 > 0$. The following properties are equivalent:*

(i) *condition (2) holds for all $\rho \in]0, \delta_1[$, $\omega_i \in \Omega_i$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $\omega_i + x_i \in B_{\delta_2}(\bar{x})$ and $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$;*

(ii) *condition (1) holds for all $\rho \in]0, \delta_1[$ and $x_i \in \delta_2\mathbb{B}$ $(i = 1, \ldots, n)$ with $\varphi(\max_{1 \leq i \leq n} d(\bar{x}, \Omega_i - x_i)) < \rho$;*

(iii) *for all $\rho \in]0, \delta_1[$ and $x, x_i \in X$ with $x + x_i \in B_{\delta_2}(\bar{x})$ $(i = 1, \ldots, n)$ and $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i - x_i)) < \rho$, it holds*

$$\bigcap_{i=1}^{n} (\Omega_i - x_i) \cap B_\rho(x) \neq \emptyset. \tag{3}$$

Moreover, if $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then properties (i)–(iii) *hold with any $\delta_1' \in]0, \delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \leq \delta_2$ in place of δ_1 and δ_2.*

If properties (i)–(iii) *hold with some $\delta_1 > 0$ and $\delta_2 > 0$, then $\{\Omega_1, \ldots, \Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2, and $\varphi-$transversal at $\bar{x}$ with any $\delta_1' \in]0, \delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \leq \delta_2$.*

Proof We first prove the equivalence of the properties (i)–(iii).

(i) $\Rightarrow$ (ii). Let $\rho \in]0,\delta_1[$ and $x_i \in \delta_2\mathbb{B}$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n} d(\bar{x},\Omega_i - x_i)) < \rho$. Choose $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ such that $\varphi(\max_{1\le i\le n}\|\bar{x}+x_i-\omega_i\|) < \rho$. Set $x_i' := \bar{x}+x_i-\omega_i$ $(i=1,\ldots,n)$. Then $\omega_i + x_i' \in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i'\|) < \rho$. By (i), condition (2) is satisfied with x_i' in place of x_i $(i=1,\ldots,n)$. This is equivalent to condition (1).

(ii) $\Rightarrow$ (iii). Let $\rho \in]0,\delta_1[$ and $x, x_i \in X$ with $x + x_i \in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(x,\Omega_i - x_i)) < \rho$. Set $x_i' := x + x_i - \bar{x}$ $(i=1,\ldots,n)$. Then $x_i' \in \delta_2\mathbb{B}$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(\bar{x},\Omega_i - x_i')) < \rho$. By (ii), condition (1) is satisfied with x_i' in place of x_i $(i=1,\ldots,n)$. This is equivalent to condition (3).

(iii) $\Rightarrow$ (i). Let $\rho \in]0,\delta_1[$, $\omega_i \in \Omega_i$ and $x_i \in X$ with $\omega_i + x_i \in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|) < \rho$. Set $x_i' := \omega_i + x_i - \bar{x}$ $(i=1,\ldots,n)$. Then, $\bar{x} + x_i' \in B_{\delta_2}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(d(\bar{x},\Omega_i - x_i')) \le \varphi(\max_{1\le i\le n}\|x_i\|) < \rho$. By (iii), condition (3) is satisfied with $\bar{x}$ and x_i' in place of x and x_i $(i=1,\ldots,n)$, respectively. This is equivalent to condition (2).

Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, and let $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ be such that $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$. Then, for all $\rho \in]0,\delta_1'[$, $\omega_i \in \Omega_i$ and $x_i \in X$ with $\omega_i + x_i \in B_{\delta_2'}(\bar{x})$ $(i=1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|) < \rho$, we have $\|\omega_i - \bar{x}\| \le \|x_i\| + \|\omega_i + x_i - \bar{x}\| < \varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$ $(i=1,\ldots,n)$. By Definition 2.1(iii), condition (2) is satisfied, and consequently, property (i) holds with δ_1' and δ_2'.

Suppose property (iii) holds with some $\delta_1 > 0$ and $\delta_2 > 0$. Setting $x_i := 0$ $(i=1,\ldots,n)$, we obtain the property in Definition 2.1(ii), i.e. $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and δ_2.

Suppose property (i) holds with some $\delta_1 > 0$ and $\delta_2 > 0$, and let $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ be such that $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$. Then, for all $\rho \in]0,\delta_1'[$, $\omega_i \in \Omega_i \cap B_{\delta_2'}(\bar{x})$ and $x_i \in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \rho$, we have $\|\omega_i + x_i - \bar{x}\| \le \|\omega_i - \bar{x}\| + \|x_i\| < \varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$ $(i=1,\ldots,n)$. By (i), condition (2) is satisfied, and consequently, $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1' and δ_2'. ☐

Corollary 2.1 *The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if and only if there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that any of the properties* (i)–(iii) *in Proposition 2.2 holds.*

The three transversality properties can be characterized in metric terms. These metric characterizations can be used as equivalent definitions of the respective properties; cf. [13, Theorem 3.1].

Theorem 2.1 *The collection $\{\Omega_1,\ldots,\Omega_n\}$ is*

(i) *$\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$ if and only if*

$$d\left(\bar{x}, \bigcap_{i=1}^{n}(\Omega_i - x_i)\right) \le \varphi\left(\max_{1\le i\le n}\|x_i\|\right)$$

for all $x_i \in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta$;

(ii) *$\varphi-$subtransversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$ if and only if*

$$d\left(x, \bigcap_{i=1}^{n}\Omega_i\right) \le \varphi\left(\max_{1\le i\le n} d(x,\Omega_i)\right)$$

for all $x \in B_{\delta_2}(\bar{x})$ with $\varphi(\max_{1\le i\le n} d(x,\Omega_i)) < \delta_1$;

(iii) *$\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$ if and only if*

$$d\left(0, \bigcap_{i=1}^{n}(\Omega_i - \omega_i - x_i)\right) \le \varphi\left(\max_{1\le i\le n}\|x_i\|\right) \tag{4}$$

for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$.

The next proposition provides alternative metric characterizations of $\varphi-$transversality corresponding to the three properties in Proposition 2.2; cf. [13, Theorem 3.2].

Proposition 2.3 *Let $\delta_1 > 0$ and $\delta_2 > 0$. Properties* (i)–(iii) *in Proposition 2.2 hold if and only if the following equivalent properties hold true:*

(i) *inequality (4) holds for all $\omega_i \in \Omega_i$ and $x_i \in X$ with $\omega_i + x_i \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$;*

(ii) *for all $x_i \in \delta_2\mathbb{B}$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n} d(\bar{x}, \Omega_i - x_i)) < \delta_1$, it holds*

$$d\left(\bar{x}, \bigcap_{i=1}^{n}(\Omega_i - x_i)\right) \le \varphi\left(\max_{1\le i\le n} d(\bar{x}, \Omega_i - x_i)\right);$$

(iii) *for all $x, x_i \in X$ with $x + x_i \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(x, \Omega_i - x_i)) < \delta_1$, it holds*

$$d\left(x, \bigcap_{i=1}^{n}(\Omega_i - x_i)\right) \le \varphi\left(\max_{1\le i\le n} d(x, \Omega_i - x_i)\right).$$

In view of Corollary 2.1, the following assertion holds true.

Corollary 2.2 *The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if and only if there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that any of the properties* (i)–(iii) *in Proposition 2.3 holds.*

The next proposition identifies important situations when, in the case of two sets, ‘restricted’ versions of the metric characterizations of the nonlinear transversality properties in Theorem 2.1 can be used: with only one set being translated in the cases of $\varphi-$semitransversality and $\varphi-$transversality, and with the point x restricted to one of the sets in the case of $\varphi-$subtransversality. The latter restricted version is of importance, for instance, when dealing with alternating projections; cf. [13, Propositions 3.1 & 3.2].

Proposition 2.4 *Let Ω_1 and Ω_2 be subsets of a normed space X, $\bar{x} \in \Omega_1 \cap \Omega_2$, $\alpha > 0$ and $\alpha' := (1+2\alpha)^{-1}$.*

(i) *If $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$, then*

$$d(\bar{x}, (\Omega_1 - x) \cap \Omega_2) \le \varphi(\|x\|) \tag{5}$$

for all $x \in X$ with $\varphi(\|x\|) < \delta$.

If $\bar{t} > 0$, $\varphi(t) \le \alpha t$ for all $t \in]0, \bar{t}]$, and condition (5) holds for all $x \in \bar{t}\mathbb{B}$, then $\{\Omega_1, \Omega_2\}$ is $\alpha'-$semitransversal at $\bar{x}$ with $\delta := \left(\alpha + \frac{1}{2}\right)\bar{t}$.

(ii) *If $\{\Omega_1, \Omega_2\}$ is $\varphi-$subtransversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then*

$$d(x, \Omega_1 \cap \Omega_2) \le \varphi(d(x, \Omega_1)) \tag{6}$$

for all $x \in \Omega_2 \cap B_{\delta_2}(\bar{x})$ with $\varphi(d(x, \Omega_1)) < \delta_1$.

If $\bar{t} > 0$, $\delta_2 > 0$, $\varphi(t) \le \alpha t$ for all $t \in]0, \bar{t}]$, and condition (6) holds for all $x \in \Omega_2 \cap B_{2\delta_2}(\bar{x})$ with $d(x, \Omega_1) < \bar{t}$, then $\{\Omega_1, \Omega_2\}$ is $\alpha'-$subtransversal at $\bar{x}$ with $\delta_1 := \left(\alpha + \frac{1}{2}\right)\bar{t}$ and δ_2.

(iii) *If* $\{\Omega_1,\Omega_2\}$ *is* $\varphi-$*transversal at* $\bar{x}$ *with some* $\delta_1>0$ *and* $\delta_2>0$, *then*

$$d\left(0,(\Omega_1-\omega_1-x)\cap(\Omega_2-\omega_2)\right)\le\varphi(\|x\|) \tag{7}$$

for all $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,2)$ *and* $x\in X$ *with* $\varphi(\|x\|)<\delta_1$.
If $\bar{t}>0$, $\delta_2>0$, $\varphi(t)\le\alpha t$ *for all* $t\in]0,\bar{t}]$, *and condition* (7) *holds for all* $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,2)$ *and* $x\in\bar{t}\mathbb{B}$, *then* $\{\Omega_1,\Omega_2\}$ *is* $\alpha'-$*transversal at* $\bar{x}$ *with* $\delta_1:=\left(\alpha+\frac{1}{2}\right)\bar{t}$ *and* δ_2.

When the sets are convex, the definitions and characterizations of the $\varphi-$semitransversality and $\varphi-$transversality properties admit simplifications. We are unsure about possible meaningful simplifications of the $\varphi-$subtransversality property.

Given a $\delta>0$, we denote by $\widehat{\mathscr{C}}_\delta$ the subfamily of functions from $\mathscr{C}$ satisfying the following property:

$$\frac{\varphi^{-1}(\rho)}{\rho}\le\frac{\varphi^{-1}(\delta)}{\delta}\quad\text{for all}\quad\rho\in]0,\delta[. \tag{8}$$

Observe that any $\varphi\in\mathscr{C}$ such that the function $t\mapsto\frac{\varphi^{-1}(t)}{t}$ is nondecreasing on $]0,\delta]$ satisfies this property. This is true (for all $\delta>0$), in particular, in the Hölder setting, i.e. when $\varphi(t):=\alpha^{-1}t^q$ $(t\ge0)$ for some $\alpha>0$ and $q\in]0,1]$.

In the convex case, the requirements that the relations in parts (i) and (iii) of Definition 2.1 hold for all small $\rho>0$ can be significantly relaxed.

Proposition 2.5 *Suppose* $\Omega_1,\ldots,\Omega_n$ *are convex,* $\delta>0$, *and* $\varphi\in\widehat{\mathscr{C}}_\delta$. *The collection* $\{\Omega_1,\ldots,\Omega_n\}$ *is*

(i) $\varphi-$*semitransversal at* $\bar{x}$ *with* δ *if and only if*

$$\bigcap_{i=1}^{n}(\Omega_i-x_i)\cap B_\delta(\bar{x})\ne\emptyset \tag{9}$$

for all $x_i\in X$ $(i=1,\ldots,n)$ *with* $\varphi(\max_{1\le i\le n}\|x_i\|)<\delta$;

(ii) $\varphi-$*transversal at* $\bar{x}$ *with* $\delta_1:=\delta$ *and some* $\delta_2>0$ *if and only if*

$$\bigcap_{i=1}^{n}(\Omega_i-\omega_i-x_i)\cap(\delta_1\mathbb{B})\ne\emptyset \tag{10}$$

for all $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ *and* $x_i\in X$ $(i=1,\ldots,n)$ *with* $\varphi(\max_{1\le i\le n}\|x_i\|)<\delta_1$.

Proof (i) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ, then, by Definition 2.1(i), for any $x_i\in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|)<\delta$, and any number ρ satisfying $\varphi\left(\max_{1\le i\le n}\|x_i\|\right)<\rho<\delta$, condition (1) holds. The latter condition obviously implies (9).

Conversely, suppose condition (9) is satisfied for all $x_i\in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|)<\delta$. Let ρ be an arbitrary number in $]0,\delta[$ and let $x_i\in X$ $(i=1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|)<\rho$. Set $t:=\varphi^{-1}(\rho)/\varphi^{-1}(\delta)$ and $x_i':=x_i/t$ $(i=1,\ldots,n)$. Then $0<t<1$ and $\|x_i'\|=\|x_i\|/t<\varphi^{-1}(\rho)/t=\varphi^{-1}(\delta)$ $(i=1,\ldots,n)$, and consequently, there exists an $x'\in\cap_{i=1}^n(\Omega_i-x_i')\cap B_\delta(\bar{x})$, i.e. $x'\in B_\delta(\bar{x})$ and $x'=\omega_i-x_i'$ for some $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$, or equivalently, $x_i=t(\omega_i-x')$ $(i=1,\ldots,n)$. In view of the convexity of the sets, we have $t\omega_i+(1-t)\bar{x}\in\Omega_i$ $(i=1,\ldots,n)$. Set $x:=\bar{x}+t(x'-\bar{x})$. We have $x=t\omega_i+(1-t)\bar{x}-t(\omega_i-x')\in\Omega_i-x_i$ $(i=1,\ldots,n)$. Moreover, in view of (8), $\|x-\bar{x}\|=t\|x'-\bar{x}\|<\varphi^{-1}(\rho)\delta/\varphi^{-1}(\delta)\le\rho$. Hence, condition (1) is satisfied. By Definition 2.1(i), $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ.

(ii) If $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with $\delta_1 := \delta$ and some $\delta_2 > 0$, then, by Definition 2.1(iii), for any $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$, and any number ρ satisfying $\varphi\left(\max_{1\le i\le n}\|x_i\|\right) < \rho < \delta_1$, condition (2) holds. The latter condition obviously implies (10).

Conversely, suppose condition (10) is satisfied for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$. Then the collection of convex sets $\Omega_i - \omega_i$ $(i = 1,\ldots,n)$, considered near their common point 0, satisfies the conditions in part (i) and is consequently $\varphi-$semitransversal at 0 with δ_1 uniformly over $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$. This means that $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with δ_1 and δ_2. ☐

Remark 2.2 (i) The 'linear' version of Proposition 2.5 was established recently in [7].

(ii) Conditions (9) and (10) are equivalent to the metric estimates $d(\bar{x}, \cap_{i=1}^n(\Omega_i - x_i)) < \delta$ and $d(0, \cap_{i=1}^n(\Omega_i - \omega_i - x_i)) < \delta_1$, respectively.

(iii) The convexity assumption as well as condition $\varphi \in \widehat{\mathscr{C}}_\delta$ in Proposition 2.5 are only needed in the sufficiency parts.

Employing the same arguments as in the proof of Proposition 2.5, it is easy to establish simplified convex case versions of the alternative representations of $\varphi-$transversality in Proposition 2.2, and the 'restricted' two-set versions of $\varphi-$semitransversality and $\varphi-$transversality in Proposition 2.4.

Proposition 2.6 *Suppose $\Omega_1,\ldots,\Omega_n$ are convex, $\delta_1 > 0$, $\delta_2 > 0$, and $\varphi \in \widehat{\mathscr{C}}_{\delta_1}$. Properties* (i)–(iii) *in Proposition 2.2 hold if and only if the following equivalent properties hold true:*

(i) *condition* (10) *holds for all $\omega_i \in \Omega_i$, $x_i \in X$ with $\omega_i + x_i \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n}\|x_i\|) < \delta_1$;*

(ii) *condition* (9) *holds with δ_1 in place of δ for all $x_i \in \delta_2\mathbb{B}$ $(i = 1,\ldots,n)$ with $\varphi(\max_{1\le i\le n} d(\bar{x}, \Omega_i - x_i)) < \delta_1$;*

(iii) *$\cap_{i=1}^n(\Omega_i - x_i) \cap B_{\delta_1}(x) \neq \emptyset$ for all $x, x_i \in X$ with $x + x_i \in B_{\delta_2}(\bar{x})$ $(i = 1,\ldots,n)$ and $\varphi(\max_{1\le i\le n} d(x, \Omega_i - x_i)) < \delta_1$.*

Proposition 2.7 *Suppose Ω_1 and Ω_2 are convex, $\bar{x} \in \Omega_1 \cap \Omega_2$, $\alpha > 0$ and $\alpha' := (1 + 2\alpha)^{-1}$.*

(i) *If $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$, then*

$$(\Omega_1 - x) \cap \Omega_2 \cap B_\delta(\bar{x}) \neq \emptyset \tag{11}$$

for all $x \in X$ with $\varphi(\|x\|) < \delta$.

Suppose $\delta > 0$, $\bar{t} := \varphi^{-1}(\delta)$, $\varphi \in \widehat{\mathscr{C}}_\delta$, and $\varphi(t) \le \alpha t$ for all $t \in]0, \bar{t}]$. If condition (11) *holds for all $x \in \bar{t}\mathbb{B}$, then $\{\Omega_1, \Omega_2\}$ is $\alpha'-$semitransversal at $\bar{x}$ with $\delta' := \left(\alpha + \frac{1}{2}\right)\bar{t}$.*

(ii) *If $\{\Omega_1, \Omega_2\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then*

$$(\Omega_1 - \omega_1 - x) \cap (\Omega_2 - \omega_2) \cap (\delta_1\mathbb{B}) \neq \emptyset \tag{12}$$

for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1, 2)$ and $x \in X$ with $\varphi(\|x\|) < \delta_1$.

Suppose $\delta_1 > 0$, $\delta_2 > 0$, $\bar{t} := \varphi^{-1}(\delta_1)$, $\varphi \in \widehat{\mathscr{C}}_{\delta_1}$, and $\varphi(t) \le \alpha t$ for all $t \in]0, \bar{t}]$. If condition (12) *holds for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ $(i = 1, 2)$ and $x \in \bar{t}\mathbb{B}$, then $\{\Omega_1, \Omega_2\}$ is $\alpha'-$transversal at $\bar{x}$ with $\delta_1' := \left(\alpha + \frac{1}{2}\right)\bar{t}$ and δ_2.*

The next statement clarifies the relationship between the nonlinear semitransversality and transversality in the convex setting.

Proposition 2.8 *Suppose $\Omega_1,\ldots,\Omega_n$ are convex.*

(i) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*semitransversal at* $\bar{x}$ *with some* $\delta>0$, *then it is* $\psi-$*transversal at* $\bar{x}$ *with any* $\psi\in\widehat{\mathscr{C}}_\delta$, $\delta_1:=\delta$ *and any* $\delta_2>0$ *such that* $\delta_2+\psi^{-1}(\delta)\le\varphi^{-1}(\delta)$.

(ii) *Suppose* $\alpha>0$, $\delta>0$, $\varphi\in\widehat{\mathscr{C}}_\delta$, *and* $\varphi^{-1}(\rho)/\rho\ge\alpha$ *for all* $\rho\in]0,\delta[$. *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*semitransversal at* $\bar{x}$ *with* δ, *then, for any* $\varepsilon\in]0,\alpha[$, *it is* $\psi-$*transversal at* $\bar{x}$ *with* $\psi\in\mathscr{C}$ *such that* $\psi^{-1}(t)=\varphi^{-1}(t)-\varepsilon t$ *if* $t\in[0,\alpha]$, $\delta_1:=\delta$ *and* $\delta_2:=\varepsilon\delta$.

Proof (i) Let $\{\Omega_1,\ldots,\Omega_n\}$ be $\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$, and let $\psi\in\widehat{\mathscr{C}}_\delta$, $\delta_1:=\delta$ and $\delta_2>0$ be such that $\delta_2+\psi^{-1}(\delta)\le\varphi^{-1}(\delta)$. Let $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ and $x_i\in X$ $(i=1,\ldots,n)$ with $\psi(\max_{1\le i\le n}\|x_i\|)<\delta$. Set $x_i':=\omega_i+x_i-\bar{x}$ $(i=1,\ldots,n)$. Then $\|x_i'\|\le\|\omega_i-\bar{x}\|+\|x_i\|<\delta_2+\psi^{-1}(\delta)\le\varphi^{-1}(\delta)$ $(i=1,\ldots,n)$, and by Proposition 2.5(i), $\cap_{i=1}^n(\Omega_i-x_i')\cap B_\delta(\bar{x})\ne\emptyset$, which is equivalent to condition (10). In view of Proposition 2.5(ii), $\{\Omega_1,\ldots,\Omega_n\}$ is $\psi-$transversal at $\bar{x}$ with δ_1 and δ_2.

(ii) Observe that $\psi^{-1}\in\mathscr{C}$, hence $\psi\in\mathscr{C}$; $\delta_2+\psi^{-1}(\delta)=\varphi^{-1}(\delta)$, and $\frac{\psi^{-1}(\rho)}{\rho}=\frac{\varphi^{-1}(\rho)}{\rho}-\varepsilon\le\frac{\psi^{-1}(\delta)}{\delta}$ for all $\rho\in]0,\delta[$. □

In the Hölder setting, the above corollary yields the following assertion.

Corollary 2.3 *Suppose* $\Omega_1,\ldots,\Omega_n$ *are convex. Let* $\alpha>0$ *and* $q\in]0,1]$. *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\alpha-$*semitransversal of order* q *at* $\bar{x}$, *then it is* $\alpha'-$*transversal of order* q *at* $\bar{x}$ *with any* $\alpha'\in]0,\alpha[$. *As a consequence,* $\{\Omega_1,\ldots,\Omega_n\}$ *is semitransversal of order* q *at* $\bar{x}$ *if and only if it is transversal of order* q *at* $\bar{x}$.

Remark 2.3 In the linear case ($q=1$), the second part of Corollary 2.3 recaptures [25, Proposition 13(iv)].

3 Slope Necessary Conditions

In this section, we formulate slope necessary conditions for the properties in Definition 2.1. They all follow the same pattern. We first establish nonlocal slope necessary conditions arising from the definitions of the respective properties. The corresponding local slope necessary conditions, their Hölder as well as simplified (δ-free) versions are formulated as corollaries. This way we expose the hierarchy of this type of conditions.

Along with the standard maximum norm on X^{n+1}, we are going to use also the following norm depending on a parameter $\gamma>0$:

$$\|(x_1,\ldots,x_n,x)\|_\gamma:=\max\left\{\|x\|,\gamma\max_{1\le i\le n}\|x_i\|\right\},\quad x_1,\ldots,x_n,x\in X. \tag{13}$$

The next theorem establishes nonlocal slope necessary conditions for the three transversality properties. The subsequent necessary conditions in this paper as well as in [12] are consequences of this theorem.

Theorem 3.1 *Suppose there exist an* $\alpha>0$ *and a* $\delta>0$ *such that* $\varphi(t)\ge\alpha t$ *for all* $t\in]0,\varphi^{-1}(\delta)[$, *and* $\gamma:=(\alpha^{-1}+1)^{-1}$.

(i) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*semitransversal at* $\bar{x}$ *with* δ, *then*

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\ne(\bar{x},\ldots,\bar{x},\bar{x})}}\frac{\varphi\left(\max\limits_{1\le i\le n}\|x_i\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma}\ge 1 \tag{14}$$

for all $x_i \in X$ $(i=1,\ldots,n)$ *satisfying*

$$0 < \max_{1\le i\le n} \|x_i\| < \varphi^{-1}(\delta). \tag{15}$$

(ii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$ *with* $\delta_1 := \delta$ *and some* $\delta_2 > 0$, *then*

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1 \tag{16}$$

for all $x \in X$ *and* $\omega_i \in \Omega_i$ $(i=1,\ldots,n)$ *satisfying*

$$\|x-\bar{x}\| < \delta_2, \quad 0 < \max_{1\le i\le n} \|\omega_i - x\| < \varphi^{-1}(\delta_1). \tag{17}$$

(iii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*transversal at* $\bar{x}$ *with* $\delta_1 := \delta$ *and some* $\delta_2 > 0$, *then*

$$\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,\bar{x})}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma} \ge 1 \tag{18}$$

for all $\omega_i \in \Omega_i$ *and* $x_i \in X$ $(i=1,\ldots,n)$ *satisfying*

$$\max_{1\le i\le n}\|\omega_i-\bar{x}\| < \delta_2, \quad 0 < \max_{1\le i\le n}\|\omega_i - x_i - \bar{x}\| < \varphi^{-1}(\delta_1). \tag{19}$$

Proof (i) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ. Let $\gamma := (\alpha^{-1}+1)^{-1}$, $x_i \in X$ $(i=1,\ldots,n)$ satisfy (15). Denote $M := \varphi(\max_{1\le i\le n}\|x_i\|) < \delta$. Then $M \ge \alpha \max_{1\le i\le n}\|x_i\|$. Let $\eta \in]0,1[$, and choose a number $\gamma' \in]\eta\gamma,\gamma[$. Then $(\gamma')^{-1}-\alpha^{-1} > 1$. Choose a $\xi > 1$ such that $\xi \le \eta^{-1}$, $\xi \le (\gamma')^{-1}-\alpha^{-1}$ and $\xi M < \delta$. By Definition 2.1(i), we have $\cap_{i=1}^n(\Omega_i - x_i)\cap B_{\xi M}(\bar{x}) \neq \emptyset$, and consequently, there exist $\hat{x} \in X$ and $\hat{\omega}_i \in \Omega_i$ $(i=1,\ldots,n)$, with $\hat{\omega}_1 - x_1 = \ldots = \hat{\omega}_n - x_n = \hat{x}$ such that $\|\bar{x}-\hat{x}\| < \xi M$. Since $\max_{1\le i\le n}\|\hat{\omega}_i - x_i - \hat{x}\| = 0$ while $\max_{1\le i\le n}\|x_i\| > 0$, we have $(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x}) \neq (\bar{x},\ldots,\bar{x},\bar{x})$. Moreover, for all $i=1,\ldots,n$,

$$\begin{aligned}\|\hat{\omega}_i-\bar{x}\| &\le \|\hat{\omega}_i - x_i - \bar{x}\| + \|x_i\|\\ &= \|\hat{x}-\bar{x}\| + \|x_i\| < \xi M + \alpha^{-1}M \le M(\gamma')^{-1} < M(\eta\gamma)^{-1},\end{aligned}$$

and consequently,

$$\begin{aligned}\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma &= \max\left\{\|\hat{x}-\bar{x}\|, \gamma\max_{1\le i\le n}\|\hat{\omega}_i-\bar{x}\|\right\}\\ &< M\max\left\{\xi,\eta^{-1}\right\} = M\eta^{-1}.\end{aligned}$$

Hence, $\varphi(\max_{1\le i\le n}\|x_i\|) = M > \eta\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma$, and consequently,

$$\begin{aligned}&\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\ (u_1,\ldots,u_n,u)\neq(\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\varphi(\max_{1\le i\le n}\|x_i\|)-\varphi(\max_{1\le i\le n}\|u_i-x_i-u\|)}{\|(u_1,\ldots,u_n,u)-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma}\\ &\qquad\ge \frac{\varphi(\max_{1\le i\le n}\|x_i\|)}{\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\bar{x},\ldots,\bar{x},\bar{x})\|_\gamma} > \eta.\end{aligned}$$

Letting $\eta \uparrow 1$, we arrive at inequality (14).

(ii) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ with δ_1 and some $\delta_2>0$. Let $\gamma:=(\alpha^{-1}+1)^{-1}$, $x\in B_{\delta_2}(\bar{x})$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ satisfy (17). Denote $M:=\varphi\left(\max_{1\le i\le n}\|\omega_i-x\|\right)<\delta_1$. Then $M\ge\alpha\max_{1\le i\le n}\|\omega_i-x\|$. Let $\eta\in]0,1[$, and choose a number $\gamma'\in]\eta\gamma,\gamma[$. Then $(\gamma')^{-1}-\alpha^{-1}>1$. Choose a $\xi>1$ such that $\xi\le\eta^{-1}$, $\xi\le(\gamma')^{-1}-\alpha^{-1}$ and $\xi M<\delta_1$. By Definition 2.1(ii), there exists an $\omega\in\cap_{i=1}^n\Omega_i$ such that $\|\omega-x\|<\xi M$. Since $\max_{1\le i\le n}\|\omega_i-x\|>0$, we have $(\omega,\ldots,\omega,\omega)\ne(\omega_1,\ldots,\omega_n,x)$. Moreover, for all $i=1,\ldots,n$,

$$\|\omega-\omega_i\|\le\|\omega-x\|+\|\omega_i-x\|<\xi M+\alpha^{-1}M\le M(\gamma')^{-1}<M(\eta\gamma)^{-1},$$

and consequently,

$$\begin{aligned}\|(\omega,\ldots,\omega,\omega)-(\omega_1,\ldots,\omega_n,x)\|_\gamma&=\max\left\{\|\omega-x\|,\gamma\max_{1\le i\le n}\|\omega-\omega_i\|\right\}\\&<M\max\left\{\xi,\eta^{-1}\right\}=M\eta^{-1}.\end{aligned}$$

Thus, $\varphi\left(\max_{1\le i\le n}\|\omega_i-x\|\right)=M>\eta\|(\omega,\ldots,\omega,\omega)-(\omega_1,\ldots,\omega_n,x)\|_\gamma$. Since $\eta\in]0,1[$ is arbitrary, we obtain

$$\begin{aligned}&\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\(u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,x)}}\frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma}\\&\qquad\ge\frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x\|\right)}{\|(\omega,\ldots,\omega,\omega)-(\omega_1,\ldots,\omega_n,x)\|_\gamma}>\eta.\end{aligned}$$

Letting $\eta\uparrow1$, we arrive at (16).

(iii) Suppose $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$. Let $\gamma:=(\alpha^{-1}+1)^{-1}$, $\omega_i\in\Omega_i$ and $x_i\in X$ $(i=1,\ldots,n)$ satisfy (19). Denote $M:=\varphi(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|)<\delta_1$. Then $M\ge\alpha\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|$. Let $\eta\in]0,1[$, and choose a number $\gamma'\in]\eta\gamma,\gamma[$. Then $(\gamma')^{-1}-\alpha^{-1}>1$. Choose a $\xi>1$ such that $\xi\le\eta^{-1}$, $\xi\le(\gamma')^{-1}-\alpha^{-1}$ and $\xi M<\delta_1$. By Definition 2.1(iii), $\cap_{i=1}^n(\Omega_i-\omega_i-x_i')\cap(\xi M)\mathbb{B}\ne\emptyset$, where $x_i':=\bar{x}+x_i-\omega_i$ $(i=1,\ldots,n)$, or equivalently, $\cap_{i=1}^n(\Omega_i-x_i)\cap B_{\xi M}(\bar{x})\ne\emptyset$. Thus, there exist $\hat{\omega}_i\in\Omega_i$ $(i=1,\ldots,n)$ and $\hat{x}\in X$ such that $\hat{\omega}_1-x_1=\ldots=\hat{\omega}_n-x_n=\hat{x}$ such that $\|\bar{x}-\hat{x}\|<\xi M$. Since $\max_{1\le i\le n}\|\hat{\omega}_i-x_i-\hat{x}\|=0$ while $\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|>0$, we have $(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})\ne(\omega_1,\ldots,\omega_n,\bar{x})$. Moreover, for all $i=1,\ldots,n$,

$$\begin{aligned}\|\hat{\omega}_i-\omega_i\|&\le\|\hat{\omega}_i-x_i-\bar{x}\|+\|x_i+\bar{x}-\omega_i\|\\&=\|\hat{x}-\bar{x}\|+\|x_i+\bar{x}-\omega_i\|<\xi M+\alpha^{-1}M\le M(\gamma')^{-1}<M(\eta\gamma)^{-1},\end{aligned}$$

and consequently,

$$\begin{aligned}\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma&=\max\left\{\|\hat{x}-\bar{x}\|,\gamma\max_{1\le i\le n}\|\hat{\omega}_i-\omega_i\|\right\}\\&<M\max\left\{\xi,\eta^{-1}\right\}=M\eta^{-1}.\end{aligned}$$

Hence, $\varphi\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)=M>\eta\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma$, and consequently,

$$\begin{aligned}&\sup_{\substack{u_i\in\Omega_i\ (i=1,\ldots,n),\ u\in X\\(u_1,\ldots,u_n,u)\ne(\omega_1,\ldots,\omega_n,\bar{x})}}\frac{\varphi(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|)-\varphi(\max_{1\le i\le n}\|u_i-x_i-u\|)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma}\\&\qquad\ge\frac{\varphi\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar{x}\|\right)}{\|(\hat{\omega}_1,\ldots,\hat{\omega}_n,\hat{x})-(\omega_1,\ldots,\omega_n,\bar{x})\|_\gamma}>\eta.\end{aligned}$$

Letting $\eta\uparrow1$, we arrive at (18). $\square$

Remark 3.1 (i) The expressions in the left-hand sides of (14), (16) and (18) are the nonlocal γ-slopes [28, p. 60] computed at respective points of the extended-real-valued function

$$\widehat{f} := f + i_{\Omega_1 \times \ldots \times \Omega_n}, \tag{20}$$

where $f : X^{n+1} \to \mathbb{R}_+$ is given for $u_1, \ldots, u_n, u \in X$ by

$$f(u_1, \ldots, u_n, u) := \varphi\left(\max_{1 \le i \le n} \|u_i - x_i - u\|\right) \tag{21}$$

in the case of (14) and (18), and by

$$f(u_1, \ldots, u_n, u) := \varphi\left(\max_{1 \le i \le n} \|u_i - u\|\right) \tag{22}$$

in the case of (16).

(ii) In view of the definition of the parametric norm (13), if inequalities (14), (16) and (18) hold with the given γ, they also hold with any $\gamma' \in]0, \gamma[$. This observation is applicable to all slope inequalities in this section.

In the Hölder setting, Theorem 3.1 yields the following statement.

Corollary 3.1 *Let $\alpha > 0$ and $q \in]0, 1]$.*

(i) *Suppose $\{\Omega_1, \ldots, \Omega_n\}$ is $\alpha-$semitransversal of order q at $\bar{x}$ with some $\delta > 0$. Set $\gamma := (\alpha^{\frac{1}{q}} \delta^{\frac{1}{q}-1} + 1)^{-1}$. Then*

$$\sup_{\substack{u_i \in \Omega_i\ (i=1,\ldots,n),\ u \in X \\ (u_1,\ldots,u_n,u) \ne (\bar{x},\ldots,\bar{x},\bar{x})}} \frac{\max\limits_{1 \le i \le n} \|x_i\|^q - \max\limits_{1 \le i \le n} \|u_i - x_i - u\|^q}{\|(u_1, \ldots, u_n, u) - (\bar{x}, \ldots, \bar{x}, \bar{x})\|_\gamma} \ge \alpha$$

for all $x_i \in X$ $(i = 1, \ldots, n)$ with $0 < \max_{1 \le i \le n} \|x_i\| < (\alpha\delta)^{\frac{1}{q}}$.

(ii) *Suppose $\{\Omega_1, \ldots, \Omega_n\}$ is $\alpha-$subtransversal of order q at $\bar{x}$. Set $\gamma := (\alpha^{\frac{1}{q}} \delta_1^{\frac{1}{q}-1} + 1)^{-1}$. Then*

$$\sup_{\substack{u_i \in \Omega_i\ (i=1,\ldots,n),\ u \in X \\ (u_1,\ldots,u_n,u) \ne (\omega_1,\ldots,\omega_n,x)}} \frac{\max\limits_{1 \le i \le n} \|\omega_i - x\|^q - \max\limits_{1 \le i \le n} \|u_i - u\|^q}{\|(u_1, \ldots, u_n, u) - (\omega_1, \ldots, \omega_n, x)\|_\gamma} \ge \alpha$$

for all $x \in B_{\delta_2}(\bar{x})$ and $\omega_i \in \Omega_i$ $(i = 1, \ldots, n)$ with $0 < \max_{1 \le i \le n} \|\omega_i - x\| < (\alpha\delta_1)^{\frac{1}{q}}$.

(iii) *Suppose $\{\Omega_1, \ldots, \Omega_n\}$ is $\alpha-$transversal of order q at $\bar{x}$. Set $\gamma := (\alpha^{\frac{1}{q}} \delta_1^{\frac{1}{q}-1} + 1)^{-1}$. Then*

$$\sup_{\substack{u_i \in \Omega_i\ (i=1,\ldots,n),\ u \in X \\ (u_1,\ldots,u_n,u) \ne (\omega_1,\ldots,\omega_n,\bar{x})}} \frac{\max\limits_{1 \le i \le n} \|\omega_i - x_i - \bar{x}\|^q - \max\limits_{1 \le i \le n} \|u_i - x_i - u\|^q}{\|(u_1, \ldots, u_n, u) - (\omega_1, \ldots, \omega_n, \bar{x})\|_\gamma} \ge \alpha$$

for all $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$ and $x_i \in X$ $(i = 1, \ldots, n)$ with $0 < \max_{1 \le i \le n} \|\omega_i - x_i - \bar{x}\| < (\alpha\delta_1)^{\frac{1}{q}}$.

Proof The assertion in part (i) is a consequence of Theorem 3.1(i) with $\varphi(t) := \alpha^{-1} t^q$ for all $t \ge 0$; then of course, $\varphi^{-1}(t) = (\alpha t)^{\frac{1}{q}}$. To prove the statement, given an α and a δ, we need to compute a lower bound $\bar{\alpha}$ for $\varphi(t)/t$ on $]0, \varphi^{-1}(\delta)[$. The function $t \mapsto \varphi(t)/t = \alpha^{-1} t^{q-1}$ is nonincreasing on $]0, +\infty[$; hence, its value at $\varphi^{-1}(\delta) = (\alpha\delta)^{\frac{1}{q}}$ provides the exact lower bound. Thus, we can take $\bar{\alpha} := \alpha^{-1} (\alpha\delta)^{\frac{q-1}{q}} = \alpha^{-\frac{1}{q}} \delta^{1-\frac{1}{q}}$. Then $\gamma := (\bar{\alpha}^{-1} + 1)^{-1} = (\alpha^{\frac{1}{q}} \delta^{\frac{1}{q}-1} + 1)^{-1}$. The rest of the proof is straightforward. The proofs for parts (ii) and (iii) are similar. □

Remark 3.2 (i) When $q = 1$, we have $\gamma := (\alpha+1)^{-1}$ in Corollary 3.1, and this value does not depend on δ. When $q < 1$, by choosing a sufficiently small δ, the value of γ can be made arbitrarily close to 1.

(ii) Part (ii) of Corollary 3.1 strengthens [30, Proposition 10], while parts (i) and (iii) are new even in the linear setting.

The next statement presents a localized version of Theorem 3.1 in the convex setting.

Corollary 3.2 *Suppose* $\Omega_1,\ldots,\Omega_n$ *and* φ *are convex,* $\varphi'_+(0) > 0$, *and* $\gamma := ((\varphi'_+(0))^{-1}+1)^{-1}$.

(i) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*semitransversal at* $\bar x$ *with some* $\delta > 0$, *then*

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar x\ (i=1,\ldots,n),\ u\to\bar x\\ (u_1,\ldots,u_n,u)\neq(\bar x,\ldots,\bar x,\bar x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|x_i\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\bar x,\ldots,\bar x,\bar x)\|_\gamma} \ge 1 \tag{23}$$

for all $x_i \in X$ $(i = 1,\ldots,n)$ *satisfying* (15).

(ii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar x$ *with some* $\delta_1 > 0$ *and* $\delta_2 > 0$, *then*

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to x\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1 \tag{24}$$

for all $x \in X$ *and* $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ *satisfying* (17).

(iii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*transversal at* $\bar x$ *with some* $\delta_1 > 0$ *and* $\delta_2 > 0$, *then*

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to\bar x\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,\bar x)}} \frac{\varphi\left(\max\limits_{1\le i\le n}\|\omega_i-x_i-\bar x\|\right)-\varphi\left(\max\limits_{1\le i\le n}\|u_i-x_i-u\|\right)}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,\bar x)\|_\gamma} \ge 1 \tag{25}$$

for all $\omega_i \in \Omega_i$ *and* $x_i \in X$ $(i = 1,\ldots,n)$ *satisfying* (19).

Moreover, if $\varphi \in \mathscr{C}^1$, *then inequalities* (23), (24) *and* (25) *in parts* (i)–(iii) *can be replaced, respectively, by*

$$\varphi'\left(\max_{1\le i\le n}\|x_i\|\right) \limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar x\ (i=1,\ldots,n),\ u\to\bar x\\ (u_1,\ldots,u_n,u)\neq(\bar x,\ldots,\bar x,\bar x)}} \frac{\max\limits_{1\le i\le n}\|x_i\|-\max\limits_{1\le i\le n}\|u_i-x_i-u\|}{\|(u_1,\ldots,u_n,u)-(\bar x,\ldots,\bar x,\bar x)\|_\gamma} \ge 1, \tag{26}$$

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x\|\right) \times \limsup_{\substack{u_i \xrightarrow{\Omega_i} \omega_i\ (i=1,\ldots,n),\ u\to x\\ (u_1,\ldots,u_n,u)\neq(\omega_1,\ldots,\omega_n,x)}} \frac{\max\limits_{1\le i\le n}\|\omega_i-x\|-\max\limits_{1\le i\le n}\|u_i-u\|}{\|(u_1,\ldots,u_n,u)-(\omega_1,\ldots,\omega_n,x)\|_\gamma} \ge 1, \tag{27}$$

$$\varphi'\left(\max_{1\le i\le n}\|\omega_i-x_i-\bar x\|\right) \times \limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar x\ (i=1,\ldots,n),\ u\to\bar x\\ (u_1,\ldots,u_n,u)\neq(\bar x,\ldots,\bar x,\bar x)}} \frac{\max\limits_{1\le i\le n}\|\omega_i-x_i-\bar x\|-\max\limits_{1\le i\le n}\|u_i-x_i-u\|}{\|(u_1,\ldots,u_n,u)-(\bar x,\ldots,\bar x,\bar x)\|_\gamma} \ge 1. \tag{28}$$

Proof In view of the convexity of φ, it holds $\varphi(t) \geq \varphi'_+(0)t$ for all $t \geq 0$. Moreover, functions (21), (22) and (20) are convex. By Proposition 1.1(ii), the left-hand sides of inequalities (14), (16) and (18) are equal to the left-hand sides of inequalities (23), (24) and (25), respectively. If $\varphi \in \mathscr{C}^1$, then, thanks to Lemma 1.1, the left-hand sides of inequalities (23), (24) and (25) are equal, respectively, to the left-hand sides of inequalities (26), (27) and (28). ☐

Remark 3.3 (i) The expressions in the left-hand sides of the inequalities (23), (24) and (25) are the γ-slopes [28, p. 61] computed at respective points of the extended-real-valued function (20), where f is defined by either (21) or (22).

(ii) The slope necessary conditions for $\varphi-$semitransversality and $\varphi-$subtransversality in parts (i) and (ii) of Corollary 3.2 are particular cases of the slope condition of $\varphi-$transversality in part (iii) of this corollary, corresponding to setting $\omega'_i := \bar{x}$ $(i = 1,\ldots,n)$ and $x_1 = \ldots = x_n$, respectively.

Sacrificing the estimates for the δ's in Theorem 3.1 and Corollary 3.2, we can formulate 'δ-free' versions of these statements.

Corollary 3.3 *Suppose* $\varphi(t) \geq \alpha t$ *for some* $\alpha > 0$ *and all* $t > 0$ *near* 0, *and* $\gamma := (\alpha^{-1} + 1)^{-1}$.

(i) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*semitransversal at* $\bar{x}$, *then inequality* (14) *holds for all* $x_i \in X$ $(i = 1,\ldots,n)$ *near* 0 *with* $\max_{1\leq i\leq n} \|x_i\| > 0$.

(ii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$, *then inequality* (16) *holds for all* $x \in X$ *near* $\bar{x}$ *and* $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ *near* $\bar{x}$ *with* $\max_{1\leq i\leq n} \|\omega_i - x\| > 0$.

(iii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*transversal at* $\bar{x}$, *then inequality* (18) *holds for all* $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ *near* $\bar{x}$ *and* $x_i \in X$ $(i = 1,\ldots,n)$ *near* 0 *with* $\max_{1\leq i\leq n} \|\omega_i - x_i - \bar{x}\| > 0$.

Corollary 3.4 *Suppose* $\Omega_1,\ldots,\Omega_n$ *and* φ *are convex,* $\varphi'_+(0) > 0$, *and* $\gamma := ((\varphi'_+(0))^{-1} + 1)^{-1}$.

(i) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*semitransversal at* $\bar{x}$, *then inequality* (23) *holds for all* $x_i \in X$ $(i = 1,\ldots,n)$ *near* 0 *with* $\max_{1\leq i\leq n} \|x_i\| > 0$.

(ii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$, *then inequality* (24) *holds for all* $x \in X$ *near* $\bar{x}$ *and* $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ *near* $\bar{x}$ *with* $\max_{1\leq i\leq n} \|\omega_i - x\| > 0$.

(iii) *If* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*transversal at* $\bar{x}$, *then inequality* (25) *holds for all* $\omega_i \in \Omega_i$ $(i = 1,\ldots,n)$ *near* $\bar{x}$ *and* $x_i \in X$ $(i = 1,\ldots,n)$ *near* 0 *with* $\max_{1\leq i\leq n} \|\omega_i - x_i - \bar{x}\| > 0$.

Moreover, if $\varphi \in \mathscr{C}^1$, *then inequalities* (23), (24) *and* (25) *in parts* (i)–(iii) *can be replaced by* (26), (27) *and* (28), *respectively.*

Remark 3.4 If $\cap_{i=1}^n \Omega_i$ is closed and $\bar{x} \in \operatorname{bd} \cap_{i=1}^n \Omega_i$, then condition $\varphi'_+(0) > 0$ in parts (ii) and (iii) of Corollaries 3.2 and 3.4 can be dropped, as in this case Proposition 2.1(iii) implies that $\varphi'_+(0) \geq 1$. Also in view of this proposition, one can suppose in parts (ii) and (iii) of Theorem 3.1 and Corollary 3.3 that $\alpha \geq 1$.

The next sufficient condition for $\varphi-$subtransversality was established in [13, Corollary 4.5].

Proposition 3.1 *Suppose* X *is Banach, and* $\Omega_1,\ldots,\Omega_n$ *are closed. The collection* $\{\Omega_1,\ldots,\Omega_n\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$ *with some* $\delta_1 > 0$ *and* $\delta_2 > 0$ *if, for some* $\gamma > 0$ *and any* $x' \in X$ *satisfying* $\|x' - \bar{x}\| < \delta_2$ *and* $0 < \max_{1\leq i\leq n} d(x',\Omega_i) < \varphi^{-1}(\delta_1)$, *there exists a* $\lambda \in]\varphi(\max_{1\leq i\leq n} d(x',\Omega_i)), \delta_1[$ *such that inequality* (24) *holds for all* $x \in X$ *and* $\omega_i, \omega'_i \in \Omega_i$ $(i = 1,\ldots,n)$ *satisfying* $\|x - x'\| < \lambda$, $\max_{1\leq i\leq n} \|\omega_i - \omega'_i\| < \lambda\gamma^{-1}$ *and* $0 < \max_{1\leq i\leq n} \|\omega_i - x\| \leq \max_{1\leq i\leq n} \|\omega'_i - x'\| < \varphi^{-1}(\lambda)$.

From Proposition 3.1 and Corollary 3.4(ii), we obtain a complete slope characterization of $\varphi-$subtransversality in the convex case.

Corollary 3.5 *Suppose X is Banach, $\Omega_1,\ldots,\Omega_n$ are closed and convex, φ is convex, $\varphi'_+(0)>0$, and $\gamma:=((\varphi'_+(0))^{-1}+1)^{-1}$. The collection $\{\Omega_1,\ldots,\Omega_n\}$ is $\varphi-$subtransversal at $\bar{x}$ if and only if inequality (24) holds for all $x\in X$ near $\bar{x}$ and $\omega_i\in\Omega_i$ $(i=1,\ldots,n)$ near $\bar{x}$ with* $\max_{1\le i\le n}\|\omega_i-x\|>0$.

Remark 3.5 Combining sufficient conditions from [13] for the other two nonlinear transversality properties with the corresponding necessary conditions from Corollary 3.4 does not lead to their complete slope characterizations.

4 Transversality and Regularity

In this section, we provide quantitative relations between the nonlinear transversality properties of collections of sets and the corresponding regularity properties of set-valued mappings, and discuss in the convex setting two nonlinear transversality properties of a set-valued mapping to a set in the range space. As before, the nonlinearity in this section is determined by a function $\varphi\in\mathscr{C}$.

Definition 4.1 Suppose X and Y are metric spaces, $F:X\rightrightarrows Y$, and $(\bar{x},\bar{y})\in\operatorname{gph}F$. The mapping F is

(i) $\varphi-$semiregular at $(\bar{x},\bar{y})$ if there exists a $\delta>0$ such that

$$d(\bar{x},F^{-1}(y))\le\varphi(d(y,\bar{y}))$$

for all $y\in Y$ with $\varphi(d(y,\bar{y}))<\delta$;

(ii) $\varphi-$subregular at $(\bar{x},\bar{y})$ if there exist $\delta_1>0$ and $\delta_2>0$ such that

$$d(x,F^{-1}(\bar{y}))\le\varphi(d(\bar{y},F(x)))$$

for all $x\in B_{\delta_2}(\bar{x})$ with $\varphi(d(\bar{y},F(x)))<\delta_1$;

(iii) $\varphi-$regular at $(\bar{x},\bar{y})$ if there exist $\delta_1>0$ and $\delta_2>0$ such that

$$d(x,F^{-1}(y))\le\varphi(d(y,F(x)))\tag{29}$$

for all $x\in X$ and $y\in Y$ with $d(x,\bar{x})+d(y,\bar{y})<\delta_2$ and $\varphi(d(y,F(x)))<\delta_1$.

In the linear case, i.e. when $\varphi(t):=\alpha^{-1}t$ with some $\alpha>0$, the properties reduce to the conventional metric *semiregularity*, *subregularity* and *regularity*, respectively; cf. [10, 17, 24, 27, 36, 42]. For discussions of the properties in Definition 4.1 in the general nonlinear setting, the reader is referred to [13, Section 5.1].

Given $n\ge 2$ subsets $\Omega_1,\ldots,\Omega_n$ of a normed space X, it is common (cf. Ioffe [22]) to study their transversality and other properties by reducing them to the corresponding regularity properties of the set-valued mapping $F:X\rightrightarrows X^n$:

$$F(x):=(\Omega_1-x)\times\ldots\times(\Omega_n-x),\quad x\in X.\tag{30}$$

Observe that

$$F^{-1}(x_1,\ldots,x_n)=(\Omega_1-x_1)\cap\ldots\cap(\Omega_n-x_n)\quad\text{for all}\quad x_1,\ldots,x_n\in X,$$

and, if $\bar{x}\in\cap_{i=1}^n\Omega_i$, then $(0,\ldots,0)\in F(\bar{x})$.

The next statement is a reformulation of the metric characterizations of the transversality properties in Theorem 2.1. It generalizes and extends the corresponding results in [7, 22, 23, 25–27, 29–33].

Theorem 4.1 *Let $\Omega_1,\dots,\Omega_n$ be subsets of a normed space X, F be defined by* (30), *$\bar{x}\in\cap_{i=1}^n\Omega_i$ and $\bar{y}:=(0,\dots,0)\in X^n$. The collection $\{\Omega_1,\dots,\Omega_n\}$ is*

(i) *$\varphi-$semitransversal at $\bar{x}$ with some $\delta>0$ if and only if F is $\varphi-$semiregular at $(\bar{x},\bar{y})$ with δ;*

(ii) *$\varphi-$subtransversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$ if and only if F is $\varphi-$subregular at $(\bar{x},\bar{y})$ with δ_1 and δ_2;*

(iii) *$\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$ if and only if*

$$d(0,F^{-1}(\omega_1+x_1,\dots,\omega_n+x_n))\le\varphi(\|y\|) \tag{31}$$

for all $\omega_i\in\Omega_i\cap B_{\delta_2}(\bar{x})$ $(i=1,\dots,n)$ and $y:=(x_1,\dots,x_n)\in X^n$ with $\varphi(\|y\|)<\delta_1$.

Observe that the condition in part (iii) of Theorem 4.1 is similar to, but not exactly the one in the definition of $\varphi-$regularity. The next statement shows that the latter corresponds to alternative metric characterizations of $\varphi-$transversality.

Proposition 4.1 *Let $\Omega_1,\dots,\Omega_n$ be subsets of a normed space X, F be defined by* (30), *$\bar{x}\in\cap_{i=1}^n\Omega_i$, $\bar{y}:=(0,\dots,0)\in X^n$, $\delta_1>0$ and $\delta_2>0$. The following properties are equivalent:*

(i) *inequality* (31) *holds for all $\omega_i\in\Omega_i$, $y:=(x_1,\dots,x_n)\in X^n$ with $\omega_i+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\dots,n)$ and $\varphi(\|y\|)<\delta_1$;*

(ii) *$d(\bar{x},F^{-1}(y))\le\varphi(d(y,F(\bar{x})))$ for all $y\in\delta_2\mathbb{B}_{X^n}$ with $\varphi(d(y,F(\bar{x})))<\delta_1$;*

(iii) *inequality* (29) *holds for all $x\in X$, $y:=(x_1,\dots,x_n)\in X^n$ with $x+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\dots,n)$ and $\varphi(d(y,F(x)))<\delta_1$;*

(iv) *the mapping F is $\varphi-$regular at $(\bar{x},\bar{y})$ with δ_1 and δ_2.*

Moreover, if $\{\Omega_1,\dots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with some $\delta_1>0$ and $\delta_2>0$, then conditions (i)–(iv) *hold with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ satisfying $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$ in place of δ_1 and δ_2.*

Conversely, if properties (i)–(iv) *hold with some $\delta_1>0$ and $\delta_2>0$, then $\{\Omega_1,\dots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ with any $\delta_1'\in]0,\delta_1]$ and $\delta_2'>0$ satisfying $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$.*

Proof With the exception of item (iv), the statement is a reformulation of Proposition 2.3 in terms of the mapping F. It is easy to see that conditions (iii) and (iv) are equivalent; cf. the hints to the proof of the linear version of this fact in [7]. □

Corollary 4.1 *Let $\Omega_1,\dots,\Omega_n$ be subsets of a normed space X, F be defined by* (30), *$\bar{x}\in\cap_{i=1}^n\Omega_i$ and $\bar{y}:=(0,\dots,0)\in X^n$. The collection $\{\Omega_1,\dots,\Omega_n\}$ is $\varphi-$transversal at $\bar{x}$ if and only if F is $\varphi-$regular at $(\bar{x},\bar{y})$.*

In the convex case, conditions (i)–(iv) in Proposition 4.1 admit simplifications.

Corollary 4.2 *Let $\Omega_1,\dots,\Omega_n$ be convex subsets of a normed space X, F be defined by* (30), *$\bar{x}\in\cap_{i=1}^n\Omega_i$, $\bar{y}:=(0,\dots,0)\in X^n$, $\delta_1>0$ and $\delta_2>0$. Conditions* (i)–(iv) *in Proposition 4.1 hold if and only if the following equivalent properties hold true:*

(i) *$F^{-1}(\omega_1+x_1,\dots,\omega_n+x_n)\cap(\delta_1\mathbb{B})\neq\emptyset$ for all $\omega_i\in\Omega_i$, $y:=(x_1,\dots,x_n)\in X^n$ with $\omega_i+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\dots,n)$ and $\varphi(\|y\|)<\delta_1$;*

(ii) *$F^{-1}(y)\cap B_{\delta_1}(\bar{x})\neq\emptyset$ for all $y\in\delta_2\mathbb{B}_{X^n}$ with $\varphi(d(y,F(\bar{x})))<\delta_1$;*

(iii) *$F^{-1}(y)\cap B_{\delta_1}(\bar{x})\neq\emptyset$ for all $x\in X$, $y:=(x_1,\dots,x_n)\in X^n$ with $x+x_i\in B_{\delta_2}(\bar{x})$ $(i=1,\dots,n)$ and $\varphi(d(y,F(x)))<\delta_1$.*

Given an arbitrary set-valued mapping $F:X\rightrightarrows Y$, one can go the other way around and reduce its regularity properties to the corresponding transversality properties of a collection of sets; cf. [13, Section 5.1]. If $(\bar{x},\bar{y})\in\operatorname{gph}F$, one can consider the two sets

$$\Omega_1:=\operatorname{gph}F\quad\text{and}\quad\Omega_2:=X\times\{\bar{y}\} \tag{32}$$

in the product space $X \times Y$. Note that $(\bar{x},\bar{y}) \in \Omega_1 \cap \Omega_2$.

The next theorem complements the relations derived in [13] and translates nonlinear transversality properties of the collection $\{\Omega_1,\Omega_2\}$ into certain metric properties of the mapping F, which can be used along with those in Definition 4.1.

Theorem 4.2 *Let X and Y be normed spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, Ω_1 and Ω_2 be defined by* (32)*.*

(i) *If $\{\Omega_1,\Omega_2\}$ is $\varphi-$semitransversal at $(\bar{x},\bar{y})$ with some $\delta > 0$, then*

$$d(\bar{x}+u, F^{-1}(\bar{y}+v)) \le \varphi\left(\max\left\{\|u\|, \|v\|/2\right\}\right) \tag{33}$$

for all $u \in X$ and $v \in Y$ with $\varphi\left(\max\left\{\|u\|, \|v\|/2\right\}\right) < \delta$.

(ii) *If $\{\Omega_1,\Omega_2\}$ is $\varphi-$subtransversal at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then*

$$d(x, F^{-1}(\bar{y})) \le \varphi\left(\max\left\{d((x,y), \operatorname{gph} F), \|y-\bar{y}\|\right\}\right) \tag{34}$$

for all $x \in B_{\delta_2}(\bar{x})$ and $y \in B_{\min\{\varphi^{-1}(\delta_1),\delta_2\}}(\bar{y})$ with $\varphi(d((x,y),\operatorname{gph} F)) < \delta_1$.

(iii) *If $\{\Omega_1,\Omega_2\}$ is $\varphi-$transversal at $(\bar{x},\bar{y})$ with some $\delta_1 > 0$ and $\delta_2 > 0$, then*

$$d(x+u, F^{-1}(y+v)) \le \varphi\left(\max\left\{\|u\|, \|v\|/2\right\}\right) \tag{35}$$

for all $(x,y) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, and $u \in X$, $v \in Y$ with $\varphi\left(\max\left\{\|u\|, \|v\|/2\right\}\right) < \delta_1$.

Moreover, if $\varphi(t) \ge t$ for all $t \in]0, \varphi^{-1}(\delta)[$ in part (i)*, or $\varphi(t) \ge t$ for all $t \in]0, \varphi^{-1}(\delta_1)[$ in parts* (ii) and (iii)*, then the respective implications hold as equivalences.*

Proof (i) Let $x_1 := (u_1, v_1)$, $x_2 := (u_2, v_2) \in X \times Y$. Then,

$$\begin{aligned}(\Omega_1 - x_1) \cap (\Omega_2 - x_2) &= (F^{-1}(\bar{y}+v_1-v_2) - u_1) \times \{\bar{y}-v_2\}, \\ d\left((\bar{x},\bar{y}), (\Omega_1 - x_1) \cap (\Omega_2 - x_2)\right) &= \max\{d(\bar{x}+u_1, F^{-1}(\bar{y}+v_1-v_2)), \|v_2\|\}.\end{aligned} \tag{36}$$

Thus, inequality

$$d\left((\bar{x},\bar{y}), (\Omega_1 - x_1) \cap (\Omega_2 - x_2)\right) \le \varphi(\max\{\|x_1\|, \|x_2\|\})$$

implies

$$d(\bar{x}+u_1, F^{-1}(\bar{y}+v_1-v_2)) \le \varphi(\max\{\|u_1\|, \|u_2\|, \|v_1\|, \|v_2\|\}), \tag{37}$$

and if $\varphi(\|v_2\|) < \delta$, the converse implication is true when $\varphi(t) \ge t$ for all $t \in]0, \varphi^{-1}(\delta)[$.

We claim that the following conditions are equivalent:

(a) inequality (33) holds for all $u \in X$ and $v \in Y$ with $\varphi\left(\max\left\{\|u\|, \|v\|/2\right\}\right) < \delta$;

(b) inequality (37) holds for all $u_1, u_2 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|, \|u_2\|, \|v_1\|, \|v_2\|\}) < \delta$.

(a) $\Rightarrow$ (b). Let $u_1, u_2 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|, \|u_2\|, \|v_1\|, \|v_2\|\}) < \delta$. Then inequality (33) holds for u_1 and $v_1 - v_2$ in place of u and v, i.e.

$$d(\bar{x}+u_1, F^{-1}(\bar{y}+v_1-v_2)) \le \varphi(\max\left\{\|u_1\|, \|v_1-v_2\|\right\}),$$

and consequently, inequality (37) holds.

(b) $\Rightarrow$ (a). Let $u \in X$ and $v \in Y$ with $\varphi\left(\max\left\{\|u\|, \|v\|/2\right\}\right) < \delta$. Then, inequality (37) holds for $u_1 := u$, $u_2 := 0$, $v_1 := v/2$ and $v_2 := -v/2$, which is equivalent to inequality (33).

Thus, (a) $\Leftrightarrow$ (b), which, in view of Theorem 2.1(i), proves the assertion.

(ii) Let $(x,y) \in X \times Y$. Thanks to (32), we have

$$d((x,y),\Omega_2) = \|y-\bar{y}\|, \;\; d((x,y),\Omega_1\cap\Omega_2) = \max\{d(x,F^{-1}(\bar{y})), \|y-\bar{y}\|\}.$$

Thus, inequality

$$d\left((x,y),\Omega_1\cap\Omega_2\right) \le \varphi(\max\left\{d((x,y),\Omega_1), d((x,y),\Omega_2)\right\}$$

implies inequality (34), and if $\varphi(\|y-\bar{y}\|) < \delta_1$, the converse implication is true when $\varphi(t) \ge t$ for all $t \in]0,\varphi^{-1}(\delta_1)[$. In view of Theorem 2.1(ii), this proves the assertion.

(iii) Let $x_1 := (u_1,v_1)$, $x_2 := (u_2,v_2)$, $(x,y) \in X \times Y$ and $z \in X$. Then,

$$\begin{aligned}
&(\Omega_1 - (x,y) - x_1) \cap (\Omega_2 - (z,\bar{y}) - x_2) \\
&\qquad\qquad = (F^{-1}(y+v_1-v_2) - x - u_1) \times \{-v_2\}, \\
&d\left((0,0), (\Omega_1 - (x,y) - x_1) \cap (\Omega_2 - (z,\bar{y}) - x_2)\right) \\
&\qquad\qquad = \max\{d(x+u_1, F^{-1}(y+v_1-v_2)), \|v_2\|\}.
\end{aligned}$$

Thus, inequality

$$d\left((0,0), (\Omega_1 - (x,y) - x_1) \cap (\Omega_2 - (z,\bar{y}) - x_2)\right) \le \varphi\left(\max\{\|x_1\|,\|x_2\|\}\right) \tag{38}$$

implies

$$d(x+u_1, F^{-1}(y+v_1-v_2)) \le \varphi(\max\{\|u_1\|,\|u_2\|,\|v_1\|,\|v_2\|\}), \tag{39}$$

and, if $\varphi(\|v_2\|) < \delta_1$, the converse implication is true when $\varphi(t) \ge t$ for all $t \in]0,\varphi^{-1}(\delta_1)[$. The same arguments as in the proof of (i) show that inequality (39) holds for all $(x,y) \in \Omega_1 \cap B_{\delta_2}(\bar{x},\bar{y})$, $u_1,u_2 \in X$ and $v_1,v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|u_2\|,\|v_1\|,\|v_2\|\}) < \delta$ if and only if inequality (35) holds for all $(x,y) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $u \in X$ and $v \in Y$ with $\varphi\left(\max\left\{\|u\|,\|v\|/2\right\}\right) < \delta$. In view of Theorem 2.1(iii), this proves the assertion. ☐

Using the estimates in the proof of Theorem 4.2, we can also translate the metric characterizations of the nonlinear transversality in Proposition 2.3 into certain metric conditions involving the set-valued mapping F.

Proposition 4.2 *Let X and Y be normed spaces, $F : X \rightrightarrows Y$, $(\bar{x},\bar{y}) \in \operatorname{gph} F$, Ω_1 and Ω_2 be defined by (32), $\delta_1 > 0$ and $\delta_2 > 0$. The following properties are equivalent:*

(i) *for all $(x,y) \in \operatorname{gph} F$, $u \in X$ and $v_1,v_2 \in Y$ with $\varphi(\max\{\|u\|,\|v_1\|\}) < \delta_1$, $\varphi(\|v_2\|) < \min\{\delta_1,\varphi(\delta_2)\}$, and $(x,y)+(u,v_1) \in B_{\delta_2}(\bar{x},\bar{y})$, it holds*

$$d(x+u, F^{-1}(y+v_1-v_2)) \le \varphi(\max\{\|u\|,\|v_1\|,\|v_2\|\}); \tag{40}$$

(ii) *for all $u \in X$ and $v_1,v_2 \in Y$ with $\max\{\|u\|,\|v_1\|\} < \delta_2$, $\varphi(\|v_2\|) < \min\{\delta_1,\varphi(\delta_2)\}$ and $\varphi(d((\bar{x},\bar{y})+(u,v_1),\operatorname{gph} F)) < \delta_1$, it holds*

$$d(\bar{x}+u, F^{-1}(\bar{y}+v_1-v_2)) \le \varphi(\max\{d((\bar{x},\bar{y})+(u,v_1),\operatorname{gph} F),\|v_2\|\}); \tag{41}$$

(iii) *for all (x,y), $(u,v_1) \in X \times Y$ and $v_2 \in Y$ with $(x,y)+(u,v_1) \in B_{\delta_2}(\bar{x},\bar{y})$, $\varphi(d\left((x,y)+(u,v_1),\operatorname{gph} F\right)) < \delta_1$, and $\varphi(\|y+v_2-\bar{y}\|) < \min\{\delta_1,\varphi(\delta_2)\}$, it holds*

$$\begin{aligned}
&d(x+u, F^{-1}(y+v_1-v_2)) \\
&\qquad\qquad \le \varphi\left(\max\left\{d\left((x,y)+(u,v_1),\operatorname{gph} F\right), \|y+v_2-\bar{y}\|\right\}\right).
\end{aligned} \tag{42}$$

Moreover, if $\{\Omega_1,\Omega_2\}$ *is* $\varphi-$*transversal at* $(\bar{x},\bar{y})$ *with some* $\delta_1>0$ *and* $\delta_2>0$*, then conditions* (i)–(iii) *hold with any* $\delta_1'\in]0,\delta_1]$ *and* $\delta_2'>0$ *satisfying* $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$ *in place of* δ_1 *and* δ_2*.*

Conversely, if properties (i)–(iii) *hold with some* $\delta_1>0$ *and* $\delta_2>0$*, and* $\varphi(t)\ge t$ *for all* $t\in]0,\varphi^{-1}(\delta_1)[$*, then* $\{\Omega_1,\Omega_2\}$ *is* $\varphi-$*transversal at* $(\bar{x},\bar{y})$ *with any* $\delta_1'\in]0,\delta_1]$ *and* $\delta_2'>0$ *satisfying* $\varphi^{-1}(\delta_1')+\delta_2'\le\delta_2$*.*

Proof (i) Given $x_1:=(u_1,v_1)$, $x_2:=(u_2,v_2)$, $(x,y)\in X\times Y$ and $z\in X$, inequality (38) implies (39), and the conditions are equivalent if $\varphi(\|v_2\|)<\delta_1$, and $\varphi(t)\ge t$ for all $t\in]0,\varphi^{-1}(\delta_1)[$. Moreover, given $x,u\in X$ and $y,v_1,v_2\in Y$, inequality (39) holds with $u_1:=u$ for all $u_2\in X$ with $\varphi(\|u_2\|)<\delta_1$ if and only if inequality (40) is satisfied. Hence, condition (i) is equivalent to the one in Proposition 2.3(i).

(ii) Given $x_1:=(u_1,v_1)$ and $x_2:=(u_2,v_2)$, inequality

$$d\left((\bar{x},\bar{y}),(\Omega_1-x_1)\cap(\Omega_2-x_2)\right)\\ \le\varphi\left(\max\{d((\bar{x},\bar{y}),\Omega_1-x_1),d((\bar{x},\bar{y}),\Omega_2-x_2)\}\right)$$

implies

$$d(\bar{x}+u_1,F^{-1}(\bar{y}+v_1-v_2))\le\varphi\left(\max\{d((\bar{x},\bar{y})+(u_1,v_1),\operatorname{gph}F),\|v_2\|\}\right),\tag{43}$$

and the converse implication is true if $\varphi(\|v_2\|)<\delta_1$ and, $\varphi(t)\ge t$ for all $t\in]0,\varphi^{-1}(\delta_1)[$. Observe that inequality (43) holds with $u_1:=u$ if and only if inequality (41) is satisfied. Hence, condition (ii) is equivalent to the one in Proposition 2.3(ii).

(iii) Given $x_1:=(u_1,v_1)$, $x_2:=(u_2,v_2)$ and $(x,y)\in X\times Y$, we have representation (36), and consequently,

$$d\left((x,y),(\Omega_1-x_1)\cap(\Omega_2-x_2)\right)\\ =\max\left\{d(x+u_1,F^{-1}(y+v_1-v_2)),\|y+v_2-\bar{y}\|\right\}.$$

Thus, inequality

$$d\left((x,y),(\Omega_1-x_1)\cap(\Omega_2-x_2)\right)\\ \le\varphi\left(\max\left\{d\left((x,y),\Omega_1-x_1\right),d\left((x,y),\Omega_2-x_2\right)\right\}\right)$$

implies inequality (42), and the converse implication is true if $\varphi(\|y+v_2-\bar{y}\|)<\delta_1$, and $\varphi(t)\ge t$ for all $t\in]0,\varphi^{-1}(\delta_1)[$. Hence, condition (iii) is equivalent to Proposition 2.3(iii).

The remaining conclusions follow from Propositions 2.2 and 2.3. □

Thanks to Proposition 2.6, we can formulate a simplified version of Proposition 4.2 for the convex case.

Corollary 4.3 *Let* X *and* Y *be normed spaces,* $F:X\rightrightarrows Y$ *have a convex graph,* $(\bar{x},\bar{y})\in\operatorname{gph}F$*,* Ω_1 *and* Ω_2 *be defined by* (32)*,* $\delta_1>0$*,* $\delta_2>0$*, and* $\varphi\in\widehat{\mathscr{C}}_{\delta_1}$*. Properties* (i)–(iii) *in Proposition 4.2 hold if and only if the following equivalent conditions hold true:*

(i) $F^{-1}(y+v_1-v_2)\cap B_{\delta_1}(x+u)\neq\emptyset$ *for all* $(x,y)\in\operatorname{gph}F$*,* $u\in X$ *and* $v_1,v_2\in Y$ *with* $\varphi(\max\{\|u\|,\|v_1\|\})<\delta_1$*,* $\varphi(\|v_2\|)<\min\{\delta_1,\varphi(\delta_2)\}$*, and* $(x,y)+(u,v_1)\in B_{\delta_2}(\bar{x},\bar{y})$*;*

(ii) $F^{-1}(\bar{y}+v_1-v_2)\cap B_{\delta_1}(\bar{x}+u)\neq\emptyset$ *for all* $u\in X$ *and* $v_1,v_2\in Y$ *with* $\max\{\|u\|,\|v_1\|\}<\delta_2$*,* $\varphi(\|v_2\|)<\min\{\delta_1,\varphi(\delta_2)\}$*, and* $\varphi(d((\bar{x},\bar{y})+(u,v_1),\operatorname{gph}F))<\delta_1$*;*

(iii) $F^{-1}(y+v_1-v_2)\cap B_{\delta_1}(x+u)\neq\emptyset$ *for all* (x,y)*,* $(u,v_1)\in X\times Y$ *and* $v_2\in Y$ *with* $(x,y)+(u,v_1)\in B_{\delta_2}(\bar{x},\bar{y})$*,* $\varphi(d((x,y)+(u,v_1),\operatorname{gph}F))<\delta_1$*, and* $\varphi(\|y+v_2-\bar{y}\|)<\min\{\delta_1,\varphi(\delta_2)\}$*.*

We finish this section by discussing nonlinear semitransversality and transversality properties of a set-valued mapping to a set in the range space under convexity assumptions. In the following definition and two statements, X and Y are normed spaces, $F : X \rightrightarrows Y$, S is a subset of Y, $(\bar{x},\bar{y}) \in \operatorname{gph} F$ and $\bar{y} \in S$.

Definition 4.2 The mapping F is

(i) $\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ if $\{\operatorname{gph} F, X \times S\}$ is $\varphi-$semitransversal at $(\bar{x},\bar{y})$, i.e. there exists a $\delta > 0$ such that

$$(\operatorname{gph} F - (u_1,v_1)) \cap (X \times (S - v_2)) \cap B_\rho(\bar{x},\bar{y}) \neq \emptyset$$

for all $\rho \in]0,\delta[$, $u_1 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}) < \rho$;

(ii) $\varphi-$transversal to S at $(\bar{x},\bar{y})$ if $\{\operatorname{gph} F, X \times S\}$ is $\varphi-$transversal at $(\bar{x},\bar{y})$, i.e. there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$(\operatorname{gph} F - (x_1,y_1) - (u_1,v_1)) \cap (X \times (S - y_2 - v_2)) \cap (\rho\mathbb{B}) \neq \emptyset$$

for all $\rho \in]0,\delta_1[$, $(x_1,y_1) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2 \in S \cap B_{\delta_2}(\bar{y})$, $u_1 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}) < \rho$.

Remark 4.1 (i) In the linear case, the property in Definition 4.2(ii) was studied by Ioffe in [23, 24].

(ii) For a similar definition of $\varphi-$subtransversality as well as primal and dual characterizations of all three properties, we refer the readers to [11–13].

The next two statements provide characterizations of the properties in Definition 4.2 in the convex case. They are direct consequences of Propositions 2.5 and 2.6, respectively.

Proposition 4.3 *Suppose* $\operatorname{gph} F$ *and* S *are convex,* $\delta > 0$, *and* $\varphi \in \widehat{\mathscr{C}}_\delta$. *The mapping* F *is*

(i) *$\varphi-$semitransversal to S at $(\bar{x},\bar{y})$ with δ if and only if*

$$(\operatorname{gph} F - (u_1,v_1)) \cap (X \times (S - v_2)) \cap B_\delta(\bar{x},\bar{y}) \neq \emptyset \tag{44}$$

for all $u_1 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}) < \delta$;

(ii) *$\varphi-$transversal to S at $(\bar{x},\bar{y})$ with $\delta_1 := \delta$ and some $\delta_2 > 0$ if and only if*

$$(\operatorname{gph} F - (x_1,y_1) - (u_1,v_1)) \cap (X \times (S - y_2 - v_2)) \cap (\delta_1\mathbb{B}) \neq \emptyset \tag{45}$$

for all $(x_1,y_1) \in \operatorname{gph} F \cap B_{\delta_2}(\bar{x},\bar{y})$, $y_2 \in S \cap B_{\delta_2}(\bar{y})$, $u_1 \in X$ and $v_1, v_2 \in Y$ with $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}) < \delta_1$.

Proposition 4.4 *Suppose* $\operatorname{gph} F$ *and* S *are convex,* $\delta_1 > 0$, $\delta_2 > 0$, *and* $\varphi \in \widehat{\mathscr{C}}_{\delta_1}$. *The following properties are equivalent:*

(i) *condition (45) holds for all $(x_1,y_1) \in \operatorname{gph} F$, $y_2 \in S$, $u_1 \in X$ and $v_1, v_2 \in Y$ with $x_1 + u_1 \in B_{\delta_2}(\bar{x})$, $y_1 + v_1, y_2 + v_2 \in B_{\delta_2}(\bar{y})$ and $\varphi(\max\{\|u_1\|,\|v_1\|,\|v_2\|\}) < \delta_1$;*

(ii) *condition (44) holds with δ_1 in place of δ for all $u_1 \in \delta_2\mathbb{B}_X$ and $v_1, v_2 \in \delta_2\mathbb{B}_Y$ with $\varphi(\max\{d((\bar{x},\bar{y}),\operatorname{gph} F - (u_1,v_1)), d(\bar{y}, S - v_2)\}) < \delta_1$;*

(iii) *for all $x, u_1 \in X$ and $y, v_1, v_2 \in Y$ such that $x + u_1 \in B_{\delta_2}(\bar{x})$, $y + v_1, y + v_2 \in B_{\delta_2}(\bar{y})$ and $\varphi(\max\{d((x,y),\operatorname{gph} F - (u_1,v_1)), d(y, S - v_2)\}) < \delta_1$, it holds*

$$(\operatorname{gph} F - (u_1,v_1)) \cap (X \times (S - v_2)) \cap B_{\delta_1}(x,y) \neq \emptyset.$$

Moreover, if F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with δ_1 and δ_2, then properties (i)–(iii) *hold with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$ in place of δ_1 and δ_2.*

Conversely, if properties (i)–(iii) *hold with δ_1 and δ_2, then F is $\varphi-$transversal to S at $(\bar{x},\bar{y})$ with any $\delta_1' \in]0,\delta_1]$ and $\delta_2' > 0$ satisfying $\varphi^{-1}(\delta_1') + \delta_2' \le \delta_2$.*

5 Conclusions and Future Work

Quantitative geometric, metric and slope necessary (in some cases also sufficient) characterizations of general nonlinear transversality properties of collections of sets have been established, particularly in the convex setting. The Hölder case has been given a special attention. Hölder subtransversality plays a crucial role in convergence analysis of numerical algorithms [5, 6, 19]. It would be interesting to check if more general nonlinear estimates could be useful. We expect characterizations of nonlinear transversality properties to be applicable in most situations where the conventional linear estimates are currently used, especially when studying problems which fail to satisfy traditional regularity assumptions. It would also be interesting to explore general nonlinear versions of other regularity properties of collections of sets, e.g., the *affine-hull regularity*, which is an important ingredient in linear convergence analysis of projection methods; cf. [14, 15]. The general approach of the current paper can be applied to the analysis of implicit multifunctions, e.g., like those studied in [8, 9].

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