Equilibrium selection for multi-portfolio optimization✩

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Abstract

This paper studies a Nash game arising in portfolio optimization. We introduce a new general multi-portfolio model and state sufficient conditions for the monotonicity of the underlying Nash game. This property allows us to treat the problem numerically and, for the case of nonunique equilibria, to solve hierarchical problems of equilibrium selection. We also give sufficient conditions for the Nash game formulation to be a potential game. Our computational experience shows that the multi-portfolio model is solvable for relevant problem sizes and substantiates the significance of the equilibrium selection.

Keywords: Game theory, Nash equilibrium problem, hierarchical optimization, variational inequalities, portfolio selection

1. Introduction

This article studies properties of a Nash game, see e.g. [2, 8, 14, 31, 32, 33, 34], arising in portfolio optimization. For the case of multiple equilibria we use its variational inequality formulation to minimize a selection function over the set of equilibria.

Portfolio optimization studies the allocation of funds among risky assets so that some utility measure is maximized. In the basic mean-variance model of portfolio theory [25, 26], Markowitz suggested a measure which accounts for the trade-off between maximizing the return and minimizing the risk. Based on this, we consider the situation of several agents who simultaneously optimize portfolios of the same risky assets, e.g., different portfolio managers in some financial institution. In our model their respective optimization problems are linked by the market impact of their transactions.

In fact, the evaluation of an investment is agent dependent, which means that, unless expectations are assumed homogeneous (e.g. CAPM), each agent will estimate future prices, returns and costs differently. As for the latter, we consider transaction costs. They can be

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divided into explicit and implicit costs [18, 38] which are, e.g., the commission applied by
the agent, which is known, and the market (liquidity) impact, respectively. Indeed, such an
impact of a single agent’s strategy on the liquidity of the assets in the market is hard to
estimate and therefore often ignored in practice [24, 39].

On the other hand, if market impact via transaction costs is considered, it links the
portfolio optimization problems of the single agents and hence gives rise to a Nash game
[29], called multi-portfolio optimization problem. Such Nash games are studied in [39] under
additional structural assumptions which lead to a potential game, i.e., the special case in
which the equilibria can be computed as solutions of a single optimization problem. The
focus of [39] is on establishing sufficient conditions for the existence and uniqueness of these
solutions.

As opposed to the latter approach, the present paper deals with multi-portfolio opti-
mization problems which are neither necessarily potential games, nor necessarily uniquely
solvable. In fact, for the case of multiple Nash equilibria we propose an equilibrium selection
model where some (upper-level) objective function is minimized over the (lower-level) set of
equilibria.

Our main contributions are the formulation of a new general multi-portfolio model, the
statement of sufficient conditions for monotonicity of the Nash game which is crucial for
its explicit numerical treatment, and new sufficient conditions for this general Nash game
formulation to be a potential game. Moreover, in the case of nonunique equilibria we consider
the hierarchical problem of equilibrium selection. A recent Tikhonov-like algorithm from [19]
allows us to present numerical results for such selection problems.

The remainder of this article is structured as follows. In Section 2 we introduce our multi-
portfolio model with market impact and derive the variational inequality formulation of the
responding Nash game. Section 3 gives sufficient conditions for monotonicity of the Nash
game as well as for the special situation of a potential game. For assets and indexes belonging
to Dow Jones Industrial Average and Euro Stoxx 50 stock markets, the equilibrium selection
problem for multi-portfolio optimization is solved numerically in Section 4 by the mentioned
method from [19]. The latter assumes the previously discussed monotonicity of the Nash
game.

2. The game theoretical model

We consider a variation of the classical Markowitz framework, where agents wish to deter-
mine their optimal portfolios based on a trade-off between maximizing expected return and
minimizing risk. To be more specific, we consider \( N \) agents indexed by \( \nu = 1, \ldots, N \). The
generic player \( \nu \) wishes to invest her budget \( b^{\nu} \in \mathbb{R}_+ \) in \( K \) assets of a market. Her decision
variables \( y^{\nu} \in Y_{\nu} \subseteq \mathbb{R}^K \) denote the fractions of \( b^{\nu} \) invested in each available asset, where
\( Y_{\nu} \) is the nonempty convex compact set of feasible portfolios, e.g. the standard simplex. Let
\( r \in \mathbb{R}^K \) denote random variables, where the entry \( r_k \) models the return of asset \( k \in \{1, \ldots, K\} \)
over a single-period investment. Then, we define \( \mu^{\nu} = \mathbb{E}^{\nu}(r) \in \mathbb{R}^K \) as agent \( \nu \)'s expectations
of the assets’ returns, as well as her covariance matrix \( \Sigma^{\nu} = \mathbb{E}^{\nu}((r - \mu^{\nu})(r - \mu^{\nu})^\top) \). Notice
that we allow different estimates for different players. We consider the following measures for
portfolio income \( I_{\nu} \) and risk \( R_{\nu} \):

\[
I_{\nu}(y^{\nu}) \triangleq b^{\nu}(\mu^{\nu})^\top y^{\nu}, \quad R_{\nu}(y^{\nu}) \triangleq \frac{1}{2} b^{\nu}(y^{\nu})^\top \Sigma^{\nu} y^{\nu}.
\]

Observe that we use the portfolio variance as risk measure.
2.1. A model for market impact

In the following we address the case in which trades are grouped and simultaneously executed. In fact, individual accounts can suffer from the market impact caused by a shortage of liquidity, which results from the fact that the joint demand of an asset can be tremendously larger than the individual demand. Subsequently we will focus on liquidity as a main factor for transaction costs, but our model also allows to use other sources of costs. We observe that, when each account is managed independently, resulting market impacts due to overall trades are naively ignored, leading to worse solutions, see [39] and our numerical evidences in Section 4.

To take account of the transaction cost effect we shall consider a market impact matrix whose entry at position \((i, j)\) models the impact of the liquidity of asset \(i\) on the liquidity of asset \(j\). Since these interrelations are hard to estimate (along with the fact that there does not even exist an univocal definition of liquidity) it is reasonable to assume that the market impact matrix \(\Omega^\nu\) is different for each agent \(\nu\). For example, this allows the single agents to introduce different estimates for possible unknown entries of the market impact matrix. It also allows different specializations like modeling the matrices \(\Omega^\nu\) as agent dependent additive perturbations of a reference matrix \(\bar{\Omega}\) (Section 3) or as agent dependent scalings or pre-multiplications of the reference matrix (Section 3.2). The latter also includes the option for single agents to neglect the presence of any transactions costs in their optimization problems and, thus, to decouple them from the Nash game. Possible sources of the differences in the agents’ transaction cost matrices \(\Omega^\nu\) will be further discussed in Section 4.

To motivate the need for not necessarily diagonal matrices \(\Omega^\nu\) in our model, note that liquidity and, more generally, trading costs show commonality (liquidity co-movements across assets or markets) [3, 5, 16, 17], which means that each action on a given security displays effects not only on that security, but also on the others. As for the specific form of \(\Omega^\nu\), we assume that the market impact factor matrix is positive semidefinite, as in [27] or [28]. This is a natural assumption, as any eigenvector associated with a negative eigenvalue would correspond to an aggregated trade with unreasonable negative transaction costs. Also note that, for the sake of simplicity, as in [1, 4] we disregard differences between buy and sell positions.

As a consequence, for each agent \(\nu\) we consider a different linear market impact unitary cost function that depends on the invested capital from the aggregated trades from all accounts, that is

\[
c^\nu(y^1, \ldots, y^N) \triangleq \Omega^\nu \sum_{\lambda=1}^N b^\lambda (y^\lambda - \bar{y}^\lambda)
\]

with a not necessarily symmetric but positive semidefinite matrix \(\Omega^\nu \in \mathbb{R}^{K \times K}\), where \(\bar{y}^\lambda \in \mathbb{R}^K\) denotes the current portfolio of any agent. We observe that the market impact unitary cost \(c^\nu\) is associated with rebalancing from the current positions \(\bar{y}^\lambda\) to the new ones \(y^\lambda\). Notice that we need to incorporate the players’ budgets \(b^\lambda\) in \(c^\nu\) because they are not necessarily identical for all the players. Summarizing, \(c^\nu\) maps the demand of all players to a unitary cost (per unit of currency invested). We thus obtain the total transaction costs (in unit of currency) of player \(\nu\) as

\[
TC^\nu(y^1, \ldots, y^N) \triangleq b^\nu (y^\nu - \bar{y}^\nu)^\top c^\nu(y^1, \ldots, y^N) = b^\nu (y^\nu - \bar{y}^\nu)^\top \Omega^\nu \sum_{\lambda=1}^N b^\lambda (y^\lambda - \bar{y}^\lambda)
\]

Invested capital  Unitary transaction costs
Hence, we arrive at the following optimization problem for each agent $\nu = 1, \ldots, N$:

$$\text{minimize}_{y^\nu} \theta_\nu(y^\nu, y^{-\nu}) \quad \text{s.t.} \quad y^\nu \in Y_\nu,$$

where

$$\theta_\nu(y^\nu, y^{-\nu}) \triangleq -I_\nu(y^\nu) + \rho_\nu R_\nu(y^\nu) + TC_\nu(y^\nu, y^{-\nu})$$

$$= -b'(\mu^\nu) y^\nu + \frac{1}{2} \rho_\nu (y^\nu)^\top \Sigma^\nu y^\nu + b'(\nu y^\nu - \bar{y}^\nu)^\top \Omega^\nu \sum_{\lambda=1}^{N} b^\lambda (y^\lambda - \bar{y}^\lambda),$$

and where the (risk aversion) parameter $\rho_\nu \in \mathbb{R}_+$ appropriately balances the two measures of return, i.e. portfolio income minus transaction costs, and risk.

**Remark 2.1** Note that, in this context, the classical Markowitz framework reads

$$\text{minimize}_{y^\nu} \{-(\mu^\nu)^\top y^\nu + \frac{1}{2} \rho_\nu (y^\nu)^\top \Sigma^\nu y^\nu \mid y^\nu \in Y_\nu\}$$

for any agent $\nu$. As major deviations from this setting we consider transaction costs as well as different budgets $b^\nu$ for the agents. This, in turn, makes it reasonable, as the forthcoming developments clarify, to consider absolute quantities instead of relative ones as commonly done in the classical Markowitz model, that is, to scale the objective function of player $\nu$ by $b^\nu$. \[ \square \]

The collection of the parametric agents’ problems $(P^\nu)$ forms a Nash Equilibrium Problem (NEP). Denoting by $y \triangleq (y^1, \ldots, y^N)$ the vector of all agents' variables and $Y \triangleq \prod_{\nu=1}^{N} Y_\nu$, we say that $\hat{y} \in Y$ is an equilibrium if, for every $\nu$,

$$\theta_\nu(\hat{y}^\nu, \hat{y}^{-\nu}) \leq \theta_\nu(y^\nu, \hat{y}^{-\nu}) \quad \forall y^\nu \in Y_\nu.$$  

We indicate with $E$ the (non-parametric) set of equilibria of the NEP:

$$E \triangleq \{ w \in Y : \theta_\nu(w^\nu, w^{-\nu}) \leq \theta_\nu(y^\nu, w^{-\nu}) \forall y^\nu \in Y_\nu, \nu = 1, \ldots, N \}.$$  

2.2. The variational inequality formulation

Each problem $(P^\nu)$ is convex since the matrices $\Sigma^\nu$ and $\Omega^\nu$ are positive semidefinite, $b^\nu$, $\rho^\nu \in \mathbb{R}_+$, and $Y_\nu$ is a convex set. As a consequence, the NEP is equivalent to the following Variational Inequality $VI(Y, F)$ [9, 10, 11, 12, 23, 35]: find $\hat{y} \in Y$ such that

$$F(\hat{y})^\top (y - \hat{y}) \geq 0 \quad \forall y \in Y,$$

where

$$F(y) \triangleq \begin{pmatrix}
\nabla_y^1 \theta_1(y) \\
\vdots \\
\nabla_y^N \theta_N(y)
\end{pmatrix}.$$  

We remark that $VI(Y, F)$ always has a solution in our framework and, thus, the NEP has an equilibrium, i.e. $E \neq \emptyset$. Moreover, the set of equilibria $E$ is convex and compact, see [12] for these basic properties of the VI reformulation of Nash games.
We further recall that crucial properties of the VI and, thus, equivalently of the NEP, are determined by the nature of the Jaco-Hessian

\[ Q = JF(y) = \begin{pmatrix} \nabla_{y_1}^T \nabla_{y_1} \theta_1(y) & \cdots & \nabla_{y_1}^T \nabla_{y_1} \theta_1(y) \\ \vdots & \ddots & \vdots \\ \nabla_{y_N}^T \nabla_{y_N} \theta_N(y) & \cdots & \nabla_{y_N}^T \nabla_{y_N} \theta_N(y) \end{pmatrix}. \]

In particular, we shall focus our attention on two fundamental properties of \( Q \), its symmetry and its positive semidefiniteness. In fact, when \( Q \) is symmetric the NEP boils down to a potential game \([12, 32]\), which in turn can be reformulated as a suitable optimization problem where the potential function

\[ p(y) \triangleq \frac{1}{2} y^T Q y + F(0)^T y \]

is minimized over \( Y \). Secondly, if \( Q \) is positive semidefinite (but possibly nonsymmetric), then the NEP is practically solvable by resorting to many effective algorithms available in the literature, see \([12]\). If \( Q \) is even positive definite, the NEP has a unique equilibrium. We recall that a positive semidefinite (definite) \( Q \) corresponds to the (strong) monotonicity of \( F \). Clearly, if \( Q \) is both symmetric and positive semidefinite, then in the resulting potential game the function \( p \) is convex. If \( Q \) is symmetric and positive definite, then \( p \) is strictly convex, and the potential game as well as the NEP possess the same unique solution.

As for our NEP, it is convenient to report the actual expression of the components of \( F \) and \( Q \):

\[ \nabla_{\nu} \theta_{\nu}(y^\nu, y^{-\nu}) = - \left( b^\nu \mu^\nu + 2(b^\nu)^2 \bar{\Omega}^\nu \bar{y}^\nu + b^\nu \Omega^\nu \sum_{\lambda \neq \nu} b^\lambda \bar{y}^\lambda \right) + (b^\nu \rho^\nu \Sigma^\nu + 2(b^\nu)^2 \Omega^\nu) y^\nu + b^\nu \Omega^\nu \sum_{\lambda \neq \nu} b^\lambda y^\lambda \]

\[ Q = \begin{pmatrix} b^1 \rho^1 \Sigma^1 + 2(b^1)^2 \Omega^1 & b^1 b^2 \Omega^1 & \cdots & b^1 b^N \Omega^1 \\ b^2 b^1 \Omega^2 & b^2 \rho^2 \Sigma^2 + 2(b^2)^2 \Omega^2 & \cdots & b^2 b^N \Omega^2 \\ \vdots & \vdots & \ddots & \vdots \\ b^N b^1 \Omega^N & b^N b^2 \Omega^N & \cdots & b^N \rho^N \Sigma^N + 2(b^N)^2 \Omega^N \end{pmatrix}. \]

In this paper we address the general case in which the NEP is not supposed to have a unique equilibrium (see Section 4 for examples). In that event, one has to discriminate among the equilibria according to some criterion, see e.g. \([37]\). More specifically, we consider the equilibrium selection problem where some convex (upper-level) merit function \( f : \mathbb{R}^{NK} \rightarrow \mathbb{R} \) is minimized over the set of equilibria \( E \):

\[ \text{minimize } f(y) \quad \text{s.t. } y \in E. \quad (3) \]

We stress that (3) is a convex problem. The upper-level objective \( f \) can be viewed either as a common additional criterion shared by all the agents, or as the cost function of an (upper-level) further player who can impose the agents’ strategies. This framework fits particularly well in our problem where some brokers interact in the same firm, and the upper-level player is possibly the manager of the firm. Some examples for the merit function \( f \) are given in Section 4.

The main difficulty in tackling (3) comes from the fact that its feasible set \( E \) is implicitly defined as the set of equilibria of a (lower-level) NEP. Rewriting the constraint of (3) by its VI formulation leads to the problem

\[ \text{minimize}_y f(y) \quad \text{s.t. } y \in Y, \quad F(y)^T (w - y) \geq 0 \quad \forall w \in Y \quad (4) \]
and shows that (3) belongs to the class of Optimization Problems with VI Constraints (OPVIC) (see, e.g. [13]). It goes without saying that, if the NEP is a potential game (see Section 3.2), then the set of equilibria $E$ boils down to the set of minimizers for the potential function $p$ over the feasible set $Y$, and problem (3) reduces to the so-called simple bilevel problem (or pure hierarchical optimization problem; see e.g. [7, 30, 36])

$$\text{minimize}_y f(y) \quad \text{s.t.} \quad y \in \arg\min_{w \in Y} p(w).$$

Hence, the OPVIC (4) generalizes the framework of simple bilevel problems to the presence of a lower-level variational inequality. However, we emphasize that, as a major departure from the more general bilevel structures (see e.g. [6, 20, 21, 22] and the references therein), here we focus only on lower-level problems that are non-parametric with respect to the upper-level variables.

### 3. A model with reference transaction costs

In practical settings and without loss of generality, any agent may consider the matrix $\Omega^\nu$ defining her transaction costs, as the additive perturbation of some nominal matrix $\overline{\Omega}$ by a matrix $\Delta^\nu$, i.e.,

$$\Omega^\nu = \overline{\Omega} + \Delta^\nu, \; \nu = 1, \ldots, N.$$  

Accordingly, we also introduce the reference transaction costs for player $\nu$,

$$\overline{TC}_\nu(y^1, \ldots, y^N) \triangleq b^\nu(y^\nu - \bar{y}^\nu)^\top \overline{\Omega} \sum_{\lambda=1}^N b^\lambda(y^\lambda - \bar{y}^\lambda).$$  

The nominal matrix $\overline{\Omega}$ could be interpreted as a reference estimate that is either commonly shared by the agents or, in our hierarchical framework, provided by the upper-level player. Further details on the interpretation of $\overline{\Omega}$ as well as sources of the agent dependent perturbations $\Delta^\nu$, $\nu = 1, \ldots, N$, will be discussed in Section 4.

Observe that neither $\overline{\Omega}$ nor $\Delta^\nu$ have to be symmetric matrices, but that it is natural to assume $\overline{\Omega}$ to be positive semidefinite. Firstly, this guarantees convex reference transaction costs $\overline{TC}_\nu$ for player $\nu$, and secondly it avoids a common bias in the perturbations $\Delta^\nu$ to make $\Omega^\nu$ positive semidefinite. In fact, while the introduction of the nominal matrix $\overline{\Omega}$ does not establish any loss of generality in our model, our standing assumption of positive semidefinite matrices $\Omega^\nu$ now depends on the perturbations $\Delta^\nu$. Positive semidefiniteness of $\Omega^\nu$ obviously follows for any positive semidefinite perturbation matrix $\Delta^\nu$, but it also results from sufficiently small perturbation matrices $\Delta^\nu$ if $\overline{\Omega}$ is positive definite. In the following we shall quantify this fact.

**Remark 3.1** The positive (semi)definiteness of a not necessarily symmetric perturbation $P$ of some not necessarily symmetric positive definite matrix $M$ may be guaranteed by the following standard result, whose brief proof we recall for completeness. We denote the symmetric part of $M$ by $\text{sym}(M) \triangleq \frac{1}{2}(M + M^\top)$. Specifically, let $M$ be a positive definite $(n, n)$-matrix whose symmetric part possesses the smallest eigenvalue $\lambda_{\min}(\text{sym}(M))$, and let $P$ be some square $(n, n)$-matrix. Then $M + P$ is positive semidefinite whenever

$$\|P\|_2 \leq \lambda_{\min}(\text{sym}(M))$$
holds. If the latter inequality is strict, then $M + P$ is positive definite. In fact, for any $x \in \mathbb{R}^n$ we have
\[
|x^\top Px| \leq \|P\|_2 \|x\|_2^2 \leq \lambda_{\min}(\text{sym}(M))\|x\|_2^2 \leq x^\top \text{sym}(M)x = x^\top Mx
\]
which entails
\[
0 \leq x^\top Mx - |x^\top Px| \leq x^\top (M + P)x
\]
and, thus, $M + P$ is positive (semi)definite. \(\square\)

The remark above entails the following result.

**Proposition 3.2** Let the matrix $\Omega$ be positive definite. Then for agent $\nu \in \{1, \ldots, N\}$ the matrix $\Omega^\nu$ is positive semidefinite whenever
\[
\|\Delta^\nu\|_2 \leq \lambda_{\min}(\text{sym}(\Omega))
\]
holds. If the inequality (7) is strict, then $\Omega^\nu$ is positive definite.

With regard to our multi-portfolio problem, in the present section we discuss how different perturbations $\Delta^\nu$ impact on the properties of the NEP. In particular, we investigate conditions that make the Tikhonov approach (from [19]) for the solution of (3) viable. In fact, we introduce sufficient conditions for monotonicity, since in [19] this has been shown to entail convergence of the latter algorithm. To complete the picture, we also give conditions implying the NEP to reduce to a potential game. This side result can help in understanding when specific perturbations make the problem easier to be dealt with.

### 3.1. Monotonicity results

To guarantee convex player transaction costs and, thus, convex players' objective functions, in view of Proposition 3.2 we assume them to satisfy $\max_{\nu=1,\ldots,N} \|\Delta^\nu\|_2 \leq \lambda_{\min}(\text{sym}(\Omega))$.

In general, the Jaco-Hessian of the game may be written as $Q = M + P$ with
\[
M = \begin{pmatrix}
b^1 \rho^1 \Sigma^1 + 2(b^1)^2 \Omega & b^1 b^2 \Omega & \cdots & b^1 b^N \Omega \\
b^2 b^1 \Omega & b^2 \rho^2 \Sigma^2 + 2(b^2)^2 \Omega & b^2 b^N \Omega \\
\vdots & \vdots & \ddots & \vdots \\
b^N b^1 \Omega & b^N b^2 \Omega & \cdots & b^N \rho^N \Sigma^N + 2(b^N)^2 \Omega
\end{pmatrix}
\]
and
\[
P = \begin{pmatrix}
2(b^1)^2 \Delta^1 & b^1 b^2 \Delta^1 & \cdots & b^1 b^N \Delta^1 \\
b^2 b^1 \Delta^2 & 2(b^2)^2 \Delta^2 & b^2 b^N \Delta^2 \\
\vdots & \vdots & \ddots & \vdots \\
b^N b^1 \Delta^N & b^N b^2 \Delta^N & \cdots & 2(b^N)^2 \Delta^N
\end{pmatrix}
\]
where \( e \) denotes the all ones vector, and \( \otimes \) is the Kronecker product. The resulting matrix \( M \) turns out to be symmetric if \( \overline{\Pi} \) happens to be symmetric. On the contrary, \( P \) is nonsymmetric even for diagonal perturbation matrices \( \Delta^\nu = \text{diag}(\delta^\nu) \) with vectors \( \delta^\nu \in \mathbb{R}^K \setminus \text{span}(e), \nu = 1, \ldots, N \). Nevertheless, we may derive sufficient conditions for the positive (semi)definiteness of such a nonsymmetric matrix \( Q = M + P \) and, thus, for the monotonicity of \( F \). Observe that, due to the nonsymmetry of \( P \), the positive semidefiniteness of the matrices \( \Delta^\nu, \nu = 1, \ldots, N \), cannot be proven to be such a sufficient condition.

**Theorem 3.3** Let \( \overline{\Pi} \) or \( \Sigma^\nu \) for all \( \nu \), be positive definite, and let the perturbation matrices \( \Delta^\nu \) satisfy

\[
\max_{\nu=1,\ldots,N} \|\Delta^\nu\|_2 \leq \min \left\{ \lambda_{\min}(\text{sym}(\overline{\Pi})), \frac{\lambda_{\min}(\text{sym}(M))}{(N+1)\|b\|_\infty^2} \right\}.
\]

Then, the Jaco-Hessian \( Q \) is positive semidefinite. If the above inequality is strict, then \( Q \) is positive definite.

**Proof.** The symmetric part of \( M \) is

\[
\text{sym}(M) = \begin{pmatrix}
(b^1)^1 \rho^1 \Sigma^1 & \cdots \\
\vdots & \ddots \\
(b^N)^N \rho^N \Sigma^N
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
(b^1)^1 I & \cdots \\
\vdots & \ddots \\
(b^N)^N I
\end{pmatrix}
\]

\[
(I + ee^\top) \otimes \text{sym}(\overline{\Pi})
\]

\[
\begin{pmatrix}
(b^1)^1 I & \cdots \\
\vdots & \ddots \\
(b^N)^N I
\end{pmatrix}.
\]

Since Kronecker products of positive (semi)definite matrices are again positive (semi)definite, \( \text{sym}(M) \) is positive definite. Hence, in view of Remark 3.1, the matrix \( Q \) is positive (semi)definite, whenever \( \|P\|_2 < (\leq)\lambda_{\min}(\text{sym}(M)) \) holds. The assertions now follow from the estimate

\[
\|P\|_2 \leq \left\| \begin{pmatrix}
(b^1)^1 I & \cdots \\
\vdots & \ddots \\
(b^N)^N I
\end{pmatrix} \right\|_2 \left\| \begin{pmatrix}
\Delta^1 & \cdots \\
\vdots & \ddots \\
\Delta^N
\end{pmatrix} \right\|_2 \left\| I + ee^\top \right\|_2 \left\| I \right\|_2 \left\| \begin{pmatrix}
(b^1)^1 I & \cdots \\
\vdots & \ddots \\
(b^N)^N I
\end{pmatrix} \right\|_2.
\]

\[
= (N+1)\|b\|_\infty^2 \max_{\nu=1,\ldots,N} \|\Delta^\nu\|_2.
\]

Observe that the matrix \( \text{sym}(M) \) is positive definite, and the inequality in Theorem 3.3 makes sense also under some weaker assumptions. For example, it is positive definite if the matrices \( \overline{\Pi} \) and \( \Sigma^\nu \) are positive semidefinite, but the kernels of the two matrices adding up to \( \text{sym}(M) \) only share the zero element.

On the other hand, a lower bound on \( \lambda_{\min}(\text{sym}(M)) \) yields the following corollary with a stronger, yet more explicit assumption than in Theorem 3.3.

**Corollary 3.4** Let \( \overline{\Pi} \) or \( \Sigma^\nu \) for all \( \nu \), be positive definite, and

\[
\max_{\nu=1,\ldots,N} \|\Delta^\nu\|_2 \leq \min \left\{ \lambda_{\min}(\text{sym}(\overline{\Pi})), \frac{\min_{\nu=1,\ldots,N}(b^\nu \rho^\nu \lambda_{\min}(\Sigma^\nu)) + \lambda_{\min}(\text{sym}(\overline{\Pi})) \min_{\nu=1,\ldots,N}(b^\nu)^2}{(N+1)\|b\|_\infty^2} \right\}.
\]

Then, the Jaco-Hessian \( Q \) is positive semidefinite. If the above inequality is strict, then \( Q \) is positive definite.
Proof. The symmetric part of $M$ may alternatively be written as
\[
\text{sym}(M) = \begin{pmatrix} b^1 \rho^1 \Sigma^1 + (b^1)^2 \text{sym}(\Omega) \\
\vdots \\
 b^N \rho^N \Sigma^N + (b^N)^2 \text{sym}(\Omega) \end{pmatrix} + bb^\top \otimes \text{sym}(\Omega)
\]
which, in view of $\lambda_{\min}(bb^\top) = 0$, yields the lower bound for its smallest eigenvalue
\[
\lambda_{\min}(\text{sym}(M)) \geq \min_{\nu=1,\ldots,N} \lambda_{\min}(b^\nu \rho^\nu \Sigma^\nu + (b^\nu)^2 \text{sym}(\Omega)) + \lambda_{\min}(bb^\top) \lambda_{\min}(\text{sym}(\Omega))
\]
\[
= \min_{\nu=1,\ldots,N} \lambda_{\min}(b^\nu \rho^\nu \Sigma^\nu + (b^\nu)^2 \text{sym}(\Omega))
\]
\[
\geq \min_{\nu=1,\ldots,N} (b^\nu \rho^\nu \lambda_{\min}(\Sigma^\nu)) + \lambda_{\min}(\text{sym}(\Omega)) \min_{\nu=1,\ldots,N} (b^\nu)^2.
\]
The assertions now follow from Theorem 3.3. □

Note that in the case $\lambda_{\min}(\text{sym}(\Omega)) \leq \min_{\nu=1,\ldots,N} (b^\nu \rho^\nu \lambda_{\min}(\Sigma^\nu))$, the following holds:
\[
\lambda_{\min}(\text{sym}(\Omega)) \leq \frac{\min_{\nu=1,\ldots,N} (b^\nu \rho^\nu \lambda_{\min}(\Sigma^\nu))}{(N+1) \max_{\nu=1,\ldots,N} (b^\nu)^2 - \min_{\nu=1,\ldots,N} (b^\nu)^2}
\]
Corollary 3.4 guarantees the monotonicity of $F$ already under our standing assumption $\max_{\nu=1,\ldots,N} \|\Delta^\nu\|_2 \leq \lambda_{\min}(\text{sym}(\Omega))$.

3.2. Potential games

A straightforward sufficient condition for the symmetry of $Q$, which is sufficient to obtain a potential game with the potential function (2), is the assumption of identical budgets $b^\nu = \bar{b}$, $\nu = 1,\ldots,N$, the absence of perturbation matrices in the definition of $\Omega^\nu$, that is, $\Delta^\nu = 0$, $\nu = 1,\ldots,N$, together with the symmetry of $\Omega$ (see Section 4 for a motivation of symmetric $\Omega$). Then we obtain $\Omega^\nu = \Omega$, $\nu = 1,\ldots,N$, and $Q$ may be written as
\[
Q = \bar{b} \begin{pmatrix} \rho^1 \Sigma^1 \\
\vdots \\
 \rho^N \Sigma^N \end{pmatrix} + \bar{b}^2 (I + ee^\top) \otimes \Omega.
\]
The symmetry of $Q$ now follows from the symmetry of the matrices $\Sigma^\nu$, $\nu = 1,\ldots,N$, and $\Omega$ together with the fact that Kronecker products of symmetric matrices are again symmetric. These assumptions basically coincide with the assumptions from [39], where a potential game approach for multi-portfolio problems with transaction costs is studied in detail. Moreover, note that in this case the matrix $Q$ is also positive semidefinite, so that the potential function (2) turns out to be convex: in fact, $Q$ is given by the sum of a positive semidefinite matrix and the Kronecker product of two symmetric positive semidefinite matrices.

However, in the following we show that under considerably weaker assumptions a potential game can still be guaranteed, including arbitrary budgets $b^\nu$ and arbitrary scaling perturbations $\Delta^\nu$, $\nu = 1,\ldots,N$, which lead to an even nonsymmetric matrix $Q$.

Proposition 3.5 Consider arbitrary budgets $b^\nu > 0$, $\nu = 1,\ldots,N$, and scaling perturbations $\Delta^\nu = a^\nu \Omega$ with $a^\nu > -1$, $\nu = 1,\ldots,N$, of a symmetric matrix $\Omega$. Then, the NEP is a potential game with potential function
\[
p(y) \triangleq \frac{1}{2} y^\top \tilde{Q} y + \tilde{d}^\top y, \tag{9}
\]
where

\[
\tilde{Q} \triangleq \begin{pmatrix}
\frac{\rho_1 b_1}{1 + a_1} \Sigma_1^1 + (b_1)^2 \bar{\Omega} & \cdots & \frac{\rho_N b_N}{1 + a_N} \Sigma_N^N + (b_N)^2 \bar{\Omega} \\
\vdots & \ddots & \vdots \\
\frac{\rho_N b_N}{1 + a_N} \Sigma_N^N + (b_N)^2 \bar{\Omega} & \cdots & \frac{\rho_1 b_1}{1 + a_1} \Sigma_1^1 + (b_1)^2 \bar{\Omega}
\end{pmatrix} + b b^\top \otimes \bar{\Omega},
\]

\[
\tilde{d} \triangleq -\begin{pmatrix}
\frac{\rho_1}{1 + a_1} (\frac{\mu_1}{1 + a_1} + 2 b_1 \bar{\Omega} y_1 + \bar{\Omega} \sum_{\lambda \neq 1} b_1^\lambda y_1^\lambda) \\
\vdots \\
\frac{\rho_N}{1 + a_N} (\frac{\mu_N}{1 + a_N} + 2 b_N \bar{\Omega} y_N + \bar{\Omega} \sum_{\lambda \neq N} b_N^\lambda y_N^\lambda)
\end{pmatrix}.
\]

Moreover, each \(\Omega^\nu\) and \(\tilde{Q}\) are positive semidefinite and \(p\) is convex.

**Proof.** The assumptions imply \(\Omega^\nu = (1 + a^\nu)\bar{\Omega}\) with \(1 + a^\nu > 0\) for each \(\nu = 1, \ldots, N\) and, clearly, the matrices \(\Omega^\nu\) are positive semidefinite. The proof relies on the fact that player \(\nu\)'s solution set is invariant if rescaling her objective function to

\[
\tilde{\theta}^\nu(y^\nu, y^{-\nu}) \triangleq \frac{1}{1 + a^\nu} \theta^\nu(y^\nu, y^{-\nu}).
\]

The Jaco-Hessian of the equivalent rescaled game then is

\[
\begin{pmatrix}
\frac{\rho_1 b_1}{1 + a_1} \Sigma_1^1 \\
\vdots \\
\frac{\rho_N b_N}{1 + a_N} \Sigma_N^N
\end{pmatrix} + \begin{pmatrix}
2 (b_1)^2 \bar{\Omega} & b_1^2 \bar{\Omega} & \cdots & b_1 b_N \bar{\Omega} \\
\vdots & \ddots & \vdots \\
b_1 b_N \bar{\Omega} & \cdots & 2 (b_N)^2 \bar{\Omega}
\end{pmatrix} = \tilde{Q}.
\]

The symmetry of \(\tilde{Q}\) now follows as above, exploiting in addition the obvious symmetry of the dyadic product \(b b^\top\). Hence, observing that

\[
\begin{pmatrix}
\frac{1}{1 + a_1} \\
\vdots \\
\frac{1}{1 + a_N}
\end{pmatrix} F(0) = \tilde{d},
\]

the game with rescaled objective functions as well as the original game are potential games with potential function \(p\). Finally, \(\tilde{Q}\) is positive semidefinite as sum of a positive semidefinite matrix and the Kronecker product of two symmetric positive semidefinite matrices. \(\square\)

We point out that the assertion of Proposition 3.5 remains true if some agents \(\nu\) only use a perturbation with \(a^\nu = -1\). In this case they do not consider any transaction costs and, thus, their optimization problems are no longer coupled to the other agents’ problems.

We remark that if the matrix \(\tilde{Q}\) is positive definite, then the potential function is strictly convex and the game has a unique Nash equilibrium. This is the case if all matrices \(\Sigma^\nu\) or \(\bar{\Omega}\) are positive definite.
4. Computational experience

In this section we explain a way to estimate the market impact matrices $\Omega^\nu$ from real-world data, we introduce different selection functions, and we use the Tikhonov approach from [19] for numerical equilibrium selection among nonunique solutions. The purpose of our study is to demonstrate that the general multi-portfolio model is solvable for relevant problem sizes and that its solutions can have significant advantages compared to the case when transaction costs are naively ignored. Moreover, we illustrate that there are situations where a selection among different equilibria is practically relevant for the outcome of the optimization process.

We consider two different data sets, the first one consisting of daily returns time series of $K = 10$ assets from banking, insurance and financial companies belonging to Euro Stoxx 50 (SX5E) (from 2/1/2019 to 31/12/2019). The second one consists in $K = 29$ assets from the Dow Jones Industrial Average (DJIA) stock markets (from 2/1/2017 to 31/12/2017). We consider $N = 20$ agents for both test cases and we set $\mu^\nu = \overline{\mu}$ and $\Sigma^\nu = \overline{\Sigma}$ for every $\nu = 1, \ldots, N$, where $\overline{\mu}$ and $\overline{\Sigma}$ denote mean and variance of the return of the $K$ assets, respectively.

As mentioned above, the market impact is not known a-priori, but has to be estimated by each agent individually. There are several approaches for measuring transaction costs, see [15] for a recent study. Our aim is to estimate transaction costs and to obtain a realistic model which closely resembles the investor’s decision problem while, at the same time, keeping the estimate simple and easy to reproduce.

Since transaction costs are largely driven by liquidity, we focus on estimating the latter. One straightforward way to measure liquidity is via traded volumes of assets. Data for traded volumes of assets from the above stock markets is publicly available, which will indeed make our measure easy to compute and to reproduce. Thus, in order to compute the entries $\Omega^\nu_{ij}$ of the transaction cost matrices, let us initially focus on obtaining estimates for the interrelation of traded volumes, $\alpha_{ij}$. Subsequently, we shall translate this measure to player-dependent transaction costs.

Let $\text{vol}_i^t$ denote the traded volume of asset $i$ at time $t$ and $\text{vol}_i$ the corresponding vector with entries $\text{vol}_i^t, t = 1, \ldots, T$. Moreover, with $\overline{\text{vol}}_i$ we denote the arithmetic mean of $\text{vol}_i$. Then we may measure the interrelation of the traded volume of asset $i$ and the traded volume of asset $j$ by their empirical correlation

$$\alpha_{ij} = \frac{\sum_{t=1}^{T} (\text{vol}_i^t - \overline{\text{vol}}_i)(\text{vol}_j^t - \overline{\text{vol}}_j)}{\sqrt{\sum_{t=1}^{T}(\text{vol}_i^t - \overline{\text{vol}}_i)^2} \sqrt{\sum_{t=1}^{T}(\text{vol}_j^t - \overline{\text{vol}}_j)^2}}$$

and obtain the empirical correlation matrix $A \triangleq (\alpha_{ij})_{i,j=1}^K$.

To account for different impacts of traded volumes correlations on transaction costs per asset, we pre-multiply $A$ by a positive definite diagonal matrix $\text{diag}(\overline{s})$. As for our numerical tests, the resulting matrix $\text{diag}(\overline{s})A$ turns out to be positive semidefinite and we take it as the nominal matrix $\Omega$:

$$\Omega = \text{diag}(\overline{s})A.$$

While $A$ and $\overline{s}$ are computable and known for all agents, their opinions about the influence of $A$ on the transactions costs may differ. We model this by setting

$$\Omega^\nu = \text{diag}(\overline{s} + s^\nu)A = \Omega + \Delta^\nu,$$
where $\Delta^{\nu} = \text{diag}(s^{\nu}) A$.

Finally, the feasible set for every agent’s problem is given by $Y_{\nu} = \{y^{\nu} \in \mathbb{R}^K : e^\top y^{\nu} = 1, y^{\nu} \geq 0\}$, thus, without loss of generality, short sales are not allowed.

Regarding the upper-level selection criterion $f$, we consider the following social welfare functions:

$$ f_{\text{overall}}(y) \triangleq \sum_{\nu=1}^{N} \tilde{\theta}_{\nu}(y) $$

where $\tilde{\theta}_{\nu}(y) \triangleq -b^{\nu}(\mu)^\top y^{\nu} + \rho^{\nu} \frac{1}{2} b^{\nu}(y^{\nu})^\top \Sigma y^{\nu} + TRC_{\nu}(y^{1}, \ldots, y^{N})$,

and

$$ f_{\text{income}}(y) \triangleq \sum_{\nu=1}^{N} -b^{\nu}(\mu)^\top y^{\nu}, $$

that is the sum of all players’ overall objectives and portfolio expected returns with reference estimates $(\bar{\mu}, \bar{\Sigma}, \bar{\Omega})$.

We consider problem instances arising from the following choices: $\bar{s} = 1e-4$, $s^{\nu}$ is generated randomly so that matrix $Q$ is positive semidefinite; budgets $b^{\nu}$’s and risk-aversion parameters $\rho^{\nu}$’s are taken randomly from the interval $[0, 1.5]$. Finally, we consider $\bar{y}^{\nu} = 0$ for every $\nu$.

In the same spirit of the analysis performed in [39], we show that the Nash model we consider actually improves on the naive approach that consists in simply ignoring the aggregate effect from other agents, and, hence, reduces to the following optimization problem for each agent $\nu$:

$$ \minimize_{y^{\nu}} b^{\nu}(\mu)^\top y^{\nu} + \rho^{\nu} \frac{1}{2} b^{\nu}(y^{\nu})^\top \Sigma y^{\nu} + (b^{\nu})^2 (y^{\nu} - \bar{y}^{\nu})^\top \Omega^{\nu} (y^{\nu} - \bar{y}^{\nu}) \quad \text{s.t.} \quad y^{\nu} \in Y_{\nu}. $$

(10)

In order to solve the problems we deal with, we rely on the projected Tikhonov-prox method. It is precisely thanks to the theoretical properties that we have established in the previous sections that one can resort to such an algorithm. More specifically, the following scheme, when applied to the problems we address, is provably convergent (see [19]) since $f$ is convex, $F$ has been shown to be monotone, and $Y$ is nonempty, convex and compact.

**Algorithm 1:** projected Tikhonov-prox

| Data: \lambda = 1, \gamma = 1, y_{0,j} = 1/K for every j = 1, \ldots, KN, w_{0} = y_{0} \in Y, i \leftarrow 1; |
| for k = 0, 1, \ldots do |
| (S.1) \tau_{k} = i, \varepsilon_{k} = 1/(\tau_{k})^{2}; |
| (S.2) y_{k+1} = P_{Y}(y_{k} - \gamma [F(y_{k}) + \frac{1}{\tau_{k}} \nabla f(y_{k}) + \lambda (y_{k} - w_{k})]); |
| (S.3) if $\|y_{k+1} - y_{k}\| \leq \varepsilon_{k}$ then |
| | $w_{k+1} = y_{k}, i = i + 1;$ |
| (S.4) else |
| | $w_{k+1} = w_{k};$ |
| end |
| end |

As commonly done for numerical scaling purposes, all the functions involved are preliminarily divided by a common Lipschitz constant.

We remark that, given the outer parameters $\tau_{k}, \varepsilon_{k}$ and $w_{k}$ the core step of the algorithm consists in the iterative (inexact) solution of the strongly monotone inner VI$(Y, F + \frac{1}{\tau_{k}} \nabla f +$
\[ \lambda(\bullet - w_k) \] until the prescribed accuracy \( \varepsilon_k \) is reached at step (S.3). As soon as this is the case, the outer parameters are updated: more specifically, \( \tau_k \) is increased, \( \varepsilon_k \) is reduced, and \( w_{k+1} \) is taken equal to \( y_k \). One can show that the sequence of outer iterates \( \{w_k\} \) produced by the algorithm converges to a solution of problem (3), see [19]. The execution of the algorithm is stopped if the condition at step (S.3) is met and the measure

\[
\frac{1}{\gamma} \|y_{k+1} - y_k\| + \frac{1}{\tau_k} \|\nabla f(y_k)\| + \lambda \|y_k - w_k\| \quad (11)
\]
is smaller than 1e-4. We observe that the first addendum in (11) accounts for the solution of the inner problem VI\((Y, F + \frac{1}{\tau_k} \nabla f + \lambda(\bullet - w_k))\) (consistently, it appears in the condition at step (S.3)). The second term is related to the upper-level objective function weighted by the Tikhonov penalty-like parameter: as typical in Tikhonov approaches, being \( \|\nabla f\| \) bounded, this addendum eventually vanishes. Finally, the last term measures the distance between two successive outer iterations and, thus roughly speaking, having it small means that the outer iterates are no more moving. We also recall from [19] that

\[
V(y_k) \overset{\Delta}{=} \|P_Y(y_k - F(y_k)) - y_k\| \leq \lambda \|y_k - w_k\| + \frac{1}{\tau_k} \|\nabla f(y_k)\| + \frac{1}{\gamma} \|y_{k+1} - y_k\|. \quad (12)
\]

Note that \( V \) is the so-called natural residual map for the lower-level VI\((Y, F)\), see [12]. We observe that \( V \) is a continuous nonnegative measure such that \( V(y) = 0 \) if and only if \( y \in \text{SOL}(Y, F) \). Furthermore, we recall that, for the problem we address, the value \( V(y) \) also gives an actual upper-bound for the distance between \( y \) and \( \text{SOL}(Y, F) \), since \( Y \) is polyhedral and \( F \) is affine (see [12, Chapter 6 and, in particular, Proposition 6.3.3]). Overall, requiring the measure in (11) to be smaller than 1e-4 entails \( V(y^k) \leq 1e-4 \) and, in turn, this means, having satisfied the condition at step (S.3), that \( w^{k+} \) is close to \( \text{SOL}(Y, F) \).

We remark that, even for the Dow Jones data set for which 29 \( \times \) 20 variables are involved, we reach termination in a few thousands of iterations. Regarding further details concerning Algorithm 1, we refer the reader to [19].

In Figure 1 we report, for each agent \( \nu \), the utility improvement entailed by our model (with \( f_{\text{overall}} \) as selection criterion) with respect to the naive approach (10), namely:

\[
- \frac{\bar{\theta}_{\nu}^{\text{overall}} - \bar{\theta}_{\nu}^{\text{naive}}}{|\bar{\theta}_{\nu}^{\text{naive}}|},
\]

where \( \bar{\theta}_{\nu}^{\text{overall}} \) denotes the value of agent \( \nu \) objective computed at the optimal solution for problem (3) with \( f = \bar{f}_{\text{overall}} \), while \( \bar{\theta}_{\nu}^{\text{naive}} \) indicates the optimal value of problem (10). Also, in the ‘Total’ bar, we report the following quantity:

\[
- \frac{\sum_{\nu=1}^{N} (\bar{\theta}_{\nu}^{\text{overall}} - \bar{\theta}_{\nu}^{\text{naive}})}{|\sum_{\nu=1}^{N} \bar{\theta}_{\nu}^{\text{naive}}|}.
\]

As it is evident from Figure 1, our approach outperforms the naive one on both data sets, since the market impact costs incurred from transactions of other accounts are properly taken into account.

In order to illustrate that equilibria are not necessarily unique, in Figure 2 we provide the quantities

\[
- \frac{\bar{\theta}_{\nu}^{\text{overall}} - \bar{\theta}_{\nu}^{\text{income}}}{|\bar{\theta}_{\nu}^{\text{income}}|}, \quad - \frac{\sum_{\nu=1}^{N} (\bar{\theta}_{\nu}^{\text{overall}} - \bar{\theta}_{\nu}^{\text{income}})}{|\sum_{\nu=1}^{N} \bar{\theta}_{\nu}^{\text{income}}|}.
\]
where \( \bar{\theta}_\nu^{\text{overall}} \) indicates, for each agent \( \nu \), the value of \( \bar{\theta}_\nu \) computed at the optimal solution for problem (3) with \( f = \bar{f}_{\text{overall}} \), while \( \bar{\theta}_\nu^{\text{income}} \) denotes, for each agent \( \nu \), the value of \( \bar{\theta}_\nu \) computed at the optimal solution for problem (3) with \( f = \bar{f}_{\text{income}} \). The results displayed in these figures show that, changing the upper-level objective \( f \) from \( \bar{f}_{\text{income}} \) to \( \bar{f}_{\text{overall}} \), we get, correspondingly, different solutions for problem (3). Hence we obtain numerical evidence that addressing the hierarchical problem (3) actually yields a selection over the lower-level equilibrium set.

5. Conclusions

Summarizing, the main contributions of this work are as follows. We formulate a new general portfolio selection model involving several decision makers whose actions are coupled due to market impact via transaction costs. The resulting Nash equilibrium problem is not assumed to be a potential game, nor to lead to a unique solution. From this respect, we
consider different merit functions that yield a selection over the set of equilibria arising from the Nash game. Overall, we obtain a hierarchical program with a lower-level Nash equilibrium problem. We thoroughly analyze the main theoretical properties of this model: specifically, we establish practical sufficient conditions that make the problem solvable and numerically tractable. In fact, it is precisely thanks to these properties that one can rely on a recently proposed Tikhonov approach to address the hierarchical program.

A fundamentally different structure is generated if some upper level player controls parameters of the lower-level players and wishes to minimize some objective depending on these parameters. In the setting of multi-portfolio optimization such parameters may be, for example, the budgets of the portfolio managers or correction terms for their transaction costs. As this structure is no longer purely hierarchical, but rather a multi-follower game, the analysis of the present paper does not apply, but we leave the investigation of such models to future research.


