Autonomous traffic at intersections: an optimization-based analysis of possible time, energy, and CO_2 savings

Do Duc Le · Maximilian Merkert · Stephan Sorgatz · Mirko Hahn · Sebastian Sager

Abstract In the growing field of autonomous driving, traffic-light controlled intersections as the nodes of large traffic networks are of special interest. We want to analyze how much an optimized coordination of vehicles and infrastructure can contribute to a more efficient transit through these bottlenecks. In addition, we are interested in the sensitivity of the results with respect to traffic density, turning behavior, or certain regulations of traffic lights. To this end, we develop a mixed-integer linear programming (MILP) model to describe the interaction between traffic-lights and discretized traffic flow. It is based on a microscopic traffic model with centrally controlled autonomous vehicles and extended formulations for different switching regulations.

We aim to determine a globally optimal traffic flow for given scenarios on a simple, but extensible, urban road network. This amounts to finding controls for the movement of each car as well as for each traffic light such that an objective function is optimized and collisions are avoided. The resulting models are very challenging to solve to global optimality, in particular when involving additional realistic traffic light regulations such as minimum red and

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green times. An evaluation of the numerical results with a traffic simulation tool indicates that the performance indicators time, energy, and emissions could be concurrently reduced by a significant amount. Potentially, the same models and algorithms might be the basis for future traffic control systems.

Keywords Traffic Optimization · Mixed-Integer Programming · Autonomous Driving · Energy-Efficient Mobility · Microscopic Traffic Modeling · Extended Formulations · Cooperative Systems

1 Introduction

Assisted and autonomous driving is a growing field of interest. Keywords such as *cooperation, connectivity, assistance,* and *automated driving* have been gaining importance in research and public awareness. Many car manufacturers, research institutes, and related industries have been researching along these lines, cf. [4, 16, 18, 19, 24], and been making efforts to develop and evaluate the impact of systems in such contexts. We want to highlight two of them: *Autonomous driving on networks* and *cooperation.* At first glance, both of them are related to each other in a sense of *smart mobility*, which is another of these – often rather fuzzily defined — keywords. We can, however, make finer distinctions between the two. Autonomous driving, which is the ultimate consequence of assisted driving and can — according to the *German Association of Automotive Industry (VDA)* – be classified into five levels, considers the movement of the individual car in the first place. Simply put, an autonomously driving car is mainly concerned with moving to its destination in an accident-free and regulation-compliant manner. Qualifying quantities such as *traffic flow* are of minor interest and therefore barely or not at all considered in current implementations. [13] suggests that traffic flow could even get worse when autonomously driving cars are introduced.

On the other hand, there are cooperative systems. They are in general independent from assisted or even autonomous driving and aim to generate benefit for the involved cars, or infrastructural devices, by exchanging information. A very simple example for a cooperative system following this definition are turn signals. The growing field of technology and devices for wireless communication promises to facilitate the process of exchanging information between traffic participants. Among other technologies, wireless LAN, which in the automotive context is often referred to as *Vehicle-to-Vehicle (V2V)* and *Vehicle-to-X (V2X)*, is of current interest. It offers sufficiently wide ranges, short delays, and direct communication between agents, cf. [1]. Putting these two major concepts together seems to be beneficial for both of them: Cooperative systems promise to be more efficient and consistent if the intended maneuvers or strategies are performed automatically. Autonomous driving will be more comfortable, safer, and potentially also more efficient in terms of traffic flow. With the technological advances in autonomous mobility, the coordination of vehicular traffic offers new possibilities for increasing traffic efficiency—especially at urban intersections which represent intrinsic bottlenecks for the movement of cars. Our goal is to quantify this anticipated potential to reduce waiting time, energy consumption, and CO_2-emissions and to analyze a variety of scenarios, including different traffic densities and traffic light regulations.

A number of contributions on coordinating autonomous vehicles at intersections has been made in recent years. The works of Dresner and Stone [2] propose a reservation scheme with a central control unit. Vehicles are treated as driver agents and send requests of timeslots to cross the intersection which are then either accepted or rejected by the central control unit based on certain rules. In [7, 8], Hult et al. present a heuristic: first, a scheduling problem is solved to find the crossing order of the vehicles, second, optimal state and control trajectories are determined for this order. In [9] this approach is applied to closed-loop, receding horizon control. Zhang, Malikopoulos, and Cassandras [25, 15] proposed a decentralized optimal control framework which minimizes energy consumption subject to maximizing the throughput using a first-in-first-out heuristic to decide on the order in which the vehicles cross the intersection. The authors in [6] construct a discrete-time discrete-space model which can be divided into cells and provide graph-based algorithms to minimize the total completion time. While most of the recent work omitted traffic lights and focussed on the collision-free traversal of the vehicles, we include these into our considerations. Due to a considerable amount of technology for the exchange of information and other purposes, traffic-light controlled intersections provide an extendable infrastructure and are therefore a reasonable setting for exemplary study.

In this paper, we study traffic at traffic-light-controlled intersections by regarding each car individually, and by centrally optimizing traffic flow to global optimality. To this end, we develop a mixed-integer linear program (MILP) with variables for states of both the cars and the traffic lights on a fixed time horizon, which will be described in Section 2. The model has been introduced in the dissertation of Sorgatz [20], on which this work in part is based. We extended it in several aspects, e.g. by incorporating right and left turns. In contrast to other publications, our model allows for an easy consideration of different traffic light regulations and we obtain globally optimal solutions for a centrally coordinated traffic flow. While the resulting MILPs turn out to be too hard to solve for real-time requirements, the offline-calculated solutions can serve as a benchmark for evaluating other approaches, e.g. heuristic solution methods, for improving traffic flow at intersections. Furthermore, they can be used to evaluate the impact of different influencing factors on the potential for improving traffic efficiency. In Section 3, we give numerical results and compare different scenarios in terms of runtime and the achieved objective value. Additionally, the resulting traffic flow is evaluated with respect to the performance indicators time, energy, and emissions with the aid of microscopic traffic simulations. Lastly, concluding remarks are given in Section 4.

2 Simple movement model and global-MILP

In this section, we develop a MILP which describes the traffic flow of all cars on a simple urban road network on a fixed time interval $\mathcal{T} := [0, T_N]$. In addition to the behavior of all cars at any timestep, we also optimize the signal states of the traffic lights in the network.

The basic structure of our scenario is as follows: Two straight roads intersect in a single intersection. Each road consists of two lanes running in opposite directions. In each lane, a traffic light tl from the set of traffic lights TL regulates traffic flow from that lane into the intersection. For simplicity, we keep to this simple example to introduce the model. Note, however, that the model can be straightforwardly extended to represent networks with multiple intersections, more than two intersecting roads and multiple lanes.

As we want to determine the optimal movement of the cars along the road offline, only the time each car enters the network, called *arrival time* $t_{\mathrm{init},c} \in \mathcal{T}$, and its velocity $v_{\mathrm{init},c} > 0$ at this time are fixed. All cars enter the network on a particular lane at the same position, which is simply labeled 0.

To obtain a microscopic traffic model, we need model components for

- the longitudinal movement of each car,
- the logic of traffic lights,
- traffic light regulations, such as minimum lengths of red and green phases,
- collision prevention on intersections,
- collision prevention on lanes, before and after possible turning maneuvers,
- a meaningful objective function.

In the following, we will discuss them one by one.

2.1 Car motion model

For the purposes of the model, a car c in the set of all cars C is a moving occupant of a stretch of road. At any given time, it moves on a road using a specific lane. A lane may be used by multiple cars. Two cars using the same lane may not simultaneously occupy the same space within that lane. We refer to our efforts to prevent this as *collision prevention.* Note that we only consider longitudinal movement, which means that overtaking maneuvers are forbidden.

The movement of any given car is governed by the following laws of motion. As friction is negligible in an urban setting, we focus on linear constraints which means a trade-off between realism and low runtimes.

$$\begin{aligned}
\dot{s}(t) &= v(t) && \forall t \in \mathcal{T}, \\
\dot{v}(t) &= a(t) && \forall t \in \mathcal{T}, \\
\dot{a}(t) &= j(t) && \forall t \in \mathcal{T}, \\
s(t_{\text{init},c}) &= 0, \\
v(t_{\text{init},c}) &= v_{\text{init},c}, \\
v(t) &\in [\underline{v}, \overline{v}] && \forall t \in \mathcal{T}, \\
a(t) &\in [\underline{a}, \overline{a}] && \forall t \in \mathcal{T}, \\
j(t) &\in [\underline{j}, \overline{j}] && \forall t \in \mathcal{T}.
\end{aligned} \tag{1}$$

In the ODE-system (1), $s(t)$ is the traveled distance of the car, measured by the position of the car's front on the lane at time t. Its velocity is encoded in $v(t)$, $a(t)$ is its acceleration, and $j(t)$ its jerk. The bounds $0 \leq \underline{v} < \overline{v}$, $\underline{a} < 0 < \overline{a}$, and $\underline{j} < 0 < \overline{j}$ are due to physical and comfort restrictions and set for each car individually. The linear ODE-system is discretized using an equidistant discretization of $\mathcal{T}$. Let $T := \{0, \dots, N\}$ and $T_c := \{t_{\text{init},c}, \dots, N\}$. The explicit Euler method on T and T_c with step length $dt := T/N$ is an appropriate choice for the rather simple motion model. This yields the following system of equations:

$$s_{c,t+1} = s_{c,t} + v_{c,t} \cdot dt \qquad \forall c \in C,\ t \in T_c \setminus \{N\}, \tag{2}$$
$$v_{c,t+1} = v_{c,t} + a_{c,t} \cdot dt \qquad \forall c \in C,\ t \in T_c \setminus \{N\}, \tag{3}$$
$$s_{t_{\text{init},c}} = 0 \qquad \forall c \in C, \tag{4}$$
$$v_{t_{\text{init},c}} = v_{\text{init},c} \qquad \forall c \in C, \tag{5}$$
$$\underline{v}_c \leq v_{c,t} \qquad \forall c \in C,\ t \in T_c, \tag{6}$$
$$v_{c,t} \leq \overline{v}_c \qquad \forall c \in C,\ t \in T_c, \tag{7}$$
$$\underline{a}_c \leq a_{c,t} \qquad \forall c \in C,\ t \in T_c, \tag{8}$$
$$a_{c,t} \leq \overline{a}_c \qquad \forall c \in C,\ t \in T_c, \tag{9}$$

as well as the inequalities:

$$\underline{j}_c \leq \frac{1}{dt} \cdot (a_{c,t+1} - a_{c,t}) \leq \overline{j}_c \quad \forall c \in C,\ t \in T_c \setminus \{N\}. \tag{10}$$

For simplicity, we demand $t_{\text{init},c} \in T$.

2.2 Traffic lights

As our main goal is to model traffic at urban intersections, traffic lights play a very important role since they manage collision prevention between cars driving on different lanes. In this section, we introduce traffic light triggers to model the regulatory effect of traffic lights. A basic insight is that an interaction between traffic lights and cars always yields constraints that are only relevant for cars that enter specific sections of the road. For instance, an intersection between two lanes is essentially a small section on either lane that cannot be driven on freely. We will refer to such a section as an *intersection area.*

2.2.1 Right of way

For each traffic light tl of the of the intersection and for each time step $t \in T$, we introduce a state variable $\chi_{tl,t}$. We can interpret a traffic light tl having a green light at time step t if the corresponding variable $\chi_{tl,t}$ equals 1. If $\chi_{tl,t} = 0$ holds, the traffic light is set to red. We now consider a single intersection with its set of traffic lights TL, where usually $|TL| = 4$. For this set, we define a decomposition into subsets

$$TL = \bigcup_{k=1}^{K} TL_k,$$

where $K \in \mathbb{N}$ and the sets TL_k contain *conflicting* traffic lights. Conflicting means that only a single traffic light $tl \in TL_k$ is allowed to be set to green in each time step. This is enforced by the constraint:

$$\sum_{tl \in TL_k} \chi_{tl,t} \leq 1 \qquad \forall t \in T,\ k \in \{1, \dots, K\}. \tag{11}$$

In particular, if the network consists of a single intersection and all traffic lights are conflicting, it holds that $K = 1$. Theoretically, it is also possible to declare traffic lights belonging to different intersections as conflicting.

2.2.2 Limiting the green and red phase

For some of our later considerations, we want to impose limits on the duration of the *green* and *red phase*. These are the time frames during which cars may or may not enter the intersection. To avoid traffic lights switching rapidly between red and green, we demand that the traffic light has to stay green for some period of time after switching from red to green and similarly has to stay red for a given minimum amount of time before it can switch to green again. The same problem appears in the unit commitment context where electrical generators for energy production are coordinated [21]. The generators need to stay active (inactive) for some duration when turned on (off). This problem is often referred to as *minimum up/down time* in the literature. We use an extended formulation which originated from Rajan and Takriti [17], based on a complete linear description of the so-called min-up/min-down polytope by Lee, Leung, and Margot [14].

The description of the problem is given by the following *turn on/off inequalities*:

$$\chi_{tl,t} = \chi_{tl,t-1} + \overline{y}_{tl,t} - \underline{y}_{tl,t} \qquad \forall tl \in TL,\ t \in \{1,\dots,N\}, \tag{12}$$

$$\sum_{i=t-L+1}^{t} \overline{y}_{tl,i} \leq \chi_{tl,t} \qquad \forall tl \in TL,\ t \in \{L,\dots,N\}, \tag{13}$$

$$\sum_{i=t-l+1}^{t} \underline{y}_{tl,i} \leq 1 - \chi_{tl,t} \qquad \forall tl \in TL,\ t \in \{l,\dots,N\}. \tag{14}$$

As stated before, the binary variable $\chi_{tl,t}$ describes the state of the traffic light tl at time step t and is 1 for green and 0 for red. Newly added binary variables $\overline{y}_{tl,t}$ and $\underline{y}_{tl,t}$ model the switching from red to green and green to red, respectively. The second inequality implies that there will be at most one switch from red to green in the last L time steps if $\chi_{tl,t} = 1$ and zero switches from red to green if $\chi_{tl,t} = 0$. This guarantees a green phase of length at least L. Similarly, the third inequality gives us a red phase of length at least l. Note that (12) – (14) represent a significant improvement over the formulation in [20] with respect to computational efficiency.

2.2.3 Traffic light triggers

In order to enable and disable constraints based on the location of the car, we use big-M formulations. Therefore, let $\underline{s}$ and $\overline{s}$ be lower and upper bounds for the distance variables $s_{c,t}$ of car c. Furthermore, we consider a traffic light trigger with associated intersection area $\left[s^{\text{start}}, s^{\text{end}}\right]$ with $\underline{s} \leq s^{\text{start}} < s^{\text{end}} \leq \overline{s}$. As we only regard a single car for our considerations in this subsection, we omit the indices for its individual variables. We do the same for indices of a single traffic light and the time interval.

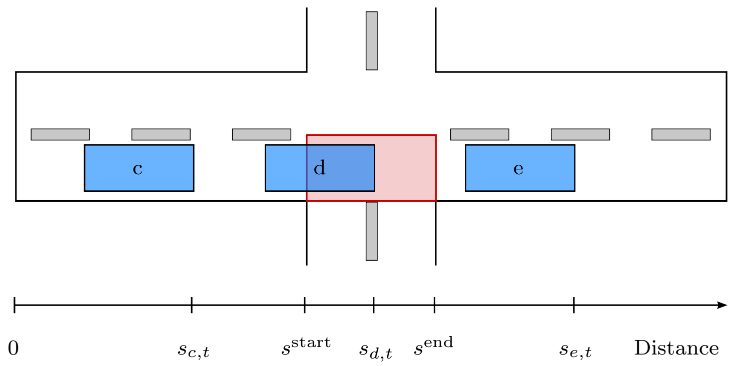

Fig. 1 Three cars c, d, e driving on a lane in the network with red intersection area. We have $(\chi_t^{\text{in}}, \chi_t^{\text{out}}) = (0,1)$ for car c, $(\chi_t^{\text{in}}, \chi_t^{\text{out}}) = (1,1)$ for car d and $(\chi_t^{\text{in}}, \chi_t^{\text{out}}) = (1,0)$ for car e.

We use an "enter-leave formulation", where we introduce two binary indicator variables χ_t^{in} and χ_t^{out} for each combination of car and traffic light. We want to ensure the relations

$$\left(\chi_t^{\text{in}} = 0\right) \Rightarrow \left(s_t \leq s^{\text{start}}\right) \quad \text{(car has not entered intersection area)}, \tag{15}$$

$$\left(\chi_t^{\text{out}} = 0\right) \Rightarrow \left(s_t \geq s^{\text{end}}\right) \quad \text{(car has already left intersection area)}. \tag{16}$$

They can be implemented by the big-M constraints

$$\left(s^{\text{start}} - \overline{s}\right) \cdot \chi_t^{\text{in}} + s_t \leq s^{\text{start}} \qquad \forall t \in T \tag{17}$$

$$\left(s^{\text{end}} - \underline{s}\right) \cdot \chi_t^{\text{out}} + s_t \geq s^{\text{end}} \qquad \forall t \in T, \tag{18}$$

where $(s^{\text{start}} - \overline{s})$ and $(s^{\text{end}} - \underline{s})$ represent feasible values for big-M, respectively, such that no restriction is imposed if the enter-leave variables are set to 1.

It follows that

$$s_t \in \left(s^{\text{start}}, s^{\text{end}}\right) \Rightarrow \left(\chi_t^{\text{in}} = 1 \wedge \chi_t^{\text{out}} = 1\right),$$

i.e. both χ_t^{in} and χ_t^{out} are equal to 1 if the car is inside the intersection area (but not necessarily vice versa). This case is only allowed if the traffic light for the corresponding lane and intersection shows green, which can be guaranteed by the constraint

$$\chi_t^{\text{in}} + \chi_t^{\text{out}} - \chi_t \leq 1 \quad \forall t \in T. \tag{19}$$

When $s_t \notin \left(s^{\text{start}}, s^{\text{end}}\right)$, it is ensured that either χ_t^{in} or χ_t^{out} can always be assigned the value 0. Since additional constraints can only deteriorate the optimal objective function value, this leads to the trigger variables acting as a proper indicator variables for the position of the car in most situations, i.e. the reverse implications in (15) and (16) are likely to also hold.

Finally, we can strengthen the linear relaxation by adding constraints

$$\chi_t^{\text{in}} + \chi_t^{\text{out}} \geq 1 \quad \forall t \in T. \tag{20}$$

Traffic light trigger variables χ_t can be shared among all cars driving towards the same traffic light. Only the binary variables χ_t^{in} and χ_t^{out} have to exist for every pair of car and traffic light. Figure 1 shows the three main configurations of trigger variables for three different car positions. Also the intersection area for the lane the cars are driving on is visualized.

2.3 Collision prevention on lanes

An essential part of modeling traffic at intersections is preventing collisions between cars. There are two types of possible collision situations we have to address in our modeling: Firstly, we have to make sure that vehicles driving on the same lane do not collide, and secondly, we have to prevent collisions inside the intersection area, where vehicles from different lanes have to interact with each other. The second type is handled by traffic lights as has been described in Section 2.2. In order to prevent collisions of cars on the same lane we have to make sure that all cars keep a safety distance to their predecessors. Due to possible turning we need to distinguish between two cases: before and after crossing the intersection.

We regard the set of relevant predecessors of a car c driving on the same lane and introduce it as C_c^{pred}. The idea is to introduce constraints of the form

$$s_{c',t} - s_{c,t} \geq l_{c'} + g_c \quad \forall c \in C,\ c' \in C_c^{\text{pred}},\ t \in T_c, \tag{21}$$

where $l_{c'}$ denotes the length of car c' and g_c is the safety gap maintained by c to cars driving in front of c in the same lane. As we are considering networks with single lanes without overtaking maneuvers, each car has only one other car to watch out for, namely its direct predecessor. However, constraint (21) is only valid if both c and c' have the same destination. Otherwise the successor-predecessor relation will change after turning and we need variations of (21) that are disabled after one car has crossed the intersection. Also, we do not know the new predecessor of c after the intersection and hence will add binary variables that encode the crossing order of cars that drive on the same lane after crossing the intersection.

2.3.1 Collision prevention before crossing the intersection

If the the car has not yet crossed the intersection, we can identify a unique direct predecessor $pred(c)$ of each car $c \in C$. Note, that the set of all cars C can be partitioned into the set C^{str} of cars going straight and the set C^{turn} of turning cars. We introduce constraints of one of the following forms depending on the turning behaviour of the vehicles c and $pred(c)$:

- If both c and $pred(c)$ have the same destination, we can use (21) with $c' = pred(c)$.
- If c and $pred(c)$ have different destinations, we use big-M constraints

$$s_{pred(c),t} - s_{c,t} + (1 - \chi^{\text{out}}_{pred(c),t}) \cdot M \geq l_{pred(c)} + g_c \quad \forall t \in T_c. \tag{22}$$

  Due to (16), the leave-trigger variable $\chi^{\text{out}}_{pred(c),t}$ takes the value of 1 if $pred(c)$ has not fully crossed the intersection at time point t (i.e. if $s_{pred(c),t} < S^{end}$), hence enforcing the required safety distance. Furthermore, it can always be set to 0 for $s_{pred(c),t} \geq S^{end}$ and hence the big-M term guarantees that the inequalities can always be satisfied if the car has already crossed the intersection for sufficiently large M (one may use the longest possible covered distance until timepoint t, i.e. the length of the path which arises if the car would have driven with maximum velocity until timepoint t).

2.3.2 Turning and collision prevention after crossing the intersection

In the case that cars have crossed the intersection the difficulty arises that the successor-predecessor relation changes and the order in which the cars cross the intersection is not known beforehand. Predecessors can come from different lanes when turning is involved.

To model the turning behaviour we introduce new binary variables $O_{c,c'}$ for each pair of cars (c, c') that drive on different lanes before but on the same lane right after crossing the intersection. It encodes the crossing order as follows:

$$O_{c,c'} = \begin{cases} 0, & \text{if } c \text{ crosses before } c' \\ 1, & \text{if } c \text{ crosses after } c' \end{cases} \tag{23}$$

Consequently, the variables $O_{c,c'}$ and $O_{c',c}$ have to satisfy

$$O_{c,c'} + O_{c',c} = 1. \tag{24}$$

With this we can now formulate the collision prevention constraints after crossing the intersection for cars c, c' as above:

$$s_{c,t} - s_{c',t} + (1 - \chi^{\text{in}}_{c,t} + O_{c,c'}) \cdot M \geq l_c + g_{c'} + \delta \quad \forall t \in T_c \cap T_{c'}, \tag{25}$$

$$s_{c',t} - s_{c,t} + (1 - \chi^{\text{in}}_{c',t} + O_{c',c}) \cdot M \geq l_{c'} + g_c + \delta \quad \forall t \in T_c \cap T_{c'}. \tag{26}$$

Equations (25) model the case where car c crosses before c', in which case c' is required to respect the safety distance requirement if c already entered the intersection area ($\chi^{\text{in}}_{c,t} = 1$, see (16)). Otherwise it can be easily satisfied thanks to big-M. Similarly, (26) is activated only if c' crosses before c and has already entered the intersection area. The adjustment parameter δ is added to project the distances of the vehicles onto the new destination lane after crossing the intersection. It depends on the length of their turning trajectories and the width of the intersection area.

Finally, the following constraints maintain the predecessor-successor relation between vehicles starting from the same lane and going to the same destination even after turning:

$$s_{PRE(c),t} - s_{c,t} \geq l_{PRE(c)} + g_c \quad \forall c \in C,\ t \in T_c, \tag{27}$$

where $PRE(c)$ is the *predecessor with the same destination* as car c. An example is shown in Figure 2. Here, vehicle c_1 turns left, c_2 goes straight and c_3 turns left, so $PRE(c_3) = c_1$. Note, that constraints (21) for the case that car c and its direct predecessor $pred(c)$ have the same destination are included in (27) with $PRE(c) = pred(c)$. Furthermore, Figure 2 demonstrates how to choose the adjustment parameter δ for vehicles turning left.

2.4 Objective function

As we are looking for an optimal traffic flow of multiple cars on a fixed time horizon, we have to decide on a suitable objective function for this purpose. Optimal traffic flow could mean that one wants to reduce the overall traveling time of all cars for reaching their destination. As the time horizon is fixed,

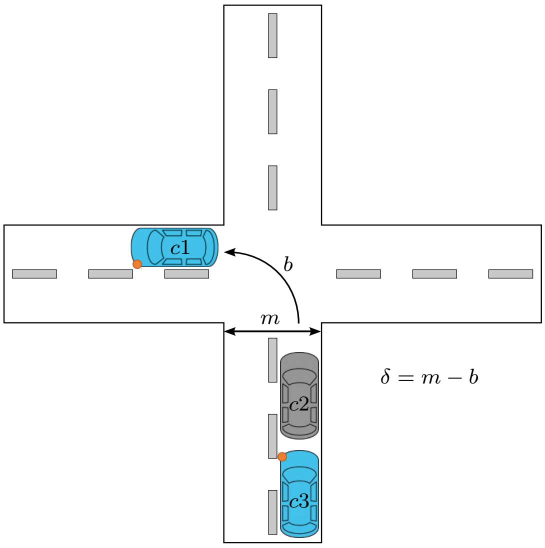

Fig. 2 Visualization of the concept of the *predecessor with the same destination* and choosing the adjustment parameter.

minimizing traveling time for a certain route is similar to maximizing the covered distance in the end of the horizon in our scenario. Therefore, we maximize the sum of the driven distances of all cars at the last time step N:

$$\max \quad \sum_{c \in C} s_{c,N}. \tag{28}$$

Thinking about economical issues one could also try to minimize the overall emissions. As the presented model does not include exhaust rates or something similar, we can approximate this with the car's squared acceleration:

$$\min \quad \sum_{\substack{c \in C, \\ t \in T}} a_{c,t}^2. \tag{29}$$

Note that this objective is nonlinear while the rest of the model is an MILP so far.

Our primary goal is to optimize the traffic flow. Only looking at objective (28), though, we observe that some vehicles do drive quite emission inefficiently in front of the intersection in cases in which it is foreseeable that they can not make it across the intersection during the next green phase. For instance, the objective value would not change if a vehicle slowly approaches the intersection with a constant speed or performs several starts and stops as long as it reaches and crosses the intersection at the same time, but the emission values would be a lot higher in the latter case. It is therefore necessary to consider both, optimizing traffic flow and reducing emissions. The standard approach would be to unify the two objectives into a single multi-objective function of the form

$$\max \quad w_1 \sum_{c \in C} s_{c,N} - w_2 \sum_{\substack{c \in C, \\ t \in T}} a_{c,t}^2 \tag{30}$$

as a weighted sum of both objectives. By adjusting the weights one explores the Pareto-optimal points, i.e. solutions with the property that none of the two individual criteria can be improved while at least maintaining the other. We tackle this problem with a slightly different view: We set our primary goal to be the optimization of overall traffic flow, and from all optimal solutions in regard to that objective we would like to obtain the one which is best in terms of emissions. This two-step approach has two advantages: Firstly, we avoid introducing a nonlinear term in the MILP model, which would make it a lot harder to solve. Secondly, after we obtain the best objective value in terms of traffic flow, we can fix all binary variables and solve the model again with the objective (29), effectively only solving a quadratic program (QP) in the second step. By fixing the binary variables, the order and times in which the vehicles cross the intersection remain unchanged, and only the driving behaviour before and after crossing the intersection are adjusted to be more emission efficient. Briefly summarized, the steps are:

1. Solve MILP with objective function (28).
2. Change the objective function to (29).
   Add constraint
   $$\sum_{c \in C} s_{c,N} \geq f^* - \varepsilon$$
   to the model, where f^* is the objective value of step 1 and ε small.
   Fix all binary variables.
3. Solve the resulting QP.

For convenience, we list all constraints describing the passage of multiple vehicles over traffic light regulated intersections below, with no indices omitted. We refer to the resulting MILP (of step 1) as *global-MILP*. Note that also more complex networks than the rather simple one which served for introducing the model can be represented. For ease of notation, we introduce the set of traffic lights, which a car c passes during its traversal of the network as TL_c. The complete MILP is stated as follows:

$$
\begin{aligned}
\max_{\substack{s,v,a,\overline{y},\underline{y},\\ \chi,\chi^{\text{in}},\chi^{\text{out}},O}} \quad & \sum_{c \in C} s_{c,N} \\
\text{s.t.} \quad & (2) - (10), && \text{(motion model)} \\
& (11), (12) - (14), && \text{(traffic light constraints)} \\
& (17) - (20), \quad \forall c \in C,\ tl \in TL_c,\ t \in T_c, \\
& (22), \quad \forall c \in C : dest(c) \neq dest(pred(c)), && \text{(collision prevention)} \\
& (24), (25), \quad \forall c, c' \in C : O_{c,c'} \text{ is defined}, \\
& (27).
\end{aligned}
$$

3 Experiments

The performance of an intersection is influenced by two factors: the operation of the traffic lights, and the behaviour of autonomous and/or human drivers. In this section, we want to investigate and quantify the potential of improvements for different scenarios, such as modifying traffic light regulations or not imposing any rules on traffic lights at all. The results allow us to determine the capacity of a road network under different conditions, which can serve as a benchmark for decentralized and heuristic apporaches.

We give some computational evidence on the performance of the global-MILP on these different scenarios considering two dimensions: performance of an MILP-solver on the testing data and resulting traffic flow according to traffic simulations.

Firstly, we introduce the four scenarios we consider:

- real-world traffic with a fixed traffic light scheme;
- all vehicles are autonomous, traffic lights have to follow a predefined, fixed scheme;
- all vehicles are autonomous, traffic lights have to ensure minimum lengths of green and red phases;
- all vehicles are autonomous, there are no restrictions on the switching schemes of the traffic lights.

For simplicity, we denote the four different scenarios as (RW), (FIX), (MIN) and (FREE).

Before we discuss the results of the experiments, we define a consistent experimental setting, present the used traffic-simulation software, and consider the representation of real-world traffic in the simulation.

3.1 Traffic simulation software

In order to analyze and rate the effects on traffic, it is necessary to realistically represent the motion of cars and the behavior of human drivers. Furthermore,

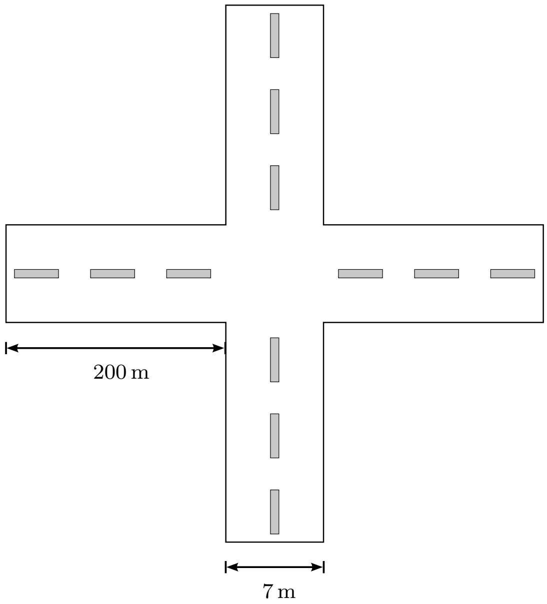

Fig. 3 Layout of the network used for numerical investigation.

the chosen software should allow control of the movements of all cars on the road and the traffic light's signal states in fine time steps. Hence, a microscopic traffic simulation software seems appropriate four our purposes. They allow the simulation of single cars and other entities in the network according to *car-following-models* that model the individual behavior of cars depending on the movement of their respective predecessor. Common examples from the literature are the models of Krauß [12], and the IDM by Treiber and Kesting, cf. [10, 22].

We use the *SUMO (Simulation of Urban MObility)* software framework, cf. [11], to analyze and visualize the computed results. SUMO is a free and open-source traffic-simulation suite which allows modeling of intermodal traffic systems including road vehicles, public transport and pedestrians. It comes with a variety of possibilities for evaluating the simulated traffic. These are, e.g. CO_2-emissions and fuel consumption according to the German Federal Environment Agency [23]. The behavior of traffic lights and cars can be accessed via *TraCI (Traffic Control Interface)* which allows to retrieve values of simulated objects and to manipulate their behaviour online.

3.2 Network and experimental setting

All of the testing instances are processed on the simple network which is depicted in Figure 3. It consists of a single intersection of two roads, each with one lane in either direction. All lanes have a width of 3.5 meters and a stretch of 200 meters of road is added in each direction from the intersection.

The fixed switching schemes of the traffic lights include a green phase and a red phase for all traffic lights of the intersection. In the real-world case a short red-amber phase is included before the green phase. Each cycle consists of four succeeding and equal phases for each lane in clockwise order. For the (MIN) case, we demand a minimal duration on the green phase. In contrast to the fixed case, the green phases can last longer and the starting times and order of the phases is not predefined. To be comparable, the minimum duration of the green phase is set to the length of the green phase of the fixed case. As mentioned before, there are no restrictions in the (FREE) case.
The arrival time of each car is fixed for each scenario and are randomly chosen according to the following principle: A mean rate of cars per lane and minute for a certain amount of minutes is fixed. The actual number of cars for each minute is randomly distributed according to a Poisson distribution, which according to Gallager [5] is suitable for generating traffic-related data. Specific arrival times are distributed equidistantly over the minute. We consider four different traffic densities. For each density five random instances are generated and proccessed for the four scenarios. The evaluated parameters are geometric means of all five instances and apply for single cars. For all methods, we set the discretization step to $dt = 0.5$ seconds. The physical bounds on the motion of the cars are chosen similarly for each method according to Table 1.

For the computed traffic of the four scenarios we use following quality measures:

- the value of the objective function, meaning the accumulated driven distance of all cars;
- the mean *travel time* and *waiting time* of all cars in the network. While travel time measures the time it takes for each car to traverse the network, the waiting time measures the difference between travel time and theoretical time for an unobstructed traversal of the network. Both values are suitable for indicating the quality of traffic flow according to Forschungsgesellschaft für Straßen- und Verkehrswesen e.V. [3].
- the mean simulated *fuel consumption* and CO_2 *emissions* of all cars in the network. We also refer to these parameters as *environmental parameters*.

All computations were performed on a server with 32 cores (Intel(R) Xeon(R) CPU E5-2640 v3 @ 2.60GHz) and 31 GB of RAM, running Debian GNU/Linux 10. CPLEX V12.9 serves as MILP-solver for all optimization problems occurring here. Preliminary experiments have shown that the solver performed better without presolve during preprocessing, which is why we disabled it for our computations. Finally, SUMO is used in version 1.1.0.

3.3 Real-world traffic

A crucial part for analyzing the effects on traffic is to simulate real-world traffic appropriately. Usually, traffic is simulated in SUMO using car-following

Table 1 Bounds on the states of the vehicles which are equal for cars and scenarios.

	Value	Description
$\underline{v}_c$	0	minimum velocity in m/s
$\overline{v}_c$	15.27	maximum velocity in m/s
$\underline{a}_c$	-7.5	minimum acceleration in m/s^2
$\overline{a}_c$	2.9	maximum acceleration in m/s^2
$\underline{j}_c$	-3	minimum jerk in m/s^3
$\overline{j}_c$	3	maximum jerk in m/s^3
l_c	4.3	length of the car in m
g_c	2.5	security gap of the car in m

Table 2 Adjustable parameters for the default car-following model in SUMO with used values.

Parameter	Value	Description
accel	2.9	acceleration ability
decel	-7.5	deceleration ability
sigma	0.5	driver imperfection $\in [0, 1]$
tau	1.0	reaction time
minGap	2.5	gap to front vehicle if halting

models. By default, the model by Krauß is enabled which provides different parameters to adapt the behaviour of cars in the simulation. That way, different types of vehicles such as passenger cars, motorcycles, trucks and busses can be modeled. Aside from the parameters shown in Table 1, additional parameters like the driver's imperfection or reaction time influence the driving behaviour. In our simulation we use the standard passenger car. Table 2 shows the adjustable parameters and used values for each of them. Later computations about emissions also use the default model for passenger cars, and are based on the emission model descriptions HBEFA3 of the German Federal Environment Agency [23].

3.4 Experimental results

In this section, we discuss the quality of solutions obtained by the global-MILP in terms of the aforementioned quality measures. We first investigate the performance of the solving process, which is depicted in Table 3.

The table lists the runtimes and objective function value (for the MILP *maximization of driven distance*) for the three scenarios (FIX), (MIN) and (FREE) for different traffic densities, averaged over five testing instances each. The column *MILP* shows the computation times for solving the MILP, while the column *QP* shows the time for solving the additional QP minimizing the sum of squared accelerations. The column *total* sums up both.

The computation times for the MILP as well as the QP increase with a higher density of cars, but are still manageable for the (FIX) and (FREE) case, ranging from around 1 second for the smallest density up to around 15

Table 3 Measurements for traffic with different densities and scope of optimization.

density [Cars/min]	T [s]	dt [s]	L [s]	computation times total [s]	MILP [s]	QP [s]	objective value MILP [m]
20.95	60	0.5	10				
fixed traffic lights				24.41	13.76	10.65	28209.2
regulated traffic lights				32533.80 (2)	32524.10 (2)	9.71	29173.1 (2)
free traffic lights				18.42	15.29	3.13	38468.4
15.06	60	0.5	10				
fixed traffic lights				9.38	4.57	4.80	20578.6
regulated traffic lights				19079.54	19075.50	4.04	21451.9
free traffic lights				10.91	9.26	1.65	27627.8
9.5	60	0.5	10				
fixed traffic lights				4.28	2.25	2.03	12891.9
regulated traffic lights				1128.13	1126.30	1.83	14056.7
free traffic lights				3.01	2.09	0.92	17392.8
5.11	60	0.5	10				
fixed traffic lights				1.73	0.89	0.84	7230.3
regulated traffic lights				111.94	111.19	0.75	7892.6
free traffic lights				1.44	1.01	0.43	9353.2

Annotations. Measurements are geometric means per vehicle of five testing instances. If only a subset of instances has been solved, the number of solved instances is given in brackets. Means were taken over all five instances; computation times for unsolved instances are set to the time limit of 10 hours.

seconds for the MILP and 10 seconds for the QP for the highest density. The MILP problem gets significantly harder when minimal green and red times, i.e. min-up-down constraints on the binary variables controlling the traffic lights, are added in the (MIN) case, though. While in the smallest case this leads to a computation time of average 111 seconds, it goes up to 19000 seconds for the second highest density, and 3 out of 5 instances could not be solved to optimality within a 10 hour time frame for the highest density. Since all the integer variables are fixed when solving the additional QP, runtimes here behave similarly to the (FIX) case.

Looking at the objective values of the MILPs, as expected, the values (total driven distance) increase when traffic lights are less regulated and have more freedom in controlling the traffic. This increase tends to be larger the higher the density is.

Considering the quality of the resulting traffic, we now concentrate on the parameters travel time, waiting time, CO_2-emissions, and fuel consumption. Going from (RW) to (FIX) to (MIN) and lastly to (FREE), the degree of freedom granted to the optimization procedure increases. The focus of this study is to determine how this increase is reflected in the quality of results for overall traffic flow as well as the environmental parameters. To this end, the offline calculated solutions via CPLEX are transferred into SUMO. The Python interface TraCI allows us to control the motion of all cars in each timestep. Thus, we can retrieve the parameters of interest from the solutions of the MILP. Table 4 shows the relevant measurements. Throughout all the problem instances we observe an improvement in all four parameters when we increase the degree of optimization. Largest improvements are achieved by going from

Table 4 Minimizing squared acceleration after maximizing driven distance. Measurements for traffic with different densities and scope of optimization.

density [Cars/min]	T [s]	dt [s]	L [s]	waiting time [s]	travel time [s]	fuel [l/100km]	CO_2 [g/km]
20.95	60	0.5	10				
real-world simulation				14.67	57.89	10.67	248.29
fixed traffic lights				8.14	39.20	9.19	213.80
regulated traffic lights				7.39	37.86	8.96	208.47
free traffic lights				0.14	26.84	6.41	149.23
15.06	60	0.5	10				
real-world simulation				12.58	50.20	10.56	245.59
fixed traffic lights				7.78	38.57	8.81	205.06
regulated traffic lights				6.84	36.89	8.61	200.21
free traffic lights				0.12	26.84	6.51	151.42
9.5	60	0.5	10				
real-world simulation				10.36	44.24	10.36	241.04
fixed traffic lights				7.83	38.75	8.59	199.84
regulated traffic lights				5.83	35.12	8.14	189.37
free traffic lights				0.08	26.79	6.55	152.30
5.11	60	0.5	10				
real-world simulation				9.03	40.62	10.23	237.95
fixed traffic lights				6.85	36.73	8.03	186.91
regulated traffic lights				4.66	33.36	7.71	179.44
free traffic lights				0.10	26.81	6.59	153.21

Annotations. Measurements are geometric means per vehicle of five testing instances.

the (RW) case to the (FIX) case, where we leave the traffic lights as they were, but optimize the traversal of the cars throughout the network, and going from the (MIN) case to the (FREE) case by removing all restricting rules on the traffic lights. Looking at the highest density traffic in more detail, we achieve a decrease in waiting times of about 44,47 % with (FIX), 49,59 % with (MIN) and 99,05 % with (FREE) compared to the (RW) case. The travel times per vehicle decrease by 32,29 %, 34,61 % and 53,63 %. Although we were primarily optimizing in terms of overall traffic flow, and considered emissions in a second step by solving the QP, we still observe a significant improvement regarding the environmental parameters. For the fuel consumption and CO_2-emissions, we register a decrease of 13,91 %, 16,06 %, 39,91 % and 13,89 %, 16,04 %, 39,90 %, respectively. Although smaller, improvements were prominent even on traffic with lower densities.

4 Conclusions

In this article, we introduced a novel mixed-integer linear program which models both the movement of individual cars on a network of roads that intersect in multiple traffic-light-controlled intersections and the switching scheme of traffic lights regulating the transit of cars. An advantage of the MILP is the presence of linear but realistic motion dynamics for each vehicle and the flexibility to model different traffic scenarios by modifying parameters such as traffic density, turning behavior and traffic light regulations. Its solution by offline calculations yields an overall optimized traffic flow. Furthermore, by solving an additional quadratic program we obtain traffic that is also energy efficient. Through simulations we analyze the influence of the aforementioned parameters on the overall traffic efficiency. Numerical results indicate that centrally optimized traffic leads to almost no waiting times (mean waiting times could be decreased by up to 99 %) and a reduction in average fuel consumption of 39 % in case of completely free traffic lights. However, also the more realistic case of controllable traffic lights that are bound to certain regulations on phase lengths leads to significant improvements in waiting times, fuel consumption and CO_2-emission when compared to non-optimized traffic.

In the growing field of assisted and autonomous driving applications, the solutions of the MILP can serve as a benchmark for suboptimal but practically implementable algorithms in cars and infrastructural devices. Computational experiments reveal that a globally optimal solution is difficult to obtain in a realistic setting. This is not only due to the relatively long solving times but also due to the fact that a resulting real-world system would be based on the premise that all cars are equipped with this system and bound to follow the system's advice. Future work will involve the development of decentralized algorithms to overcome both issues. This work allows us to validate the effectiveness of such algorithms and the quality of its solutions.

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References

[1] G. Araniti, C. Campolo, M. Condoluci, A. Iera, and A. Molinaro. LTE for Vehicular Networking: A Survey. *Communications Magazine, IEEE*, 51(5):148–157, 2013.

[2] K. Dresner and P. Stone. A Multiagent Approach to Autonomous Intersection Management. *Journal of artificial intelligence research*, 31: 591–656, 2008.

[3] Forschungsgesellschaft für Straßen- und Verkehrswesen e.V. *Handbuch für die Bemessung von Straßenverkehrsanlagen.* FGSV-Verlag, Köln, 2001.

[4] B. Friedrich, P. Wagner, W. Niebel, A. Herrmann, S. Naumann, O. Bley, O. Kutzner, M. Maurer, F. Saust, T. Schüler, H. Poppe, M. Junge, and J. Langenberg. *KOLINE: Kooperative und optimierte Lichtsignalsteuerung in städtischen Netzen: Schlussbericht zum Forschungsprojekt, Förderkennzeichen 19P9002, gefördert vom Bundesministerium für Wirtschaft und Technologie.* 2013.

[5] R. Gallager. *Stochastic Processes: Theory for Applications*. Cambridge University Press, 2013.

[6] A. Giridhar and P. Kumar. Scheduling Automated Traffic on a Network of Roads. *IEEE Transactions on Vehicular Technology*, 55(5):1467–1474, 2006. ISSN 0018-9545. doi: 10.1109/TVT.2006.877472.

[7] R. Hult, G. R. Campos, P. Falcone, and H. Wymeersch. An approximate solution to the optimal coordination problem for autonomous vehicles at intersections. *2015 American Control Conference (ACC)*, pages 763–768, 2015.

[8] R. Hult, M. Zanon, S. Gros, and P. Falcone. An MIQP-based heuristic for Optimal Coordination of Vehicles at Intersections. *2018 IEEE Conference on Decision and Control (CDC)*, pages 2783–2790, 2018.

[9] R. Hult, M. Zanon, S. Gros, and P. Falcone. Optimal Coordination of Automated Vehicles at Intersections: Theory and Experiments. *IEEE Transactions on Control Systems Technology*, 27(6):2510–2525, 2018.

[10] A. Kesting, M. Treiber, and D. Helbing. Enhanced Intelligent Driver Model to Access the Impact of Driving Strategies on Traffic Capacity. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 368(1928):4585–4605, 2010. ISSN 1364-503X. doi: 10.1098/rsta.2010.0084.

[11] D. Krajzewicz, J. Erdmann, M. Behrisch, and L. Bieker. Recent Development and Applications of SUMO-Simulation of Urban MObility. *International Journal On Advances in Systems and Measurements*, 5(3 and 4):128–138, 2012.

[12] S. Krauß. Microscopic Modeling of Traffic Flow: Investigation of Collision Free Vehicle Dynamics. PhD Thesis, 1998.

[13] S. Le Vine, A. Zolfaghari, and J. Polak. Autonomous Cars: The Tension Between Occupant Experience and Intersection Capacity. *Transportation Research Part C: Emerging Technologies*, 52:1–14, 2015. ISSN 0968090X.

[14] J. Lee, J. Leung, and F. Margot. Min-up/min-down polytopes. *Discrete Optimization*, 1(1):77–85, 2004.

[15] A. A. Malikopoulos, C. G. Cassandras, and Y. J. Zhang. A decentralized energy-optimal control framework for connected automated vehicles at signal-free intersections. *Automatica*, 93:244–256, 2018.

[16] K. Oeltze and E. Stemmler. Interdisziplinäre Bewertung der Auswirkung eines Fahrers mit Ampelassistenz auf nicht-ausgestattete Fahrer, 2015.

[17] D. Rajan and S. Takriti. Minimum up/down polytopes of the unit commitment problem with start-up costs. *IBM Res. Rep*, 23628:1–14, 2005.

[18] C. Schießl and K. Oeltze. Benefits and Challenges of Multi-Driver Simulator Studies. *IET Intelligent Transport Systems*, 9(6):618–625, 2015. ISSN 1751-956X. doi: 10.1049/iet-its.2014.0210.

[19] L. Schnieder and R. Krenkel. Betreibermodell einer Forschungsinfrastruktur für die Entwicklung intelligenter Mobilitätsdienste im realen Verkehrsumfeld. In *16. Symposium Automatisierungssysteme, Assistenzsysteme und eingebettete Systeme für Transportmittel (AAET)*, volume 16, pages 108–116. 2015.

[20] S. Sorgatz. *Optimization of Vehicular Traffic at Traffic-Light Controlled Intersections*. PhD thesis, Otto-von-Guericke University Magdeburg, 2016. URL `https://mathopt.de/PUBLICATIONS/Sorgatz2016.pdf`.

[21] S. Takriti, B. Krasenbrink, and L. S.-Y. Wu. Incorporating fuel constraints and electricity spot prices into the stochastic unit commitment problem. *Operations Research*, 48(2):268–280, 2000.

[22] M. Treiber, A. Hennecke, and D. Helbing. Congested Traffic States in Empirical Observations and Microscopic Simulations. *Physical Review E*, 62(2):1805–1824, 2000. ISSN 1063-651X. doi: 10.1103/PhysRevE.62.1805.

[23] Umweltbundesamt. *Handbuch Emissionsfaktoren des Straßenverkehrs Version 3.1*. INFRAS, Zürich, 2010.

[24] Verband der Automobilindustrie e.V. simTD - Sichere Intelligente Mobilität Testfeld Deutschland, 2014. URL `http://www.simtd.de/index.dhtml/deDE/index.html`.

[25] Y. J. Zhang, A. A. Malikopoulos, and C. G. Cassandras. Optimal Control and Coordination of Connected and Automated Vehicles at Urban Traffic Intersections. *Proceedings of the 2016 American Control Conference*, 2016.