

# DIMENSION IN POLYNOMIAL VARIATIONAL INEQUALITIES

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**Abstract.** The aim of the paper is twofold. Firstly, by using the constant rank level set theorem from differential geometry, we establish sharp upper bounds for the dimensions of the solution sets of polynomial variational inequalities under mild conditions. Secondly, a classification of polynomial variational inequalities based on dimensions of their solution sets is introduced and investigated. Several illustrative examples are provided.

**Key words.** polynomial variational inequality, polynomial fractional optimization, semialgebraic set, solution set, dimension

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**1. Introduction.** We consider the following variational inequality

$$\text{find } x \in K \text{ such that } \langle F(x), y - x \rangle \geq 0 \forall y \in K,$$

where  $K$  is a semialgebraic set in  $\mathbb{R}^n$  and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a polynomial map. The problem and its solution set are denoted by  $\text{PVI}(K, F)$  and  $\text{Sol}(K, F)$ , respectively. This problem is a natural extension of the well-known linear complementarity problem, the linear variational inequality, the tensor complementarity problem, and the polynomial complementarity problem (see, e.g., [2, 5, 7, 8]) which have received a lot of attention from researchers.

Thanks to the Tarski-Seidenberg theorem [3, Theorem 2.6],  $\text{Sol}(K, F)$  set is semialgebraic, hence that it has finitely many connected components and each component is path-connected. Furthermore, the dimension of  $\text{Sol}(K, F)$  is well-defined. The present paper focuses on this topic.

Firstly, we show a sharp upper bound for  $\dim(\text{Sol}(K, F))$  provided that the problem satisfies the Abadie constraint qualification and the constant rank condition. Results for the finiteness of  $\text{Sol}(K, F)$  are obtained. Secondly, we show that stationary points of the polynomial fractional optimization problem [10]

$$\text{minimize } \frac{p(x)}{q(x)} \text{ subject to } x \in K,$$

where  $p(x)$  and  $q(x)$  are polynomials,  $q(x) > 0$  for all  $x \in K$ , is the solution set of a certain polynomial variational inequality. Hence, an estimate for the dimension of these stationary points is established. Thirdly, based on dimensions of the solution sets, a classification of polynomial variational inequalities is introduced. We also discuss thickness of the classes.

The organization of the paper is as follows. Section 2 gives a brief introduction to semialgebraic geometry and polynomial variational inequalities. Section 3 shows upper bounds for the dimension of solution sets. Finiteness of solution sets is discussed in Section 4. Section 5 investigates stationary points in polynomial fractional optimization. A classification of polynomial variational inequalities is studied in Section 6. The last section gives some concluding remarks.

**2. Preliminaries.** In this section, we will recall some definitions, notations, and auxiliary results from semialgebraic geometry and polynomial variational inequalities.

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**2.1. Semialgebraic Sets.** Recall a subset in  $\mathbb{R}^n$  is *semialgebraic* [1, Definition 2.1.4], if it is the union of finitely many subsets of the form

$$\{x \in \mathbb{R}^n : f_1(x) = \cdots = f_\ell(x) = 0, g_{\ell+1}(x) < 0, \dots, g_m(x) < 0\},$$

where  $\ell, m$  are natural numbers, and  $f_1, \dots, f_\ell, g_{\ell+1}, \dots, g_m$  are polynomials with real coefficients.

The semialgebraic property is preserved by taking finitely union, intersection, minus and taking closure of semialgebraic sets. The well-known Tarski-Seidenberg theorem [3, Theorem 2.3] states that the image of a semialgebraic set under a linear projection is a semialgebraic set.

Let  $S_1 \subset \mathbb{R}^m$  and  $S_2 \subset \mathbb{R}^n$  be semialgebraic sets. A vector-valued map  $G : S_1 \rightarrow S_2$  is said to be semialgebraic [1, Definition 2.2.5], if its graph

$$\text{gph}(G) = \{(x, v) \in S_1 \times S_2 : v = G(x)\}$$

is a semialgebraic subset in  $\mathbb{R}^m \times \mathbb{R}^n$ .

Let  $S$  be a semialgebraic set of  $\mathbb{R}^m$ . Then there exists a decomposition of  $S$  into a disjoint union of semialgebraic subsets [1, Theorem 2.3.6]

$$S = S_1 \cup \cdots \cup S_s,$$

where each  $S_i$  is semialgebraically diffeomorphic to  $(0, 1)^{d_i}$ ,  $i \in [s]$ . Here, let  $(0, 1)^0$  be a point,  $(0, 1)^{d_i} \subset \mathbb{R}^{d_i}$  is the set of points  $x = (x_1, \dots, x_{d_i})$  such that  $x_j \in (0, 1)$  for all  $j = 1, \dots, d_i$ . The *dimension* of  $S$  is, by definition [3, Proposition 3.15],

$$\dim(S) := \max\{d_1, \dots, d_s\}.$$

The dimension is well-defined and not depends on the decomposition of  $S$ . We adopt the convention that  $\dim(\emptyset) := -\infty$ . If  $S$  is nonempty and  $\dim(S) = 0$  then  $S$  has finitely many points.

Assume that  $S \subset \mathbb{R}^m$  is a semialgebraic set and  $G : S \rightarrow \mathbb{R}^n$  is a semialgebraic map. Theorem 3.18 in [3] says that  $\dim(G(S)) \leq \dim(S)$ . Let  $S_1, \dots, S_k$  be semialgebraic sets in  $\mathbb{R}^n$ . Applying [1, Proposition 2.8.5], one has the following equality:

$$(2.1) \quad \dim(S_1 \cup \cdots \cup S_k) = \max\{\dim S_1, \dots, \dim S_k\}.$$

Let  $S_1, S_2$  are semialgebraic manifolds in  $\mathbb{R}^m$  and  $G : S_1 \rightarrow S_2$  be a smooth semialgebraic map. Assume that the rank of the Jacobian of  $G$  is  $k$  in a neighborhood of the level set  $G^{-1}(v)$ , where  $v \in S_2$  be given. The constant rank level set theorem [9, Theorem 11.2] says that  $G^{-1}(v)$  is a submanifold of  $S_1$  and  $\dim(G^{-1}(v)) = m - k$ .

**2.2. Polynomial Variational Inequalities.** Let  $K$  be a nonempty semialgebraic closed convex subset in  $\mathbb{R}^n$  and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a polynomial map. The *polynomial variational inequality* defined by  $K$  and  $F$  is the following problem:

$$\text{find } x \in K \text{ such that } \langle F(x), y - x \rangle \geq 0 \quad \forall y \in K,$$

where  $\langle x, y \rangle$  is the usual scalar product of  $x, y$  in the Euclidean space  $\mathbb{R}^n$ . We will respectively write the problem and its solution set PVI( $K, F$ ) and Sol( $K, F$ ).

Note that  $x$  solves (PVI) if and only if  $F(x) \in -N_K(x)$ , where  $N_K(x)$  is the normal cone of  $K$  at  $x \in K$  which is defined by

$$N_K(x) = \{x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle \leq 0 \quad \forall y \in K\}.$$

Clearly, if  $x$  belongs to the interior of  $K$ , then  $x \in \text{Sol}(K, F)$  if and only if  $F(x) = 0$ . When  $K = \mathbb{R}^n$ ,  $x$  solves PVI( $K, F$ ) if and only if  $x$  is a zero point of the function  $F$ .

Throughout the work, we assume that  $K$  given by finitely many convex polynomial functions  $g_i(x)$ ,  $i \in [m] := \{1, \dots, m\}$ , and finitely many affine functions  $h_j(x)$ ,  $j \in [\ell]$ , as follows

$$K = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in [m], h_j(x) = 0, j \in [\ell]\}.$$

To find the solution set of PVI( $K, F$ ), we will find the solutions on each pseudo-face of  $K$ . For every index set  $\alpha \subset [m]$ , we associate that with the following *pseudo-face*

$$K_\alpha = \{x \in \mathbb{R}^n : g_i(x) = 0, \forall i \in \alpha, g_i(x) < 0, \forall i \in [m] \setminus \alpha, h_j(x) = 0, \forall j \in [\ell]\}.$$

All pseudo-faces establish a finite disjoint decomposition of  $K$ . Therefore, we have

$$(2.2) \quad K = \bigcup_{\alpha \subset [m]} K_\alpha.$$

Since  $\text{Sol}(K, F)$  is a subset of  $K$ , from the disjoint decomposition (2.2) we have the following equality:

$$(2.3) \quad \text{Sol}(K, F) = \bigcup_{\alpha \subset [m]} [\text{Sol}(K, F) \cap K_\alpha]$$

By applying the Tarski-Seidenberg theorem with quantifiers [3, Theorem 2.6], we see that the solution set  $\text{Sol}(K, F)$  is semialgebraic in  $\mathbb{R}^n$ . From (2.3) and (2.1), one concludes that

$$(2.4) \quad \dim(\text{Sol}(K, F)) = \max_{\alpha \subset [m]} \{\dim(\text{Sol}(K, F) \cap K_\alpha)\}.$$

The Bouligand-Severi tangent cone (see, e.g., [4, p. 15]) of  $K$  at  $x \in K$ , denoted by  $T_K(x)$ , consists of the vectors  $v \in \mathbb{R}^n$ , called the tangent vectors to  $K$  at  $x$ , for which there exist a sequence of vectors  $\{y^k\} \subset K$  and a sequence of positive scalars  $\{t^k\}$  such that

$$\lim_{k \rightarrow \infty} y^k = x, \quad \lim_{k \rightarrow \infty} t^k = 0, \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{y^k - x}{t^k} = v.$$

The linearization cone (see, e.g., [4, p. 17]) of  $K$  at  $x$  is defined by

$$L_K(x) = \{v \in \mathbb{R}^n : \langle \nabla g_i(x), v \rangle \leq 0, i \in I(x), \langle \nabla h_j(x), v \rangle = 0, j \in [\ell]\}$$

with  $I(x) := \{i \in [m] : g_i(x) = 0\}$  denoting the active index set at  $x$ . One says that  $K$  satisfies the *Abadie constraint qualification* (ACQ) [4, p. 17] at  $x \in K$  if  $L_K(x) = T_K(x)$ . If the ACQ holds at every point of  $K$ , then the Karush-Kuhn-Tucker conditions can be applied to variational inequalities with the constraint set  $K$  [4, Proposition 1.3.4], i.e.,  $x \in \text{Sol}(K, F)$  if and only if there exist  $\lambda \in \mathbb{R}^m$  and  $\mu \in \mathbb{R}^\ell$  such that

$$(2.5) \quad \begin{aligned} F(x) + \sum_{i \in [m]} \lambda_i \nabla g_i(x) + \sum_{j \in [\ell]} \mu_j \nabla h_j(x) &= 0, \\ \lambda^T g(x) &= 0, \quad \lambda \geq 0, \quad g(x) \leq 0, \quad h(x) = 0. \end{aligned}$$

One says that  $K$  satisfies the *linearly independent constraint qualification*, is written LICQ for brevity, if the gradient vectors

$$\{\nabla g_i(x), \nabla h_j(x), i \in I(x), j \in [\ell]\}$$

are linearly independent, for all point  $x \in K$ . If the LICQ holds on  $K$ , then the ACQ (see, e.g. [4, p. 17]) also holds on  $K$ .

**3. Sharp Upper Bounds for Dimensions.** This section gives sharp upper bounds for the dimensions of the solution sets of polynomial variational inequalities provided that the problems satisfy the constant rank condition. Consequently, some special cases are considered.

Based on the constant-rank level set theorem [9, Theorem 11.2] from differential geometry, we first prove the following lemma which is an important tool to prove our main theorems.

**LEMMA 3.1.** *Assume that  $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a polynomial map,  $v \in \mathbb{R}^m$ , and  $P^{-1}(v) \neq \emptyset$ . If the rank of the Jacobian of  $P$ , denoted by  $\text{rank}(DP)$ , is constant on  $\mathbb{R}^m$ , then  $P^{-1}(v)$  is a semialgebraic set and its dimension satisfies the following equality:*

$$(3.1) \quad \dim(P^{-1}(v)) = m - \text{rank}(DP).$$

*Proof.* Because  $P$  is a polynomial map, the level set  $P^{-1}(v)$  is a semialgebraic set. Since  $P$  is smooth and the rank of  $DP$  is constant on  $\mathbb{R}^m$ , the equation (3.1) follows the constant rank level set theorem [9, Theorem 11.2].  $\square$

Let  $\alpha \subset [m]$  be given. Consider the function  $\Phi_\alpha$  from  $\mathbb{R}^n \times \mathbb{R}^{|\alpha|} \times \mathbb{R}^\ell$  to  $\mathbb{R}^{n+|\alpha|+\ell}$  defined by

$$\Phi_\alpha(x, \lambda_\alpha, \mu) = \left( F(x) + \sum_{i \in \alpha} \lambda_i \nabla g_i(x) + \sum_{j \in [\ell]} \mu_j \nabla h_j(x), g_\alpha(x), h(x) \right)^T,$$

where  $g_\alpha = (g_i)_{i \in \alpha}$ ,  $\lambda_\alpha = (\lambda_i)_{i \in \alpha}$ . Clearly,  $\Phi_\alpha$  is a polynomial map. The zero set  $\Phi_\alpha^{-1}(0)$  is semialgebraic in  $\mathbb{R}^{n+|\alpha|}$ . The Jacobian of  $\Phi_\alpha$  is determined as follows

$$(3.2) \quad D\Phi_\alpha(x, \lambda_\alpha) = \begin{bmatrix} DF + \sum_{i \in \alpha} \lambda_i \nabla^2 g_i(x) & \nabla g_\alpha(x) & \nabla h(x) \\ \nabla g_\alpha(x)^T & 0 & 0 \\ \nabla h(x)^T & 0 & 0 \end{bmatrix},$$

where  $DF$  is the Jacobian of  $F$ , and  $\nabla^2 g_i(x)$  is the Hessian of  $g_i(x)$ .

**REMARK 3.2.** Since  $h(x)$  is affine, the gradient  $\nabla h(x)$  is a constant vector in  $\mathbb{R}^\ell$ . We must emphasize that the polynomial matrix  $D\Phi_\alpha$  does not depends on  $\mu$ . Furthermore, if  $g_i, i \in \alpha$ , are affine then  $D\Phi_\alpha$  also does not depends on  $\lambda_\alpha$ .

The following theorem shows a upper bound for the dimension of the solution set of PVI( $K, F$ ).

**THEOREM 3.3.** *Assume that the ACQ holds on  $K$ . If the rank of  $D\Phi_\alpha$  is constant on  $\mathbb{R}^n \times \mathbb{R}^{|\alpha|}$ , for every  $\alpha \subset [m]$ , then the following inequality holds:*

$$(3.3) \quad \dim(\text{Sol}(K, F)) \leq \max_{\alpha \subset [m]} \{\min\{\dim(K_\alpha), n + |\alpha| + \ell - \text{rank}(D\Phi_\alpha)\}\}.$$

*Proof.* Let  $\alpha \subset [n]$  be given. We will prove the following inequality:

$$(3.4) \quad \dim(\text{Sol}(K, F) \cap K_\alpha) \leq \min \{ \dim(K_\alpha), n + |\alpha| + \ell - \text{rank}(D\Phi_\alpha) \}.$$

Indeed, since  $\text{Sol}(K, F) \cap K_\alpha \subset K_\alpha$ , one has  $\dim(\text{Sol}(K, F) \cap K_\alpha) \leq \dim(K_\alpha)$ . So, we need only to show the fact that

$$(3.5) \quad \dim(\text{Sol}(K, F) \cap K_\alpha) \leq n + |\alpha| + \ell - \text{rank}(D\Phi_\alpha).$$

Remind that the set of zero points  $\Phi_\alpha^{-1}(0)$  is semialgebraic. Since the rank of matrix  $D\Phi_\alpha(x, \lambda_\alpha)$  is constant on  $\mathbb{R}^n \times \mathbb{R}^{|\alpha|}$ , by applying Lemma 3.1 for the map  $\Phi_\alpha$ , we conclude that the dimension of  $\Phi_\alpha^{-1}(0)$  is  $n + |\alpha| + \ell - \text{rank}(D\Phi_\alpha)$ . From (2.5) one has

$$\text{Sol}(K, F) \cap K_\alpha \subset \pi(\Phi_\alpha^{-1}(0)),$$

where  $\pi$  is the projection  $\mathbb{R}^{n+|\alpha|} \rightarrow \mathbb{R}^n$  defined by  $\pi(x, \lambda_\alpha) = x$ . The Tarski-Seidenberg theorem says that  $\pi((\Phi_\alpha)^{-1}(0))$  is semialgebraic. It follows from [3, Theorem 3.18] that

$$\dim(\text{Sol}(K, F) \cap K_\alpha) \leq \dim(\pi(\Phi_\alpha^{-1}(0))) \leq \dim(\Phi_\alpha^{-1}(0)).$$

Therefore, (3.5) is obtained.

Note that (3.4) is true for all  $\alpha \subset [m]$ . Substituting (3.4) into (2.4), we obtain (3.3). The proof is complete.  $\square$

EXAMPLE 3.4. Consider the variational inequality  $\text{PVI}(K, F)$  given by

$$F_1(x_1, x_2) = F_2(x_1, x_2) = x_1^3 + x_1 - x_2, \quad K = \{(x_1, x_2) : x_1 + x_2 \leq 0\},$$

The Jacobian of  $F$  is defined as follows:

$$D_x F = \begin{bmatrix} 3x_1^2 + 1 & -1 \\ 3x_1^2 + 1 & -1 \end{bmatrix}.$$

One has  $K_\emptyset = \{(x_1, x_2) : x_1 + x_2 < 0\}$  and  $D\Phi_\emptyset = DF$ . Hence that,  $\text{rank}(DF) = 1$  for all  $(x_1, x_2) \in \mathbb{R}^2$ . Because of  $K_{\{1\}} = \{(x_1, x_2) : x_1 + x_2 = 0\}$  and

$$D\Phi_{\{1\}} = \begin{bmatrix} 3x_1^2 + 1 & -1 & 1 \\ 3x_1^2 + 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

we assert that  $\text{rank}(D\Phi_{\{1\}}) = 2$  for all  $(x_1, x_2) \in \mathbb{R}^2$  and  $\lambda \in \mathbb{R}$ . The problem  $\text{PVI}(K, F)$  satisfies the assumptions in Theorem 3.3. It follows that the dimension of  $\text{Sol}(K, F)$  is not greater than one. Besides, an easy computation shows that

$$\text{Sol}(K, F) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^3 + x_1, x_1 + x_2 \leq 0\},$$

hence that  $\dim(\text{Sol}(K, F)) = 1$ .

The following theorem says that the inequality (3.3) becomes an equality if the constraint set is the space  $\mathbb{R}^n$ , i.e.  $\text{PVI}(K, F)$  is an unconstrained polynomial variational inequality.

THEOREM 3.5. *Consider the case that  $K = \mathbb{R}^n$ . If  $\text{rank}(DF)$  is constant on  $\mathbb{R}^n$  and  $\text{Sol}(\mathbb{R}^n, F)$  is nonempty, then*

$$\dim(\text{Sol}(\mathbb{R}^n, F)) = n - \text{rank}(DF).$$

*Proof.* Suppose that  $\text{rank}(DF)$  is constant on  $\mathbb{R}^n$  and  $\text{Sol}(\mathbb{R}^n, F) \neq \emptyset$ . Since  $K = \mathbb{R}^n$ , the constraint set has a unique pseudo-face  $K_\emptyset = \mathbb{R}^n$ . Thus, one has  $\text{Sol}(\mathbb{R}^n, F) = F^{-1}(0)$ . Applying Lemma 3.1, the desired equality is obtained.  $\square$

When the degree of all components of  $F$  and  $g$  is small enough, the constant rank condition always is true. The following corollary considers the case that  $F$  and  $g$  are affine maps.

**COROLLARY 3.6.** *Assume that the map  $F$  is affine, i.e.  $F(x) = Mx + q$ , where  $M \in \mathbb{R}^{n \times n}$  and  $q \in \mathbb{R}^n$ . If  $K$  is polyhedral convex, i.e.  $K$  given by*

$$(3.6) \quad K = \{x \in \mathbb{R}^n : Ax \leq b\},$$

where  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$ , then (3.3) holds.

*Proof.* Suppose that  $F(x) = Mx + q$ . Its Jacobian is  $M$  for every  $x \in \mathbb{R}^n$ . Because  $g(x)$  is affine, the gradients  $\nabla g_i(x)$  are constant vectors in  $\mathbb{R}^n$ . Then  $D\Phi_\alpha$  does not depend on neither  $x$  nor  $\lambda_\alpha$ , hence that  $\text{PVI}(K, F)$  satisfies the constant rank condition. Applying Theorem 3.3, we obtain the inequality (3.3).  $\square$

**4. Finiteness of Solution Sets.** In this section, we discuss the finiteness of the solution set of a polynomial variational inequality. Remind that the solution set has finitely many points if and only if its dimension is not greater than one.

**THEOREM 4.1.** *Assume that the ACQ holds on  $K$ . If the rank of  $D\Phi_\alpha$  is  $n + |\alpha| + \ell$  on  $\mathbb{R}^n \times \mathbb{R}^{|\alpha|}$ , for all  $\alpha \subset [m]$ , then  $\text{Sol}(K, F)$  has finitely many points.*

*Proof.* Suppose that  $\text{rank}(D\Phi_\alpha) = n + |\alpha| + \ell$ , for every  $\alpha \subset [m]$ . Applying Theorem 3.3 we obtain  $\dim(\text{Sol}(K, F)) \leq 0$ . This means that  $\text{Sol}(K, F)$  has finitely many points.  $\square$

Now we consider a simpler case in which the constraint set is the nonnegative orthant of  $\mathbb{R}^n$ , i.e.  $K = \{x : -x_1 \leq 0, \dots, -x_n \leq 0\} = \mathbb{R}_+^n$ . The problem  $\text{PVI}(\mathbb{R}_+^n, F)$  becomes a *polynomial complementarity problem*, denoted by  $\text{PCP}(F)$ . We see that the ACQ holds on  $K$  and the matrix  $D\Phi_\alpha$  in (3.2) does not depend on  $\lambda_\alpha$ . For the finiteness of solution sets, the constant rank condition in Theorem 4.1 can be replaced by a simpler one.

From (3.2), now the Jacobian of  $\Phi_\alpha$  is determined as follows

$$D\Phi_\alpha = \begin{bmatrix} DF & C_\alpha \\ C_\alpha^T & 0 \end{bmatrix},$$

where  $C_\alpha = (c_{ij}) \in \mathbb{R}^{n \times |\alpha|}$  given by

$$(4.1) \quad c_{ij} = \begin{cases} 1 & \text{if } i = j, j \in \alpha, i \in [n], \\ 0 & \text{if } i \neq j, j \in \alpha, i \in [n]. \end{cases}$$

We emphasize that the polynomial matrix  $D\Phi_\alpha$  does not depend on both  $\lambda_\alpha$  and  $\mu$ . Because the rank of  $C_\alpha$  is  $|\alpha|$ ,  $\text{rank}(D\Phi_\alpha(x)) \geq |\alpha|$  for all  $x \in \mathbb{R}^n$ .

**REMARK 4.2.** Let  $A \in \mathbb{R}^{n \times n}$  be nonsingular, i.e. the kernel of the linear map  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , given by  $x \mapsto Ax$ , is trivial. We assert that the  $|\alpha| \times |\alpha|$ -matrix  $C_\alpha^T A^{-1} C_\alpha$  also is nonsingular. Indeed, suppose that there exists  $v \in \mathbb{R}^{|\alpha|}$  with  $v \neq 0$  such that  $C_\alpha^T A^{-1} C_\alpha(v) = 0$ . It follows that

$$0 = \langle C_\alpha^T A^{-1} C_\alpha(v), v \rangle = v^T C_\alpha^T A^{-1} C_\alpha v = (C_\alpha v)^T A^{-1} (C_\alpha v).$$

Since  $v \neq 0$ , from (4.1), we see that the vector  $C_\alpha v$  is nontrivial. By the nonsingularity of  $A^{-1}$ , one has  $(C_\alpha v)^T A^{-1} (C_\alpha v) \neq 0$ . It is a contradiction. The kernel of  $C_\alpha^T A^{-1} C_\alpha$  is trivial, hence that this matrix is nonsingular.

**THEOREM 4.3.** *Consider the polynomial complementarity problem  $\text{PCP}(F)$ . If  $\text{rank}(DF) = n$  on  $\mathbb{R}^n$ , then the solution set has finitely many points.*

*Proof.* Assume that  $\text{rank}(DF) = n$  on  $\mathbb{R}^n$ . Let  $\alpha \subset [n]$  be given. The pseudo-face  $K_\alpha$  defined by

$$K_\alpha = \{x \in \mathbb{R}^n : x_i = 0, i \in \alpha, x_j < 0, j \in [n] \setminus \alpha\}.$$

We will show that  $\text{rank}(D\Phi_\alpha(x)) = n + |\alpha|$  for every  $x \in \mathbb{R}^n$ . Let  $x$  be a arbitrary given. The matrix  $DF(x)$  is nonsingular. From (3.2) and the Schur determinantal formula [11, Theorem 2.2], the determinate of  $D\Phi_\alpha(x)$  is written as follows :

$$(4.2) \quad \det(D\Phi_\alpha(x)) = \det(DF(x)) \det(-C_\alpha^T (DF(x))^{-1} C_\alpha).$$

Remark 4.2 says that  $C_\alpha^T (DF(x))^{-1} C_\alpha$  is nonsingular. From (4.2), we conclude that the determinate of  $D\Phi_\alpha(x)$  is nonzero. By the sign of  $\det(DF(x))$  is nonzero constant for any  $x$ , the sign of  $\det(D\Phi_\alpha(x))$  so is. One concludes that  $\text{rank}(D\Phi_\alpha(x)) = n + |\alpha|$ .

Hence,  $\text{PCP}(F)$  satisfies the constant rank condition in Theorem 3.3. The inequality (3.3) is obtained. It is easy to check that the right hand side of (3.3) equals 0. One has  $\dim(\text{Sol}(\mathbb{R}_+^n, F)) \leq 0$ , i.e., the solution set has finitely many points.  $\square$

**EXAMPLE 4.4.** Consider the polynomial complementarity problem where the map  $F$  given by

$$F_1(x_1, x_2) = x_1 - 1, \quad F_2(x_1, x_2) = -x_1^3 - x_2 + 1.$$

The Jacobian of  $F$  is defined as follows:

$$DF(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ -3x_1^2 & -1 \end{bmatrix}.$$

Clearly,  $\text{rank}(DF(x)) = 2$  for every  $x \in \mathbb{R}^2$ . The problem  $\text{PCP}(F)$  satisfies the assumptions in Theorem 4.3, then  $\text{Sol}(\mathbb{R}_+^2, F)$  has finitely many points. Besides, an easy computation shows that the solution set has two points

$$\text{Sol}(\mathbb{R}_+^2, F) = \{(1, 0), (0, 1)\}.$$

When  $F$  is affine, the problem becomes a linear complementarity problem. The following result is a corollary of Theorem 4.3.

**COROLLARY 4.5.** *If the map  $F = Mx + q$ , where  $M \in \mathbb{R}^{n \times n}$  is nonsingular, and  $q \in \mathbb{R}^n$ , then  $\text{Sol}(\mathbb{R}_+^n, F)$  has finite points.*

*Proof.* Suppose that  $F = Mx + q$  where  $M$  is nonsingular. It follows that  $\text{rank}(DF) = \text{rank}(M) = n$  on  $\mathbb{R}^n$ . Theorem 4.3 says that  $\text{Sol}(\mathbb{R}_+^n, F)$  has finite points.  $\square$

**5. Stationary Points in Polynomial Fractional Optimization.** We discuss on dimensions of the sets of stationary points of polynomial fractional optimization problems under the constant rank condition. Consequently, some special results are obtained.

Let  $p(x)$  and  $q(x)$  be two polynomials in  $n$  variables. Assume that  $q(x) > 0$  on  $K$ . We consider the quotient function

$$f(x) = \frac{p(x)}{q(x)}, \quad x \in K.$$

The *polynomial fractional optimization problem* [10] defined by  $K$  and  $f$  is the following problem:

$$\text{minimize } f(x) \text{ subject to } x \in K.$$

We will respectively write the problem and the set of stationary points  $\text{OP}(K, f)$  and  $\text{Stat}(K, f)$ .

For each index subset  $\alpha \subset [m]$ . Consider the function  $\Psi_\alpha$  from  $\mathbb{R}^n \times \mathbb{R}^{|\alpha|} \times \mathbb{R}^\ell$  to  $\mathbb{R}^{n+|\alpha|+\ell}$  defined by

$$\Psi_\alpha(x, \lambda_\alpha, \mu) = \left( (q\nabla p - p\nabla q)(x) + \sum_{i \in \alpha} \lambda_i \nabla g_i(x) + \sum_{j \in [\ell]} \mu_j \nabla h_j(x), g_\alpha(x), h(x) \right)^T.$$

Because  $q\nabla p - p\nabla q$  is a polynomial map,  $\Psi_\alpha$  so is. Hence, the zero set  $\Psi_\alpha^{-1}(0)$  is semialgebraic in  $\mathbb{R}^{n+|\alpha|+\ell}$ . The Jacobian of  $\Psi_\alpha$  is determined as follows

$$D\Psi_\alpha(x, \lambda_\alpha) = \begin{bmatrix} Q(x) + \sum_{i \in \alpha} \lambda_i \nabla^2 g_i(x) & \nabla g_\alpha(x) & \nabla h(x) \\ \nabla g_\alpha(x)^T & 0 & 0 \\ \nabla h(x)^T & 0 & 0 \end{bmatrix},$$

where

$$(5.1) \quad Q(x) := (q\nabla^2 p - p\nabla^2 q + \nabla^T p \nabla^T q - \nabla^T q \nabla^T p)(x)$$

is the Jacobian of  $q\nabla p - p\nabla q$ .

**THEOREM 5.1.** *Assume that the ACQ holds on  $K$ . If the rank of  $D\Psi_\alpha$  is constant on  $\mathbb{R}^n \times \mathbb{R}^{|\alpha|}$ , for every  $\alpha \subset [m]$ , then the dimension of  $\text{Stat}(K, f)$  does not exceed the following number:*

$$(5.2) \quad \max_{\alpha \subset [m]} \{ \min\{ \dim(K_\alpha), n + |\alpha| + \ell - \text{rank}(D\Psi_\alpha) \} \}.$$

*Proof.* The stationary points of  $\text{OP}(K, f)$  is the solutions of the variational inequality problem defined by  $K$  and  $\nabla f$  (see, e.g., [4, Subsection 1.3.1]), where  $\nabla f$  is the gradient of  $f$ .

The  $i$ th-component of the gradient  $\nabla f(x)$  obtained by differentiating the quotient with respect to the single real variable  $x_i$ :

$$(\nabla f(x))_i = \frac{\partial}{\partial x_i} \left( \frac{p(x)}{q(x)} \right) = \frac{p'_{x_i} q(x) - q'_{x_i} p(x)}{q^2(x)}, \quad i \in [n].$$

Hence, one has

$$(5.3) \quad \nabla f = \frac{q\nabla p - p\nabla q}{q^2}.$$

We see that  $\text{Stat}(K, f)$  is semialgebraic defined by the solutions of polynomial variational inequality  $\text{PVI}(K, q\nabla p - p\nabla q)$ . Indeed, since  $q^2 > 0$  on  $K$ , from (5.3) one has

$$\langle \nabla f(x), y - x \rangle \geq 0 \quad \text{iff} \quad \langle (q\nabla p - p\nabla q)(x), y - x \rangle \geq 0,$$



for all  $x, y \in K$ . The solution sets of the two variational inequalities  $\text{PVI}(K, \nabla f)$  and  $\text{PVI}(K, q\nabla p - p\nabla q)$  are coincident. Because  $q\nabla p - p\nabla q$  is a polynomial map,  $\text{Stat}(K, f)$  is semialgebraic.

The upper bound (5.2) for the dimension of  $\text{Stat}(K, f)$  is obtained by applying Theorem 3.3 for  $\text{PVI}(K, q\nabla p - p\nabla q)$ .  $\square$

EXAMPLE 5.2. Consider the unconstrained polynomial fractional optimization problem given by

$$f(x_1, x_2) = \frac{-1}{x_1^2 - 2x_1x_2 + x_2^2 + 1}.$$

The matrix  $Q(x)$  (given by (5.1)) of  $f$  is defined as follows:

$$Q(x) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

We see that  $\text{rank}(Q(x)) = 1$  for every  $x \in \mathbb{R}^2$ . The problem  $\text{OP}(K, f)$  satisfies the assumptions in Theorem 5.1, then  $\dim(\text{Stat}(K, f)) \leq 1$ . It is not difficult to show that  $\text{Stat}(K, f)$  is the solution set of  $\text{OP}(K, f)$  with

$$\text{Stat}(K, f) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 = 0\},$$

and its dimension is one.

COROLLARY 5.3. *Assume that the ACQ holds on  $K$ . If the rank of the Jacobian  $D\Psi_\alpha$  is  $n + |\alpha| + \ell$  on  $\mathbb{R}^n \times \mathbb{R}^{|\alpha|}$ , for every  $\alpha \subset [m]$ , then  $\text{Stat}(K, f)$  has finitely many points.*

*Proof.* Suppose that  $\text{rank}(D\Psi_\alpha) = n + |\alpha| + \ell$  for every  $\alpha \subset [m]$ . Applying Theorem 5.1 we obtain  $\dim(\text{Stat}(K, f)) \leq 0$ . This means that  $\text{Stat}(K, f)$  has finitely many points.  $\square$

COROLLARY 5.4. *Assume that the set  $K$  is polyhedral convex given by (3.6). If the function  $f$  is quadratic, i.e.,*

$$f(x) = x^T Mx + Bx + c,$$

where  $M \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^n$ , and  $c \in \mathbb{R}$ , then the dimension of  $\text{Stat}(K, f)$  does not exceed the number given by (5.2).

*Proof.* Because of  $f(x) = x^T Mx + Bx + c$ , the Hessian  $\nabla^2 f = M + M^T$  does not depend on  $x$ . The constraint set given by (3.6), it follows that  $\nabla g_i$  also does not depend on  $x$ . Hence, the constant rank condition in Theorem 5.1 is obviously satisfied. The assertion is a corollary of this theorem.  $\square$

COROLLARY 5.5. *Assume that  $K$  is polyhedral convex given by (3.6). If the function  $f$  is linear fractional, i.e.,*

$$f(x) = \frac{Ax + b}{Cx + d},$$

where  $A, C \in \mathbb{R}^n$ , and  $b, d \in \mathbb{R}$ , then the dimension of  $\text{Stat}(K, f)$  does not exceed the number given by (5.2).

*Proof.* Since  $f$  is linear fractional, the matrix  $Q$  in (5.1) defined by  $Q(x) = A^T C - AC^T$  which does not depend on  $x$ . We can now proceed analogously to the proof of Corollary 5.5, then the assertion be proved.  $\square$

REMARK 5.6. Suppose that the function  $f$  is convex on  $K$ . Then the stationary points are solutions of  $\text{OP}(K, f)$ , hence that all facts in the present section can be applied for the solution set of  $\text{OP}(K, f)$ .

**6. A Classification of Polynomial Variational Inequalities.** In this section, based on the dimensions of solution sets, a classification of the polynomial variational inequalities is shown. The thickness of these classes also is discussed.

Let  $d > 0$  be given integer. Here,  $\mathcal{P}_d$  stands for the linear space of all polynomials of degree at most  $d$ . The dimension of the space  $\mathcal{P}_d$  is denoted and defined by

$$\rho_{n,d} := \dim(\mathcal{P}_d) = \binom{n+d}{d}.$$

Let  $X$  be the  $\rho_{n,d}$ -vector consist of all monomials degree at most  $d$  which is listed by the lexicographic ordering

$$X := (1, x_1, x_2, \dots, x_n, x_1^2, x_1x_2, \dots, x_1x_n, \dots, x_1^d, x_1^{d-1}x_2, \dots, x_n^d)^T.$$

For every polynomial map  $Q = (Q_1, \dots, Q_n) \in \mathcal{P}_d^n$ , there exists a unique matrix  $A \in \mathbb{R}^{n \times \rho_{n,d}}$ ,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\rho_{n,d}} \\ a_{21} & a_{22} & \cdots & a_{2\rho_{n,d}} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{n\rho_{n,d}} \end{bmatrix},$$

such that  $Q(x) = AX$ .

For each  $k \in \{-\infty\} \cup [d]$ ,  $\mathcal{D}_k$  stands for the set of all matrices  $A \in \mathbb{R}^{n \times \rho_{n,d}}$  such that  $\dim(\text{Sol}(K, AX)) = k$ .

REMARK 6.1. Since  $\text{Sol}(K, \lambda Q) = \text{Sol}(K, Q)$  for any  $\lambda > 0$ ,  $\mathcal{D}_k$  is a cone in  $\mathbb{R}^{n \times \rho_{n,d}}$  provided that this set is nonempty. Clearly, we have a disjoint decomposition of  $\mathbb{R}^{n \times \rho_{n,d}}$  by as follows:

$$\mathbb{R}^{n \times \rho_{n,d}} = \mathcal{D}_{-\infty} \cup \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_n.$$

It is of interest to know how is  $\mathcal{D}_k$  thick (or big) in  $\mathbb{R}^{n \times \rho_{n,d}}$ . From [6, Theorem 8.2], we can say that  $\mathcal{D}_{-\infty} \cup \mathcal{D}_0$  is generic in  $\mathbb{R}^{n \times \rho_{n,d}}$ , i.e.  $\mathcal{D}_{-\infty} \cup \mathcal{D}_0$  contains a countable intersection of dense and open sets in  $\mathbb{R}^{n \times \rho_{n,d}}$ , provided that the constraint set satisfies the LICQ.

The following proposition says that the cone  $\mathcal{D}_n$  is trivial when the interior of the constraint set is nonempty.

PROPOSITION 6.2. *Assume that the interior of  $K$  is nonempty. The dimension  $\text{Sol}(K, F)$  is full if and only if  $F$  is the zero polynomial.*

*Proof.* The nonemptiness of the interior of  $K$  implies [1, Proposition 2.8.4] that  $\dim(K) = n$ . Clearly, if  $F$  is the zero polynomial then  $\text{Sol}(K, F) = K$ , and hence that  $\dim(\text{Sol}(K, F)) = n$ .

Suppose that the dimension  $\text{Sol}(K, F)$  is full. By definition of  $\dim(\text{Sol}(K, F))$ , there is a nonempty open semialgebraic set  $U$  such that  $U \subset \text{Sol}(K, F)$ . Because  $U$  is open,  $U$  must be contained in the interior of  $K$ . One has  $F(x) = 0$  for all  $x \in U$ . For every  $i \in [n]$ , the zero set of  $F_i(x)$  contains the nonempty open set  $U$ , hence that  $F_i \equiv 0$ . It follows that  $F$  is the zero polynomial.  $\square$

To illustrate the thickness of  $\mathcal{D}_k$  in  $\mathbb{R}^{n \times \rho_{n,d}}$ , we investigate a special case of  $K$  which is a box in  $\mathbb{R}^n$ .

THEOREM 6.3. *Let  $\delta > 0$  be given. Assume that the constraint set is given by*

$$K = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq \delta, i \in [n]\}.$$

*Then, for each  $k \in \{0, \dots, n\}$ ,  $\mathcal{D}_k$  is nonempty and contains a semialgebraic subset  $\mathcal{E}_k \subset \mathbb{R}^{n \times \rho_{n,d}}$  such that*

$$\dim(\mathcal{E}_k) = (n - k) \times \rho_{n-k,d}.$$

*Proof.* By the nonemptiness and the compactness of  $K$ ,  $\text{Sol}(K, Q)$  is nonempty for every polynomial map  $Q$ . This implies that  $\mathcal{D}_{-\infty}$  is empty.

We first consider the case  $k = 0$ . It is easy to check that  $\text{Sol}(K, F) = \{(0, \dots, 0)\}$  when  $F(x) = (1, \dots, 1)$ . Hence,  $\mathcal{D}_0$  is nonempty. As  $K$  satisfies the LICQ, according to [6, Theorem 8.2], there is a dense and open semialgebraic subset  $\mathcal{E}_0 \subset \mathbb{R}^{n \times \rho_{n,d}}$  such that  $\dim \text{Sol}(K, AX) \leq 0$  for all  $A \in \mathcal{E}_0$ . Because of  $\mathcal{D}_{-\infty} = \emptyset$ , one has  $\dim(\mathcal{E}_0) = n \times \rho_{n,d}$ , and the assertion holds for  $k = 0$ .

We second prove the assertion with  $k = n$ . Since the interior of  $K$  is nonempty, Proposition 6.2 says that  $\mathcal{D}_n = \{0\}$ . The set  $\mathcal{E}_n$  mentioned in the theorem is  $\{0\}$  with  $\dim(\mathcal{E}_n) = 0$ .

Let  $k$  be given with  $1 \leq k \leq n-1$ . We consider the variational inequality problems in  $n - k$  variables  $(x_1, \dots, x_{n-k})$ , where the constraint set  $K'$  given by

$$K' = \{(x_1, \dots, x_{n-k}) \in \mathbb{R}^{n-k} : 0 \leq x_i \leq \delta, i \in [n - k]\}.$$

Repeating the argument in the second paragraph, we can assert that there is a dense and open semialgebraic subset  $\mathcal{E}' \subset \mathbb{R}^{(n-k) \times \rho_{n-k,d}}$  such that  $\dim \text{Sol}(K', A'X') = 0$  for all  $A' \in \mathcal{E}'$ , where  $A' \in \mathbb{R}^{(n-k) \times \rho_{n-k,d}}$  and

$$X' := (1, x_1, x_2, \dots, x_{n-k}, x_1^2, x_1 x_2, \dots, x_1 x_{n-k}, \dots, x_1^d, x_1^{d-1} x_2, \dots, x_{n-k}^d)^T.$$

Suppose that  $A' \in \mathcal{E}'$  be given and  $Q'(x) := A'X'$ . We can choose the polynomial map  $Q$  as follows

$$Q_i = \begin{cases} Q'_i & \text{if } 1 \leq i \leq n - k, \\ 0 & \text{if otherwise.} \end{cases}$$

It is not difficult to check that

$$\text{Sol}(K, Q) = \text{Sol}(K', Q') \times [0, 1]^k,$$

i.e.,  $(\bar{x}_1, \dots, \bar{x}_{n-k})$  is a solution of  $\text{PVI}(K', Q')$  iff  $(\bar{x}_1, \dots, \bar{x}_{n-k}, x_{n-k+1}, \dots, x_n)$  is a solution of  $\text{PVI}(K, Q)$  for every  $(x_{n-k+1}, \dots, x_n) \in [0, 1]^k$ . Since both solution sets are semialgebraic, by applying [1, Proposition 2.8.5], one has

$$\dim(\text{Sol}(K, Q)) = \dim(\text{Sol}(K', Q')) + k = k.$$

The set  $\mathcal{E}_k$  can be defined as the set of all block matrix  $A$ , where

$$A = \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix},$$

with  $A' \in \mathcal{E}'$ . It is clear that  $\dim(\mathcal{E}_k) = \dim(\mathcal{E}') = (n - k) \times \rho_{n-k,d}$ .

The proof is complete.  $\square$

**7. Conclusions.** Since the solution set of a polynomial variational inequality is semialgebraic, its dimension is well-defined. Theorem 3.3 gives an upper bound for the dimension provided that the constant rank condition is satisfied. It is of interest to know whether the condition can be removed, or not. We are also interested in a sharp lower bound for that dimension. A classification of polynomial variational inequalities is shown in the last section. Theorem 6.3 discussed the thickness of these classes in the parametric space of polynomial maps when the constraint set is a box.

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