Splitting a Random Pie: Nash-Type Bargaining with Coherent Acceptability Measures

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We propose an axiomatic solution for cooperative stochastic games where risk averse players bargain for the allocation of profits from a joint project that depends on management decisions by the agents. We model risk preferences by coherent acceptability functionals and show that in this setting the axioms of Pareto optimality, symmetry, and strategy proofness fully characterize a bargaining solution, which can be efficiently computed by solving a stochastic optimization problem. Furthermore, we demonstrate that there is no conflict of interest between players about management decisions and characterize special cases where random payoffs of players are simple functions of overall project profit. In particular, we show that for players with distortion risk functionals, the optimal bargaining solution can be represented by an exchange of standard options contracts with the project profit as the underlying. We illustrate the concepts in the paper by a detailed example of risk averse households that jointly invest into a solar plant.

Key words: Stochastic bargaining games; coherent risk measures; stochastic programming; photovoltaics

History:

1. Introduction

If a project is undertaken jointly by several agents and there is no efficient market for individual contributions, the question of how to distribute the project’s profits arises. The case when profits are deterministic is well studied in the field of cooperative game theory. Solution approaches include bargaining games and coalitional games.

In this paper, we focus on bargaining games, i.e., settings where agents bargain for shares of the profits of the grand coalition and cannot form sub-coalitions. There is a rich literature on theoretical properties of these games (e.g., Osborne and Rubinstein 1994, Muthoo 1998, Maschler et al. 2013) and bargaining situations are ubiquitous in economics and management. Examples include, but are not limited to, bankruptcy problems where debtors bargain for shares of the collateral (e.g., Bulow and Rogoff 1989, Dagan and Volij 1993, Stutzer 2018), individual and collective

Standard bargaining approaches assume deterministic profits for the grand coalition. However, most business decisions are characterized by uncertainty of outcomes and are taken by agents who are risk averse.

In this paper, we therefore consider the situation of multiple risk averse players who undertake a project together and bargain for the allocation of the resulting random profits. In particular, we investigate a general class of bargaining games under uncertainty where agents have risk preferences that can be expressed by coherent acceptability functionals. Based on a set of axioms, we characterize a unique distribution of acceptability values which can be efficiently computed by solving a stochastic optimization problem. Furthermore, we show that if all agents use distortion functionals, the split-up of profits can be represented by a standard call and put options contracts, which facilitates the real world implementation of the optimal allocation. We demonstrate the applicability of our approach in an example considering the joint investment in solar panels on an apartment house owned by several parties.

In the following, we briefly review the literature on stochastic cooperative game theory and point out the differences to our approach.

To the best of our knowledge, this paper is the first attempt to formulate and solve bargaining problems where project outcomes are uncertain and profits are split ex-ante. Most authors working in cooperative game theory use different solution approaches such as envy free allocations (Fragnelli and Marina 2003), maximal overall utility (Melese et al. 2017), or, most prominently, membership in the core of a coalitional game (Habis and Herings 2011, Xu and Veinott 2013, Uhan 2015, Toriello and Uhan 2017, Németh and Pintér 2017, Asimit and Boonen 2018). In contrast to our approach, which only requires the specification of the problem for the grand coalition and the reservation value of the single players, in stochastic coalitional games the payoffs of every sub-coalition have to be known.

Another difference of our approach to the extant literature is that almost all papers in stochastic cooperative game theory assume risk neutral decision making, neglecting the impact of risk aversion
on the outcome of the game (e.g., Charnes and Granot 1976, 1977, Xu and Veinott 2013, Parilina and Tampieri 2018). Among the authors that consider risk preferences, most do not consider coherent risk measures but other forms of risk quantification – most notably expected utility (Suijs et al. 1999, Fragnelli and Marina 2003, Timmer et al. 2005, Habis and Herings 2011, Melese et al. 2017, Németh and Pintér 2017). There are only a handful of papers that use coherent risk measures in cooperative games (e.g., Uhan 2015, Boonen et al. 2016, Toriello and Uhan 2017, Asimit and Boonen 2018).

Additionally, most papers focus on theoretical properties such as existence and stability of solutions, but do not derive an explicit split up of random profits (e.g., Habis and Herings 2011, Xu and Veinott 2013, Németh and Pintér 2017, Parilina and Tampieri 2018).

The papers that come closest to our approach are Uhan (2015) and Toriello and Uhan (2017), who study a special class of linear programming coalitional games, i.e., games where the payoff of players derives from the value that they have in sub-coalitions backing threats to break away from the grand coalition. The authors model risk aversion using coherent risk measures and similar to our approach derive an ex-ante split-up of random profits. The authors define the profits for all sub-games and show that the core of the game is non-empty. Apart from the fact that Uhan (2015) and Toriello and Uhan (2017) study coalitional games instead of bargaining games, these papers differ from our approach in that they impose a rather special separable, linear structure to ensure that the game can be decomposed into games played by sub-coalitions. Furthermore, the authors make limiting assumptions on the employed risk measures and probability spaces. In contrast, our approach makes virtually none of these assumptions and therefore is more widely applicable.

This paper thus contributes to the literature in the following ways:

1. We propose the first solution for bargaining games where the project profit is stochastic and risk averse players have to agree ex-ante, i.e., before outcomes are known, about a allocation of project profits in every possible future scenario.

   Our solution rests on an axiomatic framework that can be seen as a natural generalization of deterministic Nash bargaining with bargaining powers to stochastic bargaining games where players’ risk aversions are described by coherent acceptability functionals. We note that in our approach, the bargaining powers of the players follow canonically from the axioms and do not have to be determined by additional arguments outside the axiomatic foundation of the bargaining problem.

2. We show that under mild assumptions our axioms yield a unique allocation of acceptability values which can be efficiently computed as a solution of a stochastic optimization problem. In particular, our approach fully integrates with two-stage and multi-stage stochastic programming in that it does not only yield allocations of profits and acceptability values but allows for the optimization of auxiliary decisions that have to be taken in order to manage the project.
3. We propose a detailed example of a bargaining game between the owners of an apartment building who install solar panels on the roof of their jointly owned house. The question of stimulating the build-up of renewable capacity is central in the combat of climate change, and the complicated ownership structures of apartment buildings are a significant impediment to an extension of privately owned solar power beyond single-family houses. The results of the case study, in particular the findings on optimal tilt angles under risk aversion, are therefore of independent interest to a large community of researchers.

4. Our results have several managerial implications for negotiations between risk averse agents, which are of interest not only for the computation of optimal outcomes but also as guidance for negotiators involved in bargaining processes.

(a) We show that there is no conflict of interests between players on how to manage the project. Thus managerial decisions are not strategic implying that project management can conceptually, temporary, and organizationally be decoupled from the process of bargaining for the allocation of profits. This non-trivial result helps to structure real world bargaining processes.

(b) The proposed bargaining outcomes are fair and are likely to be accepted by participants. In particular, we show that the random shares of participants are comonotone, i.e., that for optimal allocations all individual profits rise with project profits and there are no situations where the gain of one player is the loss of another. Furthermore, we derive a simple and comprehensible rule for the distribution of overall acceptability of the project which takes into account the size of the contributions and therefore leads to transparent and fair allocations.

(c) Our results show that a simple ex-post proportional split up of project profits, as advocated in large parts of the literature and often used in practice, can only be optimal in degenerate cases. However, we show that under mild conditions on risk preferences, optimal allocations take the form of an exchange of standard option contracts between the players. This structure is conducive to implementing the calculated \textit{ex-ante} allocation of profits in contractual agreements and thereby facilitates the real world implementation of the bargaining outcome and allows negotiators to focus on agreements of this type, which helps to structure and simplify negotiations.

The paper is structured as follows: In Section 2, we introduce a stochastic bargaining game with a fixed set of players in which agents agree on a allocation of payoffs before the randomness realizes. The solution thus entails a distribution of profits in every possible future scenario. We impose the axioms of Pareto optimality, symmetry, and strategy proofness for a solution. While the symmetry axiom encapsulates a notion of fairness that makes our approach similar to Nash
bargaining, strategy proofness avoids the problem of strategic splitting of agents. We argue that these axioms are natural in our context and show that they uniquely characterize a bargaining solution.

In Section 3, we analyze the resulting bargaining problems using results about the sup-convolution of coherent risk measures. In particular, we obtain necessary and sufficient conditions for the existence of a solution in terms of the subgradient sets of the players’ coherent risk functionals.

We show that affine and linear allocation rules as used, e.g., in Deprez and Gerber (1985), Suijs et al. (1999), Barrieu and El Karoui (2005), Timmer et al. (2005), Baeyens et al. (2013) can only be optimal, if there is a least risk averse player. Furthermore, we prove that in the much more realistic case where the risk preferences of all agents can be described by distortion functionals, the exchange of finitely many standard option contracts achieves the optimal allocation of profits.

In Section 4, we use the problem of jointly constructing a solar roof for a shared building as an illustrative example for the proposed methods, while Section 5 concludes the paper and discusses avenues for further research.

All proofs not contained in the main document are relegated to the appendix.

2. The Bargaining Problem

In this section, we introduce an axiomatic framework for a class of bargaining problems under uncertainty. Section 2.1 outlines the general setting, introduces notation, and discusses our choice of risk preferences. In Section 2.2, we introduce and motivate three axioms for risk averse bargaining under uncertainty and show that our axiomatic framework uniquely characterizes a bargaining solution. Lastly, we show that the bargaining solution can be represented as the solution to a stochastic optimization problem, which generalizes classical Nash bargaining with bargaining power to stochastic games.

2.1. Motivation & Setting

We consider a project that yields random profits and is jointly undertaken by $n$ risk averse players. All random quantities are defined on a common probability space $\mathcal{Y} = (\Omega, \mathcal{F}, P)$.

Each participant $i \in N := \{1, \ldots, n\}$ obtains a profit $R_i : \Omega \to \mathbb{R}$ when playing alone and not as a member of the group, while the whole group of $n$ players that cooperates receives a joint random profit of $M : \mathcal{X} \times \Omega \to \mathbb{R}$. $M$ may depend on joint managerial decisions $x \in \mathcal{X}$, where $\mathcal{X}$ is the set of feasible decisions. For a fixed $x \in \mathcal{X}$, we denote by $M_x : \Omega \to \mathbb{R}$ the random variable $\omega \mapsto M(x, \omega)$. Positive $M$ and $R_i$ indicate profits, whereas negative values model losses.
When cooperating, the group decides on $x \in X$ and aims at distributing the joint outcome $M_x$ among the participants in a “fair way”, which leads to individual (random) payoffs $L(x, \omega) = (L_1(x, \omega), \ldots, L_n(x, \omega))$ such that

$$M(x, \omega) = \sum_{i=1}^{n} L_i(x, \omega), \text{ a.s.}$$

(1)

It is worthwhile to point out that in the above setting, the allocation of profits cannot be decided ex-post, i.e., after the uncertainty realizes. This is made clear by the following example.

**Example 1.** Let there be two risk averse players and two equally likely future states of the world, i.e., $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2\}$, and $P(\{\omega_1\}) = P(\{\omega_2\}) = 0.5$. Assume further that $R_1(\omega_j) = -R_2(\omega_j)$ and $R_j(\omega_1) = -R_j(\omega_2)$ for $j \in \{1, 2\}$. If the players are risk averse, they would prefer a certain payment of 0 to their endowments and thus have an incentive to agree to pool their risks by defining a payout rule

$$L_i(\omega) = \frac{1}{2}(R_1(\omega) + R_2(\omega)), \forall i \in N,$$

(2)

which yields a certain profit of 0 for both players in both states of the world.

Note that, in agreeing to (2), the players shift wealth between future states to achieve a better risk profile ex-ante, much like when entering an insurance contract. They are, however, not necessarily better off ex-post, since in each of the two states, there is one player who regrets having pooled her risks. For this reason, an ex-post allocation which is based on the allocation of wealth after the realization of randomness cannot be ex-ante optimal, as players would not agree to shift wealth between the scenarios after the randomness realized. Hence, in the above example an ex-post allocation would result in the same pay-offs as a solution without any cooperation and therefore in ex-ante welfare losses.

Consequently, we have to consider the risk preferences of agents in deciding about random ex-ante payoffs $L$. We assume that the risk preferences of player $i$ can be expressed by a coherent acceptability functional $A_i$ defined on the Lebesgue space $L^p(\mathcal{Y})$ of $p$-integrable random variables. Throughout this paper we will assume that $p \geq 1$.

**Definition 1.** $A : L^p(\mathcal{Y}) \to \mathbb{R}$ is a coherent acceptability functional if for all $X, Y \in L^p(\mathcal{Y})$, $\lambda \in \mathbb{R}^+_+$, and $c \in \mathbb{R}$ the following holds:

1. Monotonicity: If $X \leq Y$ almost surely, then $A(X) \leq A(Y)$.
2. Positive homogeneity: $A(\lambda X) = \lambda A(X)$.
3. Super-additivity: $A(X + Y) \geq A(X) + A(Y)$.
4. Translation invariance: $A(X + c) = A(X) + c$. 
While economists traditionally prefer to think about risk in terms of expected utility, coherent acceptability functionals have recently gained popularity because of their ease of interpretation and analytical tractability. In particular, coherent acceptability measures can be interpreted in terms of monetary units, which is why they sometimes also called monetary utility functions (Föllmer and Schied 2004, Pflug and Römisch 2007).

In this paper, we assume that projects can only be undertaken jointly by all players and forming sub-coalitions is either legally, organizationally, or practically impossible. In particular, we consider a bargaining setup, where players cannot obtain a higher share of $M$ based on the threat of forming sub-coalitions.

2.2. An Axiomatic Approach

This subsection develops an axiomatic theory of bargaining solutions, which is close to the Nash bargaining approach, but, as opposed to classical theory, allows profits $L_i$ and opportunity losses $R_i$ to be random and players to be risk averse.

**Definition 2.** An instance of the bargaining problem is a triple $(R, M, A)$ with $R = (R_1, \ldots, R_n)$, $R_i \in L^p(Y)$ ($i \in N$), $M : X \to L^p(Y)$, and $A = (A_1, \ldots, A_n) : \bigotimes_{i=1}^n L^p(Y) \to \mathbb{R}^n$ where $A_i : L^p(Y) \to \mathbb{R}$ are coherent acceptability measures for all $i \in N$.

For a given instance $(R, M, A)$, we define the set of feasible acceptability allocations that exceed the acceptability of the endowment for each player as

$$U(R, M, A) = \{ u \in \mathbb{R}^n | \exists x \in X \exists L : \sum_{i=1}^n L_i = M_x, L_i \in L^p(Y), A_i(R_i) \leq u_i \leq A_i(L_i), i \in N \}. \quad (3)$$

We assume that the values $v_i := A_i(R_i)$, representing the personal evaluations of the opportunity losses, are always strictly positive. The instance $(R, M, A)$ is called feasible if the set $U(R, M, A)$ is non-empty. A sufficient condition for feasibility of an instance $(R, M, A)$ is $\sum_{i=1}^n R_i \leq M_x$ almost surely for some $x$ (see Corollary 1, Equation (18) for a weaker condition that is also necessary). We restrict ourselves to feasible instances, i.e., to cases where cooperation has a non-negative value for the players.

**Definition 3.** A bargaining solution is a mapping $F$ that assigns a vector $u = (u_1, \ldots, u_n) \in U(R, M, A)$ of acceptability values to each feasible instance $(R, M, A)$.

In the following, we discuss three axioms and argue why they should be fulfilled in the context of the bargaining games outlined in this paper. The first axiom is Pareto optimality, which is a basic efficiency requirement also used in the classical Nash bargaining approach.

**Axiom 1 (PAR).** The acceptability allocation $u^* = F(R, M, A)$ prescribed by the bargaining solution is Pareto optimal. In particular, if $u \in U(R, M, A)$ is another acceptability allocation, then

$$\exists i \in N : u_i > u^*_i \Rightarrow \exists j \in N : u_j < u^*_j.$$
Clearly, any allocation that does not fulfill PAR could be improved, at least for some players, without making anybody else worse off. Choosing such an allocation is obviously wasteful and therefore undesirable.

The next axiom is symmetry, which captures the essence of the notion of fairness employed by the Nash bargaining approach. The axiom requires that if all players are indistinguishable in every relevant aspect of the game, they should obtain the same acceptability value. More formally, we can state the symmetry axiom as follows.

**Axiom 2 (SYM).** If \( A_i(R_i) = A_j(R_j) \) for all \( i, j \in N \), and if for every permutation \( \sigma \) of \( N \),

\[
(u_1, \ldots, u_n) \in U(R, M, A) \Rightarrow (u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \in U(R, M, A),
\]

then \( u^*_i = u^*_j \) for all \( i, j \in N \).

Lastly, we add an axiom that ensures what is sometimes called strategy proofness of the allocation rule. The idea is that players should not benefit from strategically splitting up or merging. This is made sure by requiring that larger investments (larger opportunity losses) should entail larger shares of the profit. Similar ideas have been applied in different forms in the literature, e.g., in the concept of the proportional Shapley value (Feldman et al. 1999, Béal et al. 2018), but also in the general context of (deterministic) allocation rules, see (see Hougaard 2009, Chapter 2).

More formally, we require the following to hold.

**Axiom 3 (STR).** If \( (R', M, A') \) is a game associated with \( (R, M, A) \) by \( N' = N \setminus \{i\} \cup \{(i, k) \mid k = 1, \ldots, \ell\} \) and

\[
R'_j = R_j \quad \forall j \neq i, \quad R'_{ik} = R_{i}/\ell \quad (k = 1, \ldots, \ell),
\]

\[
A'_j = A_j \quad \forall j \neq i, \quad A'_{ik} = A_i \quad (k = 1, \ldots, \ell),
\]

then the acceptability allocations \( u = F(R, M, A) \) and \( u' = F(R', M, A') \) are interrelated by

\[
u'_{jk} = u_j \quad \forall j \neq i, \quad u'_{ik} = u_i/\ell \quad (k = 1, \ldots, \ell).
\]

Thus, in the new game, where player \( i \) “splits herself” into \( \ell \) subplayers, splitting also her opportunity losses in equal parts, the benefits should be split as well. Note that, because of positive homogeneity of the acceptability measures \( A_i \), it makes no difference whether we split up \( L_i \) or the assigned acceptability \( u_i = A_i(L_i) \).

**Remark 1.** Hougaard (2009) uses a similar axiom called No Advantageous Splitting (NAS), which is stronger than STR as it considers the split up of players into parts of arbitrary size. The paper develops an axiomatic theory in a purely deterministic context, whereas our focus is on the stochastic situation where players use coherent risk measures. As a consequence of the deterministic modeling, Hougaard (2009) can assume additivity of the \( u_i \), which is not possible in our context, because the acceptability measures \( A_i \) are generally not linear. For this reason, we only assume
the weaker axiom STR. Our results therefore generalize the results in Hougaard (2009) to the case of coherent acceptability functionals and consequently require a technically more involved analysis to make use of STR in the axiomatic characterizations of allocations (compare, e.g., Theorem 1 below with Theorem 2.5 in Hougaard 2009).

In the following, we give a characterization of those bargaining solutions that satisfy Axioms 1–3. To this end, we will require the following assumption, which we assume to hold for the rest of the paper.

**Assumption 1.** The set $\mathcal{X}$ is a compact topological space and $x \mapsto M_x$ is a continuous mapping from $\mathcal{X}$ to $\mathcal{L}^p(\mathcal{Y})$ equipped with the weak topology $\sigma$.

Using this assumption, we are able to show the following technical lemma, which ensures that there is always a solution to every feasible bargaining game.

**Lemma 1.** Under Assumption 1, the set $\mathcal{U}(R, M, A)$ is compact.

Building on Lemma 1, we prove that $\mathcal{U}$ is a convex polyhedron in the next result.

**Lemma 2.** The set $\mathcal{U}(R, M, A)$ is the polytope

$$\{u \in \mathbb{R}^n \mid \forall i \in N : u_i \geq A_i(R_i), \sum_{i=1}^{n} u_i \leq z\}$$

with

$$z = z(R, M, A) = \max_{u, L, x} \left\{ \sum_{i=1}^{n} u_i \mid \sum_{i=1}^{n} L_i = M_x, A_i(R_i) \leq u_i \leq A_i(L_i), \forall i \in N, x \in \mathcal{X} \right\}.$$  

Before stating the main result of this section, we need the following additional technical lemma.

**Lemma 3.** With $v_i = A_i(R_i)$ for all $i \in N$ and $v = (v_1, \ldots, v_n)$, the function $v \mapsto z(R, M, A)$ is continuous.

We are now in the position to show that Axioms 1–3 characterize a unique acceptability allocation.

**Theorem 1.** A bargaining solution $F$ satisfies Axioms 1–3 if and only if for every instance $(R, M, A)$ the acceptability allocations $u = F(R, M, A)$ are given by

$$u_i = \frac{v_i}{\sum_{j=1}^{n} v_j} \cdot z(R, M, A), \quad \forall i \in N,$$

where $z$ is defined by (6) and $v_i = A_i(R_i)$ for all $i \in N$. 


Proof. Let $F$ be a bargaining solution that satisfies Axioms 1 – 3. For $u^* = F(R,M,A) \in U$, we must have $\sum_{i=1}^n u^*_i = z$ with $z = z(R,M,A)$ from (6), since $\sum_{i=1}^n u^*_i < z$ would be in contradiction to PAR. This shows that $u^* = F(R,M,A)$ must be an element of

$$U_0 = \{ u \in \mathbb{R}^n \mid u_i \geq v_i, \forall i \in N, \sum_{i=1}^n u_i = z \}.$$

To show that (7) holds first assume $v_i \in Q$ for all $i$, i.e., $v_i = p_i/q$ with $q \in \mathbb{N}$ and $p_i \in \mathbb{N}$ for all $i \in N$. Define an instance $(R',M,A')$ by splitting each player $i$ into $p_i$ players $(i,k)$ with $k = 1, \ldots, p_i$, $i \in N$, as in Axiom STR. Then all players have identical values $v'_{ik} = A_{ik}'(R'_{ik})$, since, by the positive homogeneity of the $A_i$,

$$A_{ik}'(R'_{ik}) = A_i(R_i/p_i) = \frac{1}{p_i} A_i(R_i) = \frac{v_i}{p_i} = \frac{1}{q}.$$

Successive application of Axiom STR for $i = 1, \ldots, n$ yields

$$u'_{ik} = \frac{u_i^*}{p_i} \quad \forall k = 1, \ldots, p_i, \forall i \in N.$$

Because of the identical values $v'_{ik}$, the feasible set $U'$ of $(R',M,A')$ is invariant under permutations of the coordinates. Thus both conditions of Axiom SYM are satisfied, and we can conclude that all players get the same acceptability $u'_{ik} = z \cdot \left( \sum_{j=1}^n p_j \right)^{-1} = z$ and hence

$$u_i^* = p_i \cdot z = \frac{p_i}{\sum_{j=1}^n p_j} \cdot z = \frac{v_i}{\sum_{j=1}^n v_j} \cdot z$$

as required.

For $v_i \notin Q$, the assertion follows because of the continuity of the functions on the right hand side of (7) by a straightforward convergence argument, representing $v_i \in \mathbb{R}$ as limits of rational numbers. The continuity of the quotient $v_i/\sum_j v_j$ is clear, while the continuity of $z(R,M,A)$ is proved in Lemma 3.

To show the converse, we assume that the bargaining solution $F$ satisfies (7) and show that Axioms (1) – (3) are satisfied.

Axiom PAR: Suppose that, contrary to PAR, the acceptability allocation $u^* = F(R,M,A) \in \mathcal{U}(R,M,A)$ is dominated by some other $u \in \mathcal{U}(R,M,A)$, i.e., $u_i > u_i^*$ for some $i$ and $u_j \geq u_j^*$ for all $j \in N$. Then

$$\sum_{j=1}^n u_j > \sum_{j=1}^n u_j^* = z(R,M,A),$$

which contradicts $u \in \mathcal{U}(R,M,A)$ because of (5).

Axiom SYM: In view of (5), the set $\mathcal{U}(R,M,A)$ can only be invariant under permutations of coordinates, if $v_i = v_j \forall i,j \in N$. In this case, (7) implies identical utilities $u_i = z/n \forall i \in N$. 

Axiom STR: Let the conditions of STR be satisfied for the two instances \((R, M, \mathcal{A})\) and \((R', M, \mathcal{A}')\). Using positive homogeneity and translation invariance, it is easy to see that \(z = z'\) for the corresponding solution values of (6). Then,

\[
v_i' = v_i \quad (j \neq i), \quad v_{ik}' = \frac{v_i}{\ell} \quad (k = 1, \ldots, \ell),
\]

where the last equality follows by means of the positive homogeneity of the \(A_i\). Therefore (7) implies

\[
u_j' = \frac{v_j}{\sum_{s \neq j} v_s + \sum_{s=1}^{\ell} \frac{v_s}{\ell}} \cdot z = u_j \quad \forall j : j \neq i
\]

and

\[
u_{ik}' = \frac{v_i}{\sum_{s \neq i} v_s + \sum_{s=1}^{\ell} \frac{v_s}{\ell}} \cdot z = u_{ik} \quad \forall k : k = 1, \ldots, \ell,
\]

as claimed by Axiom STR.

We call a bargaining solution \(F\) satisfying Axioms 1 – 3 a Nash-type bargaining solution or Nash bargaining solution, for short. The next theorem provides the justification for the terminology by showing that the bargaining solution \((u, L, x)\) can be obtained by solving a stochastic optimization problem with a product-form objective function similar to classical Nash bargaining without going through the stepwise procedure of first solving (6) and then applying Theorem 1 to find optimal \(u_i\).

**Theorem 2.** If the bargaining solution \(F\) fulfills Axioms 1 – 3, \(u = F(R, M, \mathcal{A})\) is the optimal solution to the stochastic optimization problem

\[
\max_{u, L, x} \prod_{i=1}^{n} (u_i - v_i) v_i \\
\text{s.t.} \quad u_i \geq v_i, \quad \forall i \in N \\
\quad u_i \leq \mathcal{A}_i(L_i), \quad \forall i \in N \\
\quad \sum_{i=1}^{n} L_i \leq M x \quad \text{almost surely,} \\
\quad x \in X.
\]

**Proof.** By monotonicity of the objective function and Lemma 2, the feasible set in (8) can be restricted to those \(u\) that satisfy \(\sum_{i=1}^{n} u_i = z\) where \(z = z(R, M, \mathcal{A})\). Except in the trivial boundary case where \(z = \sum_j v_j\), a feasible solution with \(u_i > v_i\) for all \(i\) exists. Taking the logarithm of the objective function in (8), we arrive at the following problem

\[
\max_u \sum_{i=1}^{n} v_i \log(u_i - v_i) \\
\text{s.t.} \quad u_i \geq v_i, \quad \forall i \in N \\
\quad \sum_{i=1}^{n} u_i = z.
\]

Obviously, this is a convex optimization problem, as the objective function to be maximized is concave and the constraints are linear. The Lagrange function of (9) without the inequality constraint reads

\[
\mathcal{L} = \sum_{i=1}^{n} v_i \log(u_i - v_i) - \lambda \left(\sum_{i=1}^{n} u_i - z\right).
\]
The first order condition with respect to \( u_i \) therefore yields
\[
    u_i = \frac{1 + \lambda}{\lambda} \cdot v_i, \quad \forall i \in N. \tag{10}
\]
Because of \( \sum_i u_i = z \), it follows that
\[
    \frac{1 + \lambda}{\lambda} = \frac{z}{\sum_{i=1}^{n} v_i} \geq 1 \tag{11}
\]
and therefore \( u_i \geq v_i \) is fulfilled. Plugging (11) into (10) yields
\[
    u_i = \frac{v_i}{\sum_j v_j} \cdot z,
\]
which is the solution identified by Theorem 1.

Remark 2. In Theorem 2, the parameter \( x \) is one of the decision variables the players have to decide about. However, as Theorem 1 shows, the case where \( x \) can be negotiated by the players does not add any complexity to the bargaining problem itself: Since the dependence on \( x \) only impacts the factor \( z(R, M, A) \), the optimization with respect to \( x \) can be fully decoupled from the bargaining process. In particular, there is no conflict of interest between the players over managerial decisions, since \( x \), loosely speaking, only influences the absolute size of the pie, i.e., the size of the set \( \mathcal{U} \), but not the relative shares.

This parallels Suijs and Borm (1999), Uhan (2015), Toriello and Uhan (2017) who show a similar result for allocations in the core of stochastic coalitional games between risk averse players.

Remark 3. Note that the solution to (8) need not be unique in \( L \) or \( x \), i.e., it can happen that different choices of \( L \) and/or \( x \) produce the same optimal solution \( u \). Hence, uniqueness of the bargaining solution can only be guaranteed on the level of the acceptability values \( u \).

Problem (8) is similar to the classical Nash bargaining solution adapted to our context. Apart from the stochasticity of the game entering (9) through the calculation of \( z = z(R, M, A) \), the only structural difference are the exponents \( v_i \) in the objective. This modification ensures that Axiom 3 is respected, which implies that the opportunity loss of agent \( i \) can be interpreted as her bargaining power. If agent \( i \) splits into several agents, the bargaining power decreases and the resulting new agents collectively get the same as \( i \) gets in the original game.

Beginning with Roth (1979) and Binmore (1980) there is a large literature on Nash bargaining with bargaining power. Bargaining power in this literature is represented by nonnegative numbers \( \gamma_i \) which are used as exponents in the objective function \( \prod_{i=1}^{n} (u_i - v_i)^{\gamma_i} \). In many papers on bargaining power \( \gamma_i \) are chosen more or less arbitrary. A notable exception is Binmore et al. (1986), where bargaining power is related to the bargainer’s time preferences and to the risk of a breakdown of negotiations in dynamic bargaining models. In our setup, bargaining power (as well as the
opportunity loss) is related to the acceptability $A_i(R_i)$ or lost opportunities, which is a natural point of view for investment problems.

We conclude with a discussion of the two axioms invariance with respect to affine transformations (INV) and invariance with respect to irrelevant alternatives (IIA), which are imposed in classical Nash bargaining but are absent from our approach.

We do not impose INV, since coherent risk functionals are not closed with respect to affine transformations, i.e., it does not make sense to talk about a bargaining problem with modified acceptability functionals, say, $a_iA_i(\cdot) + b_i$ (with $a \geq 0$ and $b \in \mathbb{R} \setminus \{0\}$), since the transformed functionals are not homogeneous, even if $A_i$ are homogeneous. Moreover, strategy proofness cannot be maintained simultaneously with INV: As seen from (7), in the presence of PAR and SYM, the axiom STR entails a nonlinear dependence of the values $u_i$ on the parameters $v_i$, whereas INV prescribes a linear dependence (Muthoo 1999).

We also note that IIA – the most controversial axiom in Nash bargaining (see e.g. Luce and Raiffa 1957, Hansson 1969, Tversky 1972, Kalai and Smorodinsky 1975, Saari 2006, for typical objections and counterexamples) – is not needed in our setup: The role of IIA in the classical theory is to deal with nonlinear boundaries of the bargaining set but the set of feasible acceptability values $U(R, M, A)$ has a linear border (see Lemma 2).

3. Characterizing the Optimal Allocation

So far, the solution of the bargaining game was described in terms of (6), which in consequence allowed to fully characterize the related unique acceptability allocation $u$. In the following section, we study the optimal allocation $L$ and the decision $x$. In Section 3.1, we characterize the optimal allocation of the bargaining problem using results on the sup-convolution of coherent acceptability functionals. In Section 3.2, we discuss special cases and show that affine allocations can only be optimal if there is a dominating acceptability functional. Moreover, we establish that if all $A_i$ are distortion functionals, then the form of the solution turns out to be particularly suitable for practical applications.

3.1. Characterizing the optimal allocation

We start our analysis of problem (6) by introducing the following additional running assumption.

**Assumption 2.** The coherent acceptability functionals $A_i$ are proper and upper semicontinuous. From this assumption it follows that the $A_i : \mathcal{L}^p(\mathcal{Y}) \to \mathbb{R}$ have dual representations (e.g. Pflug and Römisch 2007, Theorem 2.30)

$$A_i(X) = \inf \{E[\zeta_i X] : \zeta_i \in \mathcal{Z}_i \}, \quad (12)$$
where $Z_i$ are convex sets in $\mathcal{L}^q(\mathcal{Y})$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p \geq 1$ and
\[
Z_i \subseteq \{ \zeta_i \in \mathcal{L}^q(\mathcal{Y}) : \zeta_i \geq 0 \land E[\zeta_i] = 1 \}. \tag{13}
\]

Consequently, $A_i$ is the support function $\sigma_{Z_i}$ of $Z_i$ defined by the right hand side of (12) and is therefore completely characterized by the set $Z_i$. Assumption 2 is fulfilled for all relevant coherent acceptability functionals and guarantees that the infimum in (12) is attained.

To analyze (6), we note that the maximum sum of acceptability values is the sup-convolution $\bigstar_{i=1}^n A_i(M)$ of the individual $A_i$. In particular, for a given $M \in \mathcal{L}^p(\mathcal{Y})$, we write
\[
A_{\text{max}}(M) := \left( \bigstar_{i=1}^n A_i \right)(M) = \sup \left\{ \sum_{i=1}^n A_i(X_i) \in \mathcal{L}^p(\mathcal{Y}) : \sum_{i=1}^n X_i = M \right\}. \tag{14}
\]

We note that optimal risk sharing problems like (14) were studied by many authors in the mathematical finance and insurance literature starting in the context of classical insurance risk theory with Borch (1962), Bühlmann (1970), Gerber (1979), and subsequently in the context of convex analysis and often based on the properties of the inf-, respectively sup-convolution (e.g., Barrieu and El Karoui 2005, Jouini et al. 2008, Ludkovski and Rüschendorf 2008, Acciaio 2009, Acciaio and Svindland 2009). An excellent account of this topic can be found in Rüschendorf (2013).

With these preparation the next theorem summarizes important facts about the sup-convolution of coherent acceptability functionals.

**Theorem 3 (Sup-Convolution of Acceptability Functionals).** If
\[
Z_{\text{max}} = \bigcap_{i=1}^n Z_i \neq \emptyset, \tag{15}
\]
then:

1. The domain of $A_{\text{max}}$ is not empty and we have the following dual representation
\[
A_{\text{max}}(M) = \inf_{\zeta \in \mathcal{L}^q} \{ E[\zeta M] : \zeta \in Z_{\text{max}} \}, \quad \forall M \in \mathcal{L}^p(\mathcal{Y}). \tag{16}
\]
2. $A_{\text{max}}$ is a proper coherent upper semi-continuous acceptability functional.
3. An allocation $X = (X_1, \ldots, X_n)$ is optimal for (14) if and only if there is a $\Lambda \in \mathcal{L}^q(\mathcal{Y})$ with
\[
\Lambda \in \partial A_i(X_i) = \arg \min \{ E[\zeta_i X_i] : \zeta_i \in Z_i \}, \quad \forall i \in N. \tag{17}
\]

Conversely, for any $\Lambda \in \arg \min_{\zeta \in \mathcal{L}^q} \{ E[\zeta X] : \zeta \in Z_{\text{max}} \}$ obtained from (16) an optimal allocation $X$ can be found such that (17) is fulfilled.
4. If \( p > 1 \), the supremum in (14) is attained and there exists an optimal allocation \( X \). Under the stronger condition (and after suitable reordering) \( Z_1 \cap \bigcap_{i=2}^{n} \text{int} Z_i \neq \emptyset \), this is also true for \( p = 1 \).

5. The elements of an optimal allocation \( X = (X_1, \ldots, X_n) \) are comonotone random variables.

Note that the comonotonicity of the allocation implies that the profits of all players increase as a function of the optimal project profit \( M \), i.e., it cannot be that gain of one player leads to a loss of another. This property is arguably desirable because it enhances the perceived fairness of the solution and therefore its acceptance amongst players.

Condition (17) clearly resembles standard optimality conditions often found in economics that require the marginal utilities of all players to coincide for an optimal allocation of goods. It is also similar to the conditions found for competitive Nash equilibria in markets where agents have coherent risk preferences (e.g., Heath and Ku 2004, Ralph and Smeers 2015, Philpott et al. 2016). Furthermore, Toriello and Uhan (2017) find a similar condition for the solution of a linear stochastic coalitional game.

To relate the results in Theorem 3 to the bargaining games introduced in Section 2, we first consider the case where \( M \) does not depend on auxiliary decisions \( x \). It turns out that one may obtain the maximum \( z \) by solving the sup-convolution problem and that the optimal allocation \( L \) for the bargaining problem can be easily constructed from the optimal allocation \( X \) in (14).

**Corollary 1.** Let \( M : \Omega \to \mathbb{R} \) be the profit of the grand coalition and \( X = (X_1, \ldots, X_n) \) an optimal allocation for the sup-convolution problem (14). If

\[
A^\max(M) \geq \sum_{i=1}^{n} A_i(R_i),
\]

then

\[
z(R, M, A) = A^\max(M)
\]

and the allocation

\[
L_i = X_i + \frac{v_i}{\sum_{j=1}^{n} v_j} A^\max(M) - A_i(X_i)
\]

is both optimal for (14) and a solution of the bargaining game.

In particular,

\[
\Lambda \in \partial A_i(L_i) = \arg \min \{ \mathbb{E} [\zeta_i X_i] : \zeta_i \in Z_i \}, \quad \forall i \in N
\]

holds for any

\[
\Lambda \in \partial A^\max(M) = \arg \min \{ \mathbb{E} [\zeta_i M] : \zeta_i \in Z^\max \}
\]

and \( L_1, \ldots, L_n, M \) are comonotone random variables.

If (18) is violated, the bargaining problem has no solution.
Proof. Recall that the $A_i$ are translation equivariant. Hence, if the allocation $X_i$ is optimal for (14) then any allocation $X_i + a_i$ such that $\sum_{i=1}^n a_i = 0$ is also optimal because $\sum_{i=1}^n (X_i + a_i) = M$ and $\sum_{i=1}^n A_i(X_i + a_i) = \sum_{i=1}^n A_i(X_i)$. In particular $L_i$, defined in (20) is therefore optimal for (14) and the allocation $L_i$ leads to the acceptability allocation $u_i = \frac{u_i}{\sum_{j=1}^n v_j} A^{\text{max}}(M)$.

If (18) holds then the $u_i$ are automatically feasible for (6), $z = A^{\text{max}}(M)$, and the $u_i$ fulfill (1) such that $L, u$ is a solution of the bargaining game.

The allocation $L_i$ is also optimal for the sup-convolution problem (14), because the $L_i$ sum up to $M$ and the objective value is not changed. Therefore the fourth statement of Theorem 3 implies (21) and the fifth statement implies comonotonicity.

If on the other hand (18) does not hold, then clearly $A^{\text{max}}(M)$ cannot be allocated such that $u_i \geq A_i(R_i)$ for all $i$. \[\square\]

Remark 4. The value $z$ can be rewritten as

$$z = A^{\text{max}}(M) = \mathbb{E}[\Lambda M],$$

(23)

where $\Lambda$ is optimal for (16). This implies that the dual variable $\Lambda$ can in fact be interpreted as the (stochastic) shadow price of a change in $M$.

Next, we consider the general case, where $M$ depends on management decisions $x$. As pointed out in Remark 2, the bargaining process about acceptability values can be completely separated from the maximization with respect to $x$. Based on the sup-convolution, it is possible to make this more concrete.

Corollary 2. Let

$$A = \arg \max_{y \in X} \{A^{\text{max}}(M_y)\} \neq \emptyset$$

(24)

and

$$z = \max_{y \in X} \{A^{\text{max}}(M_y)\}.$$  

(25)

Under the assumption that an optimal allocation $X = (X_1, \ldots, X_n)$ for (25) exists, any $x \in A$ is an optimal management decision of the bargaining game, provided that the condition of Corollary 1 holds for $M = M_x$. The optimal allocation of the bargaining problem can be obtained by applying (20) and the related acceptability allocation is given by (7) with $z(R, M_x, A) = z$.

Proof. This follows directly from Lemma 2 and Corollary 1 because $x$ is optimal for $z$ and cannot be improved by any other management decision. \[\square\]

Separation of the managerial decision $x$ from the decision on the split-up of $M_x$ simplifies the decision process. In particular, the decision $x$ can be taken in order to maximize overall acceptability without any interference of strategic considerations and ensures that bargaining does not lead to suboptimal outcomes and therefore welfare losses. This is clearly a desirable property of the proposed bargaining approach.
3.2. Structure of Contracts: Affine Allocations & Options Contracts

Up to now, we analyzed general optimality conditions. In this section, we discuss two special cases of risk functionals $A_i$, which allow for simple functional characterizations of the individual profits $L_i$ in terms of $M$. The simple forms of $L$ are conducive to a real world implementation of the bargaining solution. Such an implementation would otherwise be complicated by the requirement to agree on a contract that specifies payoffs $L_i(\omega)$ for all agents $i$ in all future states $\omega$ of the world, possibly without any further structure to it. In this section, we discuss cases where optimal allocations are either affine functions of the overall profit or can be represented by options contracts with the project profits as an underlying.

We start by analyzing the case when there is an $i_{\text{max}}$ such that $Z_{i_{\text{max}}} \subseteq Z_i$ for all $i$. This, for example, occurs if the functionals $A_i$ are of the same type with the $Z_i$ controlled by a single parameter, e.g., the Average Value-at-Risk (AVaR$_{\alpha}$) with differing $\alpha$ for each participant. In this case the optimal $L_i$ are affine functions of $M$ as the next result shows.

**Corollary 3.** If there is an $i_{\text{max}} \in \{1, \ldots, n\}$ such that

$$Z_{i_{\text{max}}} \subseteq Z_i \text{ for all } i \in \{1, \ldots, n\}$$

and $H = \{i : Z_i = Z_{\text{max}}\}$, then $z(R,M,A) = A_{i_{\text{max}}}(M)$ and $L_i = a_i + b_i M$, where

$$a_i = \left(\frac{v_i}{\sum_{j=1}^{n} v_j} - b_i\right) A_{i_{\text{max}}}(M) \text{ and } b_i = \begin{cases} \beta_i & i \in H \\ 0 & \text{else} \end{cases}$$

solves the bargaining game if the condition in Corollary 1 is fulfilled and $\beta_j \geq 0$, $\sum_{j \in H} \beta_j = 1$.

**Proof.** Condition (26) implies $Z_{\text{max}} = Z_{i_{\text{max}}}$, hence $\sum_{j=1}^{n} A_j(b_j M) = A_{i_{\text{max}}}(M)$ and the allocation $X_j = b_j M$ must be optimal for the sup-convolution $A_{\text{max}}(M)$. Applying Corollary 1 directly shows that $L_i = a_i + b_i M$ solves the bargaining game.

Hence, if there are players that have higher acceptability values for any random variable, i.e., are least risk averse, these players take all the risk, while the other players receive deterministic side payments and bear no risk in the optimal allocation. An analogous result was proven by Toriello and Uhan (2017) in the context of coalitional games.

**Remark 5.** The naive solution of the allocation problem (often used in practice) would assign shares of the random profit according to the players’ share of invested capital. Moreover, many papers in the literature propose proportional allocations. Such linear rules have e.g. already been derived in Borch (1962) and Gerber (1979) in the context of classical insurance risk theory and have been generalized for dilated convex risk measures and expected risk functionals later, e.g. Deprez and Gerber (1985), Barrieu and El Karoui (2005), see also Timmer et al. (2005), Baeyens
et al. (2013). However, in the present context – even under the assumptions of Corollary 3 – such linear rules without side payments cannot be optimal as long as there is some $i \notin H$ such that $A_i(M) < A_{\text{max}}(M)$. In particular, a proportional split-up requires players with identical acceptability functional $A_i = A$ and identical threat points $v_i$ in order to be optimal.

Clearly, the case where one agent is most risk averse in the sense of (26) is a rare special case. If this condition is violated and $A_{\text{max}}(M) > \max\{A_i(M)\}$, then no affine allocation $L_i = a_i + b_i M$ can be a solution of the bargaining problem. To see this, assume that there would be such an optimal allocation. In this case, the allocation $b_i M$ would be optimal for the sup-convolution problem implying $\sum_{i=1}^n b_i = 1$, which further implies $\sum_{i=1}^n a_i = 0$. The inequality

$$\sum_{i=1}^n A_i(L_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i A_i(M) \leq \max\{A_i(M)\} < A_{\text{max}}(M)$$

then leads to a contradiction with the optimality of the allocation $L$. Hence, in most cases other types of allocations have to be considered.

Next, we consider a more general setting in which allocations take the form of standard options contracts. To that end, we make the assumption that the agents’ acceptability functionals are so called distortion functionals, and the project profit is bounded below as stated in the following assumption.

**Assumption 3.** $\inf_{x \in X} M_x = \sup \{b \in \mathbb{R} : \mathbb{P}(\{\omega \in \Omega : M_x(\omega) \leq b\}) = 0\} = C > -\infty$ for all $x \in X$ and all $A_i$ are distortion functionals, i.e., of the form

$$A_i(X) = \int_0^1 \text{AVaR}_\alpha(X) \, dm_i(\alpha),$$

where $m_i$ are arbitrary probability measures on $[0, 1]$.

In the above definition, AVaR$_\alpha$ refers to the Average Value-at-Risk, defined as

$$\text{AVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha F_X^{-1}(t) \, dt$$

with $F_X$ the cumulative distribution function of $X$. We note that the class of distortion functionals, although a proper subset of the set of coherent risk measures, contains almost all risk functionals relevant for practical applications. See Föllmer and Schied (2004), Pflug and Römisch (2007) for more details on the AVaR and distortion functionals.

In the following, we show that, under Assumption 2 and Assumption 3, any optimal allocation $X = (X_1, \ldots, X_n)$ of the sup-convolution (14) has an explicit representation that can be used to write the $X_i$ as basket of standard options. The following results therefore hold for bargaining games, but are not restricted to this setting. In particular, they extend to any Pareto optimal
allocation of profits (see Rüschendorf 2013, Chapter 8, for numerous examples). However, to maintain consistency we will still talk about the agents \(i = 1, \ldots, n\) as players throughout this section. For bargaining games, it follows as a corollary that every bargaining solution \(L\) can as well be represented by option contracts between the players plus deterministic side payments.

Using Assumption 3, we can dissect the random variables \(X_i\) into slices and then show that the acceptability of \(X_i\) can be built up from the acceptability of the slices. In particular, we establish the following lemma.

**Lemma 4.** Under Assumption 2 and Assumption 3, the following holds:
1. The optimal \(X_i\) in (14) can be written as
   \[
   X_i(M) = \xi_i(C)C + \int_{-\infty}^{\infty} \mathbb{1}_{\{x \geq s\}}(M)\xi_i(s) \, ds
   \]
   for some functions \(\xi_i : \mathbb{R} \to [0, 1]\) with \(\sum_{i=1}^{n} \xi_i(s) = 1\) for \(s \geq C\).
2. If \(A_i\) is a distortion functional, then
   \[
   A_i(X_i) = \int_{-\infty}^{\infty} A_i(\mathbb{1}_{\{x \geq s\}}(M))\xi_i(s) \, ds.
   \]

Using this representation, we can show the following theorem.

**Theorem 4.** If Assumption 2 and Assumption 3 are fulfilled, then there is always an optimal solution \(X\) of (14) for which the \(\xi_i\) in the representation (29) are of the form

\[
\xi_i(s) = \begin{cases} 
  \delta_i(s), & A_i(\mathbb{1}_{\{x \geq s\}}(M)) = \max_j A_j(\mathbb{1}_{\{x \geq s\}}(M)) \\
  0, & \text{otherwise,}
\end{cases}
\]

for almost all \(s > C\) with \(\delta_i(s) \in \{0, 1\}\) such that \(\sum_{i=1}^{n} \delta_i = 1\).

**Proof.** We first prove the statement for \(\delta_i \in [0, 1]\). Suppose that the statement is false and there exist optimal \(X_i\), which violate (30). Then for a set \(S \subseteq \mathbb{R}\) with positive probability there are players \(i\) and \(j\) such that

\[
A_i(\mathbb{1}_{\{x \geq s\}}(M)) > A_j(\mathbb{1}_{\{x \geq s\}}(M))
\]

for \(s \in S\) and \(\xi_i(s) < \xi_j(s)\).

Defining \(\xi'_i(s) = \xi_i(s) + \mathbb{1}_S(s)\xi_j(s)\) and \(\xi'_j(s) = \xi_j(s) - \mathbb{1}_S(s)\xi_j(s)\), we obtain a feasible \(X' = (X_1, \ldots, X'_i, \ldots, X'_j, \ldots, X_n)\) with

\[
\sum_{i=1}^{n} A_i(X'_i) - \sum_{i=1}^{n} A_i(X_i) = A_i(X'_i) - A_i(X_i) + A_j(X'_j) - A_j(X_j)
\]

\[
= \int_{S} \xi_j(s) \left( A_i(\mathbb{1}_{\{x \geq s\}}(M)) - A_j(\mathbb{1}_{\{x \geq s\}}(M)) \right) ds > 0,
\]
by Lemma 4, contradicting the assumption of optimality of $X$.

It follows that if for some $s > C$ the set $\mathcal{I}(s) = \arg\max_j A_j(\mathbb{1}_{\{x \geq s\}}(M))$ has only one element, i.e., $\mathcal{I}(s) = \{i\}$, then $\delta_i = 1$ and $\delta_j = 0$ for $j \neq i$. If $|\mathcal{I}(s)| > 1$, any $\xi'_i$ with $\sum_{i \in \mathcal{I}(s)} \delta_i(s) = 1$ and $\delta'_i(s) \in [0, 1]$ for $s > C$ is optimal for (14). We thus can randomly pick an $i \in \mathcal{I}$ and set $\delta'_i = 1$ to obtain an optimal solution as described in the statement of the theorem. □

The functions $M \mapsto X_i$ are thus monotonically increasing and piecewise affine with interchanging constant pieces and pieces with slope 1. The payoff can therefore be written as partition of the worst case payoff $C$ plus a portfolio of standard call and put options on $M$, which are either received by or written to the community of the other players.

The next theorem establishes that in many practically relevant cases, finitely many options are enough to represent the split $M$ into $X = (X_1, \ldots, X_n)$.

**Theorem 5.** Under Assumption 2 and Assumption 3, the players can reach an optimal allocation of payoffs $X$ by exchanging a finite number of options on $M$ as an underlying asset if

1. the measures $m_i$ are discrete with finitely many atoms or
2. the distribution of $M$ is discrete and has only finitely many atoms.

Finally, the following corollary transfers these results for the sup-convolution to allocations obtained by Nash-bargaining and thus facilitates a real world implementation of optimal allocations of the bargaining process.

**Corollary 4.** The results shown in Lemma 4, Theorem 4, and Theorem 5 for the solutions $X$ of the sup-convolution (14) also hold for solutions $L$ of bargaining games that fulfill Assumption 1 – 3 and Axioms 1 – 3.

**Proof.** As Corollary 1 shows, every bargaining solution is also a solution of the sup-convolution problem. Therefore, bargaining solutions inherit all properties shown for general solutions of (14). □

### 4. An Application Example: Construction of a Solar Roof

In this section, we illustrate the use of the proposed theory in a concrete real-world application example. As we argued in the introduction, the applications of bargaining theory where deterministic Nash bargaining is routinely applied are abundant in the literature. However, we choose a novel application from the field of energy for our example.

In the course of the global transition to clean, renewable energy, there are many situations where several small players cooperate in projects that would be too costly or otherwise too demanding for a single entity. Examples are households that jointly invest in solar power plants or community storage (Chakraborty et al. 2019a,b) or the case of companies that own renewable generation assets
and pool their production in a virtual power plant for cost efficient market access as well as for diversification of market and production risks (Baeyens et al. 2013, Nguyen and Le 2018, Kovacevic et al. 2018, Gersema and Wozabal 2018, Han et al. 2019).

Here, we analyze the problem of joint investment into solar power. In particular, we consider an investment in 7.5 kW peak (kWp) of photovoltaic (PV) panels on a duplex house near Munich that is jointly owned by a young couple and a family consisting of two kids and two parents, one of whom is working. We describe the setup, the assumptions, and the models for stochastic variables in Section 4.1 and discuss the results of the bargaining game in Section 4.2.

4.1. Parameters and Modeling

In the following, we distinguish variables between the two households by the subscripts f (family) and c (couple). For our calculation we assume that the lifetime of the panels is 20 years and that investment costs are €1300 per kWp (Kost et al. 2018). For simplicity, we assume zero maintenance cost.

The two owners of the duplex house jointly invest in the project and can both use electricity generated from the panels. Electricity that is not directly consumed is sold to the grid for a feed-in tariff $F$ that is fixed at 10 cents/kWh for the lifetime of the plant, which roughly corresponds to the current subsidy regime in Germany (see Fraunhofer ISE 2020).

Since major investments and alterations of a duplex house can only be decided unanimously by all owners, we assume that it is not possible for any of the two households to install solar panels at the roof of the building without the participation of the respective other household. Hence, if no bargaining solution is reached, no panels will be installed.

The joint profit from installing the PV panels is thus the sum of profits generated from selling to the grid for the feed-in tariff and the avoided cost of self consumption. Since we assume identical electricity prices for both households, the overall savings in the electricity bill are not affected by which of the two households consumes the electricity. Therefore, we assume that electric current obeys physical laws and flows to the households dependent solely on consumption patterns. Since the consumption of the two households is random, the savings in the electricity bills of the households are random as well and deviate from the bargaining solution. The households therefore agree to make the necessary financial transactions that ensure the agreed upon allocation of profits.

The reward of the project is random as well, since electricity production $P$ of the PV panels, average household power prices $G$, hourly demand $C_c$ and $C_f$, as well as the returns of the alternative investments $R_c$ and $R_f$ are random. We assume these factors to be independent and, for our calculations, represent them by $S = 1000$ equally probable scenarios, which are sampled from the models outlined below.
For simplicity, we assume that the investments are split up proportionately to average electricity consumption of the two households (see below), which can be seen as a proxy for apartment size. This results in investments of €4505 by the family and €5245 by the couple. The opportunity losses are defined by the 20 year random profits $R_c$ and $R_f$ from investing in a diversified portfolio of German stocks. We model the value of the portfolio as a geometric Brownian motion with a yearly drift of 6.93% and a volatility of 21.83%, which we estimate based on daily closing prices from 01/1988 to 12/2019 (ignoring missing values) for the DAX performance index "GDAXI, freely available from Yahoo! Finance.

To generate samples for average household electricity prices, we fit a GBM process to average yearly household prices from 2000 – 2018 obtained from BDEW (2019) and simulate 20 years of yearly price changes starting from a price of 30.43 cents per kWh, which was the average price in 2018.

We model a yearly profile of solar irradiation for one square meter of ground in Munich in hourly resolution following Twidell and Weir (2015), Chapter 4. In particular, given values for the total radiation $G_t$ and the diffuse radiation $G_d$, the produced energy per square meter is given by

$$\eta [(G_t - G_d) \cos \theta + G_d].$$

(31)

Here, $\eta$ denotes the efficiency of the panel and $\theta$ is the angle of incidence, i.e., the angle between the sun beam and the tilted surface of the solar panel. The angle of incidence varies with time and depends on several components:

1. The angle $\beta$ between the panel and the horizontal (tilt) where $0^\circ \leq \beta \leq 90^\circ$ if the surface is directed towards the equator and $90^\circ < \theta \leq 180^\circ$ otherwise.
2. The angle $\gamma$ between the normal to the solar panel surface, projected to the horizontal, and the local longitude meridian (azimuth). If $\gamma = 0^\circ$ the panel faces south, if $\gamma = 90^\circ$ the panel faces west, and if $\gamma = -90^\circ$ the panel faces east.
3. The latitude $\phi$ and the longitude $\psi$ of the location of the panel.
4. The declination (northern hemisphere)

$$\delta = 23.45^\circ \sin \frac{360^\circ (284 + d)}{365}$$

where $d$ denotes the day of the year with $d = 1, \ldots, 365$.
5. The rotation angle $w$ (hour angle) since the last solar noon,

$$w = (15^\circ)(t_{zone} - 12) + (\psi - \psi_{zone}).$$

Here $t_{zone}$ is the civil time in hours of the time zone containing longitude $\psi$, and $\psi_{zone}$ is the longitude where civil time $t_{zone}$ and the solar time coincide. For simplicity, we neglect the so called equation of time, which is a small correction term.
Based on these components, the angle of incidence, respectively its cosine used in (31), fulfills
\[
\cos \theta = (A - B) \sin \delta + [D \sin w + (E + F) \cos w] \cos \delta,
\]
where
\[
A = \sin \phi \cos \beta, \quad B = \cos \phi \sin \beta \cos \gamma, \quad D = \sin \beta \sin \gamma, \quad E = \cos \phi \cos \beta, \quad F = \sin \phi \sin \beta \cos \gamma.
\]

In order to calculate usable electricity from (31) for a location in Munich, we use the relevant geographic information \(\phi = 48.137154^\circ, \psi = 11.576124^\circ, \psi_{zone} = 15^\circ\). Moreover, we derive the local radiation values \(G_t\) and \(G_d\) from the global radiation maps published by the German Weather Service (DWD 2020). Finally we assume that 8m\(^2\) of area are required per kWp capacity and that the efficiency of the panels is \(\eta = 0.1\).

For a combination of angles \(x = (\beta, \gamma)\) we calculate the nominal production \(P(x)\) by (31) and the production in scenario \(s = 1, \ldots, S\) as
\[
P_s(x) = P(x) \epsilon_s
\]
where \(\epsilon_s\) are independent normally distributed errors with mean 1 and standard deviation 0.2, which model multiplicative deviations from the long-term mean.

To generate hourly electricity demands, we use the free tool LoadProfileGenerator\(^1\) using default profiles CHR33 and CHR44 for 2018 for the couple and the family, respectively. We let devices be generated randomly and assume that the couple owns an electric car which is charged at home and used for a 30km commute every day. To obtain random consumption profiles, we randomize the generated profiles by resampling days. More specifically, for a given day, we sample from the days in the generated profile that are in the same month and on the same weekday.

We assume that the risk preferences of the two households are given by
\[
\mathcal{A}_c(X) = 0.6 \mathbb{E}(X) + 0.4 \text{AVaR}_{0.1}(X), \quad \mathcal{A}_f(X) = 0.7 \mathbb{E}(X) + 0.3 \text{AVaR}_{0.05}(X).
\]
Note that the above risk preferences are distortion functionals with \(m_f(1) = 0.7, m_f(0.05) = 0.3, m_c(1) = 0.6,\) and \(m_c(0.1) = 0.4\).

\[1\]See [https://www.loadprofilegenerator.de/](https://www.loadprofilegenerator.de/)
To obtain scenarios $G_s$ for household electricity prices, we simulate trajectories $G^t_s$ for $t = 1, \ldots, 20$, $s = 1, \ldots, S$ for the 20 year lifetime of the plant from the yearly electricity price process discussed above. We then obtain scenarios for average average household prices $G_s = 20^{-1} \sum_{t=1}^{20} G^t_s$, which we use in our calculation.

Apart from the decision on the split up of project profits, the investors have to decide on the azimuth $\gamma$ and the panel tilt $\beta$ for the solar panels, which together constitute the decision $x = (\beta, \gamma)$ in our framework. We therefore use $X = [-180, 180] \times [0, 90]$ as the feasible set of our problem.

Since the production depends in a non-convex way on $x$, we discretize $X$ using a grid $X^g$ with mesh size $1^\circ$ in both dimensions and solve the problems

$$\Pi(x) = \begin{cases} \max_{L,\text{cs},\text{fs}} & A_c(L_c) + A_f(L_f) \\ \text{s.t.} & M_s = \min(P_s(x), C_{cs} + C_{fs}) + \max(P_s(x) - C_{cs} - C_{fs}, 0)F, \forall s = 1, \ldots, S \\ & M_s = L_{cs} + L_{fs}, \forall s = 1, \ldots, S \\ & u_c = \frac{v_c}{v_f + v_c}(A_c(L_c) + A_f(L_f)) \\ & u_f = \frac{v_c}{v_f + v_c}(A_c(L_c) + A_f(L_f)) \\ & A_c(L_c) \geq v_c \\ & A_f(L_f) \geq v_f \end{cases}$$

for all $x \in X^g$ and then find

$$x^* = \arg \min_{x \in X^g} \Pi(x).$$

The optimal $M$, $L$, and $u$ can then be found from the optimal solution of the problem $\Pi(x^*)$. Note that, for a fixed $x$, the above problem can be efficiently solved as a linear optimization problem.

The optimal angles are $x^* = (-18^\circ, 45^\circ)$ which deviates significantly from the angles $x^+ = (4^\circ, 41^\circ)$ maximizing overall electricity production, which is an interesting result in its own right. The production curves in summer, winter, and the transition periods (spring and autumn) are
depicted in Figure 1. Relative to $x^+$, the panels are facing eastward to receive more sunlight in the morning to match the consumption pattern of the households better. Furthermore, the higher tilt of $45^\circ$ ensures increased production in winter as compared to $x^+$. Although this choice leads to less overall production, it maximizes acceptability, since self consumption is strictly preferred to feeding into the grid.

The split up of the profits is plotted in the left panel of Figure 2. In accordance with the theory in Section 3.2, the payoffs are split up into simple options on the underlying $M$: the family receives a call option while the couple is short a put option on top of the distribution of the minimal value of $M$. Clearly, a contract specifying these payoffs as a function of $M$ is legally feasible and easily understood by the parties.

Inspecting the density plot of the profits in the right panel of Figure 2, we see that the distribution ensures that the risk averse couple gets a moderate reward for sure which is capped once $M$ reaches the strike price of the put option. Consequently, the distribution of the rewards of the couple is concentrated around its mean with less downside risk but also limited upside potential. Contrary to that, the family bears most of the downside risk of the project but also has a more pronounced upside potential due to the payout structure of the call option.

The differences in the investments and consequently in the conflict points lead to a split up of acceptability values of $u_c = 16,145$ and $u_f = 13,788$ for the couple and the family, respectively. Due to the low prices of the panels and the rather high prices of grid electricity in Germany, these acceptability values clearly exceed the acceptability values of the alternative investments which are $v_c = 6,030$ and $v_f = 5,150$. 

**Figure 2** Allocation of profits as a function of $M$ (left) and as density plots (right).
5. Conclusions

We formulate a bargaining game for risk averse players that face uncertain profits from cooperation and whose risk preferences can be described by coherent acceptability functionals. Besides the allocation of acceptability values, we put special emphasis on finding ex-ante agreements on the allocation of profits. Additionally, the players also have to agree on a usage of their resources by taking optimal managerial decisions.

We impose three axioms for a “fair solution” and show that these uniquely characterize a distribution of acceptability values. We show that bargaining solutions can be found by solving a stochastic optimization problem that can be interpreted as a generalization of Nash bargaining with bargaining powers. The individual acceptability values are fractions of the optimal overall acceptability plus side-payments between the players. As an important consequence, it follows that managerial decisions can be separated from the question of a fair allocation.

We show that in the case where agents’ risk preferences can be described by distortion risk functionals, the optimal allocation of profits can be characterized as an exchange of standard options contracts between the agents, which makes the approach practically feasible for real world applications.

In order to demonstrate the practical applicability of our approach, we present an illustrative case study of a joint investment in a solar roof on a duplex house. Two households with different consumption patters, risk preferences, and investment size optimize the alignment of the solar panels and search for a fair allocation of the profits from selling electricity to the grid and from the reduction of their energy bills by self consumption. In order to maximize self-consumption, households choose an alignment that deviates substantially from the alignment with maximal energy production. The optimal allocation of profits can be reached by fixed payments and a long position in a call option on the project profit for the player with smaller risk aversion and a short position in a put option for the more risk averse player.

Interesting topics for further research include profit sharing problems in energy applications such as jointly controlling a virtual power plant or managing a community storage, bargaining between nations for the reductions in climate gas emission, as well as problems in economics and politics where deterministic bargaining approaches are routinely applied.

References


H. Nguyen and L. Le. Sharing profit from joint offering of a group of wind power producers in day ahead markets. *IEEE Transactions on Sustainable Energy*, 9(4):1921–1934, Oct 2018. there seems to be some stochasticity here in the form of scenarios and there are also risk preferences, check that out.


Appendix A: Proofs

A.1. Proof of Lemma 1: Compactness of $\mathcal{U}(v,M,A)$

**Lemma 5.** Let $X$ and $Y$ be two topological spaces. If $Y$ is compact, then the projection

$$\text{proj}_1 : X \times Y \to X, \text{ with } \text{proj}_1(x, y) = x$$

is a closed mapping.

**Lemma 6.** For every $X$, there is a $\alpha_0(X)$ such that

$$A_i(X) = \text{AVaR}_{\alpha_0(X)}(X).$$

If $A_i \neq E$ and $X$ is not constant, then $\alpha_0(X) < 1$ is unique.

**Proof.** Note that $\alpha \mapsto \text{AVaR}_\alpha(X)$ is continuous and strictly monotonically increasing if $X$ is not almost surely constant. If $X$ is almost surely constant, then $\text{AVaR}_\alpha(X)$ is constant as well and $\alpha_0$ can be chosen arbitrarily and in particular $\text{AVaR}_1(X) = E(X) = A_i(X)$.

For the non-constant case, clearly,

$$\text{AVaR}_0(X) = \text{ess inf } X \leq A_i(X) \leq E(X) = \text{AVaR}_1(X)$$

and strict monotonicity yields a unique $\alpha_0(X)$ by the intermediate value theorem. □

**Lemma 7.** Let $A_i \neq E$ be coherent acceptability measures, then the set

$$K = \left\{ (L_1, \ldots, L_n) : \exists x \in \mathcal{X} \text{ with } \sum_i L_i \leq M, A_i(L_i) \geq v_i > -\infty \right\}$$

is relatively weakly compact in $\times_{i=1}^n \mathcal{L}^p$.

**Proof.** By the theorem of Banach-Alaoglu, $K$ is relatively weakly compact if it is norm-bounded, where we use

$$||(L_1, \ldots, L_n)|| = \sum_i ||L_i||_p$$

as the norm in $\times_{i=1}^n \mathcal{L}^p$.

Suppose there is a sequence $(L^k)_{k \in \mathbb{N}} \subseteq K$ with corresponding $(x^k)_{k \in \mathbb{N}} \subseteq \mathcal{X}$, such that the former is unbounded. Possibly by selecting a subsequence, we can find a $j$ such that $||L^k_j||_1 \to \infty$. Similarly, we can assume without loss of generality that there is a player $i$ for whom there are sets $A_i^k \subseteq \Omega$ with

$$\int_{A_i^k} L^k_i \, d\mathbb{P} \overset{n \to \infty}{\longrightarrow} -\infty. \quad (33)$$

Let $\alpha^k = \alpha_0(L^k_i, A_i)$. If $\alpha_0^k \neq 1$, then there exists an accumulation point $\bar{\alpha}_0 < 1$ and sets $B_i^k \subseteq \Omega$ such that $\mathbb{P}(A_i^k \cup B_i^k) = \alpha^k < 1$ and $L_i^k(\omega') \geq L_i^k(\omega)$ for all $\omega \in A_i^k \cup B_i^k$, $\omega' \in \Omega \setminus (A_i^k \cup B_i^k)$ and

$$\frac{1}{\alpha^k} \int_{A_i^k \cup B_i^k} L_i^k \, d\mathbb{P} \geq v_i,$$
implying that \( \int_{D_k} \frac{L_i^k}{dP} \to \infty \) and therefore for any other \( j \neq i \)

\[
A_j(L_j^k) \leq \int L_j^k \, dP \leq \int_{A_k \cup B_k} (M_{x_k} - L_i^k) \, dP + \int_{\Omega \setminus (A_k \cup B_k)} (M_{x_k} - L_i^k) \, dP
\]

\[
\leq \sup_{x \in X} E(|M_{x}|) - v_i - \int_{\Omega \setminus (A_k \cup B_k)} L_i^k \, dP \xrightarrow{n \to \infty} -\infty,
\]

since \( \sup_{x \in X} E(|M_{x}|) = \sup_{x \in X} \|M_{x}\| < \infty \) because of Assumption 1. This violates the assumption that \( A_j(L_i^k) \geq v_j > -\infty \) and thus shows that \( L_i^k \notin K \) eventually.

What is left, is the case \( \alpha^k \to 1 \). If \( E(L_i^k) \to \infty \), then clearly for \( j \neq i \)

\[
A_j(L_j^k) \leq E(L_j^k) \leq E(M_{x_k} - L_i^k) = E(M_{x_k}) - E(L_i^k) \xrightarrow{n \to \infty} -\infty < v_j.
\]

Hence, we can assume that \( E(L_i^k) \) remains bounded above and we therefore can find a finite \( C \in \mathbb{R} \), such that for every \( 0 < \alpha < 1 \) and \( 0 \leq \gamma < \alpha \)

\[
\int_{\gamma}^{\alpha} F_{L_i^k}^{-1}(t) \, dt \leq \int_{L_i^k > 0} L_i^k \, dP \leq C, \quad \forall n \in \mathbb{N}. \tag{34}
\]

Now for \( 0 < \alpha \leq 1 \)

\[
\text{AVaR}_\alpha(L_i^k) = \frac{1}{\alpha} \left( \int_{0}^{P(A_i^k)} F_{L_i^k}^{-1}(t) \, dt + \int_{P(A_i^k), \alpha} F_{L_i^k}^{-1}(t) \, dt \right) \xrightarrow{n \to \infty} -\infty,
\]

since the first term in the brackets diverges to \(-\infty\) because of (33), while the second term is bounded above by \( C \) due to (34). However, this in particular, implies that

\[
\lim_{n \to \infty} A_i(L_i^k) \leq \lim_{n \to \infty} E(L_i^k) = \lim_{n \to \infty} \text{AVaR}_\alpha(L_i^k) = -\infty
\]

and therefore \( A_i(L_i^k) \geq v_i \) is eventually violated, which leads to a contradiction to \( L_i^k \in K \), proving the claim.

□

**Proof of Lemma 1** Since \( A_i(M) \leq E(M) \), the set is clearly bounded.

To show that \( U(v, M, A) \) is closed, we write \( U(v, M, A) = \text{proj}_1(V) \) with

\[
V = \text{hypo}(A) \cap \left( \left\{ u : u_i \geq v_i \right\} \times \mathbb{R}^n \right) \cap \left( \mathbb{R}^n \times \left\{ L \in \mathbb{R}^n : \exists x \in X \text{ with } \sum_{i=1}^n L_i = M_x \right\} \right)
\]

\[
\subseteq \mathbb{R}^n \times K^\sigma.
\]

The first set is the hypograph of \( A \) which is closed in \( \prod_{i=1}^n \mathbb{L}^p \) since \( A \) is upper semi-continuous. Since the set is also convex, it is also closed in \( \mathbb{R}^n \times (\prod_{i=1}^n \mathbb{L}^p, \sigma) \), where \( \sigma \) is the weak topology in \( \prod_{i=1}^n \mathbb{L}^p \). Clearly, the second set is also closed in the same topology. To analyze the third set note that by Assumption 1 the set \( M(X) \) is weakly compact in \( \mathbb{L}^p \). Define the linear function \( f(L) = \sum_{i=1}^n L_i \) from \( \prod_{i=1}^n \mathbb{L}^p \) to \( \mathbb{L}^p \). Since \( f \) is bounded, it is weakly continuous and consequently \( f^{-1}(M(X)) \) is closed in the weak topology in \( \prod_{i=1}^n \mathbb{L}^p \).

It follows that \( V \) is closed in \( \mathbb{R}^n \times K^\sigma \) and because \( K^\sigma \) is weakly compact due to Lemma 7, Lemma 5 shows that \( U(v, M, A) \) is closed. □
A.2. Proof of Lemma 2: \( \mathcal{U} \) is a polytope

Proof of Lemma 2. The set \( \mathcal{U}(R, M, \mathcal{A}) \) defined by (3) is obviously just the \( u \)-projection of the feasible set \( \mathcal{U}^+(R, M, \mathcal{A}) \) of the maximization problem (6). Since the objective function of (6) only depends on \( u \), it is immediately seen that

\[
\sup_{u, L, x} \left\{ \sum_{i=1}^n u_i \mid (u, L, x) \in \mathcal{U}^+(R, M, \mathcal{A}) \right\} = \sup_u \left\{ \sum_{i=1}^n u_i \mid u \in \mathcal{U}(R, M, \mathcal{A}) \right\}.
\]

\( \mathcal{U}(R, M, \mathcal{A}) \) is compact due to Lemma 1, therefore (6) attains its maximum.

We show that if \( (u, L, x) \) is feasible for (6), \( u_i \geq v_i \), \( \forall i \), and \( \sum_{i=1}^n u_i' \leq \sum_{i=1}^n u_i \), then there is an \( L' \) and an \( x' \) such that also \( (u', L', x') \) is feasible. To see this, set \( u_i'' = u_i' + \frac{1}{n} \left( \sum_{j=1}^n u_i - \sum_{j=1}^n u_j' \right) \geq u_i' \) for all \( i \in N \).

Then, clearly, \( \sum_{i=1}^n u_i'' = \sum_{i=1}^n u_i \) and setting \( L_i' = L_i - u_i + u_i'' \) and \( x' = x \), we have \( \sum_{i=1}^n L_i' = M_{x'} \). Using translation invariance of \( A_i \), we get

\[
A_i(L_i') = A_i(L_i + (u_i'' - u_i)) = A_i(L_i) + (u_i'' - u_i) \geq u_i + u_i'' - u_i = u_i'' \geq u_i',
\]

which establishes feasibility of \( (u', L', x') \).

Let now \( (u^*, L^*, x^*) \) be an optimal solution of (6) with optimal value \( z \). Since \( (u^*, L^*, x^*) \) is a feasible solution, it follows from the above that every \( u \in \mathbb{R}^n \) with \( u_i \geq v_i \), \( \forall i \) and \( \sum_{i=1}^n u_i \leq \sum_{i=1}^n u^*_i = z \) can be extended by a suitable \( L \) and a suitable \( x \) to a feasible \( (u, L, x) \). Therefore, the set (5) is a subset of the \( u \)-projection of \( \mathcal{U}^+(R, M, \mathcal{A}) \). Conversely, suppose that for some feasible \( (u, L, x) \), the component \( u \) does not lie in the set (5). This could only be the case if \( \sum_{i=1}^n u_i > z \), but then there would be a better solution to the optimization problem (6) than \( (u^*, L^*, x^*) \). Thus, there cannot be a feasible \( u \) lying outside of (5).

A.3. Proof of Lemma 3: Continuity of \( z \)

Proof of Lemma 3. Starting from the definition

\[
z = \max_{L, x} \left\{ \sum_j A_j(L_j) \mid \sum_j L_j = M_x, A_j(L_j) \geq v_j \ (j \in N) \right\}
\]

for some chosen \( i \in N \) and \( \epsilon > 0 \), we define a perturbed problem

\[
z' = \max_{L, x} \left\{ \sum_j A_j(L_j) \mid \sum_j L_j = M_x, A_j(L_j) \geq v_j \ (j \neq i), A_i(L_i) \geq v_i - \epsilon \right\}
\]

Clearly, problem (36) is a relaxation of problem (35), so in particular \( z' \geq z \).

First, we show the following auxiliary result: Let \( L' \) be the optimal solution of (36). Then there is a feasible solution \( L \) of (35) with \( |A_i(L') - A_i(L)| \leq \epsilon, \forall i \in N \).

The assertion is obviously valid if \( L' \) is a feasible solution of (35). So let us assume now that this is not the case. Then, for suitable \( x \),

\[
\sum_j L_j' = M_x, \ A_j(L_j') \geq v_j \ (j \neq i), \ A_i(L_i') \geq v_i - \epsilon, \text{ and } A_i(L_i') < v_i.
\]

Setting \( u'_i = A_i(L_i') \ (j \in N) \) and \( 0 \leq \delta = v_i - u'_i \leq \epsilon \), we have

\[
\sum_{j \neq i} u'_j \geq \sum_{j \neq i} v_j + \delta,
\]
since otherwise
\[ z' = \sum_j u'_j = v_i - \delta + \sum_{j \neq i} u'_j < v_i - \delta + \sum_j v_j + \delta = \sum_j v_j, \]
i.e., \( z' \) would be smaller than \( z \), in contradiction to the fact that (36) is a relaxation of (35).

Setting \( \delta_j = u'_j - v_j \geq 0 \) \((j \neq i)\), it follows from (36) that
\[ \sum_{j \neq i} \delta_j \geq \delta. \tag{38} \]

We define \( L_i = L'_i + \delta, L_j = L'_j - \delta_j' \) for \( j \neq i \) with \( 0 \leq \delta_j' \leq \delta_j \) \((j \neq i)\) and \( \sum_{j \neq i} \delta'_j = \delta \), which is possible because of (38). Then
\[ \sum_j L_j = (L'_i + \delta) + \sum_{j \neq i} L'_j - \sum_{j \neq i} \delta_j' = \sum_j L'_j + \delta - \sum_{j \neq i} \delta_j' = \sum_j L'_j = M_x. \]

Furthermore, using translation invariance of the functionals \( A_i \), we get:
\[ A_i(L_i) = A_i(L'_i + \delta) = A_i(L'_i) + \delta = u'_i + \delta = v_i, \]
as well as
\[ A_j(L_j) = A_j(L'_j - \delta_j') = A_j(L'_j) - \delta_j' = u'_j - \delta_j' \geq u'_j - \delta_j = v_j \tag{39} \]
for each \( j \neq i \). Hence, \( L \) is feasible solution of (35), with the value for \( x \) a part of the optimal solution (36).

Finally,
\[ |A_i(L'_i) - A_i(L_i)| = |A_i(L'_i) - A_i(L'_i + \delta)| = |\delta| \leq \epsilon, \]
and because of (39), for \( j \neq i \),
\[ |A_j(L'_j) - A_j(L_j)| = |\delta_j' - \delta_j| = \delta_j' \leq \sum_{k \neq i} \delta_k' = \delta \leq \epsilon. \]

This proves the auxiliary statement.

Now suppose that some \( \epsilon > 0 \) is given, and consider the perturbed problem (36) for the given \( \epsilon \) and some component \( i \). By the auxiliary result, we can associate to the optimal solution of (36) a feasible solution of the basic problem (35) for which the objective value \( \sum_i A_i(L_i) \) is at most worse by \( n \cdot \epsilon \) compared to the solution value of (36). As problem (36) relaxes problem (35), this means that the solution values \( z \) and \( z' \) can differ by not more than \( n \cdot \epsilon \).

Let us write \( \zeta(v) \) for \( z(R, M, A) \) with \( A_i(R_i) = v_i \) \((i \in R)\). Then what has been shown above is that for two vectors \( v \) and \( v' \),
\[ v_j = v'_j \quad \forall j \neq i \text{ and } |v_i - v'_i| \leq \epsilon \Rightarrow |\zeta(v) - \zeta(v')| \leq n \epsilon. \]

Consider now, to given \( v \), a vector \( \bar{v} \) with \( |v_j - \bar{v}_j| \leq \epsilon \) for all \( j \). Then
\[ |\zeta(v) - \zeta(\bar{v})| \leq |\zeta(v) - \zeta(\bar{v}_1, v_2, \ldots, v_n)| + |\zeta(\bar{v}_1, v_2, \ldots, v_n) - \zeta(\bar{v}_1, \bar{v}_2, v_3, \ldots, v_n)| + |\zeta(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n-1}, v_n) - \zeta(\bar{v})| \leq n^2 \epsilon, \]
showing the continuity of \( \zeta(v) \) as claimed. \( \square \)
A.4. Proof of Theorem 3: Properties of the sup-convolution

In order to characterize solutions of (14), we require a technical lemma and define the following dual cones for sets $Z$

$$Z^*(\Lambda) = \{ X \in L^p(Y) : E[X(Z - \Lambda)] \geq 0 \text{ for all } Z \in Z_i \}. \quad (40)$$

For the proof of the main theorem of this section, we will require the following Lemma.

**Lemma 8.** Let $A_i : L^p(Y) \to \mathbb{R}$ with $1 < p < \infty$, then

$$(Z^{\text{max}})^* (\zeta) = \left( \bigcap_{i=1}^n Z_i \right)^* (\zeta) = \sum_{i=1}^n Z_i^* (\zeta), \quad (41)$$

for any $\zeta \in Z^{\text{max}} \neq \emptyset$.

**Proof.** By Assumption 2, the functionals $A_i$ are upper semicontinuous and concave. Hence, their hypographs are (norm) closed and convex. By the Hahn-Banach separation theorem, it follows that they are weakly-closed as well. Since $1 < p < \infty$, $L^p(Y)$ is reflexive and the weak topology is equivalent to the weak* topology. Therefore the hypographs are weak*-closed.

The (positive homogeneous) functionals $A_i$ are by definition identical with the support functions $\sigma_{Z_i}$ of the defining sets $Z_i$. Because the hypographs of the individual support functions are weak*-closed, the sum

$$\sum_{i=1}^n \text{hypo } \sigma_{Z_i}, \text{ is weak*-closed.} \quad (42)$$

Finally, by Theorem 3.1 in Burachik and Jeyakumar (2005), (41) follows from (42).

**Proof of Theorem 3** The first point follows directly from Rüschendorf (2013), Proposition 11.1 (for the nonempty domain) and Acciaio (2009), Theorem 2.1 (for the dual representation).

To prove the second point, note that the fact that $A^{\text{max}}$ is a coherent acceptability measure directly follows from the dual representation (16). Upper semi-continuity follows by applying Theorem 2 and Theorem 4 of Section 6.2 in Chapter IV of Bourbaki (1989) to the family of upper semicontinuous (in fact even continuous) functionals $X \mapsto E[\zeta X]$ for $\zeta \in Z^{\text{max}}$. Finally, $A^{\text{max}}$ is proper, since all $A_i$ are proper by Rüschendorf (2013), Proposition 11.1.

The third statement directly follows from Proposition 11.4. and Theorem 11.5 in Rüschendorf (2013).

In order to prove the fourth point for $p > 1$, note first that because of the properties of $A$ in the second point, by Kaina and Rüschendorf (2009) Theorem 2.9, there exist $\Lambda \in Z^{\text{max}} \neq \emptyset$ such that

$$\Lambda \in \arg \min_{\zeta \in L^q} \{ E[\zeta M] : \zeta \in Z^{\text{max}} \}. \quad (43)$$

Assume now $1 < p < \infty$, $Z^{\text{max}} \neq \emptyset$ and consider any $\Lambda$, fulfilling (43), which can be rewritten as

$$M \in Z^{\text{max}}^*(\Lambda). \quad (44)$$

Applying Lemma 8 then yields

$$M \in \sum_{i=1}^n Z_i^*(\Lambda), \quad (45)$$
which shows that for any Λ that fulfills (43) there exist $X_i \in Z^*_i(\Lambda)$, which sum up to $M$. Hence, there exist Λ such that (43) holds for these $X_i$, which are consequently optimal for the sup-convolution problem (14). If $p = \infty$, we apply the above argument for an arbitrary $1 < p' < \infty$. Since $M \in L^{p'}(\mathcal{Y})$ and (45) ensures that the resulting $L_i$ are in $L^\infty(\mathcal{Y})$, the argument still goes through.

Existence of an allocation for $p = 1$ (in fact, for $1 \leq p \leq \infty$) under the stronger condition was shown in (Rüschendorf 2013), Theorem 11.3.

Finally, to prove the last statement, we show that for any pair of optimal assignments $X_i, X_j$ the set

$$A(i, j) = \{\omega \in \Omega : \exists \omega' \in \Omega : X_j(\omega') \geq X_j(\omega') \wedge X_i(\omega') < X_i(\omega')\}$$

has probability zero.

To this end, assume that $\mathbb{P}(A(i, j)) > 0$ and consider optimal $X_i, X_j$, an arbitrary $\omega \in A(i, j)$, and a related $\omega'$ fulfilling the defining condition of $A(i, j)$, i.e.,

$$X_j(\omega) \geq X_j(\omega') \wedge X_i(\omega) < X_i(\omega').$$

Consider the conjugate representations (12) and the related optimal dual variables $\zeta_i$ and $\zeta_j$ for $X_i$ and $X_j$. From (13) we know that $E(\zeta_i) = E(\zeta_j) = 1$ and $\zeta_i \geq 0$ and $\zeta_j \geq 0$ must hold almost surely. It follows from the optimality of $\zeta_i, \zeta_j$ and (47) that $\zeta_j(\omega) \geq \zeta_j(\omega')$ and $\zeta_i(\omega) < \zeta_i(\omega')$. This holds for almost all $\omega \in A(i, j)$, which has positive probability as assumed above. But this violates (17) which states that $\zeta_i = \zeta_j$ almost surely, implying that $X_i, X_j$ cannot be optimal. It follows that $A(i, j)$ must have probability 0 and because $i, j$ were arbitrary, the same is true for the set $A(j, i)$. Together this yields that $X_i$ and $X_j$ are comonotone.

Note that the comonotonicity of optimal allocations resulting from the sup-convolution of coherent risk measures has been derived earlier, based on arguments on comonotone improvement in convex-order arguments (see Ludkovski and Rüschendorf 2008). However, since our proof is more direct and much less involved, we decided to state it above.

### A.5. Proof of Lemma 4: Representation of Distortion Functionals

**Proof of Lemma 4.** By the comonotonicity of the $X_i$ shown in Theorem 3.5 and by Proposition 5.16 in McNeil et al. (2005), $X_i$ can be written as a monotonous function of $M$, i.e., $X_i = g_i(M)$ with $g_i : \mathbb{R} \to \mathbb{R}$ monotonously increasing. Since $M = \sum_{i=1}^n X_i$, the mappings $g_i$ have to be absolutely continuous.

Therefore, it follows from the fundamental theorem of calculus that

$$X_i(\omega) = g_i(M(\omega)) = \int_C^{M(\omega)} g_i'(s) \, ds + g_i(C) = \int_{-\infty}^{\infty} 1_{\{s \geq C\}}(M(\omega)) \xi_i(s) \, ds + g_i(C)$$

with

$$\xi_i(s) = \begin{cases} g_i'(s), & \text{if } s > C \\ 0, & \text{if } s < C. \end{cases}$$

Also note that by the condition $M = \sum_{i=1}^n X_i$ it follows that for $s \geq C$, $\xi_i(s) \geq 0$ and $\sum_{i=1}^n \xi_i(s) = 1$ almost surely, establishing the first part of the theorem.
To prove the second part, note the random variables \( 1_{x \geq s}(M) \) are comonotone for all \( s \in \mathbb{R} \). Now define a set of points \((s_i)_{i \in \mathbb{N}}\) such that

\[
\sum_{i=1}^{N} 1_{(x \geq s_i)}(M) \xi_i(s_i) \xrightarrow{N \to \infty} \int_{-\infty}^{\infty} 1_{(x \geq s)}(M) \xi_i(s) \, ds.
\]

We then get

\[
\mathcal{A}_i(X_i) = \mathcal{A}_i \left( \lim_{N \to \infty} \sum_{i=1}^{N} 1_{(x \geq s_i)}(M) \xi_i(s_i) + \xi_i(C) C \right) = \lim_{N \to \infty} \mathcal{A}_i \left( \sum_{i=1}^{N} 1_{(x \geq s_i)}(M) \xi_i(s_i) \right) + \xi_i(C) C
\]

where the second equality follows from Theorem 3.2 in Wozabal and Wozabal (2009) and the third one by comonotone additivity of \( \mathcal{A}_i \) (e.g., Pflug and Römisch 2007, Proposition 2.49).

### A.6. Proof of Theorem 5: Representation as finitely many option contracts

**Proof of Theorem 5.** To prove the first point note that for any \( \alpha \)

\[
\text{AVaR}_\alpha(1_{(x \geq s)}(M)) = \frac{1}{\alpha} \int_0^\alpha F_{1_{(x \geq s)}}^{-1}(t) \, dt = \max \left( \frac{F_{1_{(x \geq s)}}^{-1}(\alpha) - s}{\alpha}, 0 \right). \tag{48}
\]

It follows that if \( m_i \) has only finitely many atoms \( \alpha_1, \ldots, \alpha_k \) with probabilities \( m_1, \ldots, m_k \), then

\[
\mathcal{A}_i(1_{(x \geq s)}(M)) = \sum_{i=1}^{k} m_i \text{AVaR}_\alpha_i(1_{(x \geq s)}(M)),
\]

which together with (48) implies that \( s \mapsto \mathcal{A}_i(1_{(x \geq s)}(M)) \) are piecewise affine functions with finitely many pieces. Since these functions can cross only finitely often, \( \mathbb{R} \) can be divided into finitely many intervals where \( i^* = \arg \max_j \mathcal{A}_j(1_{(x \geq s)}(M)) \) remains constant. In these intervals player \( i^* \) receives all the additional payments from the project, i.e., \( M \mapsto X_{i^*}(M) \) has slope 1 while \( X_i(M) \) has slope 0 for all \( i \neq i^* \). This structure of payments can be achieved by combining finitely many standard call and put options.

Similarly, to prove the second point, note that if \( M \) has finite support there are only finitely many distinct \( 1_{(x \geq s)} \), i.e., the sets \( \arg \max_i \mathcal{A}_i(1_{(x \geq s)}(M)) \) can only change finitely often, leading to \( L \) which can be expressed by finitely many options. \qed