An MISOCP-Based Solution Approach to the Reactive Optimal Power Flow Problem

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Abstract

In this letter, we present an alternative mixed-integer non-linear programming formulation of the reactive optimal power flow (ROPF) problem. We utilize a mixed-integer second-order cone programming (MISOCP) based approach to find global optimal solutions of the proposed ROPF problem formulation. We strengthen the MISOCP relaxation via the addition of convex envelopes and cutting planes. Computational experiments on challenging test cases show that the MISOCP-based approach yields promising results compared to a semidefinite programming based approach from the literature.

1 Introduction

The reactive optimal power flow (ROPF) problem is a variant of the well-known optimal power flow (OPF) problem in which additional discrete decisions, such as shunt susceptance and tap ratio, are considered. Due to the presence of these discrete variables in the ROPF problem, it can be formulated as a mixed-integer non-linear programming (MINLP) problem. This letter utilizes the recent developments in the OPF problem to propose an efficient way of solving the ROPF problem.

OPF is one of the most studied problems in the area of power systems and a variety of solution approaches have been proposed in the literature. Local methods such as the interior point method try to solve the OPF problem but they do not provide any assurances of global optimality. In recent years, convex relaxations of the OPF problem have drawn considerable research interest since the convexity property promises a globally optimal solution under certain conditions. Several approaches have been developed based on convex quadratic, semidefinite programming (SDP), second order cone programming (SOCP) and convex-distflow formulations. The ROPF problem has a similar structure with the OPF problem, except the inclusion of shunt susceptance and tap ratio variables, which are typically modelled as discrete variables. The resulting MINLP problem is difficult to solve and the literature has primarily focused on various heuristic methods [2]. The systematic treatment of the ROPF problem is limited to an SDP-based relaxation called tight-and-cheap relaxation (TCR) proposed in [1].

This letter proposes a new MINLP formulation for the ROPF problem along with its mixed-integer second-order cone programming (MISOCP) relaxation and an improved MISOCP relaxation with convex envelopes and cutting planes. We also test the accuracy and efficiency of our approach with the TCR method from the literature on difficult test cases and obtain promising results.

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2 Mathematical Model

2.1 MINLP Formulation

Consider a power network \( \mathcal{N} = (\mathcal{B}, \mathcal{L}) \), where \( \mathcal{B} \) and \( \mathcal{L} \) denote the set of buses and the set of transmission lines respectively. Let \( \mathcal{G} \subseteq \mathcal{B} \), \( \mathcal{S} \subseteq \mathcal{B} \) and \( \mathcal{T} \subseteq \mathcal{L} \) respectively denote the set of generators connected to the grid, the buses with a variable shunt susceptance and the lines with a variable tap ratio. Rest of the parameters are given as follows:

- For each bus \( i \in \mathcal{B} \); \( p_i^d \) and \( q_i^d \) are the real and reactive power load, \( V_i \) and \( \overline{V}_i \) are the bounds on the voltage magnitude, \( \delta(i) \) is the set of neighbors and \( \{ b_{ki}^k : k \in \mathcal{S}_i \} \) is the set of allowable shunt susceptances.
- For each generator located at bus \( i \in \mathcal{G} \); active and reactive outputs must be in the intervals \([p_i^{\min}, p_i^{\max}]\) and \([q_i^{\min}, q_i^{\max}]\), and we have \( p_i^{\min} = p_i^{\max} = q_i^{\min} = q_i^{\max} = 0 \) for \( i \in \mathcal{B} \setminus \mathcal{G} \).
- For each line \((i,j) \in \mathcal{L} \); \( G_{ij} \) and \( B_{ij} \) are conductance and susceptance, \( \{ \tau_{ij}^l : l \in \mathcal{T}_{ij} \} \) is the set of allowable tap ratios, \( S_{ij} \) is the apparent power flow limit and \( \theta_{ij} \) is the bound on the phase angle.

We define the following decision variables:

- For each bus \( i \in \mathcal{B} \), \( |V_i| \) and \( \theta_i \) are the voltage magnitude and phase angle, \( b_{ii} \) is the shunt susceptance, \( \alpha_{ki} \) is one if \( b_{i} = b_{ki}^k \) and zero otherwise.
- For each generator located at bus \( i \in \mathcal{G} \); \( p_i^g \) and \( q_i^g \) are the real and reactive power output.
- For each line \((i,j) \in \mathcal{L} \); \( p_{ij} \) and \( q_{ij} \) are the real and reactive power flow, \( \tau_{ij} \) is the tap ratio, \( \beta_{ij}^l \) is one if \( \tau_{ij} = \tau_{ij}^l \) and zero otherwise.

Then, the ROPF problem can be modeled as the following MINLP:

\[
\min \sum_{i \in \mathcal{G}} f(p_i^g) \tag{1}
\]

\[
p_i^g - p_i^d = g_i |V_i|^2 + \sum_{j \in \delta(i)} p_{ij} \quad i \in \mathcal{B} \tag{2}
\]

\[
q_i^g - q_i^d = -b_{ii} |V_i|^2 + \sum_{j \in \delta(i)} q_{ij} \quad i \in \mathcal{B} \tag{3}
\]

\[
p_{ij} = G_{ij}(|V_i|/\tau_{ij})^2 + (|V_i|/\tau_{ij})|V_j|\left[(G_{ij} \cos(\theta_i - \theta_j)
- B_{ij} \sin(\theta_i - \theta_j))\right] \quad (i,j) \in \mathcal{L} \tag{4}
\]

\[
q_{ij} = -B_{ij}(|V_i|/\tau_{ij})^2 - (|V_i|/\tau_{ij})|V_j|\left[B_{ij} \cos(\theta_i - \theta_j)
+ G_{ij} \sin(\theta_i - \theta_j)\right] \quad (i,j) \in \mathcal{L} \tag{5}
\]

\[
V_i \leq |V_i| \leq \overline{V}_i \quad i \in \mathcal{B} \tag{6}
\]

\[
\sum_{k \in \mathcal{S}_i} b_{ki}^k \alpha_{ki} = b_{ii} \quad i \in \mathcal{B}, \quad \sum_{l \in \mathcal{T}_{ij}} \tau_{ij}^l \beta_{ij}^l = \frac{1}{\tau_{ij}} \quad (i,j) \in \mathcal{L} \tag{7}
\]

\[
\sum_{k \in \mathcal{S}_i} \alpha_{ki} = 1 \quad i \in \mathcal{B}, \quad \sum_{l \in \mathcal{T}_{ij}} \beta_{ij}^l = 1 \quad (i,j) \in \mathcal{L} \tag{8}
\]
\[ \alpha_i^k \in \{0,1\} \quad i \in \mathcal{B}, \quad \beta_j^l \in \{0,1\} \quad (i,j) \in \mathcal{L} \quad (9) \]

\[ b_{ii} = 0 \quad i \not\in \mathcal{S}, \quad \tau_{ij} = 1 \quad (i,j) \not\in \mathcal{T} \quad (10) \]

\[ q_i^\text{min} \leq q_i^g \leq q_i^\text{max}, \quad p_i^\text{min} \leq p_i^d \leq p_i^\text{max} \quad i \in \mathcal{G} \quad (11) \]

\[ p_{ij}^2 + q_{ij}^2 \leq S_{ij}^2, \quad |\theta_i - \theta_j| \leq \bar{\theta}_{ij} \quad (i,j) \in \mathcal{L}. \quad (12) \]

Here, the objective function (1) minimizes the total real power generation cost subject to the following constraints: real and reactive power flow balance at bus \( i \) (2)–(3), real and reactive power flow from \( i \) to \( j \) (4)–(5), shunt susceptance selection for bus \( i \) and tap ratio selection for line \( (i,j) \) (7), voltage magnitude bounds at bus \( i \) (6), binary restrictions (8)–(9), reactive and active power output of generator \( i \) (11), apparent flow and phase angle limit for each line \( (i,j) \) (12).

### 2.2 An Alternative MINLP

In this section, we propose an alternative MINLP formulation of the ROPF problem motivated by [3]. Let us define a set of new decision variables \( c_{ii}, c_{ij}, s_{ij} \), respectively representing the quantities \( |V_i|^2 \), \( |V_i||V_j|\cos(\theta_i - \theta_j) \) and \( s_{ij} := -|V_i||V_j|\sin(\theta_i - \theta_j) \) for \( i \in \mathcal{B} \) and \( (i,j) \in \mathcal{L} \). We denote the lower (upper) bounds of variables \( c_{ii}, c_{ij}, s_{ij} \) as \( \underline{c}_{ii}, \underline{c}_{ij}, \bar{s}_{ij} \) \( (\underline{\tau}_{ii}, \underline{\tau}_{ij}, \bar{s}_{ij}) \) and set them as follows:

\[
\underline{c}_{ii} := V_i^2, \quad \bar{c}_{ii} := V_i^2 \quad i \in \mathcal{B} \\
\underline{c}_{ij} := \sqrt{V_i V_j \cos(\bar{\theta}_{ij})}, \quad \bar{c}_{ij} := \sqrt{V_i V_j} \quad (i,j) \in \mathcal{L} \\
\underline{s}_{ij} := -\sqrt{V_i V_j \sin(\bar{\theta}_{ij})}, \quad \bar{s}_{ij} := \sqrt{V_i V_j} \sin(\bar{\theta}_{ij}) \quad (i,j) \in \mathcal{L}.
\]

We will now discuss the constraints in the alternative formulation and their relations with the MINLP in Section 2.1. The updated version of the real power flow balance constraint (2) is given as:

\[ p_i^g - p_i^d = g_i c_{ii} + \sum_{j \in \delta(i)} p_{ij} \quad i \in \mathcal{B}. \quad (13) \]

Since the variable \( b_{ii} \) can be eliminated from the formulation by substituting \( \sum_{k \in \mathcal{S}_i} b_{ii}^k \alpha_i^k \), the reactive power flow equation (3) is first rewritten as follows:

\[ q_i^g - q_i^d = - \left( \sum_{k \in \mathcal{S}_i} b_{ii}^k \alpha_i^k \right) |V_i|^2 + \sum_{j \in \delta(i)} q_{ij} \quad i \in \mathcal{B}. \quad (14) \]

Then, we define a new variable \( \Gamma_i^k := c_{ii} \alpha_i^k \) to linearize (14) and include additional constraints as follows:

\[ q_i^g - q_i^d = - \sum_{k \in \mathcal{S}_i} b_{ii}^k \Gamma_i^k + \sum_{j \in \delta(i)} q_{ij} \quad i \in \mathcal{B} \]

\[ \underline{c}_{ii} \alpha_i^k \leq \Gamma_i^k \leq \bar{c}_{ii} \alpha_i^k, \quad c_{ii} = \sum_{k \in \mathcal{S}_i} \Gamma_i^k \quad i \in \mathcal{B}. \quad (15) \]

We now update power flow constraints using a similar procedure. In particular, we substitute \( 1/\tau_{ij} \) with \( \sum_{l \in \mathcal{T}_{ij}} \beta_{ij}^l/\tau_{ij} \) into constraints (4) and (5). After defining the new variables \( \Phi_{ij}^l := c_{ii} \beta_{ij}^l \),
\( \Phi_{ij} := c_{ij} \beta_{ij} \) and \( \Psi_{ij} := s_{ij} \beta_{ij} \), we rewrite the real and reactive power flow constraints (4)–(5) together with other equations necessary for the linearization as follows:

\[
\begin{align*}
    p_{ij} &= \sum_{l \in T_{ij}} G_{ij} \left( c_{ii} \beta_{ij} \left( \frac{\Phi_{ij}}{\tau_{ij}} \right)^2 + \frac{\Phi_{ij}}{\tau_{ij}} \right) - B_{ij} \Psi_{ij} \frac{\Phi_{ij}}{\tau_{ij}} \quad (i, j) \in \mathcal{L} \\
    q_{ij} &= \sum_{l \in T_{ij}} -B_{ij} \left( \frac{\Phi_{ij}}{\tau_{ij}} \right)^2 - G_{ij} \frac{\Phi_{ij}}{\tau_{ij}} \quad (i, j) \in \mathcal{L} \\
    c_{ii} \beta_{ij} &\leq \Phi_{ij} \leq \tau_{ii} \beta_{ij}, \quad c_{ii} = \sum_{l \in T_{i,j}} \Phi_{ij} \quad (i, j) \in \mathcal{L} \tag{16} \\
    c_{ij} \beta_{ij} &\leq \Psi_{ij} \leq \tau_{ij} \beta_{ij}, \quad c_{ij} = \sum_{l \in T_{i,j}} \Psi_{ij} \quad (i, j) \in \mathcal{L} \\
    s_{ij} \beta_{ij} &\leq \Psi_{ij} \leq \bar{s}_{ij} \beta_{ij}, \quad s_{ij} = \sum_{l \in T_{i,j}} \bar{s}_{ij} \beta_{ij} \quad (i, j) \in \mathcal{L}.
\end{align*}
\]

We also update the constraint on voltage magnitude bounds (6) as follows:

\[
\begin{align*}
    V_i^2 &\leq c_{ii} \leq \bar{V}_i^2 \quad i \in \mathcal{B} \tag{17}.
\end{align*}
\]

Finally, we define the following consistency constraints for each line \((i, j)\):

\[
\begin{align*}
    c_{ij}^2 + s_{ij}^2 &= c_{ii}c_{jj} \quad (i, j) \in \mathcal{L} \tag{18} \\
    (\Phi_{ij}^l)^2 + (\Psi_{ij}^l)^2 &= \bar{\Phi}_{ij}^l c_{jj} \quad (i, j) \in \mathcal{L} \tag{19} \\
    \theta_j - \theta_i &= \arctan(s_{ij}/c_{ij}) \quad (i, j) \in \mathcal{L} \tag{20}.
\end{align*}
\]

Equation (18) preserves the trigonometric relation between the variables \(c_{ii}, c_{ij}\) and \(s_{ij}\). If we multiply (18) by \(\beta_{ij}\), we can get a similar condition for the variables \(\Phi_{ij}^l, \Phi_{ij}^l\) and \(\Psi_{ij}^l\).

The alternative formulation minimizes (1) subject to constraints (8)–(13) and (15)–(20).

### 2.3 MISOCPL Relaxation

The feasible region of the alternative MINLP formulation is non-convex due to constraints (18)–(20). Let us relax these constraints as follows:

\[
\begin{align*}
    c_{ij}^2 + s_{ij}^2 &\leq c_{ii}c_{jj} \quad (i, j) \in \mathcal{L} \tag{21} \\
    (\Phi_{ij}^l)^2 + (\Psi_{ij}^l)^2 &\leq \bar{\Phi}_{ij}^l c_{jj} \quad (i, j) \in \mathcal{L}.
\end{align*}
\]

Then, an MISOCPL relaxation is obtained as (1), (8)–(13), (15)–(17) and (21).

To strengthen the MISOCPL relaxation, we also consider an outer-approximation of the arctangent constraint (20). This is achieved by the inclusion of four hyperplanes as described in [4]. We will use the abbreviation MISOCPLA to refer to this stronger relaxation. Additionally, we generate cutting planes for each cycle in the cycle basis using a method called SDP Separation, more details can be found in [3]. We denote this further improved relaxation as MISOCPLA+.
Table 1: Computational results (time is measured in seconds).

<table>
<thead>
<tr>
<th>Case</th>
<th>TCR2</th>
<th>MISOCPA+</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LB Time UB %Gap</td>
<td>LB Time UB %Gap</td>
</tr>
<tr>
<td>3lmbd</td>
<td>5769.87 0.65 5812.64 0.74</td>
<td>5783.94 0.53 5812.64 0.49</td>
</tr>
<tr>
<td>5pjm</td>
<td>15313.38 0.72 17551.89 12.75</td>
<td>16395.73 0.22 17551.89 6.59</td>
</tr>
<tr>
<td>30ieee</td>
<td>205.19 1.13 205.64 0.22</td>
<td>205.15 1.38 205.25 0.05</td>
</tr>
<tr>
<td>118ieee</td>
<td>3695.39 4.66 3720.08 0.66</td>
<td>3684.68 12.89 3714.91 0.81</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>1.79</strong> <strong>3.59</strong></td>
<td><strong>3.75</strong> <strong>1.99</strong></td>
</tr>
<tr>
<td>3lmbd_api</td>
<td>363.00 0.66 367.74 1.29</td>
<td>362.92 0.56 367.74 1.31</td>
</tr>
<tr>
<td>6ww_api</td>
<td>273.76 0.53 273.76 0.00</td>
<td>273.66 0.38 273.76 0.04</td>
</tr>
<tr>
<td>14ieee_api</td>
<td>319.12 0.93 323.29 1.29</td>
<td>318.65 0.76 321.09 0.76</td>
</tr>
<tr>
<td>30as_api</td>
<td>559.96 2.38 571.13 1.96</td>
<td>556.71 0.92 571.13 2.52</td>
</tr>
<tr>
<td>30fsr_api</td>
<td>213.93 2.16 372.14 42.51</td>
<td>227.57 0.95 372.11 38.84</td>
</tr>
<tr>
<td>39epri_api</td>
<td>7333.40 2.59 7466.25 1.78</td>
<td>7259.19 11.63 7480.45 2.96</td>
</tr>
<tr>
<td>118ieee_api</td>
<td>5932.26 4.47 10258.47 42.17</td>
<td>5910.20 14.23 10158.62 41.82</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>1.96</strong> <strong>13.00</strong></td>
<td><strong>4.20</strong> <strong>12.61</strong></td>
</tr>
<tr>
<td>3lmbd_sad</td>
<td>5831.07 0.57 5992.72 2.70</td>
<td>5867.46 0.53 5992.72 2.09</td>
</tr>
<tr>
<td>4gs_sad</td>
<td>321.55 0.58 324.02 0.76</td>
<td>323.65 0.16 324.02 0.12</td>
</tr>
<tr>
<td>5pjm_sad</td>
<td>25560.36 0.62 26423.33 3.27</td>
<td>26419.23 0.21 26423.32 0.02</td>
</tr>
<tr>
<td>9wscc_sad</td>
<td>5521.49 0.54 5590.09 1.23</td>
<td>5589.54 0.20 5590.09 0.01</td>
</tr>
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<td>36270.50 2.77 45886.11 20.96</td>
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<tr>
<td>30as_sad</td>
<td>903.09 2.32 914.44 1.24</td>
<td>906.96 1.11 914.44 0.82</td>
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<tr>
<td>30ieee_sad</td>
<td>205.30 0.96 205.79 0.24</td>
<td>205.27 1.51 205.37 0.05</td>
</tr>
<tr>
<td>118ieee_sad</td>
<td>3869.62 4.66 4323.91 10.51</td>
<td>4003.35 9.30 4258.72 6.00</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>1.68</strong> <strong>6.69</strong></td>
<td><strong>1.97</strong> <strong>3.76</strong></td>
</tr>
<tr>
<td>Overall Average</td>
<td><strong>1.81</strong> <strong>8.36</strong></td>
<td><strong>3.17</strong> <strong>6.64</strong></td>
</tr>
</tbody>
</table>

3 Computational Experiments

3.1 Algorithm

We first solve the continuous relaxation of the MISOCPA formulation by relaxing the integrality of $\alpha^k_i$ and $\beta^l_{ij}$ variables. Then, for each cycle in the cycle basis, we use the SDP separation method to generate cutting planes to separate this continuous relaxation solution from the feasible region of the SDP relaxation of the cycle. The separation process is parallelized over cycles. We repeat this procedure five times consecutively. Then, we solve the final MISOCPA+ relaxation to obtain a lower (LB) bound, and then fix the binary variables in the MINLP formulation to obtain an upper bound (UB) from the remaining NLP using a local solver. The optimality gap is computed as $\%Gap = 100 \times (1 - \text{LB/UB})$.

3.2 Results

We compare the percentage optimality gap and the computational time of the MISOCPA+ approach with the publicly available implementation of TCR relaxation of Type 2 (TCR2) from [1]. All computational experiments have been carried out on a 64-bit desktop with Intel Core i7 CPU with 3.20GHz processor and 64 GB RAM. Our code is written in Python language using Spyder environment. The solvers Gurobi, IPOPT and MOSEK are used to solve the MISOCPA+ relaxation,
NLP and separation problems, respectively.

For the computational experiments, we use the OPF instances from the NESTA library; typical operating conditions, congested operating conditions (API) and small angle difference conditions (SAD). We only consider difficult instances in which the SOCP optimality gap is more than 1% [3].

The sets of the discrete values are determined as $b^k_i \in \{0, 1\}$ for $i \in S$ and $\tau_{ij} \in \{0.9, 0.95, 1, 1.05, 1.1\}$ for $(i, j) \in T$, which represent the on/off status of the shunt susceptance and values of the tap ratio, respectively.

The results of our computational experiments are reported in Table 1. We observe that MISOCPA+ has smaller optimality gap in fourteen out of nineteen instances, and better or same upper bound in eighteen of these instances (except 39epri_api). If we compare the averages of optimality gap, MISOCPA+ outperforms TCR2 in all types of NESTA instances. MISOCPA+ has the best performance on SAD instances and dominates TCR2 in all of them. Overall, we note that MISOCPA+ relaxation has more accurate solutions with 6.64% optimality gap, on average, than TCR with 8.36%. In terms of computational time, MISOCPA+ is slower with 3.17 seconds, on average, than TCR2 with 1.81.

4 Conclusion

In this letter, we propose an MISOCP-based approach, namely MISOCPA+, to approximate globally optimal solutions of the ROPF problem. The accuracy and efficiency of this approach are compared with TCR2 using difficult OPF instances from the NESTA library. The computational results indicate that MISOCPA+ is quite promising to solve any type of instances accurately, especially the ones with small angle conditions.

References


