A New Coherent Multivariate Average-Value-at-Risk

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Abstract

A new multivariate performance measure Average-Value-at-Risk, $m\text{AVaR}_\alpha$ evaluating the sum of $N$ risky assets composing the portfolio of an investor with respect to risk $N$-dimensional risk level vector $\alpha$ is proposed. We show that the proposed operator satisfies the four axioms of a coherent risk measure, while reducing to the one variable Average-Value-at-Risk $\text{AVaR}$, in case $N = 1$. In that respect, $m\text{AVaR}_\alpha$ is the natural extension of $\text{AVaR}$ to $N$ dimensional case, while maintaining its axiomatic properties. $m\text{AVaR}_\alpha$ is flexible by giving the investor the choice of choosing the risk level $\alpha_i$ of each risky asset differently. This flexibility is novel and can not be achieved applying univariate $\text{AVaR}_\alpha$ with corresponding risk level $\alpha$ to the sum of the risk of portfolio. The framework is applicable for Gaussian mixture models with dependent marginals that are naturally used in financial and actuarial modelling. A multivariate tail variance and its connection with $m\text{AVaR}_\alpha$ is also presented via a Chebyshev inequality for tail events. Explanatory examples with explicit solutions are also illustrated throughout the manuscript.

Keywords: Multivariate Coherent Risk Measures, Risk Management, Average-Value-at-Risk

1 Introduction

Coherent risk measures have been introduced in [1] to quantify univariate risk (loss) in an axiomatic framework. These axioms have been monotonicity, translation invariance, convexity (diversification of risk) and positive homogeneity. Since its introduction of this concept, they have seen huge development both in theory and practice. One of the fundamental coherent

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risk measures is the so-called Average-Value-at-Risk (also called Tail Conditional Expectation, Conditional Tail Expectation, Expected Shortfall, Tail Value-at-Risk, Conditional-Value-at-Risk) of random loss $X$ for a predetermined risk level $\alpha$, denoted by $\text{AVaR}_\alpha(X)$. Beside its acceptance in Solvency II regulations, $\text{AVaR}_\alpha(X)$ has been rigorously studied in academia in many papers (see [12, 13, 14, 15, 16, 17] among others). Roughly speaking, $\text{AVaR}_\alpha(X)$ evaluates random loss $X$ by taking average of worst case scenarios from confidence level $\alpha$ onward.

On the other hand, risk managers are not only dealing with one source of loss but rather a portfolio of risk. In particular, they are dealing with random vectors representing different sources of risks. This implies evaluating the possible outcomes of multivariate random vector using univariate $\text{AVaR}_\alpha$ with a single risk level $\alpha$ level is not adequate. To remedy this shortcoming, several extensions of $\text{AVaR}$ has been introduced. Some of the works in this direction are [6, 3, 7, 8, 9, 11, 10, 18]. In this paper, we focus on evaluating the sum of the risk of the portfolio of risk composed of $N$ different risk/loss sources. In particular, we introduce an actual extension of univariate $\text{AVaR}$ to multivariate Average-Value-at-Risk $\text{mAVaR}_\alpha(X)$ that preserves the axioms of $\text{AVaR}$ by taking $N$ dimensional risk portfolio $X$ and maps to $\mathbb{R}$ using $N$-dimensional risk level $\alpha = (\alpha_1, \ldots, \alpha_N)$. We show that $\text{mAVaR}_\alpha$ reduces to univariate $\text{AVaR}$ in case $N = 1$ by preserving the axioms of a coherent risk measure. In that sense, $\text{mAVaR}_\alpha$ is the natural extension of $\text{AVaR}$. Furthermore, at the same time, $\text{mAVaR}_\alpha$ is flexible of choosing $N$ dimensional risk level $\alpha$ giving the option to the risk manager to assign different risk levels $\alpha_i$ for each source $i = 1, \ldots, N$. We show that $\text{mAVaR}_\alpha(X)$ can handle dependent multivariate Gaussian mixture $X$ that is prevalent in actuarial and financial modelling. $\text{mAVaR}_\alpha$ evaluates multivariate Gaussian mixtures by keeping the corresponding axioms of coherent risk measures.

The rest of the paper is as follows. In Section 2, we introduce the framework and background theory regarding Average-Value-at-Risk along with the frequent notation used in the rest of the paper. In Section 3, we define multivariate Average-Value-at-Risk and its regularity properties along with the examples shedding light on its properties. In Section 4, we give examples of use of $\text{mAVaR}_\alpha$ in Gaussian mixture models with possibly dependent marginals. In Section 5, we give a Chebychev type stability result for $\text{mAVaR}_\alpha$ by introducing and using the corresponding tail variance and give an example in multivariate Gaussian setting.
2 Framework and Notation

We fix an integer $N \geq 1$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ for $i = 1, \ldots, N$ be the fixed atomless probability space. We denote $\mathcal{X}_i \triangleq L^1(\Omega, \mathcal{F}, \mathbb{P})$, where an $\mathcal{F}$ measurable random variable $X_i \in \mathcal{X}_i$ means $\|X_i\| \triangleq \mathbb{E}^\mathbb{P}[|X_i|] < \infty$ and $\mathbb{E}^\mathbb{P}[-]$ stands for the expectation with respect to $\mathbb{P}$. For $X_i, Y_i \in \mathcal{X}_i$, we identify $X_i$ with $Y_i$, if $\mathbb{P}(X_i = Y_i) = 1$. $\mathcal{X}_i$ stands for the riskiness of the financial random position. Namely, positive values of $X_i \in \mathcal{X}_i$ represent costs/loss, whereas negative values of it stand for profits/gain. Let $\mathcal{X}^N \triangleq \mathcal{X}_1 \times \ldots \times \mathcal{X}_N$ with $\Omega \triangleq \otimes_{i=1}^N \Omega_i$, $\mathbb{P} \triangleq \prod_{i=1}^N \mathbb{P}_i$ and $\mathcal{F} \triangleq \otimes_{i=1}^N \mathcal{F}_i$. We note here that any $(X_1, \ldots, X_N) \in \mathcal{X}^N$ is jointly independent with respect to $\mathbb{P}$. $\mathbb{E}[]$ stands for the expectation taken with respect to $\mathbb{P}$, whereas $\mathbb{E}^Q[]$ stands for the specific probability measure $Q$ that is taken with respect to in its context. Moreover, we denote $X \triangleq (X_1, \ldots, X_N) \in \mathcal{X}^N$ be the corresponding portfolio of risky assets with $\|X\| \triangleq \sum_{i=1}^N |X_i|$ and denote by $\mathcal{M}_1(\mathcal{X}_i)$ for $i = 1, \ldots, N$ and $\mathcal{M}_1(\mathcal{X}^N)$ the set of all probability measure that are absolutely continuous with respect to $\mathbb{P}_i$ and $\mathbb{P}$, respectively. Namely, taking $\frac{1}{p} + \frac{1}{q} = 1$ with $q = \infty$ for $p = 1$, we let

$$\mathcal{M}_1^i(\mathcal{X}_i) \triangleq \left\{ Q_i \in \mathcal{M}_1^i(\Omega, \mathcal{F}_i, \mathbb{P}_i) : \frac{dQ_i}{d\mathbb{P}_i} \in L^\infty(\Omega, \mathcal{F}_i, \mathbb{P}_i) \right\},$$

$$\mathcal{M}_1(\mathcal{X}^N) \triangleq \left\{ Q \in \mathcal{M}_1(\Omega, \mathcal{F}, \mathbb{P}) : \frac{dQ}{d\mathbb{P}} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \right\},$$

where $\frac{dQ_i}{d\mathbb{P}_i}$ stands for the Radon-Nikodym density of $Q_i$ with respect to $\mathbb{P}_i$.

Further, we denote $\alpha \triangleq (\alpha_1, \ldots, \alpha_N)$, where $\alpha_i \in [0, 1)$ for $i = 1, \ldots, N$ standing for the risk level of $N$ dimensional loss vector. Given $X_i \in \mathcal{X}_i$ for $i = 1, \ldots, N$, we denote

$$x_{\alpha_i} \triangleq \inf\{t \in \mathbb{R} : \mathbb{P}(X_i \leq t) \geq \alpha_i\}$$

as the corresponding $\alpha_i$ quantile of $X_i$. Furthermore, letting $X_1 \triangleq (X_1, \ldots, X_N) \in \mathcal{X}^N$ and $X_2 \triangleq (Y_1, \ldots, Y_N) \in \mathcal{X}^N$, $X_1 \leq X_2$ stands for $X_i \leq Y_i$, $\mathbb{P}_i$ almost surely (a.s.) for all $1 \leq i \leq N$. Similarly, $X_1 + X_2$ stands for $(X_1 + Y_1, \ldots, X_N + Y_N)$. For $\lambda \in \mathbb{R}$ and $X \triangleq (X_1, \ldots, X_N) \in \mathcal{X}^N$, $\lambda X \triangleq (\lambda X_1, \ldots, \lambda X_N)$. We say $X_n$ converges pointwise to $X$ i.e. $X_n \rightarrow X$ $\mathbb{P}$-a.s., if each $X^n_i \rightarrow X_i$ $\mathbb{P}_i$ almost surely as $n \rightarrow \infty$ for $i = 1, \ldots, N$.

**Definition 2.1.** A multivariate risk measure $\rho$ for portfolio vectors is any map from $\mathcal{X}^N$ to $\mathbb{R}$. Given a portfolio $X \in \mathcal{X}^N$, the expression $\rho(X)$ stands for the amount of risk capital that the holder of portfolio has to invest additionally into its portfolio $X$ such that the portfolio is acceptable. Furthermore, letting $X, X_1, X_2 \in \mathcal{X}^N$ and $c = (c_1, \ldots, c_N)$ with $c_i \in \mathbb{R}$ for $i = 1, \ldots, N$, a multivariate risk measure $\rho : \mathcal{X}^N \rightarrow \mathbb{R}$ is called coherent, if $\rho$ satisfies the following axioms:
(A1) **Monotonicity:** If $X_1 \leq X_2$, then $\rho(X_1) \leq \rho(X_2)$.

(A2) **Translation invariance:** $\rho(X + c) = \rho(X) + c_1 + \ldots + c_N$.

(A3) **Convexity:** $\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho(X_1) + (1 - \lambda)\rho(X_2)$.

(A4) **Positive Homogeneity:** $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \geq 0$.

Furthermore, a coherent multivariate risk measure $\rho$ is called law invariant, if $X_1, X_2 \in X$ and $P(X_1 \leq z) = P(X_2 \leq z)$ for all $z \in \mathbb{R}^N$ implies $\rho(X_1) = \rho(X_2)$. Namely, if $X_1$ and $X_2$ are equal in distribution, denoted by $\rho(X_1) \overset{d}{=} \rho(X_2)$, their evaluated losses are equal to each other.

A fundamental law-invariant coherent risk measure in univariate case is Average-Value-at-Risk with a prespecified risk level $\alpha_i$ of a random loss $X$ denoted by $\text{AVaR}_{\alpha_i}(X)$. We recall that for $0 \leq \alpha_i < 1$ and $X_i \in X_i$, $\text{AVaR}_{\alpha_i}(X_i)$ is defined as

\[
\text{AVaR}_{\alpha_i}(X_i) \triangleq \mathbb{E}_{P_i}[X_i | X_i \geq x_{\alpha_i}],
\]

which is equal to

\[
\text{AVaR}_{\alpha_i}(X_i) \triangleq \sup_{Q_i \in \mathcal{D}_{\alpha_i}} \mathbb{E}[Q_i X_i]
\]

where

\[
\mathcal{D}_{\alpha_i} \triangleq \left\{ Q_i \in \mathcal{M}_i^1 : 0 \leq \frac{dQ_i}{dP_i} \leq \frac{1}{1 - \alpha_i} \right\}.
\]

(See [5] for the equivalence of (2.2) and (2.3).)

### 3 Multivariate Average-Value-at-Risk

**Definition 3.1.** The multivariate Average-Value-at-Risk $m\text{AVaR}_\alpha : X^N \rightarrow \mathbb{R}$ for risk level $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $0 \leq \alpha_i < 1$ for $i = 1, \ldots, N$ and $X \in X^N$ is defined as

\[
m\text{AVaR}_\alpha(X) \triangleq \mathbb{E} \left[ \sum_{i=1}^{N} X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right],
\]

where, $x_{\alpha_i}$ is the $\alpha_i$ quantile of $X_i$ for $i = 1, \ldots, N$ as in (2.1).
Remark 3.1. By independence of $X_i$’s with respect to $\mathbb{P}$, (3.4) translates into

$$m\text{AVaR}_\alpha(X) = \sum_{i=1}^N \mathbb{E}[X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}]$$

(3.5)

$$= \sum_{i=1}^N \mathbb{E}^{P_i}[X_i | X_i \geq x_{\alpha_i}]$$

$$= \sum_{i=1}^N \text{AVaR}_{\alpha_i}(X_i).$$

Lemma 3.1. $m\text{AVaR}_\alpha : \mathcal{X}^N \to \mathbb{R}$ is a multivariate law invariant coherent risk measure.

Proof. Let $X_1 = (X_1, \ldots, X_N) \in \mathcal{X}^N$ and $X_2 = (Y_1, \ldots, Y_N) \in \mathcal{X}^N$ with $X_1 \overset{d}{=} X_2$. Then $\sum_{i=1}^N X_i \overset{d}{=} \sum_{i=1}^N Y_i$ by (3.5). Hence, law invariance of $m\text{AVaR}_\alpha$ follows. We prove next that $m\text{AVaR}_\alpha$ satisfies the four axioms of multivariate coherent risk measure. Let $X$ be in $\mathcal{X}^N$ and $c \in \mathbb{R}^N$ and $\lambda \geq 0$ as in Definition 3.2.

(i) $m\text{AVaR}_\alpha$ is sublinear.

$$m\text{AVaR}_\alpha(X_1 + X_2) = \sum_{i=1}^N \text{AVaR}_{\alpha_i}(X_i + Y_i)$$

$$\leq \sum_{i=1}^N \text{AVaR}_{\alpha_i}(X_i) + \text{AVaR}_{\alpha_i}(Y_i)$$

$$= m\text{AVaR}_\alpha(X_1) + m\text{AVaR}_\alpha(X_2)$$

(ii) $m\text{AVaR}_\alpha$ is monotone. If $X_1 \leq X_2$, then

$$m\text{AVaR}_\alpha(X_1) = \sum_{i=1}^N \text{AVaR}_{\alpha_i}(X_i)$$

$$\leq \sum_{i=1}^N \text{AVaR}_{\alpha_i}(Y_i)$$

$$= m\text{AVaR}_\alpha(X_2)$$
(iii) $\text{mAVaR}_\alpha$ is translation invariant.

\[
\text{mAVaR}_\alpha(X + c) = \sum_{i=1}^{N} \text{AVaR}_{\alpha_i}(X_i + c_i) \\
= \sum_{i=1}^{N} \text{AVaR}_{\alpha_i}(X_i) + c_i \\
= \text{mAVaR}_\alpha(X) + \sum_{i=1}^{N} c_i
\]

(iv) $\text{mAVaR}_\alpha$ is positively homogeneous.

\[
\text{mAVaR}_\alpha(\lambda X) = \sum_{i=1}^{N} \text{AVaR}_{\alpha_i}(\lambda X_i) \\
= \lambda \sum_{i=1}^{N} \text{AVaR}_{\alpha_i}(X_i) \\
= \lambda \text{mAVaR}_\alpha(X)
\]

\[\square\]

**Theorem 3.1.** [2, 3] Let $X = (X_1, \ldots, X_N) \in \mathcal{X}^N$. Any multivariate law- invariant coherent risk measure $\rho : \mathcal{X}^N \to \mathbb{R}$ is of the form

\[
\rho(X) = \sup_{(Q_1, \ldots, Q_N) \in \mathcal{D}} \left( \mathbb{E}^{Q_1}[X_1] + \ldots + \mathbb{E}^{Q_N}[X_N] \right),
\]

for some $\mathcal{D} \subset \prod_{i=1}^{N} \mathcal{M}_1(X_i)$.

By Theorem 3.1 and Lemma 3.1, we get the following representation of $\text{mAVaR}_\alpha$.

**Lemma 3.2.** Let $\alpha = (\alpha_1, \ldots, \alpha_N) \in [0, 1)^N$, $X \in \mathcal{X}^N$ and $\text{mAVaR}_\alpha(X)$ be as defined in Definition 3.4. Then,

\[
\text{mAVaR}_\alpha(X) = \sup_{Q \in \mathcal{D}_\alpha} \mathbb{E}^Q \left[ \sum_{i=1}^{N} X_i \right] \\
= \sum_{i=1}^{N} \sup_{D_{\alpha_i}} \mathbb{E}^{D_i}[X_i],
\]
where \( \mathcal{D}_\alpha \in \mathcal{M}_1(\mathcal{X}^N) \) of the form \( \prod_{i=1}^N \mathcal{D}_{\alpha_i} \subset \prod_{i=1}^N \mathcal{M}_1(X_i) \) with

\[
\mathcal{D}_{\alpha_i} \triangleq \left\{ Q_i \in \mathcal{M}_1 : 0 \leq \frac{dQ_i}{dP_i} \leq \frac{1}{1 - \alpha_i} \right\}, \quad \text{for } i = 1, \ldots, N.
\]

**Proof.** By Lemma 3.1, we have that \( \text{mAVaR}_\alpha(X) \) is a multivariate coherent risk measure. Then, by Theorem 3.1, we have that

\[
\text{mAVaR}_\alpha(X) = \sup_{Q \in \mathcal{D}_\alpha} \mathbb{E}\left[ \sum_{i=1}^N X_i \right], \tag{3.6}
\]

for some \( \mathcal{D}_\alpha \subset \mathcal{M}_1(\mathcal{X}^N) \). We also have using (2.3) and (3.5) that

\[
\text{mAVaR}_\alpha(X) = \sum_{i=1}^N \text{AVaR}_{\alpha_i}(X_i) \tag{3.7}
\]

\[
= \sum_{i=1}^N \sup_{Q_i \in \mathcal{D}_{\alpha_i}} \mathbb{E}^Q[X_i],
\]

Hence, by (3.6) and (3.7), the result follows. \( \square \)

The next example will shed light into flexibility of \( \text{mAVaR}_\alpha \) compared to the sum of \( N \) independent random variables \( \sum_{i=1}^N X_i \) applied to univariate \( \text{AVaR}_{\tilde{\alpha}} \) with the corresponding risk level \( \tilde{\alpha} \).

**Example 3.1.** Let \( X = (X_1, \ldots, X_N) \in \mathcal{X}^N \) be a random vector with independent marginals and fix \( \alpha \in (0, 1) \). Define \( Z = \sum_{i=1}^N X_i \) on \( \mathcal{X}^N \) and denote the \( \alpha \) quantile of \( Z \) as \( z_\alpha \) as in (2.1). Note that

\[
\text{AVaR}_\alpha(Z) = \mathbb{E}[Z \mid Z \geq z_\alpha]
\]

\[
= \sup_{Q \in \mathcal{D}_{\text{AVaR}_\alpha}} \mathbb{E}^Q\left[ \sum_{i=1}^N X_i \right],
\]

where

\[
\mathcal{D}_{\text{AVaR}_\alpha} = \left\{ Q \in \mathcal{M}_1(\mathcal{X}^N) : 0 \leq \frac{dQ}{dP} \leq \frac{1}{1 - \alpha} \right\}.
\]

On the other hand, letting \( \tilde{\alpha} \triangleq (1 - \alpha)^{1/N} \), we have

\[
\text{AVaR}_{\tilde{\alpha}}\left( \sum_{i=1}^N X_i \right) = \sum_{i=1}^N \sup_{Q_i \in \mathcal{D}_{\tilde{\alpha}}} \mathbb{E}^{Q_i}[X_i]
\]
with
\[ D_{\tilde{\alpha}} = \left\{ Q_i \in M_1(X_i) : 0 \leq \frac{dQ_i}{dp_i} \leq \frac{1}{\tilde{\alpha}} \right\}. \]

Thus, choosing \( \tilde{\alpha} = (\tilde{\alpha}, \ldots, \tilde{\alpha}) \), we have
\[
mAVaR_{\tilde{\alpha}}(X) = \sum_{i=1}^{N} \text{AVaR}_{\tilde{\alpha}}(X_i)
= \text{AVaR}_{\alpha}\left(\sum_{i=1}^{N} X_i\right)
\]

Thus, only if the risk level \( \alpha_i \) for each \( X_i \) is equal to each other for each \( i \in [1, \ldots, N] \), then we can represent \( mAVaR_{\alpha}(X) \) via one-variate AVaR choosing the corresponding risk level \( \tilde{\alpha} \).

On the other hand, consider \( mAVaR_{\alpha} \) with \( \alpha = (\alpha_1, \ldots, \alpha_N) \) with \( \alpha_i \neq \alpha_j \) for some \( i, j \in [1, \ldots, N] \). Then, \( mAVaR_{\alpha} \) can not be represented using \( N \) independent \( X_i \)'s and a fixed \( \alpha \) risk level. This shows that \( mAVaR_{\alpha} \) achieves a bigger flexibility than sum of \( N \) independent risks required in multidimensional setting, while at the same time keeping the axioms of a multivariate coherent risk measure.

The next example shows that conditional expectation of a sum of dependent random variables conditioned on its corresponding quantiles might fail to be a coherent risk measure by violating the convexity axiom of a coherent risk measure.

**Example 3.2.** Consider a random variable \( X \in X^i \) having a continuous distribution and taking positive values with positive measure and let \( Y = \frac{X}{2} \). Fix \( \alpha \in (0, 1) \) such that \( x_\alpha > 0 \) is the corresponding \( \alpha \) quantile of \( X \). Choose \( \tilde{\alpha} \) such that \( \mathbb{P}_i(Y \leq x_\alpha) = \alpha \), i.e. \( \mathbb{P}_i(X \geq 2x_\alpha) = \tilde{\alpha} \), and denote the corresponding quantile as \( y_\tilde{\alpha} \). Then,
\[
\mathbb{E}_i[X + \frac{X}{2} | X \geq x_\alpha, X \geq 2y_\tilde{\alpha}] = \mathbb{E}_i\left[X + \frac{X}{2} | X \geq x_\alpha, X \geq 2x_\alpha\right]
= \frac{3}{2} \mathbb{E}[X | X \geq 2x_\alpha]
\]

Hence,
\[
\mathbb{E}_i[3/2X | X \geq 2x_\alpha] > \mathbb{E}_i[X | X \geq x_\alpha] + \mathbb{E}_i[1/2X | X \geq 2x_\alpha]
\mathbb{E}_i[X | X \geq 2x_\alpha] > \mathbb{E}_i[X | X \geq x_\alpha]
\]

In particular, convexity assumption (A3) in Definition 2.1 can be violated, in case the random variables are dependent. Note that the same violation is not to be observed, if the random variables are independent.
On the other hand, mAVaR$\alpha$ with $N = 1$ reduces to AVaR$\alpha$. Indeed, for $N = 1$, we have $\alpha = \alpha_1$, $X_1 = X$ and $X_2 = Y$. Denote $X + Y = Z$ such that the corresponding quantile of $Z$ as $z_\alpha$ with $\inf\{t \in \mathbb{R} : P(Z \leq z) \geq \alpha\}$. Hence,

\[
\mathbb{E}[3/2X|Z \geq z_\alpha] = \text{AVaR}_\alpha(3/2X)
\]

\[
\text{AVaR}_\alpha(3/2X) \leq \text{AVaR}_\alpha(X) + \text{AVaR}_\alpha(Y)
\]

\[
= \mathbb{E}[X|X \geq x_\alpha] + \mathbb{E}[Y|Y \geq y_\alpha]
\]

In particular, mAVaR$\alpha$ for $N = 1$ preserves convexity axiom of a univariate coherent risk measure. This also indicates that mAVaR$\alpha$ is the natural extension of AVaR$\alpha$ for $N > 1$.

### 3.1 Continuity Properties of mAVaR$\alpha$

We next introduce the relevant continuity notions in the following definition that we will verify for mAVaR$\alpha$.

**Definition 3.2.** Let $\rho : \mathcal{X}^N \to \mathbb{R}$ be a multivariate coherent risk measure, $(X_n \triangleq (X_n^1, \ldots, X_n^N))_{n \geq 1}$ be a sequence in $\mathcal{X}^N$ and $X \triangleq (X_1, \ldots X_N) \in \mathcal{X}^N$. Then,

- $\rho$ is called continuous from above, if $X_n \downarrow X$ implies that $\lim \rho(X_n) = \lim \rho(X)$.
- $\rho$ is called continuous from below, if $X_n \uparrow X$ implies that $\lim \rho(X_n) = \lim \rho(X)$.
- $\rho$ is said to have Fatou property, if for every $i = 1, \ldots, N$, $|X_i^n| \leq Y_i$ $\mathbb{P}_i$ a.s. for some $Y_i \in \mathcal{X}_i$ and $X_i^n \to X_i$ $\mathbb{P}_i$ a.s. imply that $\rho(X_n) \leq \lim \inf \rho(X)$.
- $\rho$ is called Lebesgue-continuous, if $X_n \to X$, $\mathbb{P}$-a.s. and $\|X_n\| \leq \|Y\|$ for some $Y \in \mathcal{X}^N$, then $\lim_n \rho(X_n) = \lim \rho(X)$.

We next recall a result for Average-Value-at-Risk in univariate case, i.e. $N = 1$.

**Theorem 3.2.** [17, 2] Let $\alpha \in [0, 1]$ be fixed. Then, AVaR$_\alpha : \mathcal{X}_i \to \mathbb{R}$ is

(i) continuous from below,

(ii) continuous from above,

(iii) has Fatou property,

(iv) is Lebesgue continuous.

We immediately get the following corollary for mAVaR$\alpha$.

**Corollary 3.1.** mAVaR$\alpha$ has the properties stated in Definition 3.2.

**Proof.** The result follows by the representation (3.7) and Equation (3.5). \qed
4 Applications to Normal Mixture Models

Although by construction $X \in \mathcal{X}^N$ is a random vector with independent marginals, we show that for multivariate Normal mixture random vectors with dependent marginals, $\text{mAVaR}_\alpha$ can still be used as a multivariate coherent risk measure while keeping its axioms. Namely, specific to multivariate Normal mixtures $X \in \mathcal{X}^N$, independence of the marginals of $X$ are not required. (We refer the reader to [20] for this class of Normal mixture multivariate distributions and its use in financial and actuarial modelling.) This is of fundamental importance, where the risk of the portfolio with dependent components is evaluated while keeping the axiomatic features of a coherent risk measure. This flexibility is due to peculiarity that any multivariate Gaussian random vector with possibly dependent Gaussian marginals can be represented by jointly independent Normal random variables.

Example 4.1. (Multivariate Gaussian Distribution) Consider $X = (X_1, \ldots, X_N) \in \mathcal{X}^N$ that is multivariate Gaussian distributed with covariance matrix

$$
\Sigma = \begin{bmatrix}
\sigma_{11}^2 & \cdots & \sigma_{1N} \\
\sigma_{21} & \cdots & \sigma_{2N} \\
\vdots & \ddots & \vdots \\
\sigma_{N1} & \cdots & \sigma_{NN}^2
\end{bmatrix}
$$

and mean vector $\mu = (\mu_1, \ldots, \mu_N)$. Then, this implies that each $X_i$ is of the form

$$
X_i = \mu_i + \sum_{j=1}^{N} \lambda_{ij}^i Y_j \text{ for } i = 1, \ldots, N,
$$

where $Y = (Y_1, \ldots, Y_N)$ is a Gaussian standard normal random vector with independent marginals for some scalars $\lambda_{ij}^i$ for $i, j = 1, \ldots, N$ (see e.g. [4]). Namely, $X = \mu + AY$ where

$$
A = \begin{bmatrix}
\lambda_{11}^1 & \cdots & \lambda_{N1}^1 \\
\lambda_{12}^2 & \cdots & \lambda_{N2}^2 \\
\vdots & \ddots & \vdots \\
\lambda_{1N}^N & \cdots & \lambda_{NN}^N
\end{bmatrix}
$$

is an $\mathbb{R}^{N \times N}$ matrix and $\mu \in \mathbb{R}^N$ is the $N$-dimensional mean vector. Suppose one wants to evaluate for $\alpha = (\alpha_1, \ldots, \alpha_N) \in [0, 1)^N$, the total sum of the cost of the portfolio

$$
\text{mAVaR}_\alpha(X) = \mathbb{E} \left[ \sum_{i=1}^{N} X_i \mid X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right].
$$
Then, one solves the linear system

\[
\begin{align*}
\mu_1 + \lambda_1^1 y_{a_1} + \lambda_1^2 y_{a_2} + \ldots + \lambda_N^1 y_{a_N} &= x_{a_1} \\
\mu_2 + \lambda_1^2 y_{a_1} + \lambda_2^2 y_{a_2} + \ldots + \lambda_N^2 y_{a_N} &= x_{a_2} \\
\vdots & \quad \ldots \\
\mu_N + \lambda_1^N y_{a_1} + \lambda_2^N y_{a_2} + \ldots + \lambda_N^N y_{a_N} &= x_{a_N}
\end{align*}
\]

for \( y_{a_1}, y_{a_2}, \ldots, y_{a_N} \) for given quantiles \( x_{a_1}, x_{a_2}, \ldots, x_{a_N} \). Note that the system (4.9) would reasonably have a unique solution. Otherwise, one of \( X_i \), e.g. \( X_1 \), could be defined by \( N-1 \) other risks \( X_2, \ldots, X_N \). But then, this would imply that \( X_1 \) is a linear combination of \( X_2, \ldots, X_N \). In particular, we would have \( N-1 \) conditions and the conditional information would be formed by \( X_2 \geq x_{a_2}, \ldots, X_N \geq x_{a_N} \). Thus, we would use \( N-1 \) dimensional mAVaR with \( \alpha = (\alpha_2, \ldots, \alpha_N) \). In particular, matrix \( A \) in (4.8) is invertible. Thus, the linear system (4.9) has a unique solution with \( N \) equations and \( N \) unknowns, and we get

\[
mAVaR_\alpha(X) = \sum_{i=1}^N \mu_i + \sum_{i=1}^N \mathbb{E} \left[ \sum_{j=1}^N \lambda_i^j Y_j | Y_1 \geq y_{a_1}, \ldots, Y_N \geq y_{a_N} \right]
\]

\[
= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \sum_{j=1}^N \lambda_i^j \mathbb{E}^{P_i} \left[ Y_j | Y_i \geq y_{a_i} \right]
\]

\[
= \sum_{i=1}^N \mu_i + \sum_{i=1}^N \sum_{j=1}^N \lambda_i^j \frac{\phi(y_{a_i})}{1 - \alpha_i},
\]

where \( \phi(\cdot) \) is the probability density function of a standard normal random variable. In (4.10), we used the closed form formula for the standard normal random variable \( Y_i \) with with risk level \( \alpha_i \) and \( y_{a_i} \) as the corresponding quantile, whose derivation is

\[
\text{AVaR}_{\alpha_i}(Y_i) = \mathbb{E}^{P_i}[Y_i | Y_i \geq y_{a_i}]
\]

\[
= \frac{1}{\mathbb{P}^i(Y_i \geq y_{a_i})} \frac{1}{\sqrt{2\pi}} \int_{y_{a_i}}^\infty ye^{-\frac{y^2}{2}} dy
\]

\[
= \frac{\phi(y_{a_i})}{1 - \alpha_i}.
\]

Based on Example 4.1, we will next evaluate the tail risk for losses \( X \in \mathcal{X}^N \) with dependent marginals, where \( X \) is assumed to have a Gaussian variance mixture model using mAVaR. Gaussian variance mixtures are generalizations of multivariate Gaussian random vectors generated by incorporating randomness into the covariance matrix of a Gaussian random vector. To be more specific, \( X \in \mathcal{X}^N \) is said to have a normal variance mixture, if

\[
X = \mu + \sqrt{W} AZ,
\]

(4.11)
where

- \( Z \) is multivariate Gaussian random vector in \( \mathcal{X}^N \).
- \( A \) is an \( \mathbb{R}^{n \times n} \) invertible matrix as in (4.8).
- \( W \geq 0 \) is a nonnegative real valued univariate random variable independent of \( Z \).

Note that if we condition on \( W \), then \( X \) is multivariate Gaussian random vector. This observation is the key to to evaluate mAVaR_\( \alpha \) on any Gaussian variance mixture models. In the next example, we will model the \( N \) dimensional portfolio \( X \) having a multivariate \( t \)-distribution.

**Example 4.2. (Multivariate \( t \)-Distribution)** Let \( X = (X_1, \ldots, X_N) \in \mathcal{X}^N \) be multivariate \( t \)-distribution. Namely, \( X \) is as in (4.11) where \( W \) has inverse Chi-square distribution for a prespecified degree of freedom \( \nu > 2 \), i.e.

\[
f(x; \nu) = \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} x^{-\nu/2} e^{-\nu/(2x)} \quad \text{for } x > 0,
\]

where \( \Gamma(\cdot) \) is the Gamma function (see e.g. [19]). Then, conditioned on \( W \), we are back in the setting of Example 4.1. In particular, given \( W \), we choose heuristically the risk levels \( \alpha = (\alpha_1, \ldots, \alpha_N) \) and the corresponding quantiles \( (x_{\alpha_1}, \ldots, x_{\alpha_N}) \) of multivariate Gaussian random vector \( WX | W = w \) such that using (4.9), (4.10) and (4.11) we get

\[
\mathbb{E} \left[ \sum_{i=1}^{N} X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}, W = w \right]
\]

\[
= \sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda^j_i \mathbb{E} \left[ Y_i | Y_i \geq y_{\alpha_i} \right]
\]

\[
= \sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} \sqrt{w} \lambda^j_i \frac{\phi(y_{\alpha_i})}{1 - \alpha_i}.
\]

Thus, we conclude the explicit expression using (4.12) analogous to (4.10)

\[
\text{mAVaR}_\alpha(X) = \mathbb{E} \left[ \sum_{i=1}^{N} X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right]
\]

\[
= \sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda^j_i \frac{\phi(y_{\alpha_i})}{1 - \alpha_i} \int_0^\infty \sqrt{w} \frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} w^{-\nu/2} e^{-\nu/(2x)} dw
\]
Next example will demonstrate that mAVaR\(_\alpha\) can handle Gaussian mean variance mixture models, as well, which are prevalent in financial and actuarial modelling. The motivation behind application of Gaussian mean variance mixture models in finance is that it is observed in stock returns negative returns (losses) have heavier tails than positive returns (gains). In particular, we need to introduce asymmetry by mixing normal distributions with different means as well as different variances giving the class of multivariate normal mean-variance mixtures so that we have a skewed distribution.

**Example 4.3. (Generalised Hyperbolic Model)** Let \(X \in \mathcal{X}^N\) be of the form

\[
X = \mu + W \gamma + \sqrt{W} AZ,
\]

where \(\mu, A, Z\) are as in (4.11). \(\gamma \in \mathbb{R}^N\) is an \(N\) dimensional vector giving the mixture in mean term \(\mu\). Note that we are back in the setting (4.11) when \(\gamma = (0, \ldots, 0)\). \(W\) is assumed to be Gamma distributed, i.e. its density function is of the

\[
\frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, \text{ for } x > 0,
\]

where \(b > 0\) and \(a \in \mathbb{R}\) are scalars. As in Example 4.2, we use the observation \(X\) conditioned on \(W\) is multivariate Gaussian. Hence, we have

\[
\begin{align*}
\mathbb{E} \left[ \sum_{i=1}^{N} \mu_i + W \sum_{i=1}^{N} \gamma_i + \sqrt{w} \sum_{i=1}^{N} X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}, W = w \right] \\
= \sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} \gamma_i \mathbb{E}[W] + \mathbb{E}[\sqrt{w} \sum_{i=1}^{N} X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}, W = w] \\
= \sum_{i=1}^{N} \mu_i + ab \sum_{i=1}^{N} \gamma_i + \mathbb{E}[\sqrt{w} \sum_{i=1}^{N} X_i | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}, W = w] \\
= \sum_{i=1}^{N} \mu_i + ab \sum_{i=1}^{N} \gamma_i + \sum_{i=1}^{N} \sqrt{w} \sum_{j=1}^{N} \lambda_i \frac{\phi(y_{\alpha_i})}{1 - \alpha_i},
\end{align*}
\]

where \(\mathbb{E}[W] = ab\) stands for the expected value of a Gamma\((a, b)\) distribution. Thus, following the lines as in 4.2, we have

\[
\text{mAVaR}_\alpha(X) = \sum_{i=1}^{N} \mu_i + ab \sum_{i=1}^{N} \gamma_i + \mathbb{E}\left[\sqrt{W}\right] \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \frac{\phi(y_{\alpha_i})}{1 - \alpha_i} \\
= \sum_{i=1}^{N} \mu_i + ab \sum_{i=1}^{N} \gamma_i + \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \frac{\phi(y_{\alpha_i})}{1 - \alpha_i} \right) \int_0^\infty \sqrt{w} \frac{b^a}{\Gamma(a)} w^{a-1} e^{-bw} dw
\]
5 A Tail Inequality for mAVaR

In this section, we assume that for $X = (X_1, \ldots, X_N)$, each $X_i \in \mathcal{X}_i$ is further an element of $L^2(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, i.e. $\mathbb{E}^\mathbb{P}_i[(X_i)^2] < \infty$ for all $i = 1, \ldots, N$.

**Definition 5.1.** (i) Let $X_i \in \mathcal{X}_i$ and $\alpha_i \in [0, 1)$ be the risk level. We define the tail variance of $X_i$ with risk level $\alpha_i$ as

$$t\text{Var}_{\alpha_i}(X_i) \triangleq \mathbb{E}^\mathbb{P}_i[(X_i - \text{AVaR}_{\alpha_i}(X_i))^2 | X_i \geq x_{\alpha_i}]$$

(ii) Let $X \in \mathcal{X}_N$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in [0, 1)^N$ be the risk ratio. We define the tail variance of total sum of risk $X$ with risk ratio $\alpha$ as

$$m\text{tVar}_\alpha(X) \triangleq \mathbb{E}[(\sum_{i=1}^N X_i - \text{mAVaR}_\alpha(X))^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}]$$

We next list the properties that reveal the analogy between variance and multivariate tail conditional variance.

**Lemma 5.1.** Let $\alpha = (\alpha_1, \ldots, \alpha_N) \in [0, 1)^N$, $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$ and $c = (c_1, \ldots, c_N) \in \mathbb{R}^N$ and let $X = (X_1, \ldots, X_N) \in \mathcal{X}_N$. Then, $m\text{tVar}_\alpha$ satisfies the following properties.

(i) $m\text{tVar}_\alpha(X) = \sum_{i=1}^N t\text{Var}_{\alpha_i}(X_i)$

**Proof.** We have

$$\mathbb{E}[(\sum_{i=1}^N X_i - \text{AVaR}_\alpha(\sum_{i=1}^N X_i))^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}]$$

$$= \sum_{i=1}^N \mathbb{E}[(X_i - \text{AVaR}_{\alpha_i}(X_i))^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}]$$

$$+ 2 \sum_{1 \leq i < j \leq N} \mathbb{E}[(X_i - \text{AVaR}_{\alpha_i}(X_i))(X_j - \text{AVaR}_{\alpha_j}(X_j))]$$

$$= \sum_{i=1}^N \mathbb{E}[(X_i - \text{AVaR}_{\alpha_i}(X_i))^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}]$$

$$= \sum_{i=1}^N t\text{Var}_{\alpha_i}(X_i),$$

where the last two equalities are implied by joint independence of $X_1, \ldots, X_N$. □
(ii) $\text{mtVar}_\alpha(\lambda X) = \sum_{i=1}^{N} \lambda_i^2 \text{tVar}_{\alpha_i}(X_i)$

Proof. The result follows by

$$\text{mtVar}_\alpha(\lambda X) = \sum_{i=1}^{N} \mathbb{E}[\left( \sum_{i=1}^{N} \lambda_i X_i - \text{AVaR}_{\alpha_i}(\lambda_i X_i) \right)^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}]$$

$$= \sum_{i=1}^{N} \lambda_i^2 \mathbb{E}[\left( \sum_{i=1}^{N} X_i - \text{AVaR}_{\alpha_i}(X_i) \right)^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}]$$

$$= \sum_{i=1}^{N} \lambda_i^2 \mathbb{E}[\left( \sum_{i=1}^{N} X_i - \text{AVaR}_{\alpha_i}(X_i) \right)^2 | X_i \geq x_{\alpha_i}]$$

\[\square\]

(iii) $\text{mtVar}_\alpha(X + c) = \text{mtVar}_\alpha(X)$

Proof. The result follows by

$$\text{mtVar}_\alpha(X + c) = \sum_{i=1}^{N} \mathbb{E}[\left( X_i + c_i - \text{AVaR}_{\alpha_i}(X_i + c_i) \right)^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}]$$

$$= \sum_{i=1}^{N} \mathbb{E}[\left( X_i - \text{AVaR}_{\alpha_i}(X_i) \right)^2 | X_i \geq x_{\alpha_i}, \ldots, X_N \geq x_{\alpha_N}]$$

\[\square\]

Next, we introduce the following tail Chebyshev inequality giving the connection between $\text{mtVar}_\alpha(\cdot)$ and $\text{mAVaR}_\alpha(\cdot)$. In particular, as in the classical case, scaling by $\sqrt{\text{mtVar}_\alpha}$ and shifting by $\text{mAVaR}_\alpha$, one can get the confidence interval of how much $X \in \mathcal{X}^N$ deviates from $\text{mAVaR}_\alpha(X)$ in the tail event $\{X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N}\}$ and how reliable $\text{mAVaR}_\alpha$ is.

Lemma 5.2. Let $k > 0$ with $X = (X_1, \ldots, X_N)$. Then, we have

$$\mathbb{P}\left( \frac{|\sum_{i=1}^{N} X_i - \text{mAVaR}_\alpha(X)|}{\sqrt{\text{mtVar}_\alpha(X)}} \geq k | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right) \leq \frac{1}{k^2}$$
Proof. The proof follows the same lines as in classical Chebyshev inequality. We have

\[ k^2 \mathbb{P}\left( \left( \sum_{i=1}^{N} X_i - \text{AVaR}_{\alpha_i}(X_i) \right)^2 \geq k^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right) \]

\[ \leq \mathbb{E}\left[ \left( \sum_{i=1}^{N} X_i - \text{AVaR}_{\alpha_i}(X_i) \right)^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right] \]

Thus, we have

\[ \mathbb{E}\left[ \sum_{i=1}^{N} (X_i - \text{AVaR}_{\alpha_i}(X_i))^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right] \]

\[ + 2 \sum_{1 \leq i < j \leq N} \mathbb{E}\left[ (X_i - \text{AVaR}_{\alpha_i}(X_i))(X_j - \text{AVaR}_{\alpha_j}(X_j)) | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right] \]

\[ = \sum_{i=1}^{N} \mathbb{E}\left[ (X_i - \text{AVaR}_{\alpha_i}(X_i))^2 | X_i \geq x_{\alpha_i} \right] \]

\[ = \sum_{i=1}^{N} \text{tVar}_{\alpha_i}(X_i) \]

\[ = \text{mtVar}_{\alpha}(X), \tag{5.13} \]

where (5.13) follows by joint independence of \((X_1, \ldots, X_n)\). Hence, we conclude the proof. □

We next give an application of Lemma 5.2 following Example 4.1. Derivations on multivariate Gaussian mixtures as in Example 4.2 and Example 4.3 are analogous and skipped.

Example 5.1. Let \(X \in X^N\) be the multivariate Gaussian vector with covariance matrix \(\Sigma\) and \(\mu\) as in Example 4.1. Suppose one wants to evaluate

\[ \text{mtVar}_{\alpha}(X) = \mathbb{E}\left[ \left( \sum_{i=1}^{N} X_i - \text{mAVaR}_{\alpha}(X) \right)^2 | X_1 \geq x_{\alpha_1}, \ldots, X_N \geq x_{\alpha_N} \right]. \tag{5.14} \]

Then, representing each \(X_i\) via

\[ X_i = \mu_i + \sum_{j=1}^{N} \lambda_j^i Y_j, \quad \text{for } i = 1, \ldots, N, \]

we have

\[ \text{mAVaR}_{\alpha}(X) = \sum_{i=1}^{N} \mu_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}\left[ \lambda_j^i Y_j | Y_i \geq y_{\alpha_i} \right]. \]
Then, we rewrite (5.14)

\[
\text{mtVar}_\alpha(X) = \mathbb{E} \left[ \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i^j Y_j - \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}[\lambda_i^j Y_j | Y_1 \geq y_1, \ldots, Y_N \geq y_N] \right)^2 | Y_1 \geq y_1, \ldots, Y_N \geq y_N \right]
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i^j)^2 \text{tVar}_{\alpha_j}(Y_j) + 2 \sum_{1 \leq i < j \leq N} \mathbb{E} \left[ \left( \sum_{k=1}^{N} \lambda_k^i Y_k - \mathbb{E} \left[ \sum_{k=1}^{N} \lambda_k^i Y_k | Y_1 \geq y_{a_1}, \ldots, Y_N \geq y_{a_N} \right] \right) \right. \\
\times \left. \left( \sum_{k=1}^{N} \lambda_k^j Y_k - \mathbb{E} \left[ \sum_{k=1}^{N} \lambda_k^j Y_k | Y_1 \geq y_{a_1}, \ldots, Y_N \geq y_{a_N} \right] \right) \right] \left[ Y_1 \geq y_{a_1}, \ldots, Y_N \geq y_{a_N} \right]
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} (\lambda_i^j)^2 \text{tVar}_{\alpha_j}(Y_j) + 2 \sum_{1 \leq i < j \leq N} \mathbb{E} \left[ \left( \sum_{k=1}^{N} \lambda_k^i Y_k - \sum_{k=1}^{N} \lambda_k^i \text{AVaR}_{\alpha_k}(Y_k) \right) \right. \\
\times \left. \left( \sum_{k=1}^{N} \lambda_k^j Y_k - \sum_{k=1}^{N} \lambda_k^j \text{AVaR}_{\alpha_k}(Y_k) \right) \right] \left[ Y_1 \geq y_{a_1}, \ldots, Y_N \geq y_{a_N} \right]
\]

(5.15)

Here, we use the

\[
\text{tVar}_{\alpha_j}(Y_j) = \mathbb{E}[Y_j^2 | Y_j \geq y_{a_j}] - \text{AVaR}_{\alpha_j}^2(Y_j)
\]

\[
= \alpha_j \phi(\alpha_j) + 1 - \alpha_j - \left( \frac{\phi(y_{a_j})}{1 - \alpha_j} \right)^2
\]

Further, we use

\[
\mathbb{E}[g(Y_1, \ldots, Y_N) | Y_1 \geq y_{a_1}, \ldots, Y_N \geq y_{a_N}] = \left( \prod_{i=1}^{N} \frac{1}{1 - \alpha_i} \right) \frac{1}{(2\pi)^{n/2}} \\
\times \int_{y_{a_1}}^{\infty} \cdots \int_{y_{a_N}}^{\infty} g(y_1, \ldots, y_N) e^{-\frac{\sum_{i=1}^{N} y_i^2}{2}} dy_1 \cdots dy_N,
\]

where

\[
g(y_1, \ldots, y_N) = \left( \left( \sum_{k=1}^{N} \lambda_k^i y_k - \sum_{k=1}^{N} \lambda_k^i \phi(y_{a_k}) \right) \right) \\
\times \left( \sum_{k=1}^{N} \lambda_k^j y_k - \sum_{k=1}^{N} \lambda_k^j \phi(y_{a_k}) \right)
\]

Hence, we reach to the explicit solution of (5.15).
References


