Distributionally robust zero-sum games

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Abstract

We consider a two player zero-sum game with stochastic linear constraints. The probability distributions of the vectors associated with the constraints are partially known. The available information with respect to the distribution is based mainly on the two first moments. In this vein, we formulate the stochastic linear constraints as distributionally robust chance constraints. We consider three different types of moments based uncertainty sets. For each uncertainty set, we show that there exists a saddle point equilibrium of the game. The latter corresponds to the optimal solution of a primal-dual pair of second order cone programs. In order to illustrate our theoretical results, we consider an instance of a zero-sum game together with distributionally robust chance constraints. A comparison with expected value constraints zero-sum game is drawn to illustrate the effectiveness of our approach.

Keywords: Stochastic programming, Distributionally robust chance constraints, Zero-sum game, Saddle point equilibrium, Second-order cone program.

1. Introduction

In two player zero-sum games, the gain of one player is the loss of the other player. We typically represent a zero-sum game with a payoff matrix where the rows and the columns are the actions of player 1 and player 2, respectively. Neumann \cite{32} showed that there exists a mixed strategy saddle point equilibrium

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(SPE) of a zero-sum game. Dantzig [13] showed the equivalence of SPE and the solution of a primal-dual pair of linear programs. Adler [1] studied the equivalence between linear programming problems and zero-sum games. Charnes [7] generalized the zero-sum game considered in [13, 32]. He introduced linear inequality constraints for both the players, and showed the equivalence of SPE of a constrained zero-sum game and the solution of a primal-dual pair of linear programs. Nash [24] introduced the equilibrium concept for a finite number of rational players where each player has finite number of actions. He showed that there exists a mixed strategy Nash equilibrium for finite strategic games. Since then, general strategic games have been extensively studied in the literature [2, 14, 17].

The games discussed in the above-mentioned papers have deterministic strategy set and payoff function for each player. However, the decision making process is fed by input parameters which are usually subject to uncertainties [12, 23]. The model uncertainties can be accounted for by the expected value approach in case of risk neutral decision makers. Ravat and Shanbhag [26] considered the stochastic Nash games where each player optimizes his expected value payoff subject to his expected value constraints. For risk averse players, the payoff criterion with the risk measure CVaR [20, 26] and the variance was considered in the literature [11]. Singh et al. [28, 29, 30] introduced a risk averse payoff criterion based on the chance constraint programming to model finite strategic games with random payoffs. Such games are called chance-constrained games. As for the elliptically distributed random payoffs, the authors showed the existence of a Nash equilibrium for a chance-constrained game [28], and proposed an equivalent mathematical program to compute the Nash equilibria of the game [30]. In [29], the authors considered the games where the probability distributions of the players’ payoffs are partially known, and belong to given distributional uncertainty sets. They formulated these games via a distributionally robust chance constraint, and showed the existence of a Nash equilibrium. Moreover, they proposed an equivalent mathematical program based method to come-up with Nash equilibria. There are several zero-sum chance-constrained games studied in past literature [3, 6, 8, 10].

The chance-constrained games in the above-mentioned papers considered the case where players’ payoffs are random variables and the strategy sets are deterministic in nature. To the best of our knowledge, the research on the games with stochastic strategy sets defined by chance constraints is very scarce [25, 31]. Peng et al. [25] considered an $n$-player general sum game with joint chance constraints for each player. They showed the existence of a Nash equilibrium when the row vectors of the stochastic constraints are independent, and follow a multi-
variate normal distribution. Singh and Lisser [31] considered a stochastic version of a two player constrained zero-sum game studied in [7] where each player has individual chance constraints. They showed the equivalence of an SPE and the optimal solution of a primal-dual pair of a second order cone programs (SOCPs) when the random row vectors of the constraints of each player follow a multivariate elliptically symmetric distribution. Unlike stochastic two players constrained zero-sum game [31], we have no further information on the probability distribution of the random constraint vectors. However, we know that the distribution of these random vectors belong to a given uncertainty set. In this paper, we consider three different types of well-known uncertainty sets [9, 15, 16] based on the first and second order moments of the stochastic linear constraint random vectors. For each type of uncertainty set, we show that there exists an SPE obtained from the optimal solutions of a primal-dual pair of SOCPs. In order to illustrate our theoretical results, we compare the distributionally robust chance constraints case with the expected value constraints case.

The structure of the rest of the paper is as follows. Section 2 contains the definition of a zero-sum game with distributionally robust chance constraints. Section 3 presents the reformulation of distributionally robust chance constraints as second order cone constraints under three different uncertainty sets. Section 4 outlines a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game. We present our numerical results in Section 5, and conclude the paper in Section 6.

2. The Model

We consider a two-player zero-sum game defined by an \( n_1 \times n_2 \) matrix \( G \), where \( n_1 \) and \( n_2 \) denote the number of pure strategies of player 1 and player 2, respectively. The set of mixed strategies of player \( i, i = 1, 2 \), is given by \( X^i = \left\{ x^i \in \mathbb{R}^{n_i} \middle| \sum_{j=1}^{n_i} x^i_j = 1, x^i_j \geq 0, \forall j = 1, 2, \ldots, n_i \right\} \). Each component of the matrix \( G \) corresponds to the payoff for player 1 and the cost for player 2. For a given strategy pair \( (x^1, x^2) \in X^1 \times X^2 \), \( (x^1)^T G x^2 \) represents the payoff of player 1 and the cost of player 2; \( T \) denotes the transposition. For a given \( x^1 \in X^1 \) (resp., \( x^2 \in X^2 \)), player 2 (resp., player 1) chooses an optimal strategy by minimizing (resp., maximizing) \( (x^1)^T G x^2 \) over all \( x^2 \in X^2 \) (resp., \( x^1 \in X^1 \)). A strategy pair \((x^1, x^2)\) is an SPE of the zero-sum game if \( x^1 \) (resp., \( x^2 \)) is an optimal strategy of player 1 (resp., player 2) for a fixed strategy \( x^2 \) (resp., \( x^1 \)) of player 2 (resp., player 1). An SPE of a zero-sum game exists in mixed strategies [32], and can be computed via a primal-dual pair of linear programs [13]. In certain cases,
the players' strategies are further restricted by linear inequalities. For example, a mixed strategy \( x^1 \) of player 1 is subject to the following linear constraints

\[
A^1 x^1 \leq b^1, \quad (2.1)
\]

whilst the mixed strategy \( x^2 \) of player 2 satisfies the linear constraints given by

\[
A^2 x^2 \geq b^2, \quad (2.2)
\]

where \( A^1 \) is a \( p \times n_1 \) matrix and \( A^2 \) is a \( q \times n_2 \) matrix. We denote \( A^1 = [a^1_1, a^1_2, \ldots, a^1_p]^T \) and \( A^2 = [a^2_1, a^2_2, \ldots, a^2_q]^T \), where \( a^i_k \) represents the \( k \)th row of the matrix \( A^i \). Let \( J_1 = \{1, 2, \ldots, p\} \) and \( J_2 = \{1, 2, \ldots, q\} \) be the index sets for the constraints of player 1 and player 2, respectively.

A zero-sum game where \( A^1 \) and \( A^2 \) are deterministic matrices is considered by Charnes [7]. He showed that an SPE of a constrained zero sum game is an optimal solution to a primal-dual pair of linear programs. Singh and Lisser [31] considered the case when \( A^1 \) and \( A^2 \) are random matrices where each row vector follows a multivariate elliptically symmetric distribution. They formulated each stochastic linear constraint from (2.1) and (2.2) as a chance constraint. Such games are called zero-sum chance-constrained games. They showed that an SPE of a zero-sum chance-constrained game is an optimal solution to a primal-dual pair of SOCPs.

In this paper, we consider the case where we have no distributional knowledge of the probability distributions of the random vectors of \( A^1 \) and \( A^2 \) except their first two moments which leads to optimize over an uncertainty set. When considering the worst case scenario, we use distributionally robust framework to formulate the stochastic linear constraints (2.1) and (2.2). The distributionally robust chance constraints of player 1 are given by

\[
\inf_{F^1_k \in \mathcal{D}^1_k} \mathbb{P} \left( a^1_k x \leq b^1_k \right) \geq \alpha^1_k, \quad \forall \, k \in J_1, \quad (2.3)
\]

where \( F^1_k \) is a probability distribution of \( a^1_k \), \( \mathcal{D}^1_k \) is the uncertainty set corresponding to the probability distribution of random vector \( a^1_k \), and \( \alpha^1_k \) is the confidence level of player 1 for the \( k \)th constraint. Similarly, the distributionally robust chance constraints of player 2 are given by

\[
\inf_{F^2_l \in \mathcal{D}^2_l} \mathbb{P} \left( -a^2_l y \leq -b^2_l \right) \geq \alpha^2_l, \quad \forall \, l \in J_2, \quad (2.4)
\]
where $F^2_l$, $D^2_l$ and $\alpha^2_l$ are analogously defined. Therefore, for $\alpha^1 = (\alpha^1_k)_{k \in J_1}$ and $\alpha^2 = (\alpha^2_l)_{l \in J_2}$ the feasible strategy sets of player 1 and player 2 are given by

$$S^1_{\alpha^1} = \left\{ x^1 \in X^1 \mid \inf_{F^1_k \in D^1_k} \mathbb{P}\{ a^1_k x^1 \leq b^1_k \} \geq \alpha^1_k, \ \forall \ k \in J_1 \right\}, \quad (2.5)$$

and

$$S^2_{\alpha^2} = \left\{ x^2 \in X^2 \mid \inf_{F^2_l \in D^2_l} \mathbb{P}\{ -a^2_l x^2 \leq -b^2_l \} \geq \alpha^2_l, \ \forall \ l \in J_2 \right\}. \quad (2.6)$$

We call the matrix game $G$ with the strategy set $S^1_{\alpha^1}$ for player 1 and the strategy set $S^2_{\alpha^2}$ for player 2 as a zero-sum game with distributionally robust chance constraints. We denote this game by $Z\alpha$. A strategy pair $(x^1, x^2) \in S^1_{\alpha^1} \times S^2_{\alpha^2}$ is called an SPE of the game $Z\alpha$ at $\alpha = (\alpha^1, \alpha^2) \in [0, 1]^p \times [0, 1]^q$, if

$$(x^1)^T G x^2 \leq (x^1)^T G x^2 \leq (x^1)^T G x^2,$$

for all $x^1 \in S^1_{\alpha^1}, x^2 \in S^2_{\alpha^2}$.

3. Reformulation of distributionally robust chance constraints

We consider the case where we only have full/partial information about the mean vectors and covariance matrices of the random vectors $a^1_k$, $k \in J_1$, and $a^2_l$, $l \in J_2$. We consider three different uncertainty sets based on the available information about the mean vectors and the covariance matrices. For each uncertainty set, chance constraints (2.3) and (2.4) are reformulated as second order cone constraints.

3.1. Moments based uncertainty sets

We consider three different types of uncertainty sets based on first and second order moments information. First, we consider the case where the first two order moments are known. The uncertainty set of player $i$, $i = 1, 2$, defined by the mean vector $\mu^i_k$ and the covariance matrix $\Sigma^i_k$ of $(a^i_k)^T$, is given by

$$D^i_k (\mu^i_k, \Sigma^i_k) = \left\{ F^i_k \mid E_{F^i_k} \left[ (a^i_k)^T \right] = \mu^i_k \right\}, \quad (3.1)$$

for all $k \in J_i$. These uncertainty sets are considered in [5, 16]. As for the second uncertainty set, we assume that the first moment is known whilst the second moment is unknown. The uncertainty set of player $i$, $i = 1, 2$, defined by the mean
vector $\mu^i_k$ and the upper bound $\Sigma^i_k$ on covariance matrix of $(a^i_k)^T$, is given by

$$D^i_k (\mu^i_k, \Sigma^i_k) = \left\{ F^i_k \mid E_{F^i_k}[(a^i_k)^T] = \mu^i_k, \right. \left. \mathrm{COV}_{F^i_k}[(a^i_k)^T] \preceq \gamma^i_{k1}\Sigma^i_k \right\}, \quad (3.2)$$

for all $k \in J_i$. These uncertainty sets are considered in [9]. The last uncertainty set is based on unknown first two moments. The uncertainty set of player $i, i = 1, 2$, where the mean vector of $(a^i_k)^T$ lies in an ellipsoid of size $\gamma^i_{k1} \geq 0$ centered at $\mu^i_k$ and the covariance matrix of $(a^i_k)^T$ lies in a positive semidefinite cone defined with a matrix inequality, is given by

$$D^i_k (\mu^i_k, \Sigma^i_k) = \left\{ F^i_k \mid \left( \mathbb{E}_{F^i_k}[(a^i_k)^T] - \mu^i_k \right)^\top \left( \Sigma^i_k \right)^{-1} \left( \mathbb{E}_{F^i_k}[(a^i_k)^T] - \mu^i_k \right) \leq \gamma^i_{k1}, \right. \left. \mathrm{COV}_{F^i_k}[(a^i_k)^T] \preceq \gamma^i_{k2}\Sigma^i_k \right\}, \quad (3.3)$$

for all $k \in J_i$. $\mathrm{COV}_{F^i_k}$ is a covariance operator under probability distribution $F^i_k$. These uncertainty sets are considered in [15].

### 3.2. Second order cone constraint reformulation

It follows from [5, 9, 16] that the chance constraints (2.3) and (2.4) for uncertainty sets (3.1) and (3.2) can be reformulated as

$$(\mu^1_k)^T x^1 + \sqrt{\frac{\alpha^1_k}{1 - \alpha^1_k}} \left\| \left( \Sigma^1_k \right)^{1/2} x^1 \right\| \leq b^1_k, \quad \forall k \in J_1, \quad (3.4)$$

and

$$-(\mu^2_l)^T x^2 + \sqrt{\frac{\alpha^2_l}{1 - \alpha^2_l}} \left\| \left( \Sigma^2_l \right)^{1/2} x^2 \right\| \leq -b^2_l, \quad \forall l \in J_2, \quad (3.5)$$

where $\| \cdot \|$ is the Euclidean norm. Based on the structure of the uncertainty set (3.3), the constraint (2.3) can be written as

$$\inf_{(\mu, \Sigma) \in U^1_k} \inf_{F^i_k \in \mathcal{D}(\mu, \Sigma)} \mathbb{P}\{a^i_k x^1 \leq b^1_k\} \geq \alpha^1_k,$$

where

$$\mathcal{D}(\mu, \Sigma) = \left\{ F^i_k \mid E_{F^i_k}[(a^i_k)^T] = \mu, \mathrm{COV}_{F^i_k}[(a^i_k)^T] = \Sigma \right\}$$

and

$$U^1_k = \left\{ (\mu, \Sigma) \mid (\mu - \mu^1_k)^\top \left( \Sigma^1_k \right)^{-1} (\mu - \mu^1_k) \leq \gamma^1_{k1}, \Sigma \preceq \gamma^2_{k2}\Sigma^1_k \right\}.$$
According to one-sided Chebyshev inequality [22, 27], we have
\[
\inf_{F_k^1 \in \mathcal{D}(\mu, \Sigma)} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} = \begin{cases} 
1 - \frac{1}{1 + \frac{1}{(x^1)^T \Sigma x^1}}, & \text{if } \mu^T x^1 \leq b_k^1, \\
0, & \text{otherwise}. 
\end{cases}
\]

For the case \(\mu^T x^1 > b_k^1\),
\[
\inf_{F_k^1 \in \mathcal{D}(\mu, \Sigma)} \mathbb{P}\{a_k^1 x^1 \leq b_k^1\} = 0,
\]
which leads to constraint (2.3) to be infeasible. When \(\mu^T x^1 \leq b_k^1\), the constraint (2.3) is equivalent to
\[
\inf_{(\mu, \Sigma) \in \mathcal{U}_k^1} 1 - \frac{1}{1 + (\mu^T x^1 - b_k^1)^2/((x^1)^T \Sigma x^1)} \geq \alpha_k^1,
\]
which can be reformulated as
\[
h_k^1(x^1) \geq \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}, \quad (3.6)
\]
where
\[
h_k^1(x^1) = \begin{cases} 
\min_{\mu, \Sigma} \frac{b_k^1 - \mu^T x^1}{\sqrt{(x^1)^T \Sigma x^1}} \\
\text{s.t. } (i) \ (\mu - \mu_k^1)^T (\Sigma_k^1)^{-1} (\mu - \mu_k^1) \leq \gamma_k^1, \\
(ii) \ \Sigma \preceq \gamma_k^2 \Sigma_k^1. 
\end{cases} \quad (3.7)
\]
For the sake of simplicity, we separate problem (3.7) into two optimization problems
\[
h_k^1(x^1) = \frac{b_k^1 + v_1(x^1)}{\sqrt{v_2(x^1)}},
\]
where
\[
v_1(x^1) = \begin{cases} 
\min_{\mu} -\mu^T x^1 \\
\text{s.t. } (\mu - \mu_k^1)^T (\Sigma_k^1)^{-1} (\mu - \mu_k^1) \leq \gamma_k^1, 
\end{cases}
\]
\[
v_2(x^1) = \begin{cases} 
\max_{\Sigma} \ (x^1)^T \Sigma x^1 \\
\text{s.t. } \Sigma \preceq \gamma_k^2 \Sigma_k^1. 
\end{cases}
\]
Then, we apply the KKT conditions, and get $v_1(x) = -(\mu_1^k)^T x - \sqrt{\gamma_{k1}^1} \sqrt{(x^1)^T \Sigma k^1 x^1}$. Since, $u^T \Sigma u \leq u^T \gamma_{k2}^1 \Sigma u$ for any $u \in \mathbb{R}^n$, then, $v_2(x) = \gamma_{k2}^1 (x^1)^T \Sigma k^1 x^1$. Therefore, using (3.6) we have the following reformulation of (2.3)

$(\mu_1^k)^T x + \left( \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}} \sqrt{\gamma_{k2}^1} + \sqrt{\gamma_{k1}^1} \right) \| (\Sigma_k^1)^{\frac{1}{2}} x^1 \| \leq b_k^1,$ (3.8)

for all $k \in I_1$. Similarly, the reformulation of (2.4) is given by

$- (\mu_2^l)^T x + \left( \sqrt{\frac{\alpha_l^2}{1 - \alpha_l^2}} \sqrt{\gamma_{l2}^1} + \sqrt{\gamma_{l1}^2} \right) \| (\Sigma_l^2)^{\frac{1}{2}} x^2 \| \leq - b_l^2,$ (3.9)

for all $l \in I_2$. The reformulation of feasible strategy sets (2.5) and (2.6) for uncertainty sets (3.1), (3.2), and (3.3) can be written as

$S_{\alpha_1}^1 = \left\{ x^1 \in X^1 \mid (\mu_1^k)^T x + \kappa_{\alpha_k} \| (\Sigma_k^1)^{\frac{1}{2}} x^1 \| \leq b_k^1, \ k \in I_1 \right\},$ (3.10)

and

$S_{\alpha_2}^2 = \left\{ x^2 \in X^2 \mid - (\mu_2^l)^T x + \kappa_{\alpha_l} \| (\Sigma_l^2)^{\frac{1}{2}} x^2 \| \leq - b_l^2, \ l \in I_2 \right\},$ (3.11)

where $\kappa_{\alpha_k} = \sqrt{\frac{\alpha_k}{1 - \alpha_k}}, \ i = 1, 2$, represents the reformulation under uncertainty sets (3.1) and (3.2), and $\kappa_{\alpha_l} = \left( \sqrt{\frac{\alpha_l}{1 - \alpha_l}} \sqrt{\gamma_{l2}^1} + \sqrt{\gamma_{l1}^2} \right), i = 1, 2$, represents the reformulation for uncertainty set (3.3). We assume that the strategy sets (3.10) and (3.11) satisfy the strict feasibility condition given by Assumption 1.

**Assumption 1.**

1. There exists an $x^1 \in S_{\alpha_1}^1$ such that the inequality constraints of $S_{\alpha_1}^1$ defined by (3.10) are strictly satisfied.

2. There exists an $x^2 \in S_{\alpha_2}^2$ such that the inequality constraints of $S_{\alpha_2}^2$, defined by (3.11) are strictly satisfied.

4. **Existence and characterization of saddle point equilibrium**

In this section, we show that there exists an SPE of the game $Z_{\alpha}$ if the probability distributions of the random vectors of the constraints of both the players belong to the uncertainty sets defined in Section 3.1. We further propose a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game $Z_{\alpha}$.
Theorem 4.1. Consider the game $Z_\alpha$ where the probability distributions of the row vectors $a_{ik}, k \in J_i, i = 1, 2$, belong to the uncertainty sets described in Section 3.1. Then, there exists an SPE of the game for all $\alpha \in (0, 1)^p \times (0, 1)^q$.

Proof. Let $\alpha \in (0, 1)^p \times (0, 1)^q$. For uncertainty sets described in Section 3.1, the strategy sets $S_{\alpha_1}$ and $S_{\alpha_2}$ are given by (3.10) and (3.11), respectively. It is easy to see that $S_{\alpha_1}$ and $S_{\alpha_2}$ are convex and compact sets. The function $(x_1^T G x_2^2)$ is a bilinear and continuous function. Hence, there exists an SPE from the minimax theorem of von Neumann [32].

4.1. Equivalent primal-dual pair of second order cone programs

From the minimax theorem [32], $(x_1^*, x_2^*)$ is an SPE for the game $Z_\alpha$ if and only if

$$x_1^* \in \arg \max_{x_1 \in S_{\alpha_1}} \min_{x_2 \in S_{\alpha_2}} (x_1^T G x_2^2), \quad (4.1)$$

$$x_2^* \in \arg \min_{x_2 \in S_{\alpha_2}} \max_{x_1 \in S_{\alpha_1}} (x_1^T G x_2^2). \quad (4.2)$$

We start with $\min_{x_2 \in S_{\alpha_2}} \max_{x_1 \in S_{\alpha_1}} (x_1^T G x_2^2)$ problem. The inner optimization problem $\max_{x_1 \in S_{\alpha_1}} (x_1^T G x_2^2)$ can be equivalently written as

$$\max_{x_1, (t_k^1)_{k \in J_1}} (x_1^T G x_2^2)$$

subject to:

$$(i) - (x_1^T \mu_k^1 - \kappa_{\alpha_k}^T t_k^1) + b_k^1 \geq 0, \quad \forall \ k \in J_1$$

$$(ii) t_k^1 - (\Sigma_k^1)^{1/2} x_1 = 0, \quad \forall \ k \in J_1$$

$$(iii) \sum_{j=1}^{n_1} x_j^1 = 1,$$

$$(iv) x_j^1 \geq 0, \quad \forall \ j = 1, 2, \ldots, n_1.$$  \(4.3\)

Let $\lambda_k^1 = (\lambda_k^1)_{k \in J_1} \in \mathbb{R}^p, \delta_k^1 \in \mathbb{R}^{n_1}, \ k \in J_1, \text{ and } v_1 \in \mathbb{R}$ be the Lagrange multipliers of constraints $(i), (ii),$ and $(iii)$ of $(4.3)$, respectively. Then, the Lagrangian dual problem of the SOCP $(4.3)$ is an SOCP [4, 21]. Moreover, the duality gap is zero.
according to Assumption [1]. Therefore, \( \min_{x^2 \in S^2} \max_{x^1 \in S^1} (x^1)^T G x^2 \) problem is equivalent to the following SOCP

\[
\begin{align*}
\min_{x^2, v^1, (\delta_k^1)_{k \in J_1}, (\lambda_k^1)_{k \in J_1}} & \quad v^1 + \sum_{k \in J_1} \lambda_k^1 b_k^1 \\
\text{s.t.} & \quad (i) \ G x^2 - \sum_{k \in J_2} \lambda_k^1 \mu_k^1 - \sum_{k \in J_1} (\Sigma_k^1)^{\frac{1}{2}} \delta_k^1 \leq v^1 1_{n_1}, \\
& \quad (ii) \ - (\mu_l^2)^T x^2 + \kappa_{\alpha_l^2} \left\| (\Sigma_l^2)^{\frac{1}{2}} x^2 \right\| \leq -b_l^2, \quad \forall \ l \in J_2, \\
& \quad (iii) \ ||\delta_k^1|| \leq \lambda_k^1 \kappa_{\alpha_k^1}, \forall \ k \in J_1, \\
& \quad (iv) \ sum_{j=1}^{n_2} x_j^2 = 1, \\
& \quad (v) \ x_j^2 \geq 0, \ \forall \ j = 1, 2, \ldots, n_2 \\
& \quad (vi) \ \lambda_k^1 \geq 0, \ \forall \ k \in J_1,
\end{align*}
\]

where \( 1_{n_1} \) is an \( n_1 \times 1 \) vector of ones. Similarly, \( \max_{x^1 \in S^1} \min_{x^2 \in S^2} (x^1)^T G x^2 \) problem is equivalent to the following SOCP

\[
\begin{align*}
\max_{x^1, v^2, (\delta_l^2)_{l \in J_2}, (\lambda_l^2)_{l \in J_2}} & \quad v^2 + \sum_{l \in J_2} \lambda_l^2 b_l^2 \\
\text{s.t.} & \quad (i) \ G^T x^1 - \sum_{l \in J_2} \lambda_l^2 \mu_l^2 - \sum_{l \in J_2} (\Sigma_l^2)^{\frac{1}{2}} \delta_l^2 \geq v^2 1_{n_2}, \\
& \quad (ii) \ (\mu_k^1)^T x^1 + \kappa_{\alpha_k^1} \left\| (\Sigma_k^1)^{\frac{1}{2}} x^1 \right\| \leq b_k^1, \forall \ k \in J_1, \\
& \quad (iii) \ ||\delta_l^2|| \leq \lambda_l^2 \kappa_{\alpha_l^2}, \forall \ l \in J_2, \\
& \quad (iv) \ sum_{j=1}^{n_1} x_j^1 = 1, \\
& \quad (v) \ x_j^1 \geq 0, \ \forall \ j = 1, 2, \ldots, n_1, \\
& \quad (vi) \ \lambda_l^2 \geq 0, \ \forall \ l \in J_2.
\end{align*}
\]

It follows from the duality theory of second order cone programming problem that \( (P) \) and \( (D) \) form a primal-dual pair [4, 21].
Remark 4.2. For $\kappa_{\alpha_i}^k = \sqrt{\frac{\alpha_i^k}{1 - \alpha_i^k}}, i = 1, 2$, $[\mathbf{P}]$ and $[\mathbf{D}]$ represent the primal-dual pair of SOCPs for the uncertainty sets defined by (3.1) and (3.2). For $\kappa_{\alpha_i}^k = \left(\sqrt{\frac{\alpha_i^k}{1 - \alpha_i^k}} \sqrt{\gamma_{k2}^i} + \sqrt{\gamma_{k1}^i}\right), i = 1, 2$, $[\mathbf{P}]$ and $[\mathbf{D}]$ represent the primal-dual pair of SOCPs for the uncertainty set defined by (3.3).

Next, we show that the optimal solutions of (P) and (D) give an SPE of the game $Z_\alpha$.

**Theorem 4.3.** Consider the game $Z_\alpha$ where the probability distributions of the row vectors $a_i^k, k \in J, i = 1, 2$, belong to the uncertainty sets defined by (3.1), (3.2), (3.3). Let Assumption 1 holds. Then, for a given $\alpha \in (0,1)^p \times (0,1)^q$, $(x^1, x^2)$ is an SPE of the game $Z_\alpha$ if and only if there exists $(v^1, (\delta^1_i^k)_{k \in J_1}, \lambda^1)$ and $(v^2, (\delta^2_i^k)_{l \in J_2}, \lambda^2)$ such that $(x^2, v^1, (\delta^1_i^k)_{k \in J_1}, \lambda^1)$ and $(x^1, v^2, (\delta^2_i^k)_{l \in J_2}, \lambda^2)$ are optimal solutions of $[\mathbf{P}]$ and $[\mathbf{D}]$ respectively.

**Proof.** Let $(x^1, x^2)$ be an SPE of the game $Z_\alpha$. Then, $x^1$ and $x^2$ are the solutions of (4.1) and (4.2) respectively. Therefore, there exists $(v^1, (\delta^1_i^k)_{k \in J_1}, \lambda^1)$ and $(v^2, (\delta^2_i^k)_{l \in J_2}, \lambda^2)$ such that $(x^2, v^1, (\delta^1_i^k)_{k \in J_1}, \lambda^1)$ and $(x^1, v^2, (\delta^2_i^k)_{l \in J_2}, \lambda^2)$ are optimal solutions of $[\mathbf{P}]$ and $[\mathbf{D}]$ respectively.

Let $(x^2, v^1, (\delta^1_i^k)_{k \in J_1}, \lambda^1)$ and $(x^1, v^2, (\delta^2_i^k)_{l \in J_2}, \lambda^2)$ be optimal solutions of $[\mathbf{P}]$ and $[\mathbf{D}]$ respectively. Under Assumption 1, $[\mathbf{P}]$ and $[\mathbf{D}]$ are strictly feasible. Therefore, strong duality holds for primal-dual pair $[\mathbf{P}]$-[\mathbf{D}]$. Then, we have

$$v^1 + \sum_{k \in J_1} \lambda^1_k b^1_k = v^2 + \sum_{l \in J_2} \lambda^2_l b^2_l. \quad (4.4)$$

Take $x^1 \in S_{\alpha_i}^1$ and multiply the constraint (i) of $[\mathbf{P}]$ by $(x^1)^T$. Then, by using Cauchy-Schwartz inequality, we have

$$(x^1)^T G x^2 \leq v^1 + \sum_{k \in J_1} \lambda^1_k b^1_k, \forall x^1 \in S_{\alpha_i}^1. \quad (4.5)$$

Similarly, we have

$$(x^1)^T G x^2 \geq v^2 + \sum_{l \in J_2} \lambda^2_l b^2_l, \forall x^2 \in S_{\alpha_i}^2. \quad (4.6)$$
Take \( x^1 = x^{1*} \) and \( x^2 = x^{2*} \) in (4.5) and (4.6), then from (4.4), we get
\[
(x^{1*})^T G x^{2*} = v^{1*} + \sum_{k \in J_1} \lambda^{1*}_k b^1_k = v^{2*} + \sum_{l \in J_2} \lambda^{2*}_l b^2_l. \tag{4.7}
\]

It follows from (4.5), (4.6), and (4.7) that \((x^{1*}, x^{2*})\) is an SPE of the game \( Z_\alpha \).

5. Numerical Results

For illustration purpose, we consider an instance of a zero-sum game with random constraints. We compute the saddle point equilibria of the game by solving the SOCPs (P) and (D). We use the convex programs solver CVX on MATLAB for solving the SOCPs [18, 19]. Consider a zero-sum game described by the following \( 4 \times 4 \) payoff matrix \( G \)
\[
G = \begin{pmatrix}
1 & 4 & 4 & 2 \\
5 & 4 & 4 & 2 \\
3 & 5 & 4 & 3 \\
3 & 2 & 3 & 1
\end{pmatrix}.
\]

We consider the stochastic linear constraints defined by \( 3 \times 4 \) random matrices \( A^1 \) and \( A^2 \). The probability distributions of the row vectors of \( A^1 \) and \( A^2 \) are partially known. We only have full/partial information about the mean vectors and covariance matrices of the row vectors of \( A^1 \) and \( A^2 \). The data are summarized as
Table 1 summarizes the saddle point equilibria of the game $Z_\alpha$ for various values of $\alpha$ for all three uncertainty sets defined in Section 3.1. For uncertainty set (3.3), the SOCPs might be infeasible when the value of $\kappa_\alpha$ is relatively high. We also study zero-sum game problem under expected value constraints for the same data set. The constraints of each player are linear in this case. An SPE for these games can be computed by solving a primal-dual pair of linear programs given in [7]. An SPE for this game is $(x_{1avg}^*, x_{2avg}^*) = ((0, 0.25, 0.75, 0), (0.2917, 0.1250, 0.5833))$.

We compare the saddle point equilibria of both the problems by generating 100 scenarios. A scenario is termed as violated if any of the three constraints is violated. We generate the scenarios via multivariate normal and multivariate Laplace random generators where the mean and covariance matrix are the same as in the example. The simulations results are shown in Figures 1 and 2. For the SPE of the game $Z_\alpha$ at $\alpha = 0.7$ corresponding to uncertainty set (3.1), the constraints of player 1 and player 2 are violated for 5 and 4 scenarios, respectively. However, for
the SPE \((x_{1\text{avg}}^1, x_{2\text{avg}}^2)\) of zero-sum game under expected value constraints, there are 42 and 54 violated scenarios for players 1 and 2, respectively. This shows that the saddle point equilibria of the game \(Z_{\alpha}\) are more reliable compared to the zero-sum game under expected value constraints.

Table 1: Saddle point equilibrium for different uncertainty sets

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>Saddle Point Equilibrium</th>
<th>Value of the game</th>
<th>Uncertainty set</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0.5, 0.5, 0.5))</td>
<td>((0.5, 0.5, 0.5))</td>
<td>((0.3070, 0.3555, 0.0565, 0.2810))</td>
<td>((0.0216, 0.0.5398, 0.4386))</td>
</tr>
<tr>
<td>((0.6, 0.6, 0.6))</td>
<td>((0.6, 0.6, 0.6))</td>
<td>((0.0858, 0.4166, 0.4976, 0))</td>
<td>((0.0226, 0.5992, 0.3782))</td>
</tr>
<tr>
<td>((0.7, 0.7, 0.7))</td>
<td>((0.7, 0.7, 0.7))</td>
<td>((0.1706, 0.4884, 0.3410, 0))</td>
<td>((0.0967, 0.6276, 0.2757))</td>
</tr>
<tr>
<td>((0.8, 0.8, 0.8))</td>
<td>((0.8, 0.8, 0.8))</td>
<td>((0.3070, 0.3555, 0.0565, 0.2810))</td>
<td>((0.2941, 0.1603, 0.5151, 0.0305))</td>
</tr>
<tr>
<td>((0.5, 0.5, 0.5))</td>
<td>((0.5, 0.5, 0.5))</td>
<td>((0.0847, 0.4138, 0.5015, 0))</td>
<td>((0.0226, 0.5953, 0.3821))</td>
</tr>
<tr>
<td>((0.6, 0.6, 0.6))</td>
<td>((0.6, 0.6, 0.6))</td>
<td>((0.1002, 0.4555, 0.4443, 0))</td>
<td>((0.0233, 0.6519, 0.3248))</td>
</tr>
<tr>
<td>((0.7, 0.7, 0.7))</td>
<td>((0.7, 0.7, 0.7))</td>
<td>((0.2183, 0.4867, 0.2336, 0.0614))</td>
<td>((0.1371, 0.6435, 0.2194))</td>
</tr>
<tr>
<td>((0.8, 0.8, 0.8))</td>
<td>((0.8, 0.8, 0.8))</td>
<td>Infeasible</td>
<td>Infeasible</td>
</tr>
</tbody>
</table>

6. Conclusions

We show the existence of a mixed strategy SPE for a two player zero-sum game with distributionally robust chance constraints under three different uncertainty sets. The saddle point equilibria of these games can be obtained from the optimal solutions of a primal-dual pair of SOCPs. We compute the saddle point equilibria of an instance of the game \(Z_{\alpha}\) by solving SOCPs \((P)\) and \((D)\). For the same instance of the game, we compared the SPE of \(Z_{\alpha}\) with the SPE of the zero-sum game with expected value constraints by generating 100 scenarios. We find that the number of feasible scenarios for the game \(Z_{\alpha}\) is far larger than the game with expected value constraints.

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(a) Chance constraints case  (b) Expected value constraints case

Figure 1: Violated constraints for player 1

(a) Chance constraints case  (b) Expected value constraints case

Figure 2: Violated constraints for player 2

References


