# Regret Minimization and Separation in Multi-Bidder Multi-Item Auctions

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We study a robust auction design problem with a minimax regret objective, where a seller seeks a mechanism for selling multiple items to multiple anonymous bidders with additive values. The seller knows that the bidders' values range over a box uncertainty set but has no information about their probability distribution. This auction design problem can be viewed as a zero-sum game between the seller, who chooses a mechanism, and a fictitious adversary or 'nature,' who chooses the bidders' values from within the uncertainty set with the aim to maximize the seller's regret. We characterize the Nash equilibrium of this game analytically. The Nash strategy of the seller is a mechanism that sells each item via a separate auction akin to a second price auction with a random reserve price. The Nash strategy of nature is mixed and constitutes a probability distribution on the uncertainty set under which each bidder's values for the items are comonotonic.

## 1. Introduction

Consider the problem of designing an auction for selling $J$ items to $I$ bidders. The bidders assign each item a private value, which captures the maximum amount of money they would be willing to pay for this item. The set of values that a bidder assigns to *all* items is referred to as his value profile. We assume that the bidders' preferences are quasilinear and additively separable, that is, the bidders assign any bundle of items a value equal to the sum of the values of its constituents.

In the standard Bayesian setting, the seller's beliefs about the bidders' value profiles are modeled via a commonly known probability distribution, and it is assumed that the seller aims to maximize her expected revenues. If there is only one item ($J=1$), the optimal mechanism is well-understood under relatively general conditions, see, *e.g.*, Myerson (1981) and Cremer and McLean (1988). If there are multiple items ($J>1$), on the other hand, computing the optimal mechanism is #P-hard even in unrealistically simple situations (Daskalakis et al. 2014). Even though Daskalakis et al. (2017) and Cai et al. (2019) recently proposed duality schemes for solving multi-item auction design problems, closed-form solutions remain limited to special probabilistic models and/or small numbers of items, see, *e.g.*, Daskalakis et al. (2013) or Giannakopoulos and Koutsoupias (2014).

Assuming that the probability distribution of the bidder's values is commonly known not only renders the mechanism design problem intractable, but it is also difficult to justify in practice. Instead, it is natural to seek mechanisms that are optimal under limited distributional information. When the probability distribution of the bidders' values is ambiguous, the term 'optimal' becomes ambiguous itself. The literature on (distributionally) robust mechanism design regards an auction as optimal if it maximizes the worst-case expected revenues in view of all possible distributions consistent with the information available. The bulk of this literature focuses on single-item auctions, see, *e.g.*, Bose et al. (2006), Bei et al. (2019), Koçyiğit et al. (2020) and Suzdaltsev (2020). As a notable exception, Bandi and Bertsimas (2014) propose a numerical procedure to solve a robust multi-item auction design problem with budget constraints. Carroll (2017) explicitly characterizes the optimal mechanism of a correlation-robust screening problem, where the marginal distributions of the agent's multidimensional type are precisely known to the principal, while their joint distribution remains unknown. The multidimensional monopoly pricing problem, which is equivalent to the single-bidder multi-item auction design problem, constitutes a special case of this screening problem. For this special case, Carroll (2017) shows that it is optimal to sell the items separately. Gravin and Lu (2018) then demonstrate that this separation result remains valid even if the bidder is subject to a budget constraint. Koçyiğit et al. (2018) consider a variant of the multidimensional monopoly pricing problem with a minimax regret objective, where the seller has no knowledge of the value distribution apart from its support. They analytically characterize the best randomized as well as the best deterministic mechanism. In both cases, the optimal mechanism sells the items separately via single-item mechanisms that were first characterized by Bergemann and Schlag (2008).

The separation results reviewed above are not easily generalized to multi-bidder auctions. In this paper, we consider the multi-bidder extension of the mechanism design problem studied by Koçyiğit et al. (2018). Specifically, we assume that the seller perceives each bidder's value profile as an uncertain parameter that is only known to range over a rectangular uncertainty set spanned by the origin and a vector of nonnegative upper bounds. In addition, we assume that the bidders are anonymous, which implies that the vectors of item-wise upper bounds are identical for all bidders. When aiming to maximize the worst-case revenue, the seller faces a special case of the robust mechanism design problem studied by Bandi and Bertsimas (2014). Under the box uncertainty set considered here, however, the set of optimal mechanisms is very rich and contains naïve mechanisms that have little practical appeal. For example, it is optimal for the seller to keep all items. This prompts us to adopt a minimax regret objective, that is, we assume in this paper that the seller seeks a mechanism that minimizes her worst-case regret. The regret of a mechanism is defined as the difference between the revenues that could have been achieved under full knowledge of the bidders' value profiles and the actual revenues generated by the mechanism. The worst-case regret is obtained by maximizing the realized regret across all possible value profiles of the bidders. Caldentey et al. (2017) as well as Poursoltani and Delage (2019) argue that, in a general robust optimization context, minimizing the worst-case regret results in less conservative decisions than maximizing the worst-case revenue. The main contributions of this paper are listed below.

- We interpret the multi-bidder multi-item auction design problem with minimax regret objective as a zero sum game between the seller, who chooses a mechanism to auction the items, and a fictitious adversary or 'nature,' who chooses the bidders' value profiles from within a box uncertainty set with the aim to maximize the seller's regret. We characterize the Nash equilibrium of this game analytically and prove that the seller's Nash strategy is a mechanism under which each item is auctioned separately. The separate mechanisms for the individual items can be interpreted as second price auctions with random reserve prices. If there is only one bidder, these separate mechanisms reduce to randomized posted price mechanisms that were first described by Bergemann and Schlag (2008) in the context of the monopoly pricing of a single item.

- We show that nature's Nash strategy is mixed and thus represents a probability distribution on the uncertainty set. Under this distribution, each bidder's values for the items are comonotic, and any bidder's value profile can be non-zero only if all other bidders' value profiles vanish.

The mechanism design model studied in this paper requires no distributional information except for an upper bound on each bidder's value for each item. This model is relevant if there is no trustworthy distributional information or if any distributional information is costly or time-consuming to acquire. Such a situation could arise, for example, when firms use auctions for initial public offerings. In this case, there is indeed no distributional information available about the bidders' values for the offered shares. On the other hand, the model studied here may be overly conservative when data is abundant, as is typically the case in online advertisement, where auctions for ad placements are held in real time within fractions of seconds. To our best knowledge, this paper establishes the first non-trivial robust optimality guarantee for a separable mechanism involving multiple bidders as well as multiple items. We expect that the insights distilled in this paper will pave the way towards more general separation results with a broader range of applications.

This paper also relates to the literature on approximately optimal mechanism design, see, *e.g.*, Dhangwatnotai et al. (2015), Hart and Nisan (2017), Allouah and Besbes (2020) and the references therein. Under this modeling paradigm, the seller aims to identify a mechanism for which some objective function (*e.g.*, the expected revenue) is guaranteed to be close to a full information benchmark value (*e.g.*, the maximum expected revenue achievable) under every probability distribution consistent with the assumptions made. The vast majority of the existing approximation results critically rely on certain independence assumptions (*e.g.*, the values must be independent across bidders or items). In the context of a monopoly pricing problem with a single buyer it has been shown, for example, that if the buyer's values for the items are independent, then simple mechanisms (such as selling the goods separately or as a single grand bundle at deterministic posted prices) provide constant-factor approximations to the expected revenue of the unknown optimal mechanism (Hart and Nisan 2017). However, if the buyer's values are correlated, these approximation guarantees cease to hold (Hart and Nisan 2019). An important advantage of the robust approach adopted in this paper is its ability to account for correlations and to provide optimality guarantees for simple mechanisms even if the bidders' values may be dependent.

**Notation.** For any closed set $\mathcal{A} \subseteq \mathbb{R}^n$, we denote by $\Delta(\mathcal{A})$ the family of all probability distributions on $\mathcal{A}$, and for any $\mathbb{P} \in \Delta(\mathcal{A})$, $\text{supp}(\mathbb{P})$ represents the support of $\mathbb{P}$. The set of all Borel-measurable functions from a Borel set $\mathcal{D} \subseteq \mathbb{R}^n$ to a Borel set $\mathcal{R} \subseteq \mathbb{R}^m$ is denoted by $\mathcal{L}(\mathcal{D}, \mathcal{R})$. Random variables are designated by tildes (*e.g.*, $\tilde{\boldsymbol{v}}$), and their realizations are denoted by the same symbols without tildes (*e.g.*, $\boldsymbol{v}$). For a logical expression $\mathcal{E}$, we define $\mathbb{1}_{\mathcal{E}} = 1$ if $\mathcal{E}$ is true and $\mathbb{1}_{\mathcal{E}} = 0$ otherwise. Throughout the paper, bidders are indexed by superscripts and items by subscripts.

## 2. Problem Setup

We consider the problem of designing a mechanism for selling $J$ different items to $I \geq 2$ bidders. The sets of items and bidders are denoted by $\mathcal{J} = \{1, 2, \ldots, J\}$ and $\mathcal{I} = \{1, 2, \ldots, I\}$, respectively. Each bidder $i \in \mathcal{I}$ assigns each item $j \in \mathcal{J}$ a value $v_j^i$ that reflects his willingness to pay. In the following we denote by $\boldsymbol{v}^i = (v_1^i, \ldots, v_J^i)$ the row vector of the values that bidder $i$ assigns to all items and by $\boldsymbol{v}_j = (v_j^1, \ldots, v_j^I)^\top$ the column vector of all bidders' values for item $j$. In addition, we let $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_J)$ be the matrix of all bidders' values for all items. While bidder $i$ has full knowledge of his value profile $\boldsymbol{v}^i$, the seller perceives the matrix $\boldsymbol{v}$ as uncertain. For each $j \in \mathcal{J}$, we assume that the seller only knows a common upper bound $\overline{v}_j > 0$ on the value $v_j^i$ for all $i \in \mathcal{I}$. This assumption is a manifestation of the anonymity of the bidders. In the following we denote by $\overline{\boldsymbol{v}} = (\overline{v}_1, \ldots, \overline{v}_J)$ the vector of all item-wise upper bounds. The seller has no other information about the distribution of $\boldsymbol{v}$ or suspects that any available information is not trustworthy. For ease of exposition, we assume that the seller incurs no costs for supplying any of the items to any of the bidders. In the following, we denote by $\mathcal{V} = \times_{j\in\mathcal{J}}[0,\overline{v}_j]$ the uncertainty set of the value profiles of any fixed bidder $i \in \mathcal{I}$ and by $\mathcal{V}^I = \times_{j\in\mathcal{J}}[0,\overline{v}_j]^I$ the uncertainty set of the value profiles of all bidders, which is symmetric under permutations of the bidders. We also let $\mathcal{W}_j^i$, $i \in \mathcal{I}$ and $j \in \mathcal{J}$, be any partition of the hypercube $[0,\overline{v}_j]^I$ such that $\mathcal{W}_j^i$ contains only scenarios $\boldsymbol{v}_j$ for which bidder $i$ is among the highest bidders for item $j$. In other words, $\boldsymbol{v}_j \in \mathcal{W}_j^i$ implies that $i \in \arg\max_{k\in\mathcal{I}} v_j^k$. If there are multiple highest bidders, an arbitrary tie-breaking rule is used (*e.g.*, the lexicographic tie-breaker assigns $\boldsymbol{v}_j$ to $\mathcal{W}_j^i$ if $i = \min\arg\max_{k\in\mathcal{I}} v_j^k$).

An auction mechanism $(\boldsymbol{q},\boldsymbol{m})$ consists of an allocation rule $\boldsymbol{q} \in \mathcal{L}(\mathcal{V}^I,\mathbb{R}_+^{I\times J})$ and a payment rule $\boldsymbol{m} \in \mathcal{L}(\mathcal{V}^I,\mathbb{R}^I)$. Given a matrix $\boldsymbol{v} \in \mathcal{V}^I$ of value profiles reported by all bidders, the mechanism $(\boldsymbol{q},\boldsymbol{m})$ outputs the allocation probabilities of the items to the bidders as well as the payments charged to the bidders. Specifically, in scenario $\boldsymbol{v}$, the seller allocates item $j$ to bidder $i$ with probability $q_j^i(\boldsymbol{v})$ and charges this bidder the amount $m^i(\boldsymbol{v})$. As a result, the utility of bidder $i$ evaluates to $\sum_{j\in\mathcal{J}} q_j^i(\boldsymbol{v})v_j^i - m^i(\boldsymbol{v})$ and is therefore quasilinear and additively separable across the items.

A (dominant strategy) incentive compatible and (ex-post) individually rational mechanism $(\boldsymbol{q},\boldsymbol{m})$ satisfies the following constraints.

$$\sum_{j\in\mathcal{J}} q_j^i(\boldsymbol{v})v_j^i - m^i(\boldsymbol{v}) \geq \sum_{j\in\mathcal{J}} q_j^i(\boldsymbol{w}^i,\boldsymbol{v}^{-i})v_j^i - m^i(\boldsymbol{w}^i,\boldsymbol{v}^{-i}) \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I, \forall \boldsymbol{w}^i \in \mathcal{V} \quad \text{(IC)}$$

$$\sum_{j\in\mathcal{J}} q_j^i(\boldsymbol{v})v_j^i - m^i(\boldsymbol{v}) \geq 0 \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}^I \quad \text{(IR)}$$

$$\sum_{i\in\mathcal{I}} q_j^i(\boldsymbol{v}) \leq 1 \quad \forall j \in \mathcal{J}, \forall \boldsymbol{v} \in \mathcal{V}^I \quad \text{(Inv)}$$

The incentive compatibility constraint (IC) ensures that each bidder maximizes his utility by reporting his true value profile irrespective of the values reported by the other bidders. The individual rationality constraint (IR) ensures that the bidders earn nonnegative utilities from participating in the mechanism $(\boldsymbol{q},\boldsymbol{m})$ under truthful reporting. Finally, the inventory constraint (Inv) ensures that the seller allocates each item $j \in \mathcal{J}$ with a probability of at most one. Note that the inequality expresses the possibility that the seller may keep any item $j$ to herself with a positive probability.

Remark 1. Incentive compatibility and individual rationality constraints are routinely used in mechanism design and may be imposed essentially without any loss of generality thanks to the revelation principle (Krishna 2009, Chapter 5). Throughout this paper, we assume that the seller restricts attention to dominant strategy incentive compatible and ex-post individually rational mechanisms. In contrast, the Bayesian mechanism design literature typically studies Bayesian incentive compatibility and interim individual rationality, see, *e.g.*, Myerson (1981). While less restrictive, these constraints can only be enforced if the seller is able to assign a crisp probability distribution to the bidders' values. In this paper, we assume that the seller lacks the relevant information to do this. It is thus natural to focus on dominant strategy incentive compatibility and ex-post individual rationality, which do not require any distributional information. □

The seller's ex-post regret is defined as the difference between the maximum profit that could have been realized under complete information about $\boldsymbol{v}$ and the profit earned with the mechanism $(\boldsymbol{q}, \boldsymbol{m})$. If the seller was fully aware of the bidders' values $\boldsymbol{v}$, she would sell item $j$ at the price $\max_{i\in\mathcal{I}} v_j^i$ to any bidder $i \in \arg\max_{i\in\mathcal{I}} v_j^i$. The maximum profit under complete information can thus be expressed as $\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}} v_j^i)$. The profit earned with mechanism $(\boldsymbol{q}, \boldsymbol{m})$, on the other hand, amounts to $\sum_{i\in\mathcal{I}} m^i(\boldsymbol{v})$. In summary, the ex-post regret thus equals $\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} m^i(\boldsymbol{v})$, and the worst-case regret is obtained by maximizing the ex-post regret over all value profiles $\boldsymbol{v} \in \mathcal{V}^I$.

Throughout this paper we assume that the seller aims to design an incentive compatible and individually rational mechanism that minimizes her worst-case regret. This mechanism design problem can be formalized as the following robust optimization problem.

$$
\begin{aligned}
z^\star = \inf_{\boldsymbol{q},\boldsymbol{m}} \quad & \sup_{\boldsymbol{v}\in\mathcal{V}^I} \sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} m^i(\boldsymbol{v}) \\
\text{s.t.} \quad & \boldsymbol{q} \in \mathcal{L}(\mathcal{V}^I, \mathbb{R}_+^{I\times J}),\, \boldsymbol{m} \in \mathcal{L}(\mathcal{V}^I, \mathbb{R}^I) \\
& \text{(IC), (IR), (Inv)}
\end{aligned}
\tag{1}
$$

From now on, we use the shorthand $\mathcal{X}$ to denote the set of all mechanisms feasible in (1).

## 3. Optimal Mechanism

One particularly simple policy for the seller would be to auction each item individually. Any such mechanism is separable in view of the following definition.

Definition 1 (Separability). A mechanism $(\boldsymbol{q},\boldsymbol{m})$ is called separable if there exists an item-wise allocation rule $\hat{\boldsymbol{q}}_j \in \mathcal{L}([0,\overline{v}_j]^I, \mathbb{R}_+^I)$ and an item-wise payment rule $\hat{\boldsymbol{m}}_j \in \mathcal{L}([0,\overline{v}_j]^I, \mathbb{R}^I)$ for all $j \in \mathcal{J}$ such that $\boldsymbol{q}(\boldsymbol{v}) = (\hat{\boldsymbol{q}}_1(\boldsymbol{v}_1), \ldots, \hat{\boldsymbol{q}}_J(\boldsymbol{v}_J))$ and $\boldsymbol{m}(\boldsymbol{v}) = \sum_{j\in\mathcal{J}} \hat{\boldsymbol{m}}_j(\boldsymbol{v}_j)$ for all $\boldsymbol{v} \in \mathcal{V}^I$.

We now investigate the separable mechanism $(\boldsymbol{q}^\star,\boldsymbol{m}^\star)$ consisting of the single-item mechanisms $(\hat{\boldsymbol{q}}_j, \hat{\boldsymbol{m}}_j)$, $j \in \mathcal{J}$, defined through

$$\hat{q}_j^i(\boldsymbol{v}_j) = \begin{cases} 1 + \log(\frac{v_j^i}{\overline{v}_j}) & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \geq \frac{\overline{v}_j}{e} \\ 0 & \text{otherwise} \end{cases} \tag{2a}$$

and

$$\hat{m}_j^i(\boldsymbol{v}_j) = \begin{cases} v_j^i + (\max_{k\neq i} v_j^k)\log(\max_{k\neq i} \frac{v_j^k}{\overline{v}_j}) & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \geq \max_{k\neq i} v_j^k \geq \frac{\overline{v}_j}{e} \\ v_j^i - \frac{\overline{v}_j}{e} & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \geq \frac{\overline{v}_j}{e} > \max_{k\neq i} v_j^k \\ 0 & \text{otherwise} \end{cases} \tag{2b}$$

for all $\boldsymbol{v}_j \in [0,\overline{v}_j]^I$. Under this mechanism, the probability of allocating item $j \in \mathcal{J}$ to bidder $i \in \mathcal{I}$ can be strictly positive only if bidder $i$ has the highest value for item $j$ among all bidders and if that value exceeds $\frac{\overline{v}_j}{e}$. We emphasize that the payment rule $\hat{\boldsymbol{m}}_j$ is deterministic even though the allocation of item $j$ is randomized under $\hat{\boldsymbol{q}}_j$. Specifically, bidder $i \in \mathcal{I}$ always pays a nonnegative amount for each item $j \in \mathcal{J}$, and this amount is strictly positive only when the corresponding allocation probability is strictly positive, *i.e.*, when $v_j^i > \frac{\overline{v}_j}{e}$.

In the remainder of the paper we will show that the mechanism $(\boldsymbol{q}^\star,\boldsymbol{m}^\star)$ is optimal in (1). To this end, we will first construct a new mechanism $(\boldsymbol{q}',\boldsymbol{m}')$ equivalent to $(\boldsymbol{q}^\star,\boldsymbol{m}^\star)$, under which the seller implements a separate second price auction for each item $j \in \mathcal{J}$ with a random reserve price $\tilde{r}_j$ governed by the probability distribution $\mathbb{Q}_j \in \Delta([0, \overline{v}_j])$ defined via

$$\mathbb{Q}_j(\tilde{r}_j \le x) = \begin{cases} 1 + \log(\frac{x}{\overline{v}_j}) & \text{if } \frac{\overline{v}_j}{e} \le x \le \overline{v}_j \\ 0 & \text{if } 0 \le x < \frac{\overline{v}_j}{e}. \end{cases}$$

For each item $j \in \mathcal{J}$, the respective second price auction proceeds as follows. First, the reserve price $r_j$ is sampled from the distribution $\mathbb{Q}_j$. Note that the smallest possible value of $r_j$ under this distribution is $\frac{\overline{v}_j}{e}$. The seller then asks the bidders to report their bids for item $j$. After collecting all bids, the seller allocates item $j$ to the highest bidder provided that his bid exceeds the reserve price $r_j$, and the winner pays an amount equal to the maximum of the second highest bid and $r_j$. In the case of ties, item $j$ is given to the unique bidder whose index $i$ satisfies $\boldsymbol{v}_j \in \mathcal{W}_j^i$.

By construction, the mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ is separable and can formally be described via the single-item mechanisms $(\hat{\boldsymbol{q}}'_j, \hat{\boldsymbol{m}}'_j)$, $j \in \mathcal{J}$, defined through

$$(\hat{q}')_j^i(\boldsymbol{v}_j, r_j) = \begin{cases} 1 & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \ge r_j \\ 0 & \text{otherwise} \end{cases} \tag{3a}$$

and

$$(\hat{m}')_j^i(\boldsymbol{v}_j, r_j) = \begin{cases} \max\{\max_{k \neq i} v_j^k, r_j\} & \text{if } \boldsymbol{v}_j \in \mathcal{W}_j^i \text{ and } v_j^i \ge r_j \\ 0 & \text{otherwise} \end{cases} \tag{3b}$$

for all $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$. Note that $(\boldsymbol{q}', \boldsymbol{m}')$ is manifestly randomized because it depends on the realizations of the random reserve prices. Note also that, unlike under $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, under $(\boldsymbol{q}', \boldsymbol{m}')$ a bidder pays for an item only if he receives it. We next show that the randomized mechanism $(\boldsymbol{q}', \boldsymbol{m}')$ is equivalent to the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ if it is averaged across the random reserve prices.

Proposition 1. *We have* $\mathbb{E}_{\mathbb{Q}_j}[(\hat{q}')_j^i(\boldsymbol{v}_j, \tilde{r}_j)] = \hat{q}_j^i(\boldsymbol{v}_j)$ *and* $\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')_j^i(\boldsymbol{v}_j, \tilde{r}_j)] = \hat{m}_j^i(\boldsymbol{v}_j)$ *for all* $i \in \mathcal{I}$, $j \in \mathcal{J}$ *and* $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$.

*Proof.* Fix an arbitrary $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$. If $\boldsymbol{v}_j \notin \mathcal{W}_j^i$, then both $\mathbb{E}_{\mathbb{Q}_j}[(\hat{q}')_j^i(\boldsymbol{v}_j, \tilde{r}_j)]$ and $\hat{q}_j^i(\boldsymbol{v}_j)$ evaluate to 0 and are thus equal. If $\boldsymbol{v}_j \in \mathcal{W}_j^i$, on the other hand, one readily verifies that

$$\mathbb{E}_{\mathbb{Q}_j}[(\hat{q}')_j^i(\boldsymbol{v}_j, \tilde{r}_j)] = \mathbb{E}_{\mathbb{Q}_j}[\mathbb{1}_{(\tilde{r}_j \le v_j^i)}] = \mathbb{Q}_j(\tilde{r}_j \le v_j^i) = \hat{q}_j^i(\boldsymbol{v}_j).$$

Consider now the expected payment $\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)]$ of bidder $i$ for item $j$ in scenario $\boldsymbol{v}_j$. If $\boldsymbol{v}_j \notin \mathcal{W}^i_j$, then both $\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)]$ and $\hat{m}^i_j(\boldsymbol{v}_j)$ evaluate to 0 and are thus equal. If $\boldsymbol{v}_j \in \mathcal{W}^i_j$ and $\max_{k\neq i} v^k_j \geq \frac{\overline{v}_j}{e}$, however, then we have

$$\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)] = \mathbb{E}_{\mathbb{Q}_j}\left[\max\{\max_{k\neq i} v^k_j, \tilde{r}_j\}\mathbb{1}_{(\tilde{r}_j \leq v^i_j)}\right] = \int_{\frac{\overline{v}_j}{e}}^{\max_{k\neq i} v^k_j} (\max_{k\neq i} v^k_j)\,\frac{1}{x}\,\mathrm{d}x + \int_{\max_{k\neq i} v^k_j}^{v^i_j} x\,\frac{1}{x}\,\mathrm{d}x$$
$$= v^i_j + (\max_{k\neq i} v^k_j)\log\left(\max_{k\neq i}\frac{v^k_j}{\overline{v}_j}\right) = \hat{m}^i_j(\boldsymbol{v}_j).$$

Finally, if $\boldsymbol{v}_j \in \mathcal{W}^i_j$ and $v^i_j \geq \frac{\overline{v}_j}{e} > \max_{k\neq i} v^k_j$, then the reserve price exceeds the second highest bid with probability 1, which implies that

$$\mathbb{E}_{\mathbb{Q}_j}[(\hat{m}')^i_j(\boldsymbol{v}_j, \tilde{r}_j)] = \mathbb{E}_{\mathbb{Q}_j}\left[\max\{\max_{k\neq i} v^k_j, \tilde{r}_j\}\mathbb{1}_{(\tilde{r}_j \leq v^i_j)}\right] = \int_{\frac{\overline{v}_j}{e}}^{v^i_j} x\,\frac{1}{x}\,\mathrm{d}x = v^i_j - \frac{\overline{v}_j}{e} = \hat{m}^i_j(\boldsymbol{v}_j).$$

As the above arguments hold for each $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$, the claim follows. □

Remark 2. All second price auctions with deterministic reserve prices are (dominant strategy) incentive compatible (Krishna 2009, Chapter 2). Thus, $(\boldsymbol{q}', \boldsymbol{m}')$ is incentive compatible for all realizations of the random reserve prices. As incentive compatibility is enforced via linear inequalities, it is preserved by averaging $(\boldsymbol{q}', \boldsymbol{m}')$ with respect to the distributions of the reserve prices, which then implies via Proposition 1 that $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is incentive compatible, too. Under $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, the bidders thus have a weak preference to report their true values. All second price auctions with reserve prices are also (ex-post) individually rational because the bidders' utilities are always nonnegative. Indeed, under truthful bidding, a bidder pays at most his own bid. Using similar arguments as above, one can thus show that $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is also individually rational. As the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ ostensibly satisfies the inventory constraint (Inv), we thus conclude that it is feasible in (1). □

Proposition 2. *The worst-case regret of the separable mechanism* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *corresponding to the single-item mechanisms* (2) *is given by* $\sum_{j\in\mathcal{J}} \frac{\overline{v}_j}{e}$.

*Proof.* To evaluate the worst-case regret of $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$, we will first compute the realized regret $(\max_{i\in\mathcal{I}} v^i_j) - \sum_{i\in\mathcal{I}} \hat{m}^i_j(\boldsymbol{v}_j)$ of selling item $j \in \mathcal{J}$ in scenario $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$. To this end, fix an arbitrary $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$ and note that there is a unique $i' \in \mathcal{I}$ with $\boldsymbol{v}_j \in \mathcal{W}^{i'}_j$. If $\frac{\overline{v}_j}{e} > v^{i'}_j$, then $\hat{m}^i_j(\boldsymbol{v}_j) = 0$

for all $i \in \mathcal{I}$, that is, no bidder is charged. We thus have $(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) = v_j^{i'} < \frac{\overline{v}_j}{e}$. If $v_j^{i'} \ge \frac{\overline{v}_j}{e} > \max_{k\neq i'} v_j^k$, however, then $\hat{m}_j^{i'}(\boldsymbol{v}_j) = v_j^{i'} - \frac{\overline{v}_j}{e}$, and all other bidders pay nothing. Thus, $(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) = v_j^{i'} - (v_j^{i'} - \frac{\overline{v}_j}{e}) = \frac{\overline{v}_j}{e}$. Finally if $v_j^{i'} \ge \max_{k\neq i'} v_j^k \ge \frac{\overline{v}_j}{e}$, then $\hat{m}_j^{i'}(\boldsymbol{v}_j) = v_j^{i'} + (\max_{k\neq i'} v_j^k)\log(\max_{k\neq i'} \frac{v_j^k}{\overline{v}_j})$, and all other bidders pay nothing. This implies that

$$(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) = v_j^{i'} - \left( v_j^{i'} + (\max_{k\neq i'} v_j^k)\log\left(\max_{k\neq i'} \frac{v_j^k}{\overline{v}_j}\right)\right)$$
$$= -(\max_{k\neq i'} v_j^k)\log\left(\max_{k\neq i'} \frac{v_j^k}{\overline{v}_j}\right) \le \max_{x\in[\overline{v}_j/e,\overline{v}_j]} -x\log\left(\frac{x}{\overline{v}_j}\right) = \frac{\overline{v}_j}{e},$$

where the inequality follows from the assumption that $\max_{k\neq i'} v_j^k \ge \frac{\overline{v}_j}{e}$, and the last equality holds because $-x\log(\frac{x}{\overline{v}_j})$ is monotonically decreasing in $x \ge \frac{\overline{v}_j}{e}$.

The above reasoning implies that the worst-case regret of the individual mechanism $(\hat{\boldsymbol{q}}_j, \hat{\boldsymbol{m}}_j)$ for selling item $j \in \mathcal{J}$ amounts to $\sup_{\boldsymbol{v}_j\in\mathcal{V}}(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j) = \frac{\overline{v}_j}{e}$. This worst-case regret is attained by any scenario $\boldsymbol{v}_j \in [0,\overline{v}_j]^I$ with $v_j^{i'} \ge \frac{\overline{v}_j}{e} \ge \max_{k\neq i'} v_j^k$. The worst-case regret of the separable mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is thus given by

$$\sup_{\boldsymbol{v}\in\mathcal{V}^I} \sum_{j\in\mathcal{J}} (\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} (m^\star)^i(\boldsymbol{v}) = \sup_{\boldsymbol{v}\in\mathcal{V}^I} \sum_{j\in\mathcal{J}} \left( (\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j)\right)$$
$$= \sum_{j\in\mathcal{J}} \left( \sup_{\boldsymbol{v}_j\in\mathcal{V}} (\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}} \hat{m}_j^i(\boldsymbol{v}_j)\right) = \sum_{j\in\mathcal{J}} \frac{\overline{v}_j}{e},$$

where the second equality follows from the rectangularity of the uncertainty set $\mathcal{V}^I$, which implies that $\boldsymbol{v}_j \in \mathcal{V}$ for all $j \in \mathcal{J}$ if and only if $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_J) \in \mathcal{V}^I$. □

To show that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ solves (1), we first reformulate the minimax problem (1) as an infinite-dimensional convex program. To this end, note that problem (1) can be interpreted as a zero-sum game between the seller, who chooses the mechanism $(\boldsymbol{q}, \boldsymbol{m})$, and some fictitious adversary that one may think of as 'nature,' who chooses the bidders' value profiles $\boldsymbol{v}$ with the goal to inflict maximum damage to the seller. As the allocation probabilities may be fractional, the seller plays a mixed strategy chosen from the convex feasible set $\mathcal{X}$. Nature's feasible set coincides with the box uncertainty set $\mathcal{V}^I$ and is also convex. While the objective of the zero-sum game constitutes an affine function of the payment rule $\boldsymbol{m}$ for every fixed scenario $\boldsymbol{v}$, however, it is generically non-concave in $\boldsymbol{v}$ for fixed mechanisms $(\boldsymbol{q},\boldsymbol{m})$. To convert this zero-sum game to a convex-concave saddle point problem, we should allow nature to play mixed strategies corresponding to distributions $\mathbb{P}\in\Delta(\mathcal{V}^I)$. With this standard trick, problem (1) can be reformulated as

$$\inf_{(\boldsymbol{q},\boldsymbol{m})\in\mathcal{X}} \sup_{\mathbb{P}\in\Delta(\mathcal{V}^I)} \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}}\tilde{v}^i_j)-\sum_{i\in\mathcal{I}}m^i(\tilde{\boldsymbol{v}})\right]. \tag{4}$$

We will now show that the zero-sum game (4) admits a Nash equilibrium in mixed strategies, which can be evaluated analytically. Specifically, we will prove that the seller's Nash strategy is given by the separable mechanism $(\boldsymbol{q}^\star,\boldsymbol{m}^\star)$. In order to construct nature's Nash strategy, we introduce a marginal distribution $\hat{\mathbb{P}}\in\Delta(\mathcal{V})$ for the value profile $\tilde{\boldsymbol{v}}^1$, which is defined through

$$\hat{\mathbb{P}}(\tilde{\boldsymbol{v}}^1\le\boldsymbol{v}^1) = \begin{cases} \min_{j\in\mathcal{J}}\left(1-\frac{1}{e}\left(\frac{\overline{v}_j}{v^1_j}\right)\right)^+ & \text{if } \boldsymbol{v}^1\in\mathcal{V}\setminus\{(\overline{v}_1,\ldots,\overline{v}_J)\}\\ 1 & \text{if } \boldsymbol{v}^1=(\overline{v}_1,\ldots,\overline{v}_J). \end{cases} \tag{5}$$

Note that $\hat{\mathbb{P}}$ is known to represent nature's Nash strategy in problem (4) if there is only one bidder, that is, in the special case where $I=1$ and the auction design problem collapses to a monopoly pricing problem (Koçyiğit et al. 2018, Theorem 2). Note also that the values $(\tilde{v}^1_1,\ldots,\tilde{v}^1_J)$ of bidder 1 for the different items under $\hat{\mathbb{P}}$ are comonotonic. In the following, we will use $\hat{\mathbb{P}}$ explicitly to construct a Nash strategy for nature when $I>1$. To this end, define for each $i\in\mathcal{I}$ a probability distribution $\hat{\mathbb{P}}^i\in\Delta(\mathcal{V}^I)$ of the random matrix $\tilde{\boldsymbol{v}}$ through the relations

$$\hat{\mathbb{P}}^i(\tilde{\boldsymbol{v}}^i\le\boldsymbol{w})=\hat{\mathbb{P}}(\tilde{\boldsymbol{v}}^1\le\boldsymbol{w})\ \ \forall\boldsymbol{w}\in\mathcal{V} \quad\text{and}\quad \hat{\mathbb{P}}^i(\tilde{\boldsymbol{v}}^k=\boldsymbol{0})=1\ \ \forall k\neq i.$$

Note that under $\hat{\mathbb{P}}^i$, the marginal distribution of the value profile $\tilde{\boldsymbol{v}}^i$ coincides with $\hat{\mathbb{P}}$, while the marginal distributions of the value profiles $\tilde{\boldsymbol{v}}^k$, $k\neq i$, are all equal to the Dirac distribution that concentrates unit mass at $\boldsymbol{0}$. As $\hat{\mathbb{P}}$ constitutes a comonotonic distribution, the values of bidder $i$ for the items are comonotonic under $\hat{\mathbb{P}}^i$. The support of the distribution $\hat{\mathbb{P}}^i$ is given by

$$\mathrm{supp}(\hat{\mathbb{P}}^i)=\left\{\boldsymbol{v}\in\mathcal{V}^I:\boldsymbol{v}^i=s\overline{\boldsymbol{v}}\text{ for some } s\in[\tfrac{1}{e},1]\text{ and }\boldsymbol{v}^k=\boldsymbol{0}\ \ \forall k\neq i\right\}.$$

Finally, define $\mathbb{P}^\star\in\Delta(\mathcal{V}^I)$ as the average of the distributions $\hat{\mathbb{P}}^i$ across $i\in\mathcal{I}$, that is, set

$$\mathbb{P}^\star=\frac{1}{I}\sum_{i\in\mathcal{I}}\hat{\mathbb{P}}^i. \tag{6}$$

The support of $\mathbb{P}^\star$ can therefore be expressed as $\text{supp}(\mathbb{P}^\star) = \cup_{i\in\mathcal{I}}\text{supp}(\hat{\mathbb{P}}^i)$. Specifically, under $\mathbb{P}^\star$ the highest bidder's value profile exceeds the positive threshold $\frac{1}{e}\overline{\boldsymbol{v}}$ almost surely, while all other bidders' value profiles are almost surely equal to $\mathbf{0}$.

We will show that the separable mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ corresponding to the single-item mechanisms (2) and the probability distribution $\mathbb{P}^\star$ defined in (6) represent the Nash strategies of the seller and of nature in problem (4), respectively. To simplify the subsequent discussion, we denote by

$$z(\boldsymbol{m}, \mathbb{P}) = \mathbb{E}_{\mathbb{P}}\left[\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}}\tilde{v}_j^i) - \sum_{i\in\mathcal{I}} m^i(\tilde{\boldsymbol{v}})\right]$$

the expected regret of the mechanism $(\boldsymbol{q}, \boldsymbol{m})$ under the probability distribution $\mathbb{P}\in\Delta(\mathcal{V}^I)$.

Theorem 1 **(Nash Equilibrium)**. *The separable mechanism* $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ *corresponding to the single-item mechanisms* (2) *and the distribution* $\mathbb{P}^\star$ *defined in* (6) *satisfy the saddle point condition*

$$\sup_{\mathbb{P}\in\Delta(\mathcal{V}^I)} z(\boldsymbol{m}^\star, \mathbb{P}) \leq z(\boldsymbol{m}^\star, \mathbb{P}^\star) \leq \inf_{(\boldsymbol{q},\boldsymbol{m})\in\mathcal{X}} z(\boldsymbol{m}, \mathbb{P}^\star). \tag{7}$$

To prove Theorem 1, we will first show that the problem on the left-hand side of (7) is solved by $\mathbb{P}^\star$ and attains an optimal value of $\sum_{j\in\mathcal{J}}\frac{\overline{v}_j}{e}$. Next, we relax the problem on the right-hand side of (7) to a single-buyer multi-item pricing problem with the objective of minimizing the regret under the distribution $\hat{\mathbb{P}}$. This relaxation is facilitated by the symmetric construction of the distribution $\mathbb{P}^\star$. We know from Theorem 2 by Koçyiğit et al. (2018) that the optimal value of the resulting pricing problem amounts to $\sum_{j\in\mathcal{J}}\frac{\overline{v}_j}{e}$. The observation that $z(\boldsymbol{m}^\star, \mathbb{P}^\star) = \sum_{j\in\mathcal{J}}\frac{\overline{v}_j}{e}$ then completes the proof.

*Proof of Theorem 1.* We first show that $\mathbb{P}^\star$ solves the problem on the left-hand side of (7) (Step 1), and then we prove that $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ solves the problem on the right-hand side of (7) (Step 2).

*Step 1.* By Proposition 2, the worst-case regret of the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is given by

$$\sup_{\boldsymbol{v}\in\mathcal{V}^I}\sum_{j\in\mathcal{J}}(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}}(m^\star)^i(\boldsymbol{v}) = \sum_{j\in\mathcal{J}}\frac{\overline{v}_j}{e}. \tag{8}$$

The proof of Proposition 2 further reveals that the worst-case regret of the individual mechanism $(\hat{\boldsymbol{q}}_j, \hat{\boldsymbol{m}}_j)$ for selling item $j\in\mathcal{J}$ amounts to $\sup_{\boldsymbol{v}_j\in\mathcal{V}}(\max_{i\in\mathcal{I}} v_j^i) - \sum_{i\in\mathcal{I}}\hat{m}_j^i(\boldsymbol{v}_j) = \frac{\overline{v}_j}{e}$ and that this worst-case regret is attained by any scenario $\boldsymbol{v}_j \in [0, \overline{v}_j]^I$ for which there exists $i \in \mathcal{I}$ with $v_j^i \geq \frac{\overline{v}_j}{e} \geq \max_{k \neq i} v_j^k$. As the uncertainty set $\mathcal{V}^I$ is rectangular, the worst-case regret (8) of the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ is thus attained by any scenario $\boldsymbol{v} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_J) \in \mathcal{V}^I$ for which $\boldsymbol{v}_j$ satisfies the above condition for each $j \in \mathcal{J}$. This implies that the worst-case regret (8) is attained if

$$\boldsymbol{v} \in \left\{ \boldsymbol{v} \in \mathcal{V}^I : \exists i \in \mathcal{I} \text{ with } \boldsymbol{v}^i \geq \frac{1}{e} \overline{\boldsymbol{v}} \text{ and } \boldsymbol{v}^k = \boldsymbol{0} \;\; \forall k \neq i \right\} \supseteq \operatorname{supp}(\mathbb{P}^\star),$$

where the subset relation follows readily from the construction of $\mathbb{P}^\star$. Fix now an arbitrary distribution $\mathbb{P} \in \Delta(\mathcal{V}^I)$. Then, the expected regret of $\boldsymbol{m}^\star$ under $\mathbb{P}$ satisfies

$$z(\boldsymbol{m}^\star, \mathbb{P}) = \mathbb{E}_{\mathbb{P}} \left[ \sum_{j \in \mathcal{J}} (\max_{i \in \mathcal{I}} \tilde{v}_j^i) - \sum_{i \in \mathcal{I}} (m^\star)^i(\tilde{\boldsymbol{v}}) \right] \leq \sum_{j \in \mathcal{J}} \frac{\overline{v}_j}{e},$$

where the inequality follows from (8). The inequality is tight for $\mathbb{P}^\star$ because the ex-post regret in any scenario $\boldsymbol{v} \in \operatorname{supp}(\mathbb{P}^\star)$ equals $\sum_{j \in \mathcal{J}} \frac{\overline{v}_j}{e}$. Thus, $\mathbb{P}^\star$ solves the problem on the left-hand side of (7).

*Step 2.* Consider now the expected regret minimization problem on the right-hand side of (7), which can be expressed more explicitly as

$$\begin{aligned} \inf_{\boldsymbol{q}, \boldsymbol{m}} \quad & \mathbb{E}_{\mathbb{P}^\star} \left[ \sum_{j \in \mathcal{J}} (\max_{i \in \mathcal{I}} \tilde{v}_j^i) - \sum_{i \in \mathcal{I}} m^i(\tilde{\boldsymbol{v}}) \right] \\ \text{s.t.} \quad & \boldsymbol{q} \in \mathcal{L}(\mathcal{V}^I, \mathbb{R}_+^{I \times J}), \, \boldsymbol{m} \in \mathcal{L}(\mathcal{V}^I, \mathbb{R}^I) \\ & \text{(IC), (IR), (Inv).} \end{aligned} \tag{9}$$

To prove that $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ solves problem (9), we first relax this problem by replacing its objective function with a lower bound and by reducing the uncertainty sets of its robust constraints (Step 2.a). We then aggregate the constraints of the resulting problem across the bidders to obtain an even looser relaxation of (9), which turns out to be equivalent to a multi-item pricing problem involving a single bidder (Step 2.b). By leveraging Theorem 2 of Koçyiğit et al. (2018), we then show that this problem's optimal value matches the objective value of $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ in (9).

*Step 2.a.* To construct a relaxation of problem (9), we first establish a lower bound on its objective function. Indeed, for any fixed mechanism $(\boldsymbol{q}, \boldsymbol{m})$ we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^\star} \left[ \sum_{j \in \mathcal{J}} (\max_{i \in \mathcal{I}} \tilde{v}_j^i) - \sum_{i \in \mathcal{I}} m^i(\tilde{\boldsymbol{v}}) \right] &= \frac{1}{I} \sum_{k \in \mathcal{I}} \mathbb{E}_{\hat{\mathbb{P}}^k} \left[ \sum_{j \in \mathcal{J}} (\max_{i \in \mathcal{I}} \tilde{v}_j^i) - \sum_{i \in \mathcal{I}} m^i(\tilde{\boldsymbol{v}}) \right] \\ = \frac{1}{I} \sum_{k \in \mathcal{I}} \mathbb{E}_{\hat{\mathbb{P}}^k} \left[ \sum_{j \in \mathcal{J}} \tilde{v}_j^k - \sum_{i \in \mathcal{I}} m^i(\tilde{\boldsymbol{v}}) \right] &\geq \frac{1}{I} \sum_{k \in \mathcal{I}} \mathbb{E}_{\hat{\mathbb{P}}^k} \left[ \sum_{j \in \mathcal{J}} \tilde{v}_j^k - m^k(\tilde{\boldsymbol{v}}) \right], \end{aligned}$$

where the first equality follows from the definition of $\mathbb{P}^\star$, while the second equality holds because $\tilde{\boldsymbol{v}}^i = \mathbf{0}$ for all $i \neq k$ and $\tilde{\boldsymbol{v}}^k \geq \frac{1}{e}\overline{\boldsymbol{v}} > \mathbf{0}$ almost surely under $\hat{\mathbb{P}}^k$. The inequality exploits the individual rationality constraint (IR), which implies that $m^i(\tilde{\boldsymbol{v}}) \leq 0$ almost surely under $\hat{\mathbb{P}}^k$ for all $i \neq k$.

For each bidder $i \in \mathcal{I}$, define $\mathcal{S}^i = \{\boldsymbol{v} \in \mathcal{V}^I : \boldsymbol{v}^k = \mathbf{0} \ \ \forall k \neq i\}$. Next, we relax the incentive compatibility constraint (IC) and the individual rationality constraint (IR) for any bidder $i \in \mathcal{I}$ by enforcing them only for scenarios $\boldsymbol{v} \in \mathcal{S}^i \subseteq \mathcal{V}^I$. The resulting relaxations are thus representable as

$$\begin{aligned}&\sum_{j\in\mathcal{J}} q^i_j(\boldsymbol{v}^i,\boldsymbol{v}^{-i})v^i_j - m^i(\boldsymbol{v}^i,\boldsymbol{v}^{-i})\\ &\geq \sum_{j\in\mathcal{J}} q^i_j(\boldsymbol{w}^i,\boldsymbol{v}^{-i})v^i_j - m^i(\boldsymbol{w}^i,\boldsymbol{v}^{-i}) \quad \forall i\in\mathcal{I}, \forall \boldsymbol{v}\in\mathcal{S}^i, \forall \boldsymbol{w}^i\in\mathcal{V}\end{aligned} \tag{$\widehat{\text{IC}}$}$$

and

$$\sum_{j\in\mathcal{J}} q^i_j(\boldsymbol{v}^i,\boldsymbol{v}^{-i})v^i_j - m^i(\boldsymbol{v}^i,\boldsymbol{v}^{-i}) \geq 0 \quad \forall i\in\mathcal{I}, \forall \boldsymbol{v}\in\mathcal{S}^i. \tag{$\widehat{\text{IR}}$}$$

Similarly, we note that the original inventory constraint (Inv) implies the relaxation

$$\begin{aligned}&\sum_{i\in\mathcal{I}} q^i_j(\boldsymbol{v}) \leq 1 \quad \forall j\in\mathcal{J}, \forall k\in\mathcal{I}, \forall \boldsymbol{v}\in\mathcal{S}^k\\ &\implies \quad q^k_j(\boldsymbol{v}) \leq 1 \quad \forall j\in\mathcal{J}, \forall k\in\mathcal{I}, \forall \boldsymbol{v}\in\mathcal{S}^k,\end{aligned} \tag{$\widehat{\text{Inv}}$}$$

where the second implication holds because the allocation probabilities are nonnegative on $\mathcal{V}^I$.

In summary, we obtain the following relaxation of problem (9).

$$\begin{aligned}\inf_{\boldsymbol{q},\boldsymbol{m}} \quad & \frac{1}{I}\sum_{k\in\mathcal{I}} \mathbb{E}_{\hat{\mathbb{P}}^k}\left[\sum_{j\in\mathcal{J}} \tilde{v}^k_j - m^k(\tilde{\boldsymbol{v}})\right]\\ \text{s.t.} \quad & \boldsymbol{q}\in\mathcal{L}(\mathcal{V}^I,\mathbb{R}^{I\times J}_+),\ \boldsymbol{m}\in\mathcal{L}(\mathcal{V}^I,\mathbb{R}^I)\\ & (\widehat{\text{IC}}),\ (\widehat{\text{IR}}),\ (\widehat{\text{Inv}})\end{aligned} \tag{10}$$

*Step 2.b.* We now use constraint aggregation to construct a relaxation of problem (10), which constitutes another—even looser—relaxation of problem (9). To this end, define for any $i \in \mathcal{I}$ the linear embedding $E^i \in \mathcal{L}(\mathcal{V},\mathcal{V}^I)$ via

$$E^i(\boldsymbol{v}) = (\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{i-1},\boldsymbol{v}^\top,\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{I-i})^\top,$$

where, by slight abuse of notation, $\boldsymbol{v} \in \mathcal{V}$ denotes any value profile of a fixed bidder (recall also that the elements of $\mathcal{V}$ constitute row vectors). The proposed aggregation averages all constraint of (10) across the bidders and expresses the resulting optimization problem in terms of the new auxiliary variables $\boldsymbol{f} \in \mathcal{L}(\mathcal{V}, \mathbb{R}_+^J)$ and $g \in \mathcal{L}(\mathcal{V}, \mathbb{R})$ defined via

$$f_j(\boldsymbol{v}) = \frac{1}{I} \sum_{i \in \mathcal{I}} q_j^i(E^i(\boldsymbol{v})) \;\; \forall j \in \mathcal{J} \quad \text{and} \quad g(\boldsymbol{v}) = \frac{1}{I} \sum_{i \in \mathcal{I}} m^i(E^i(\boldsymbol{v})),$$

respectively, where $\boldsymbol{v} \in \mathcal{V}$ again denotes any value profile of a fixed bidder.

Thanks to the definition of the set $\mathcal{S}^i$ introduced in Step 2.a, the relaxed incentive compatibility constraint ($\widehat{\text{IC}}$) can be expressed as

$$\sum_{j \in \mathcal{J}} q_j^i(E^i(\boldsymbol{v}))v_j - m^i(E^i(\boldsymbol{v})) \geq \sum_{j \in \mathcal{J}} q_j^i(E^i(\boldsymbol{w}))v_j - m^i(E^i(\boldsymbol{w})) \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V},$$

where we use again $\boldsymbol{v}$ and $\boldsymbol{w}$ to denote arbitrary value profiles of a fixed bidder. By averaging the above inequality across all bidders $i \in \mathcal{I}$, we obtain the following aggregate constraint, which can be reformulated in terms of the new decision variables $\boldsymbol{f}$ and $g$.

$$\begin{aligned} &\frac{1}{I} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} q_j^i(E^i(\boldsymbol{v}))v_j - \frac{1}{I} \sum_{i \in \mathcal{I}} m^i(E^i(\boldsymbol{v})) \geq \frac{1}{I} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} q_j^i(E^i(\boldsymbol{w}))v_j - \frac{1}{I} \sum_{i \in \mathcal{I}} m^i(E^i(\boldsymbol{w})) \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V} \\ \iff \quad &\sum_{j \in \mathcal{J}} f_j(\boldsymbol{v})v_j - g(\boldsymbol{v}) \geq \sum_{j \in \mathcal{J}} f_j(\boldsymbol{w})v_j - g(\boldsymbol{w}) \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V} \end{aligned} \tag{$\widehat{\text{IC}}'$}$$

Similarly, we can reformulate the relaxed individual rationality constraint ($\widehat{\text{IR}}$) as

$$\sum_{j \in \mathcal{J}} q_j^i(E^i(\boldsymbol{v}))v_j - m^i(E^i(\boldsymbol{v})) \geq 0 \quad \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}.$$

By averaging the resulting inequality across all bidders $i \in \mathcal{I}$, we obtain the following aggregate constraint and its reformulation in terms of $\boldsymbol{f}$ and $g$.

$$\begin{aligned} &\frac{1}{I} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} q_j^i(E^i(\boldsymbol{v}))v_j - \frac{1}{I} \sum_{i \in \mathcal{I}} m^i(E^i(\boldsymbol{v})) \geq 0 \quad \forall \boldsymbol{v} \in \mathcal{V} \\ \iff \quad &\sum_{j \in \mathcal{J}} f_j(\boldsymbol{v})v_j - g(\boldsymbol{v}) \geq 0 \quad \forall \boldsymbol{v} \in \mathcal{V} \end{aligned} \tag{$\widehat{\text{IR}}'$}$$

Finally, the relaxed inventory constraint ($\widehat{\text{Inv}}$) can be formulated as

$$q_j^i(E^i(\boldsymbol{v})) \leq 1 \quad \forall j \in \mathcal{J}, \forall i \in \mathcal{I}, \forall \boldsymbol{v} \in \mathcal{V}.$$

Averaging the above inequality across all bidders $i \in \mathcal{I}$, we obtain

$$\begin{aligned} &\frac{1}{I}\sum_{i\in\mathcal{I}} q_j^i(E^i(\boldsymbol{v})) \leq 1 \quad \forall j \in \mathcal{J}, \forall \boldsymbol{v} \in \mathcal{V} \\ \iff \quad & f_j(\boldsymbol{v}) \leq 1 \quad \forall j \in \mathcal{J}, \forall \boldsymbol{v} \in \mathcal{V}. \end{aligned} \tag{$\widehat{\text{Inv}}'$}$$

We can also re-express the decision-dependent part of the objective function of problem (10) in terms of the new variables as

$$\frac{1}{I}\sum_{k\in\mathcal{I}} \mathbb{E}_{\hat{\mathbb{P}}^k}\left[m^k(\tilde{\boldsymbol{v}})\right] = \frac{1}{I}\sum_{k\in\mathcal{I}} \mathbb{E}_{\hat{\mathbb{P}}^k}\left[m^k(E^k(\tilde{\boldsymbol{v}}^k))\right] = \frac{1}{I}\sum_{k\in\mathcal{I}} \mathbb{E}_{\hat{\mathbb{P}}}\left[m^k(E^k(\tilde{\boldsymbol{v}}^1))\right] = \mathbb{E}_{\hat{\mathbb{P}}}\left[g(\tilde{\boldsymbol{v}}^1)\right],$$

where the first equality holds because $\tilde{\boldsymbol{v}}^i = \boldsymbol{0}$ for all $i \neq k$ almost surely under $\hat{\mathbb{P}}^k$, while the second equality holds because the marginal distribution of $\tilde{\boldsymbol{v}}^k$ under $\hat{\mathbb{P}}^k$ is given by $\hat{\mathbb{P}}$.

The resulting aggregation of problem (10) can now be represented as

$$\begin{aligned} \inf_{\boldsymbol{f},g} \quad & \mathbb{E}_{\hat{\mathbb{P}}}\left[\sum_{j\in\mathcal{J}} \tilde{v}_j^1 - g(\tilde{\boldsymbol{v}}^1)\right] \\ \text{s.t.} \quad & \boldsymbol{f} \in \mathcal{L}(\mathcal{V}, \mathbb{R}_+^J),\, g \in \mathcal{L}(\mathcal{V}, \mathbb{R}) \\ & (\widehat{\text{IC}}'), (\widehat{\text{IR}}'), (\widehat{\text{Inv}}'). \end{aligned} \tag{11}$$

By construction, the problems (10) and (11) constitute two increasingly loose relaxations of problem (9). Moreover, problem (11) constitutes a multi-item pricing problem involving a single bidder ($I = 1$) that minimizes the expected regret under the distribution $\hat{\mathbb{P}}$. The decision variables $\boldsymbol{f}$ and $g$ can be interpreted as the allocation and payment rules of the sales mechanism, respectively. The optimal value of this problem amounts to $\sum_{j\in\mathcal{J}} \frac{\overline{v}_j}{e}$ (Koçyiğit et al. 2018, Theorem 2). As problem (11) is a relaxation of problem (9) and as $z(\boldsymbol{m}^\star, \mathbb{P}^\star) = \sum_{j\in\mathcal{J}} \frac{\overline{v}_j}{e}$ by Step 1, we can conclude that the mechanism $(\boldsymbol{q}^\star, \boldsymbol{m}^\star)$ solves problem (9). This observation completes the proof. □

By Theorem 1, the optimal value of (1) is given by $\sum_{j\in\mathcal{J}} \frac{\overline{v}_j}{e}$ and does therefore not depend on the number of bidders. This insight culminates in the following corollary.

COROLLARY 1. *The optimal value of* (1) *is independent of the number $I$ of bidders.*

Corollary 1 implies that the seller incurs the same worst-case regret irrespective of the number of bidders participating in the auction.

## 4. Concluding Remarks

We studied a robust auction design problem with minimax regret objective, where the seller only knows that the bidders' values range over a box uncertainty set. We interpreted this problem as a zero-sum game between the seller and nature, and we showed that this game admits a Nash equilibrium in mixed strategies that can be characterized in closed form. The seller's Nash strategy is a separable mechanism consisting of item-wise second price auctions with (random) reserve prices. Nature's Nash strategy is a distribution on the uncertainty set under which the values of the items are comonotonic for any fixed bidder and under which only the highest bidder assigns a positive value to any fixed item. Under this distribution the second highest bid for any item falls almost surely below the reserve price, which implies that the worst-case regret remains constant in the number of bidders. Our proof critically relies on the permutation symmetry of the bidders, which is a manifestation of their anonymity, and the rectangularity of the uncertainty set. We hope, however, that similar techniques can be used to solve auction design problems in which the seller has more information about the distribution of the bidders' values (*e.g.*, about their moments). Specifically, we hope that this paper will provide insights and motivation to prove separation results for more general robust auction design problems with different informational assumptions.

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