Portfolio Optimization with Drift Uncertainty

Kerem Uğurlu

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Department of Mathematics, Nazarbayev University.
e-mail:kerem.ugurlu@nu.edu.kz

Abstract

We study the utility maximization problem of a portfolio of one risky asset, a stock, and one riskless asset, a bond, under Knightian uncertainty on the drift term representing the long term growth rate of the risky asset. We further assume that the investor has a prior estimate about the drift term, so that we incorporate into the model a penalty term for deviating from the prior about the mean. We provide explicit solutions, when the investor has logarithmic, power and exponential utility functions. A numerical case study with respect to different utilities has also been presented.

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1 Introduction

The basic question in mathematical finance is how to model the stock movements. Starting with pioneering works, \[1,2\], the stocks are modelled as geometric Brownian motions, where there exists a fixed underlying reference probability measure \(\mathbb{P}\) that is retrieved from historical data of the price movements. However, it is impossible to precisely identify \(\mathbb{P}\). Hence, model ambiguity, also called Knightian uncertainty, \[3\], is inevitably taken into consideration in mathematical finance. Namely, the investor is uncertain about the underlying dynamics of the model, and takes a robust approach to the utility maximization problem.

One of the directions in optimization under Knightian uncertainty is the Bayesian approach. In this direction, \[4\] studies an option pricing problem, where the parameters
are learned as information increases. [5] studies a utility maximization problem applying Bayesian techniques to learn the uncertain parameters as information increases. In [11], a Markowitz portfolio selection problem with unknown drift vector in the multi-dimensional framework has been studied, where a Bayesian approach from filtering theory is used to learn the posterior distribution about the drift given the observed market data of the assets. Another direction in optimization under Knightian uncertainty is the max min approach, where the investor minimizes over the priors, corresponding to different scenarios, and then maximizes over the admissible investment strategies. As described in [9], this max min approach can be perceived as a non-cooperative game between two agents, namely the investor and the fictitious agent, nature. The investor tries to maximize his expected utility of terminal wealth by choosing an admissible policy judiciously, while nature is competing with the investor by choosing the parameters of the underlying dynamics to minimize the terminal wealth. We refer the reader to [12] for a thorough discussion on max min and Bayesian approaches in utility maximization. Furthermore, an axiomatic approach to Knightian uncertainty has been introduced in [6], where a family of risk evaluation operators, so called coherent and convex risk measures, is introduced. Here, coherent risk measures can be incorporated into the max min framework, where the decision taker behaves according to a family of alternative scenarios, whereas convex risk measures incorporate a penalty factor into the max min framework that allow to put more weight into one of the alternative scenarios while keeping the properties of a coherent risk measure except positive homogeneity. Based on [6], [7] applies convex risk measures on a portfolio maximization problem. Another work in this direction has been [13], where Markowitz-type classic portfolio selection problem is studied using convex risk measures.

In this paper, we are studying a utility maximization problem. We assume that the investor has one stock, risky asset, and one bond, riskless asset. We take that there is uncertainty on the long term growth rate of the stock represented by some random process $\mu_t$, whereas the volatility has been estimated to be a constant denoted by $\sigma$. Assuming constant $\sigma$ corresponds to estimating quadratic variation of a single stock. This is in concurrence with the practice. On the other hand, estimating the drift of a stock with reasonable precision is seriously difficult, if not impossible. It requires extraordinarily long time series corresponding to decades, which is rarely available (see [12] for a discussion on this). Based on that, inspired by convex risk measures, we take a max min approach and also incorporate a penalty factor into the model. The penalty factor quantifies the uncertainty of the underlying model. As it increases, the reliance on the model decreases. Analogously, as the penalty factor decreases, the reliance on the model increases. The penalty factor applied on the model differs for each utility function. In particular, the penalty factor is applied in multiplicative or additive
fashion depending on the specific chosen utility function. This allows the model to be consistent with the classical utility maximization, when there is no Knightian uncertainty.

The rest of the paper is as follows. In Section 2, we introduce the model dynamics and financial scenario, and present the investor’s value function with respect to different utility functions. In Section 3, we introduce three utility functions that are assumed to be utilised by the investor. These are logarithmic, power and exponential utility functions. We give the explicit solutions along with the optimal parameters in this setting using these utility functions. In Section 4, we present numerical cases on these three cases and conclude the paper.

2 Model Dynamics and Investor’s Value Function

2.1 Model Dynamics

We work on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})\) satisfying the usual assumptions supporting a standard Brownian motion \(W\). We consider a market consisting of a risky asset \(S_t\) and one riskless asset \(R_t\). We assume \(S_t\) and \(R_t\) satisfy the following dynamics

\[
dR_t = rR_t dt \\
S_0 = s_0 \\
dS_t = S_t(\mu_t dt + \sigma dW_t), \quad \mathbb{P}\text{-a.s.,}
\]

where, \(S_0 = s_0 > 0\) is the value of the stock at \(t = 0\).

Assumption 2.1. We assume that \(r > 0\) is the risk-free interest rate and \(\sigma > 0\) is the constant volatility of the stock. Further, we take that \((\mu_t)_{t\in[0,T]}\) is a \(\mathcal{B}([0,T])\) measurable deterministic bounded process representing the long term growth rate of the stock \(S_t\).

In this framework, we consider the problem of an agent investing in a risky asset \(S\) and one riskless asset \(R\) for a given initial endowment \(x_0 > 0\), where the investor trades in a self financing way. We denote \((\hat{\pi}_t)_{t\in[0,T]}\) as an \(\mathcal{B}([0,T]) \otimes \mathcal{F}_T\) progressively measurable stochastic process, which stands for the total amount of money invested in the risky asset \(S_t\) at time \(t \in [0,T]\). Then, we have for initial wealth \(X_0 = x_0 > 0\)

\[
d\hat{X}^\pi = \hat{\pi}_t S_t^{-1} dS_t + (\hat{X}^\pi_t - \hat{\pi}_t) r dt, \\
d\tilde{X}^\pi = \hat{\pi}_t (\mu_t dt + \sigma dW_t) + (\tilde{X}^\pi_t - \hat{\pi}_t) r dt, \quad \mathbb{P}\text{-a.s.}
\]

We further represent the amount of money invested in the risky assets as a fraction of current wealth via \(\hat{\pi}_t = \hat{X}^\pi_t \pi_t\) for \(t \in [0,T]\), where \(\pi_t\) stands for the corresponding fraction at time
\( t \in [0, T] \), and take the discounted wealth \( X^{\pi, \eta}_t = e^{-rt} \hat{X}^{\pi, \eta}_t \). Hence, for \( X^{\pi, \eta}_t = x \), the dynamics of wealth in this setting are given by

\[
dX^{\pi, \eta}_t = X^{\pi, \eta}_t \pi_t ((\mu_t - r) dt + \sigma dW_t)
\]

\[
X^{\pi, \eta}_T = x \exp \left( \int_0^T (\mu_u - r) - \frac{1}{2} \pi_u^2 \sigma^2 du + \int_0^T \pi_u \sigma dW_u \right)
\]

**Definition 2.1.** Let \((\eta_t)_{t \in [0, T]}\) and \((\pi_t)_{t \in [0, T]}\) denote the \( \mathcal{B}([0, T]) \otimes \mathcal{F}_T \) progressively measurable process representing the uncertainty in long term growth rate \( \mu = (\mu_t)_{t \in [0, T]} \) and the cash-value allocated in the risky asset, respectively. We call \((\eta_t)_{t \in [0, T]}\) and \((\pi_t)_{t \in [0, T]}\) admissible and denote it by \( \eta \in \Theta_{\text{ad}} \) and \( \pi \in \Pi_{\text{ad}} \), respectively, if they satisfy

(i) \( \mathbb{P}(X^{\pi, \eta}_t > 0, t \in [0, T]) = 1 \).

(ii) \( \mathbb{P}(|\eta_t| + |\pi_t| \leq C, t \in [0, T]) = 1 \), for some \( C > 0 \).

In particular, for a fixed \( \eta \) and \( \pi \), the wealth of the investor must stay positive. Moreover, the total amount of money invested in risky asset \( \pi \) both short and long positions and the uncertainty in the growth rate \( \eta \) are bounded throughout \([0, T]\).

Let \( \eta = (\eta_t)_{t \in [0, T]} \) be as in Definition 2.1 Then, we define a new measure \( \mathbb{P}^{\eta} \) that is equivalent to \( \mathbb{P} \) via

\[
dW^{\eta}_t \triangleq dW_t + \eta_t dt
\]

\[
d\frac{\mathbb{P}^{\eta}}{\mathbb{P}} \triangleq \exp \left( - \int_0^T \eta_t dW_t - \frac{1}{2} \int_0^T \eta_t^2 dt \right)
\]

\[
= \exp \left( - \int_0^T \eta_t dW^{\eta}_t + \frac{1}{2} \int_0^T \eta_t^2 dt \right)
\]

\[
d\frac{\mathbb{P}^{\eta}}{d\mathbb{P}} \triangleq \exp \left( - \int_t^T \eta_u dW^{\eta}_u - \frac{1}{2} \int_t^T \eta_u^2 du \right)
\]

such that (2.1) reads as

\[
dX^{\pi, \eta}_t = \pi_t X^{\pi, \eta}_t ((\mu_t - \sigma \eta_t - r) dt + \sigma dW^{\eta}_t),
\]

where the additional term \(- \sigma \eta_t\) gives the uncertainty on drift term. Furthermore, we denote \( \mathbb{E}^\mathbb{P}_t[\cdot] \triangleq \mathbb{E}^\mathbb{P}[\cdot|\mathcal{F}_t] \) as the conditional expectation that is taken with respect to measure \( \mathbb{P} \). Similarly, \( \mathbb{E}^{\eta}_t[\cdot] \triangleq \mathbb{E}^{\eta}[\cdot|\mathcal{F}_t] \) stands for the conditional expectation that is taken with respect to \( \mathbb{P}^{\eta} \) as defined in (2.2).
2.2 Investor’s Value Function

The investor utilizes the classical Merton problem, but he is uncertain about the underlying dynamics of the drift \((\mu_t)_{t \in [0,T]}\) of the stock. The investor takes a robust approach, where he evaluates his terminal wealth \(X^{\pi,\eta}_T\) using the maximum of the admissible policies \(\Pi_{ad}\) among the least favourable \(\Theta_{ad}\) parameters using a utility function by assuming a prior on \(\mu = (\mu_t)_{t \geq 0}\). For deviating from his assumption on the drift term \(\mu\), he puts a penalizing function \(\alpha(t, \eta, \phi)\) that depends on time \(t\), deviation term \(\eta\) and \(\phi > 0\) quantifying his reliance on the prior for the model. Hence, the investor’s value function for \(x > 0\) depending on the logarithmic \(\log(x)\), power \(x^\gamma\) with \(0 < \gamma < 1\) and exponential \(-\beta e^{-\beta x}\) with \(\beta > 0\) utility functions read as

\[
V_{\log}(t, x, \mu) \triangleq \sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}_t^P[\log(X^{\pi,\eta}_T) + \alpha_{\log}(t, \eta, \phi)] \tag{2.4}
\]

\[
V_{\text{pow}}(t, x, \mu) \triangleq \sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}_t^P[(X^{\pi,\eta}_T)^\gamma \alpha_{\text{pow}}(t, \eta, \phi)] \tag{2.5}
\]

\[
V_{\text{exp}}(t, x, \mu) \triangleq \sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}_t^P[-\beta \exp(-\beta(X^{\pi,\eta}_T)) \alpha_{\text{exp}}(t, \eta, \phi)] \tag{2.6}
\]

3 Explicit Solutions with Specific Utility Functions

In this section, we thoroughly elaborate the value functions in (2.4), (2.5) and (2.6) along with the optimal controls and give explicit representations for these three specific utility functions.

3.1 Logarithmic Utility

We are going to use the utility function \(\log(x)\) for \(x > 0\) in this section. Using (2.1) and (3.1), we have

\[
\log(X^{\pi,\eta}_T) = \log(x) + \int_t^T \pi_u (\mu_u - r) - \frac{1}{2} \pi_u^2 \sigma^2 du + \int_t^T \pi_u \sigma dW_u
\]

\[
\log(X^{\pi,\eta}_T) = \log(x) + \int_t^T \pi_u (\mu_u - r - \sigma \eta_u) - \frac{1}{2} \pi_u^2 \sigma^2 du + \int_t^T \pi_u \sigma dW_u^n \tag{3.1}
\]

Next, for \(\phi > 0\), we define the time dependent penalty term for deviating from the prior on \(\mu\) via

\[
\alpha_{\log}(t, \eta, \phi) \triangleq \frac{dP_t^n}{dP_t} \log \left( \frac{dP_t^n}{dP_t} \right) = \frac{dP_t^n}{dP_t} \left( \frac{1}{2\phi} \int_t^T \eta_u^2 du - \int_t^T \eta_u dW_u^n \right)
\]
The value function of the investor then reads as

\[
V_{\log}(t, x, \mu) \triangleq \sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}_t^\pi \left[ \log(X_T^{\pi,\eta}) + \alpha_{\log}(t, \eta, \phi) \right]
= \log(x) + \sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}_t^\eta \left[ \int_t^T \pi_u (\mu_u - r - \sigma \eta_u) \right.
- \left. \frac{1}{2} \pi_u^2 \sigma^2 + \frac{1}{2} \phi \eta_u^2 du \right]
\]  

(3.2)

In (3.2), we used the Bayes formula for conditional expectation to change from \(\mathbb{E}_t^P[\cdot]\) to \(\mathbb{E}_t^\eta[\cdot]\) (see e.g. [10]).

Remark 3.1. Note that as \(\phi \to \infty\), the penalty term \(\alpha_{\log}(t, \eta, \phi)\) vanishes, and the problem turns into a max min approach

\[
\sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}_t^\pi \left[ \log(X_T^{\pi,\eta}) \right].
\]

This implies the prior belief on \(\mu_t\) for \(S_t\) does not have an effect in investor’s decisions. Analogously, as \(\phi \to 0\), the value function reads as

\[
\sup_{\pi \in \Pi_{ad}} \mathbb{E}_t^\pi \left[ \log(X_T^{\pi}) \right],
\]

where the investor behaves as if his prior on \(\mu_t\) is the real long term growth of the stock. In particular, the problem turns into classical Merton problem [1].

Theorem 3.1. Let \(\Theta_{ad}\) and \(\Pi_{ad}\) be as in Definition 2.1 and the value function \(V_{\log}(t, x, \mu)\) be as in (3.2) for a fixed \(\phi > 0\). Then, under Assumption 2.1 for \(t \in [0, T]\), the optimal parameters are

\[
\eta^*_t = \frac{\phi (\mu_t - r)}{\sigma (\phi + 1)}, \\
\pi^*_t = \frac{\mu_t - r}{\sigma^2 (\phi + 1)}.
\]  

(3.3) (3.4)

The optimal value function for \(t \in [0, T]\) and \(x > 0\) reads as

\[
V_{\log}(t, x) = \log(x) + \frac{1}{2\sigma^2 (\phi + 1)} \int_t^T (\mu_u - r)^2 du.
\]  

(3.5)

Proof. By (3.2), consider the mapping

\[
f(\eta_u) \triangleq \pi_u (\mu_u - r \sigma \eta_u) - \frac{1}{2} \pi_u^2 \sigma^2 + \frac{1}{2 \phi} \eta_u^2
\]  

(3.6)
We note that $f$ is convex in $\eta$, and for any fixed $\pi_u$, it is enough to check first order condition to find the infimum. Hence,

$$\eta_u^* = \pi_u \sigma \phi \tag{3.7}$$

Then, plugging (3.7) into (3.6), we get

$$g(\pi_u) \triangleq \pi_u (\mu_u - r) - \sigma \pi_u (\pi_u \phi) - \frac{1}{2} \pi_u^2 \sigma^2 + \frac{1}{\phi} \pi_u^2 \sigma^2 \phi^2$$

$$= \pi_u (\mu_u - r) - \frac{1}{2} \pi_u^2 \sigma^2 (\phi + 1)$$

Then, analogously, $g$ is concave in $\pi_u$, and again it is enough to check first order condition to find the supremum. In particular,

$$0 = \mu_u - r - \pi_u^* \sigma^2 (\phi + 1),$$

$$\pi_u^* = \frac{\mu_u - r}{\sigma^2 (\phi + 1)}.$$

Thus, (3.7) reads as

$$\eta_u^* = \frac{\phi (\mu_u - r)}{\sigma (\phi + 1)}.$$  

Hence, plugging the optimal controls $\pi_u^*$ and $\eta_u^*$ back into (3.2), we get the value function as

$$V_{\log}(t, x) = \log(x) + \frac{1}{2 \sigma^2 (\phi + 1)} \int_t^T (\mu_u - r)^2 du$$

Hence, we conclude the proof. \qed

**Remark 3.2.** We, indeed, see that $\phi > 0$ determines how much the investor takes his prior $\mu_t$ into consideration while taking decisions. The optimal parameters and the value function are in concurrence with these, as needed. When $\phi = 0$, the investor solves the classical Merton problem with $\mu_t$ with $\pi_t^* = (\mu_t - r)/\sigma_u^2$ and $V_{\log}(t, x) = \log(x) + \frac{1}{2} (\mu_t - r)^2$. Analogously, as $\phi \to \infty$, the investor does not take his prior about $\mu$ into consideration and takes a max min approach. This implies $\pi^* \equiv 0$, meaning the investor puts no amount of his wealth into risky asset and the whole discounted wealth does not differ from the value $\log(x)$ at time $t$ with $V_{\log}(t, x) = \log(x)$. We further remark that for any $\phi > 0$, the optimal controls in (3.3) and (3.4) are admissible, i.e. $\pi^* \in \Pi_{ad}$ and $\eta^* \in \Theta_{ad}$.
3.2 Power Utility

We are going to use the utility function $U(x) = x^\gamma$ for $x > 0$ and $0 < \gamma < 1$ in this section. Using (2.1), we have

$$(X_T^{\pi,\eta})^\gamma = x^\gamma \exp \left( \int_t^T \gamma (\pi_u (\mu_u - r) - \sigma \eta_u) - \frac{1}{2} \pi_u^2 \sigma^2 \right) du + \int_t^T \gamma \pi_u \sigma dW^\eta_u$$

We define the penalty function of the power utility for $\phi > 0$ as

$$\alpha_{pow}(t, \eta, \phi) \triangleq \exp \left( - \int_t^T \eta_u dW_u - \frac{1}{2} \int_t^T \eta_u^2 du \right) \exp \left( \int_t^T \frac{1}{2\phi} \eta_u^2 du \right)$$

Hence, the value function at $t \in [0, T]$ and $x > 0$ reads as

$$V_{pow}(t, x) \triangleq \sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}^P_0 \left[ (X_T^{\pi,\eta})^\gamma \alpha_{pow}(t, \eta, \phi) \right]$$

(3.8)

$$= x^\gamma \sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}^P_0 \left[ \exp \left( \int_t^T (\gamma \pi_u (\mu_u - r) - \sigma \eta_u) - \frac{\gamma}{2} \pi_u^2 \sigma^2 \right) + \frac{1}{2\phi} \eta_u^2 du \right]$$

(3.9)

$$+ \int_t^T \pi_u \sigma dW^\eta_u$$

Theorem 3.2. Let $\Theta_{ad}$ and $\Pi_{ad}$ be as in Definition 2.1. Let the value function be as in (3.8) for a fixed $\phi > 0$. Then, under Assumption 2.1, for $t \in [0, T]$, the optimal parameters for $t \in [0, T]$ are

$$\eta^*_t = \frac{\phi \gamma (\mu_t - r)}{(\gamma (\phi - 1) + 1)\sigma}$$

(3.10)

$$\pi^*_t = \frac{\mu_t - r}{(\gamma (\phi - 1) + 1)\sigma}.$$  

(3.11)

The optimal value function for $t \in [0, T]$ and $x > 0$ reads as

$$V_{pow}(t, x, \mu) = x^\gamma \exp \left( \frac{\gamma}{2\sigma^2 (\gamma (\phi - 1) + 1)} \int_t^T (\mu_u - r) du \right).$$

(3.12)

Proof. Let

$$Z^n_\pi \triangleq \exp \left( \int_0^T \gamma \sigma \pi_u dW^\eta_u - \frac{1}{2} \gamma^2 \int_0^T \sigma^2 \pi_u^2 du \right),$$

such that, by Definition 2.1, it defines the new measure $Q_\pi$ on $\mathcal{F}_T$ by

$$dQ_\pi(\omega) \triangleq Z^n_\pi(\omega)dP_0(\omega).$$
Then, we rewrite (3.9) as

\[
V(t, x, \mu) = x^\gamma \sup_{\pi \in \Pi_{ad}} \inf_{\eta \in \Theta_{ad}} \mathbb{E}^Q_{\pi} \left[ \exp \left( \int_t^T \gamma \pi_u (\mu_u - r - \sigma \eta_u) \right. \right.
\]

\[
- \frac{1}{2} \gamma (1 - \gamma) \sigma^2 \pi_u^2 + \frac{1}{2 \phi} \eta_u^2 du \left. \right]\]

Here, \( \mathbb{E}^Q_{\pi} [\cdot] \triangleq \mathbb{E}^Q_{\pi} [\cdot | \mathcal{F}_t] \) stands for the conditional expectation taken with respect to the measure \( Q_{\pi} \). We note that for fixed \( \pi \in \Pi_{ad} \), the expression inside integral of

\[
\eta \to \exp \left( \int_t^T \gamma \pi_u (\mu_u - r - \sigma \eta_u) - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 \pi_u^2 + \frac{1}{2 \phi} \eta_u^2 \right)
\]

is convex in \( \eta \in \Theta_{ad} \). Taking the integral and exponential preserves convexity. Hence, it is enough to find the minimizer among \( \eta \in \Theta_{ad} \)

\[
\eta_u \to \left( \gamma \pi_u (\mu_u - r - \sigma \eta_u) - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 \pi_u^2 + \frac{1}{2 \phi} \eta_u^2 \right).
\]

(3.13)

Thus, to find the minimizer \( \eta^* \), checking the first order condition in (3.13) is enough with

\[
-\gamma \pi_u \sigma + \frac{1}{\phi} \eta_u^* = 0
\]

\[
\phi \sigma \gamma \pi_u = \eta_u^*
\]

(3.14)

Then, plugging \( \eta^* \) back into (3.13), we have

\[
f(\pi) = \gamma \pi_u (\mu_u - r - \sigma \phi \gamma \pi_u) - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 \pi_u^2 + \frac{1}{2 \phi} \sigma^2 \gamma^2 \pi_u^2
\]

(3.15)

\[
= \gamma \pi_u (\mu_u - r) - \sigma^2 \phi \gamma^2 \pi_u^2 - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 \pi_u^2 + \frac{1}{2} \phi \sigma^2 \gamma^2 \pi_u^2
\]

\[
= \gamma \pi_u (\mu_u - r) - \frac{1}{2} \phi \sigma^2 \gamma^2 \pi_u^2 - \frac{1}{2} \gamma (1 - \gamma) \sigma^2 \pi_u^2
\]

By noting that \( f(\pi_u) \) is a concave function, integrating and exponentiating is monotone increasing, we conclude first order condition is again enough to find the maximizer \( \pi_u^* \). Thus,

\[
0 = \gamma (\mu_u - r) - \phi \sigma^2 \gamma^2 \pi_u - \gamma (1 - \gamma) \sigma^2 \pi_u
\]

\[
= \mu_u - r - \phi \sigma^2 \gamma \pi_u - (1 - \gamma) \sigma^2 \pi_u
\]

\[
= \mu_u - r - \pi_u (\phi \sigma^2 \gamma + (1 - \gamma) \sigma^2)
\]

with

\[
\pi_u^* = \frac{\mu_u - r}{\sigma^2 (\gamma (\phi - 1) + 1)}, \ u \in [0, T]
\]
Hence, plugging $\pi_u^*$ back into (3.14), we find the optimal $(\eta_u^*)_{u \in [0,T]}$

$$\eta_u^* = \frac{\phi \gamma (\mu_u - r)}{(\gamma (\phi - 1) + 1) \sigma}, \ u \in [0,T].$$

Then, plugging $\pi_u^*$ back to $f(\pi_u^*)$ in (3.15), we get after some simple algebraic manipulations

$$f(\pi_u^*) = \frac{1}{2} \frac{\gamma (\mu_u - r)^2}{\sigma^2 (\gamma (\phi - 1) + 1)}.$$

In particular, the value function as in (3.8) reads as

$$V_{pow}(t, x) = x^\gamma \exp \left( \frac{\gamma}{2 \sigma^2 (\gamma (\phi - 1) + 1)} \int_t^T (\mu_u - r) du \right).$$

Hence, we conclude the proof. □

**Remark 3.3.** The penalty function $\alpha_{pow}$ with the prespecified level $\phi > 0$ gives the same effect on the optimal parameters and the value function as in logarithmic case. In particular, as $\phi = 0$, we have the benchmark case of classical Merton problem with power utility. As $\phi \to \infty$, the prior $\mu_t$ becomes irrelevant, the investor refrains completely from investing into risky asset with $\pi \triangleq 0$ and $V_{pow} = x^\gamma$. We emphasize here we use the penalty function in a multiplicative way compared to the additive application with logarithmic utility. We further remark that for any $\phi > 0$, the optimal controls are admissible, i.e. $\pi^* \in \Pi_{ad}$ and $\eta^* \in \Theta_{ad}$.

### 3.3 Exponential Utility

We next analyze the problem for the exponential utility $u(x) = -\beta e^{-\beta x}$ with $x > 0$ and $\beta > 0$. We take $\hat{\pi}_t = X_t^{\pi, \eta} \pi_t$ in (2.1) such that

$$dX_T^{\hat{\pi}, \eta} = \hat{\pi}_t(\mu_t - r)dt + \hat{\pi}_t \sigma dW_t, \ \mathbb{P} - \text{a.s.}$$

We define the penalty function for the exponential case for a fixed $\phi > 0$ as

$$\alpha_{exp}(t, \eta, \phi) \triangleq \exp \left( - \int_t^T \eta_u dW_u - \frac{1}{2} \int_t^T \eta_u^2 du \right) \exp \left( - \int_t^T \frac{1}{2 \phi} \eta_u^2 du \right)$$

$$= \exp \left( - \int_t^T \eta_t dW^\eta_u + \frac{1}{2} \int_t^T \eta_u^2 du \right) \exp \left( - \int_t^T \frac{1}{2 \phi} \eta_u^2 du \right).$$
Then, the value function at $t \in [0, T]$ reads as

$$V_{\text{exp}}(t, x) \triangleq \sup_{\pi \in \Pi_{\text{ad}}} \inf_{\eta \in \Theta_{\text{ad}}} \mathbb{E}_{t}^{\mathbb{P}_{0}}[-\beta e^{-\beta x} \exp(-\beta X_{T}^{\pi, \eta}) \alpha_{p}(t, \eta, \phi)]$$

$$= \sup_{\pi \in \Pi_{\text{ad}}} \inf_{\eta \in \Theta_{\text{ad}}} \mathbb{E}_{t}^{\eta} \left[-\beta e^{-\beta x} \exp \left(-\beta \int_{t}^{T} \pi_{u}(\mu_{u} - r - \sigma \eta_{u}) du - \beta \int_{t}^{T} \sigma \pi_{u} dW^{\eta} \right) \times \exp \left(-\frac{1}{2\phi} \int_{t}^{T} \eta_{u}^2 du \right) \right]$$

$$= -\beta e^{-\beta x} \inf_{\pi \in \Pi_{\text{ad}}} \mathbb{E}_{0}^{\pi} \left[\exp \left(-\beta \int_{t}^{T} \pi_{u}(\mu_{u} - r - \sigma \eta_{u}) du - \beta \int_{t}^{T} \sigma \pi_{u} dW^{\eta} \right) \times \exp \left(-\frac{1}{2\phi} \int_{t}^{T} \eta_{u}^2 du \right) \right]$$

$$= -\beta e^{-\beta x} \inf_{\pi \in \Pi_{\text{ad}}} \mathbb{E}_{0}^{\pi} \left[\exp \left(-\beta \int_{t}^{T} \pi_{u}(\mu_{u} - r - \sigma \eta_{u}) du - \beta \int_{t}^{T} \sigma \pi_{u} dW^{\eta} \right) \right]$$

(3.16)

**Theorem 3.3.** Let $\Theta_{\text{ad}}$ and $\Pi_{\text{ad}}$ be as in Definition 2.1. Let the value function be as in (3.8) for a fixed $\phi > 0$. Then, for $t \in [0, T]$, the optimal parameters for $t \in [0, T]$ are

$$\eta_{t}^* = \frac{\phi (\mu_{t} - r)}{\sigma (\phi + 1)},$$

$$\pi_{t}^* = \frac{\mu_{t} - r}{\beta \sigma^2 (\phi + 1)}.$$  

(3.17)

The optimal value function for $t \in [0, T]$ and $x > 0$ reads as

$$V_{\text{exp}}(t, x) = -\beta e^{-\beta x} \exp \left(-\frac{1}{2} \int_{t}^{T} \frac{(\mu_{u} - r)^2}{\sigma^2 (\phi + 1)} du \right).$$  

(3.18)

**Proof.** Similar, to power utility, we first change the underlying measure of the problem. We define

$$Z_{\pi} \triangleq \exp \left(-\beta \int_{0}^{T} \pi_{u} \sigma dW^{\eta} - \frac{1}{2} \int_{0}^{T} \beta^2 \pi_{u}^2 \sigma^2 du \right)$$

$$dQ_{\pi} = Z_{\pi} dP^{\eta},$$

with $P^{\eta}$ as in (2.2) such that (3.16) reads as

$$-\beta e^{-\beta x} \inf_{\pi \in \Pi_{\text{ad}}} \sup_{\eta \in \Theta_{\text{ad}}} \mathbb{E}_{t}^{Q^{\pi}} \left[\exp \left(\int_{t}^{T} -\beta \pi_{u}(\mu_{u} - r - \sigma \eta_{u}) du + \frac{1}{2} \beta^2 \pi_{u}^2 \sigma^2 - \frac{1}{2\phi} \eta_{u}^2 du \right) \right]$$

We first note that

$$\eta \rightarrow -\beta \pi_{u}(\mu_{u} - r - \sigma \eta_{u}) + \frac{1}{2} \beta^2 \pi_{u}^2 \sigma^2 - \frac{1}{2\phi} \eta_{u}^2$$  

(3.19)
is concave in $\eta$. By monotonicity of taking integral and exponential function, first order condition is enough to find the maximizer $\eta^* \in \Theta_{\text{ad}}$, which reads as

$$\pi_u \beta \sigma - \frac{1}{\phi} \eta^*_u = 0$$

$$\beta \phi \pi_u \sigma = \eta^*_u$$  \hfill (3.20)

Plugging $\eta^*_u$ back into (3.19), we have

$$-\beta \pi_u (\mu_u - r - \sigma^2 \beta \phi \pi_u) + \frac{1}{2} \beta^2 \pi_u^2 \sigma^2 - \frac{1}{2} \beta^2 \phi \pi_u^2 \sigma^2$$

$$= -\beta \pi_u (\mu_u - r) + \frac{1}{2} \beta^2 \pi_u^2 \sigma^2 \phi + \frac{1}{2} \beta^2 \pi_u^2 \sigma^2$$

Then,

$$\pi_u \to -\beta \pi_u (\mu_u - r) + \frac{1}{2} \beta^2 \pi_u^2 \sigma^2 \phi + \frac{1}{2} \beta^2 \pi_u^2 \sigma^2$$

being convex in $\pi_u$, we have the minimizer $\pi^*_u \in \Pi_{\text{ad}}$ with

$$-\beta (\mu_u - r) + \beta^2 \pi^*_u \sigma^2 \phi + \beta^2 \pi^*_u \sigma^2 = 0$$

$$\pi^*_u = \frac{\mu_u - r}{\beta \sigma^2 (\phi + 1)}.$$  \hfill (3.21)

Plugging back into (3.20), we have

$$\eta^*_u = \frac{\phi (\mu_u - r)}{\sigma (\phi + 1)}$$  \hfill (3.22)

Thus, the value function reads as

$$V_{\text{exp}}(t, x) = -\beta e^{-\beta x} \exp \left( -\frac{1}{2 \sigma^2 (\phi + 1)} \int_t^T (\mu_u - r)^2 du \right).$$

Hence, we conclude the proof. \hfill \Box

**Remark 3.4.** As in logarithmic and utility cases, the parameter $\phi$ quantifies how much the investor relies on his model. For $\phi = 0$, (3.22) and (3.21) read as $\pi^*_u = (\mu_u - r)/\beta \sigma^2$ and $\eta^*_u = 0$ for $u \in [0, T]$, respectively, which are the optimal parameters for the classical Merton problem with exponential utility. Similarly, as $\phi \to \infty$, $\eta^*_u = (\mu_u - r)/\sigma$ implying $\pi \equiv 0$ and refraining from risky asset altogether. We further remark that for any $\phi > 0$, the optimal controls are admissible, i.e. $\pi^* \in \Pi_{\text{ad}}$ and $\eta \in \Theta_{\text{ad}}$ as in logarithmic and power utilities.
4 Numerical Case Study

In this section, we give numerical examples with explanatory simulations by discussing the effects of the parameters in the model. Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be the filtered probability space supporting a Wiener process \(W = (W_t)_{t \geq 0}\) as described in model dynamics along with the assumptions for the underlying dynamics. Let terminal time be \(T = 10\) with \(t = 0\). For simplicity, we take that the prior on long term growth rate is a scalar with \(\mu = 0.3\). The risk free interest rate is assumed to be \(r = 0.05\). The volatility of the stock is estimated to be \(\sigma = 0.5\). We further consider for \(\phi > 0\) the penalty functions of the form (2.4), (2.5) and (2.6)

\[
\phi \in [10^{-4}, 10^{-2}, 10^{-1}, 1, 10, 10^2, 10^4]. \tag{4.1}
\]

4.1 Logarithmic Utility

By (3.4), the simulations of the optimal portion to invest into risky asset with respect to (4.1) are as in Figure 1. We see that as the reliance on the prior increases, namely as \(\phi\) decreases, the optimal \(\pi^*\) converges to the value \(\frac{\mu-r}{\sigma^2} = 1\). Similarly, as the reliance on the prior decreases, namely as \(\phi\) increases, the optimal \(\pi^*\) converges to 0. The value function of (3.5) reads as

\[
V_{\log}(t, x) = \log(x) + \frac{(\mu-r)^2(T-t)}{2\sigma^2(\phi+1)},
\]

whose plot with respect to \(\phi\) values as in (4.1) with \(t = 0\) and \(x = 1\) is to be seen in Figure 2. We note that as \(\phi \to 0\), namely as the reliance on our prior increases, and hence the uncertainty decreases, the value function increases to the classical Merton value. On the
other hand, as the reliance on our prior decreases and hence, the uncertainty increases, the value function decreases to 0.

4.2 Power Utility

For the power utility, we choose $\gamma$ in $\{0.01, 0.99\}$. By (3.10) and (3.11), the simulations of the optimal portion to invest into risky asset with respect to (4.1) are as in Figure 3. Compared to the logarithmic case, in power utility, we have additional risk awareness parameter $\gamma$. We see the reliance on the model does not have a strong effect, when the investor uses a utility with $\gamma = 0.99$, compared to the case with $\gamma = 0.01$. Namely, when the investor is already risk averse with $\gamma = 0.01$, the investment into risky asset does not decrease as strong and immediate as in the case, when the investor uses less risk averse utility with $\gamma = 0.99$. The
value function of \((3.12)\) reads as

\[
V_{\text{pow}}(t, x) = x^\gamma \exp \left( \frac{(T - t) \gamma (\mu - r)^2}{2\sigma^2(\gamma(\phi - 1) + 1)} \right)
\]

We plot (4.2) with respect to different \(\phi\)'s in (4.1) with \(t = 0\) and \(x = 1\) in Figure 4. We see the same pattern in value function for power utility, as well. When, the investor is already risk averse, the effect of parameter uncertainty is not as immediate as in the latter case with less risk-averse investor with \(\gamma = 0.99\). We see that the value function is affected both by \(\gamma\) and \(\phi\). If the investor has a more risk averse utility with \(\gamma = 0.99\), then the effect of uncertainty on the model is not as immediate as in the case when the investor uses utility function with \(\gamma = 0.01\).

### 4.3 Exponential Utility

For the exponential utility, we take \(\beta > 0\) among \(\beta \in \{10^{-4}, 10^4\}\). By (3.17), the simulations of the optimal portion to invest into the risky asset with respect to (4.1) are as in Figure 5. We see that contrary to the power utility case, the shape of the graph does not change with respect to different \(\beta\) values. The value function in (3.18) reads as

\[
V_{\text{exp}}(t, x) = -\beta e^{-\beta x} \exp \left( -\frac{(\mu - r)^2(T - t)}{2\sigma^2(\phi + 1)} \right)
\]

We present the plot with respect to \(\phi\) values in (4.1) in Figure 6. Here, we note that the shape of the graphs does not change with respect to small and large \(\beta\) values as for the portion invested into the risky asset with respect to \(\beta = 10^{-4}\) and \(\beta = 10^4\).
Figure 5: Portion Invested Into Risky Asset with Exponential Utility

(a) $\beta = 10^{-4}$  
(b) $\beta = 10^4$

Figure 6: Value with Exponential Utility

(a) $\beta = 10^{-4}$  
(b) $\beta = 10^4$
References


