Necessary and sufficient conditions for rank-one generated cones

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Abstract

A closed convex conic subset $S$ of the positive semidefinite (PSD) cone is rank-one generated (ROG) if all of its extreme rays are generated by rank one matrices. The ROG property of $S$ is closely related to the exactness of SDP relaxations of nonconvex quadratically constrained quadratic programs (QCQPs) related to $S$. We consider the case where $S$ is obtained as the intersection of the PSD cone with finitely many homogeneous linear matrix inequalities and conic constraints and identify sufficient conditions that guarantee that $S$ is ROG. Our general framework allows us to recover a number of well-known results from the literature. In the case of two linear matrix inequalities, we also establish the necessity of our sufficient conditions. This extends one of the few settings from the literature—the case of one linear matrix inequality and the S-lemma—where an explicit characterization for the ROG property exists. Finally, we show how our ROG results on homogeneous cones can be translated into inhomogeneous SDP exactness results and convex hull descriptions in the original space of a QCQP. We close with a few applications of these results; specifically, we recover the well-known perspective reformulation of a simple mixed-binary set via the ROG toolkit.

1 Introduction

Let $S^n$ denote the real vector space of $n \times n$ real symmetric matrices and $S^n_+$ the cone of positive semidefinite matrices. We will say that a closed convex set $S \subseteq S^n_+$ is rank-one generated (ROG) if

$$S = \text{conv}(S \cap \{xx^\top : x \in \mathbb{R}^n\}),$$

where $\text{conv}(\cdot)$ is the closed convex hull operation. In words, $S$ is ROG if and only if it is equal to the closed convex hull of its rank-one matrices.

In most applications, the set $S \subseteq S^n_+$ will be represented as the intersection of $S^n_+$ with a (possibly infinite) system of linear matrix inequalities (LMIs). Note also that any closed convex set $S \subseteq S^n_+$ can be expressed in this form. An obvious question then is: What does the ROG property of $S$ correspond to in terms of the system of LMIs defining $S$?

In this paper, we will mainly restrict our attention to closed convex cones and explore necessary and/or sufficient conditions on $M \subseteq S^n$ for the set

$$S(M) := \{X \in S^n_+ : \langle M, X \rangle \geq 0, \forall M \in M\}$$

to be ROG. Nevertheless, our results also have implications in the more general setting of arbitrary closed convex sets $S \subseteq S^n_+$ and their defining LMIs.

1.1 Motivation

The ROG property is important in studying semidefinite program (SDP) relaxations of quadratically constrained quadratic programs (QCQPs).
QCQPs are a fundamental class of optimization problems that arise naturally in many areas. Indeed, many problems including binary integer linear programs, max-cut, max-clique, certain robust optimization problems and polynomial optimization problems can be readily recast as QCQPs (see [2, 5, 18] and references therein).

It is well known that any QCQP can be reformulated as an SDP in a lifted space with an additional nonconvex rank constraint. Dropping this rank constraint leads to the standard SDP relaxation [23]. A general QCQP and its SDP relaxation are given by

\[
\inf_{y \in \mathbb{R}^{n-1}} \{ q_0(y) : q_i(y) \geq 0, \forall i \in [m] \} = \inf_{x \in \mathbb{R}^n} \left\{ x^\top M_0 x : \begin{array}{l} x^\top M_i x \geq 0, \forall i \in [m] \\ x_1^2 = 1 \end{array} \right\} \geq \inf_{X \in \mathbb{S}_+^n} \left\{ \langle M_0, X \rangle : \begin{array}{l} \langle M_i, X \rangle \geq 0, \forall i \in [m] \\ X_{1,1} = 1 \end{array} \right\}. \tag{1}
\]

Here, the \( q_i \)s are quadratic functions of the form \( q_i(y) = y^\top A_i y + 2 b_i^\top y + c_i \), \( x \) should be thought of as \( (\frac{1}{y}, 1) \), and the \( M_i \)s are matrices of the form \( M_i := \begin{pmatrix} c_i & b_i^\top \\ b_i & A_i \end{pmatrix} \).

In general, it is NP-hard to determine whether the SDP relaxation of a given QCQP is exact, i.e., when equality holds in (1) (see [19]). Nevertheless, sufficient conditions that ensure equality in (1) are of great interest, and thus establishing such conditions has attracted a lot of attention in the literature.

Geometrically, SDP exactness occurs if and only if there exist rank one matrices in the feasible domain of the SDP approaching its optimum. The ROG property is a similar but stronger notion of exactness. Specifically, the set

\[
\left\{ X \in \mathbb{S}_+^n : \begin{array}{l} \langle M_i, X \rangle \geq 0, \forall i \in [m] \\ X_{1,1} = 1 \end{array} \right\}
\]

is ROG if and only if there exist rank one matrices in the feasible domain of the SDP approaching its optimum for every choice of \( M_0 \) in (1). In other words, this set is ROG if and only if equality holds in (1) for every choice of objective function.

The ROG property is a natural strengthening of SDP exactness. Consider, for example, the problem of minimizing an arbitrary quadratic function over an ellipsoid. The celebrated S-lemma [26] guarantees that the SDP relaxation of this problem is exact regardless of the choice of objective function—this is precisely the ROG property. From a different perspective, the ROG property of spectrahedra can be thought of as an analogue of the integrality property of polyhedra for linear programming relaxations of integer programs. While there are well-known sufficient conditions such as total unimodularity or total dual integrality for polyhedra (see [10] for recent developments and earlier references), the research on sufficient conditions for the ROG property of spectrahedra is much more recent and limited.

The ROG property is also relevant in the context of sum-of-squares (SOS) programming. Consider a real homogeneous quadratic variety \( V := \{ x \in \mathbb{R}^n : x^\top M_i x = 0, \forall i \in [m] \} \). Let \( \mathcal{P}_V \) denote the set of nonnegative quadratic forms on \( V \), i.e., \( \mathcal{P}_V := \{ M \in \mathbb{S}^n : x^\top M x \geq 0, \forall x \in V \} \). Let \( \Sigma_V \) denote the set of quadratic forms that are “immediately nonnegative” on \( V \), i.e., \( \Sigma_V := \mathbb{S}_+^n + \text{span} \{ M_i : i \in [m] \} \).

It is clear that \( \Sigma_V \subseteq \mathcal{P}_V \). A direct calculation shows that the dual cones of \( \mathcal{P}_V \) and \( \Sigma_V \) are given by

\[
\mathcal{P}_V^* = \text{conv} \{ xx^\top : \langle M_i, xx^\top \rangle = 0, \forall i \in [m] \} \quad \text{and} \quad \Sigma_V^* = \{ X \in \mathbb{S}_+^n : \langle M_i, X \rangle = 0, \forall i \in [m] \},
\]

respectively. Therefore, \( \Sigma_V = \mathcal{P}_V \) if and only if \( \Sigma_V^* = \mathcal{P}_V^* \), which holds if and only if \( \Sigma_V^* \) is rank one generated. In other words, every nonnegative quadratic form on \( V \) is “immediately nonnegative” if and only if \( \Sigma_V^* \) is ROG. See [6, Section 6] for further connections and applications of the ROG property in the context of real algebraic geometry and statistics.

1.2 Related literature

Bounds on the rank of extreme points of general spectrahedra. A rich line of research has proven optimal worst-case bounds on the rank of extreme points of a spectrahedron in terms of the number of its
defining linear matrix equalities (LMEs) [3, 20]; see also [4, Chapter II.13]. It is known that given \( m \) LMEs, if there exists a positive semidefinite (PSD) solution to the LMEs, then there also exists a PSD solution with rank at most \( r \) for any integral \( r \) such that

\[
m < \binom{r + 2}{2}.
\]

From this, we may deduce\(^1\) that any spectrahedron defined by \( m \) LMEs has only extreme points of rank at most \( r \) for any integral \( r \) satisfying \( m + 1 < \binom{r + 2}{2} \). In particular, taking \( r = 1 \), this bound implies that any spectrahedron defined by a single LME is ROG. Unfortunately, this bound does not shed much light onto (even the existence of) ROG spectrahedra in the case where \( m > 1 \). Although this bound is tight in general, it does not exploit potential structure in the defining LMEs. In other words, it is possible to achieve stronger bounds on the rank of extreme points of spectrahedra with additional structure. Our work complements this line of research by examining properties of systems of LMEs and LMIs that guarantee the ROG property beyond the case of \( m = 1 \).

**SDP exactness.** The question of when equality holds in (1) has attracted significant interest. Within this line of research, a number of papers study the classical trust region subproblem (TRS)—the problem of minimizing a nonconvex quadratic function over an ellipsoid—and its variants, and identify cases under which an exact SDP reformulation is possible. This line of work can be traced back to Yakubovich’s S-procedure [13, 26] (also known as the S-lemma) and the work of Sturm and Zhang [24]. We refer the interested readers to the excellent survey by Burer [7] and references therein.

It is worth noting that although the results in [7] are stated in terms of the exactness of (strengthened) SDP relaxations, the underlying arguments in fact establish the ROG property for the corresponding SDP feasible domains. For example, the domain of the SDP relaxation associated with the classical TRS is the intersection of \( \mathbb{S}^n_+ \) with a single LMI, which is well known to be ROG via S-lemma. In the other variants of TRS examined in [7], the domain of the associated exact SDP reformulation involves at least one problem specific conic constraint (in fact a second-order cone constraint), and consequently is described by an infinite family of well-structured LMIs.

These lines of work can be thought of as addressing the special case where there are only a few (usually one or two) nonconvex quadratic functions in the QCQP on the left of (1). In contrast, Burer and Ye [8] and Wang and Kilinc-Karzan [25] recently introduced more general sufficient conditions for SDP exactness which do not make explicit assumptions on the number of nonconvex quadratic functions. As an example, it can be shown that SDP exactness holds whenever a natural symmetry parameter of the QCQP is large enough [25]. These sufficient conditions for SDP exactness have also been shown to guarantee that the (projection of the) epigraph of the SDP relaxation coincides exactly with the convex hull of the epigraph of the QCQP. In particular, the convex hulls of epigraphs of “highly-symmetric” QCQPs are semidefinite-representable. Results in this line of work generally depend heavily on how the objective function interacts with the constraints. Our work complements this line of research by establishing conditions for SDP exactness which are oblivious to the objective function.

**Algebro-geometric properties of ROG spectrahedra.** The ROG property has also been studied from a more algebro-geometric perspective [6, 17].

Hildebrand [17] studies algebraic properties of ROG cones obtained by adding homogeneous LMEs to \( \mathbb{S}^n_+ \), and proves important facts about their representations. The study begins by exploring the minimal defining polynomials and facial structure of ROG cones. These properties are then used to build the main contribution of [17]: The geometry of an ROG cone determines its representation as a linear section of a PSD cone (of any dimension) uniquely up to an isomorphism on the underlying vector space. Additional results in this paper include a complete classification of ROG cones of degree\(^2\) at most four as well as a number of operations on ROG cones (the direct product, full extension, and intertwining operations) that preserve the ROG property.

\(^1\)After taking into account an additional LME due to the objective function and applying Straszielewicz Theorem (see [22, Theorem 18.6]).

\(^2\)This is the degree of the minimal defining polynomial. This quantity is shown to be equivalent to the maximum rank over matrices in the ROG cone.
In Section 3, we establish a number of new sufficient conditions for the ROG property. As an example, with an outline of the paper, is as follows: 

V is generated in degree at most (a) In Section 2, we introduce our main terminology and basic tools. Specifically, we show how the ROG semidefinite cone with a set of homogeneous LMIs is an ROG cone. A summary of our contributions, along with a number of examples to demonstrate that even simple extensions of our sufficient conditions are not possible. We conclude this section by recovering the well-known result that the SDP relaxation strengthened with a second-order cone reformulation-linearization technique (SOC-RLT) inequality is exact for a variant of the TRS with a single linear inequality constraint.

In contrast to [6, 17], our results deal with possibly infinitely many linear matrix inequalities. The ROG property of such sets is not obvious and does not follow immediately from the ROG property of spectrahedral cones defined by LMIs. Indeed, we will see that both replacing equalities with inequalities (Remark 8) and lifting inequalities to equalities (Example 5) can destroy the ROG property of a spectrahedral cone. In addition, our more general setup allows us to handle additional interesting spectrahedral cones that have conic constraints, for example those arising from variants of the TRS. We also discuss implications of the ROG property in terms of the exactness of SDP relaxations of QCQPs and explicit convex hull characterizations of sets defined by quadratic inequality constraints.

ROG spectrahedra arising from PSD matrix completion. The ROG property has also been studied for spectrahedra arising in the matrix completion literature. PSD matrix completion arises in a number of areas—for example in statistics, this problem is related to maximum likelihood estimation in Gaussian graphical models [11]. Let E denote the edge set of an undirected graph on n vertices that contains all self-loops. Let K ⊆ S^n denote the projection of S^n on the indices in E. Then, a matrix Y that is specified only on E has a PSD completion if and only if it lies in the cone K. A short calculation shows that

\[ K = \left\{ Y \in S^n : Y_{i,j} = 0, \forall (i,j) \notin E \right\}, \quad \text{where} \quad S = \left\{ X \in S^n : X_{i,j} = 0, \forall (i,j) \notin E \right\}. \]

Consequently, the condition that every fully specified submatrix of Y is positive semidefinite is necessary and sufficient for Y to have a PSD completion if and only if S is ROG. It is well-known that S is ROG if and only if E is the edge set of a chordal graph4 on n vertices [1, 15, 21].

1.3 Overview and outline of the paper

In this paper, we study necessary and/or sufficient conditions under which the intersection of the positive semidefinite cone with a set of homogeneous LMIs is an ROG cone. A summary of our contributions, along with an outline of the paper, is as follows:

(a) In Section 2, we introduce our main terminology and basic tools. Specifically, we show how the ROG property behaves when we switch from linear matrix inequalities (LMIs) to linear matrix equalities (LMEs) and how the ROG property for LMEs is characterized by the existence of solutions of quadratic systems. In Section 2.5, using our basic tools, we recover the well-known ROG set defined by a single LMI/LME (i.e., the S-lemma) and discuss a few implications for a simple sufficient condition in the case of two LMIs/LMEs.

(b) In Section 3, we establish a number of new sufficient conditions for the ROG property. As an example, we show that S is ROG when S = \{X ∈ S^n : Xc ∈ K\} for a fixed vector c and an arbitrary closed convex cone K. We also provide a number of examples to demonstrate that even simple extensions of our sufficient conditions are not possible. We conclude this section by recovering the well-known result that the SDP relaxation strengthened with a second-order cone reformulation-linearization technique (SOC-RLT) inequality is exact for a variant of the TRS with a single linear inequality constraint.

3A real projective variety V satisfies property N_{2,p} for an integer p ≥ 1 if the jth syzygy module of the homogeneous ideal of V is generated in degree at most j + 2 for all j < p.

4A graph is chordal if every minimal cycle in the graph has at most 3 edges.
(c) A well-known consequence of the S-lemma is that the set $S(M)$ is ROG whenever $M = \{ M \}$ is a single LMI; see e.g., Ye and Zhang [27, Lemma 2.2]. In Section 4, we give a complete characterization of ROG cones defined by two LMIs. One of our main results states a necessary and sufficient condition on the matrices $M_1$ and $M_2$ which ensures that the set $S$ is ROG. In particular, we establish in Theorem 3 that such a set is ROG if and only if $M_1$ and $M_2$ either “do not interact” inside $S^n$ or have a specific rank-two structure. We conclude that in the case of $m = 2$, there exist simple certificates of the ROG property.

(d) In Section 5, we give a few applications of ROG cones. In particular, we show how homogeneous ROG results can be translated into inhomogeneous SDP exactness results and SDP-based convex hull descriptions. We then apply our ROG-based sufficient condition for exactness of the SDP relaxation to a simple set involving binary and continuous variables linked through a complementarity constraint. This gives a new method for deriving the well-known perspective reformulation for the convex hull of this set. We close this section by presenting a number of examples that highlight how our ROG-based sufficient conditions for the SDP exactness and convex hull descriptions differ from other SDP exactness conditions in the literature. The results in this section are self-contained and serve as additional motivation for the main study.

We will compare our results with the literature in further detail in the sections as outlined above.

1.4 Notation

For a positive integer $m$, let $[m] := \{1, \ldots, m\}$. Let $\mathbb{R}^n_+$ denote the nonnegative vectors in $\mathbb{R}^n$. For $i \in [n]$, let $e_i \in \mathbb{R}^n$ denote the $i$th standard basis vector. Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. Let $S^n$ denote the vector space of $n \times n$ real symmetric matrices and $S^n_+$ the cone of positive semidefinite matrices. We write $M \geq 0$ (respectively $M > 0$) if $M$ is positive semidefinite (respectively positive definite). For $M \in \mathbb{R}^{n \times n}$, let $\text{Sym}(M) := (M + M^\top)/2 \in S^n$. For $x \in \mathbb{R}^n$, let $\text{Diag}(x) \in S^n$ denote the diagonal matrix with $\text{Diag}(x)_{ii} = x_i$ for all $i \in [n]$. For a positive integer $n$, let $I_n$ denote the $n \times n$ identity matrix. When the dimension is clear from context, we will simply write $I$ instead of $I_n$. For $M \in S^n$, let $\text{range}(M)$, $\text{ker}(M)$, $\text{rank}(M)$, $\text{det}(M)$, $\text{tr}(M)$ denote the range, kernel, rank, determinant and trace of $M$, respectively. Let $E$ denote an arbitrary Euclidean space. Given a subset $M \subseteq E$, let $\text{int}(M)$, $\text{bd}(M)$, $\text{conv}(M)$, $\text{cone}(M)$, $\text{roag}(M)$, and $\text{span}(M)$ denote the interior, boundary, convex hull, closed convex hull, conic hull, closed conic hull, and span (linear hull) of $M$, respectively. For $M \subseteq E$, let $M^\perp$ denote the subspace orthogonal to $M$. For a subspace $W \subseteq E$, let $\dim(W)$ denote its dimension. For a cone $K \subseteq E$, let $\text{extr}(K)$ denote its extreme rays and define $K^* := \{ y \in E : \langle x, y \rangle \geq 0, \forall x \in K \}$ to be the dual cone of $K$. Given a subspace $W \subseteq \mathbb{R}^n$, we will identify $W$ with $\mathbb{R}^{\dim(W)}$. Let $S^W$ denote $S^{\dim(W)}$ identifed with the linear subspace of $S^n$ given by $\{ X \in S^n : \text{range}(X) \subseteq W \}$. For $x \in \mathbb{R}^n$, let $x_W \in W$ denote the projection of $x$ onto $W$. For $M \in S^n$, let $M_W \in S^W$ denote the restriction of $M$ to $W$, i.e., $M_W := U^\top MU$, where $U : W \to \mathbb{R}^n$ is the inclusion map. When there is no confusion, let $0_x$ denote either the zero vector in $\mathbb{R}^n$ or the zero matrix in $S^n$. Similarly, let $0_W$ denote either the zero vector in $W$ or the zero matrix in $S^W$. For $x \in W$ and $y \in W^\perp$, let $x \oplus y$ denote their direct sum. For $X \in S^W$ and $Y \in S^W^\perp$, let $X \oplus Y$ denote their direct sum, i.e., the unique matrix in $S^n$ such that $(x \oplus y)^\top (X \oplus Y) (x \oplus y) = x^\top X x + y^\top Y y$ for all $x \in W$ and $y \in W^\perp$.

2 Properties of ROG cones

2.1 Definitions

Given $M \subseteq S^n$, define

$$S(M) := \{ X \in S^n_+ : \langle M, X \rangle \geq 0, \forall M \in M \}. \tag{11}$$

Note that $S(M)$ is a closed convex cone. We are interested in the following property of such sets.

**Definition 1.** A closed convex cone $S \subseteq S^n_+$ is rank-one generated (ROG) if

$$S = \text{conv}(S \cap \{ xx^\top : x \in \mathbb{R}^n \}).$$

\[ \square \]
Remark 1. Note that when $S \subseteq S^n_+$ is a closed convex cone, we have $\text{conv}(S \cap \{xx^T : x \in \mathbb{R}^n\}) = \text{conv}(S \cap \{xx^T : x \in \mathbb{R}^n\})$. □

We will make extensive use of the following definitions and basic facts.

**Definition 2.** For $X \in S^n$ nonzero, the ray spanned by $X$ is

$$\mathbb{R}_+X := \{\alpha X : \alpha \geq 0\}.$$ 

Let $S \subseteq S^n_+$ be a closed convex cone and suppose $X \in S$ is nonzero. We say that $\mathbb{R}_+X$ is an extreme ray of $S$ if for any $Y,Z \in S$ that $X = (Y + Z)/2$, we must have $Y,Z \in \mathbb{R}_+X$. □

**Fact 1.** Let $X \in S^n_+$. Then $x \in \text{range}(X)$ if and only if there exists $\epsilon > 0$ such that $X - \epsilon xx^T \in S^n_+$. □

**Fact 2.** Let $S \subseteq S^n_+$ be a closed convex cone. Then, for $X \neq 0$, $\mathbb{R}_+X$ is an extreme ray of $S$ if and only if for every $Y$,

$$[X - Y, X + Y] \subseteq S \implies \exists \alpha \in \mathbb{R} \text{ such that } Y = \alpha X.$$

The following fact follows immediately from Facts 1 and 2.

**Fact 3.** Let $S \subseteq S^n_+$ be a closed convex cone. If $X \in S$ has rank($X$) = 1, then $\mathbb{R}_+X$ is an extreme ray of $S$.

**Lemma 1.** Let $S \subseteq S^n_+$ be a closed convex cone. Then $S$ is ROG if and only if for each extreme ray $\mathbb{R}_+X$ of $S$ we have rank($X$) = 1.

**Proof.** The statement holds trivially when $S = \{0\}$. We will suppose that $S \neq \{0\}$. As $S$ is a closed convex cone contained in $S^n_+$, it has a nonempty compact base given by $B = S \cap \{X \in S^n_+ : \langle I, X \rangle = 1\}$.

$(\Leftarrow)$ Let $X$ be an extreme point of $B$. Then $\mathbb{R}_+X$ is an extreme ray of $S$, whence $X$ has rank one. Recalling that a finite-dimensional compact convex set is the convex hull of its extreme points, we deduce that $B = \text{conv}(B \cap \{xx^T : x \in \mathbb{R}^n\})$, whence $S = \text{conv}(S \cap \{xx^T : x \in \mathbb{R}^n\})$.

$(\Rightarrow)$ It suffices to show that the extreme points of $B$ have rank one. Let $X$ be an exposed point of $B$. Then $X \neq 0$ and $\mathbb{R}_+X$ is an exposed ray of $S$. Let $M \in S^n$ expose $\mathbb{R}_+X$. In other words $M$ satisfies: $\langle M, X' \rangle \geq 0$ for all $X' \in S$ and equality holds if and only if $X' \in \mathbb{R}_+X$. Assuming that $S$ is ROG, we can write $X = \sum_{i=1}^k x_ix_i^T$ where $x_ix_i^T \in S$. We deduce that $\langle M, x_ix_i^T \rangle \geq 0$ for all $i \in [k]$ since $x_ix_i^T \in S$. Then, from $0 = \langle M, X \rangle = \sum_{i=1}^k \langle M, x_ix_i^T \rangle$, we conclude that for all $i \in [k]$ we have $\langle M, x_ix_i^T \rangle = 0$, which implies $x_ix_i^T \in \mathbb{R}_+X$. Hence, $X$ has rank one. In particular, all exposed points of $B$ have rank one. Finally, applying Straszewicz Theorem [22, Theorem 18.6]|—the set of exposed points of a closed convex set is dense in its extreme points—and noting that the rank function is lower semicontinuous, we deduce that all extreme points of $B$ have rank one. □

### 2.2 Relating LMIs to LMEs

Given a set $M \subseteq S^n$, we will quickly switch from studying $S(M)$ to sets defined by LMEs, i.e., sets of the form

$$\mathcal{T}(M) := \{X \in S^n_+ : \langle M, X \rangle = 0, \forall M \in M\}.$$ 

Sets of the form $\mathcal{T}(M)$ are simpler to analyze than sets of the form $S(M)$.

**Remark 2.** It is clear that given any $M \subseteq S^n$, we have $S(M) = S(\text{cone}(M))$ and $\mathcal{T}(M) = \mathcal{T}(\text{span}(M))$. In particular, we may without loss of generality assume that $M$ is finite when analyzing sets of the form $\mathcal{T}(M)$—simply replace $M$ with a finite basis of span($M$). On the other hand, $\text{cone}(M)$ is not necessarily finitely generated.
We now present a series of lemmas relating $S(M)$ and $T(M)$ and their facial structures in terms of the ROG property. These results are particularly instrumental when we analyze the spectrahedral sets defined by finitely many LMIs/LMEs.

**Lemma 2.** For any set $M \subseteq \mathbb{S}^n$, the following are equivalent:

1. $S(M)$ is ROG.
2. Every face of $S(M)$ is ROG.
3. $S(M) \cap T(M')$ is ROG for every $M' \subseteq M$.

**Proof.** (1. $\Rightarrow$ 2.) Note that every extreme ray of a face of $S(M)$ is also an extreme ray of $S(M)$.

(2. $\Rightarrow$ 3.) First, suppose $M' = \emptyset$. Then, $T(M') = \mathbb{S}^n_+$ and thus $S(M) \cap T(M') = S(M)$. Since $S(M)$ is a face of itself, by part 2. we deduce it is ROG. Now consider any $\emptyset \neq M' \subseteq M$. Note that $T(M')$ only depends on the linear span of $M'$, thus without loss of generality we may assume that $M'$ is a basis of $\text{span}(M')$. Take $Y$ to be the average of $M'$, i.e., $Y = \frac{1}{|M'|} \sum_{M \in M'} M$. Note that $Y \in \text{cone}(M')$ so that $Y \in S(M)^+$.

We claim that $S(M) \cap T(M') = S(M) \cap Y^\perp$. Indeed, for all $X \in S(M)$, we have that $\langle Y, X \rangle = 0$ if and only if $\langle Y, M' \rangle = 0$ for all $M \in M'$ if and only if $X \in T(M')$. We deduce that $S(M) \cap T(M')$ is a face of $S(M)$, and thus it is ROG.

(3. $\Rightarrow$ 1.) Take $M' = \emptyset$.

We have the following immediate corollary of Lemma 2.

**Corollary 1.** For any set $M \subseteq \mathbb{S}^n$, if $S(M)$ is ROG then $T(M)$ is ROG.

**Proof.** Take $M' = M$ in Lemma 2.

Informally, an extreme ray of $S(M)$ should also be an extreme ray of $S(M')$ for $M' \subseteq M$ as long as $M'$ contains the “relevant” inequalities in $M$. The following technical lemma makes this notion precise.

**Lemma 3.** Let $M \subseteq \mathbb{S}^n$ and let $\mathbb{R}_+ X$ be an extreme ray of $S(M)$. Let $M' \subseteq M$ contain all of the constraints that are tight at $X$, i.e., $\{ M \in M : \langle M, X \rangle = 0 \} \subseteq M'$. If $M \setminus M'$ is compact, then $\mathbb{R}_+ X$ is an extreme ray of $S(M')$. If additionally $M' = \{ M \in M : \langle M, X \rangle = 0 \}$, then $\mathbb{R}_+ X$ is an extreme ray of $T(M')$.

**Proof.** Suppose $Y \in \mathbb{S}^n$ is such that $[X - Y, X + Y] \subseteq S(M')$. By compactness of $M \setminus M'$, we have that $\langle M, X \rangle$ achieves a positive minimum value on $M \setminus M'$. Furthermore, by compactness, $\langle M, Y \rangle$ is bounded on $M \setminus M'$. In particular, there exists $\epsilon > 0$ small enough guaranteeing that $\langle M, X \pm \epsilon Y \rangle > 0$ for all $M \in M \setminus M'$. This together with $[X - Y, X + Y] \subseteq S(M')$ implies that $[X - \epsilon Y, X + \epsilon Y] \subseteq S(M)$. Thus, we conclude that $Y = \alpha X$ for some $\alpha \in \mathbb{R}$, i.e., $\mathbb{R}_+ X$ is extreme in $S(M')$.

The second statement follows by replacing $S(M')$ with $T(M')$ in the argument above.

This lemma allows us to strengthen Lemma 2 in a few ways.

**Lemma 4.** Let $M \subseteq \mathbb{S}^n$ be compact. Then, $S(M)$ is ROG if and only if $S(M) \cap T(M')$ is ROG for every $\emptyset \neq M' \subseteq M$.

**Proof.** ($\Rightarrow$) This direction follows Lemma 2.

($\Leftarrow$) Let $\mathbb{R}_+ X$ be an extreme ray of $S(M)$ and define $M' = \{ M \in M : \langle M, X \rangle = 0 \}$. First suppose $M' \neq \emptyset$. As $\mathbb{R}_+ X$ is also an extreme ray of $S(M) \cap T(M')$, which by assumption is ROG, we have that $\text{rank}(X) = 1$. Now suppose $M' = \emptyset$. By Lemma 3 and the assumption that $M$ is compact, we deduce that $\mathbb{R}_+ X$ is an extreme ray of $T(\emptyset) = \mathbb{S}^n_+$. We conclude that $\text{rank}(X) = 1$. 


We now present a few lemmas that are useful in reasoning about extreme rays of $\mathcal{M}$. We conclude that given Lemma 4, it may be tempting to try to strengthen the third condition in Lemma 2 to the condition that $S(\mathcal{M}) \cap T(\mathcal{M}')$ is ROG for every $\emptyset \neq \mathcal{M}' \subseteq \mathcal{M}$. The following example shows that this is not possible without making the compactness assumption of Lemma 4.

**Example 1.** Suppose $n = 2$ and $\mathcal{M} = \bigcup_{i \in [4]} \mathcal{M}_i$, where

\[
\mathcal{M}_1 = \left\{ \begin{pmatrix} 1 \\ -1 + \epsilon \end{pmatrix} : \epsilon > 0 \right\}, \quad \mathcal{M}_2 = \left\{ \begin{pmatrix} -1 \\ 1 + \epsilon \end{pmatrix} : \epsilon > 0 \right\}, \\
\mathcal{M}_3 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} : \epsilon > 0 \right\}, \quad \mathcal{M}_4 = \left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix} : \epsilon > 0 \right\}.
\]

Noting that $S(\mathcal{M})$ is unchanged upon taking the closure of $\mathcal{M}$ and that for all $i \in [4]$ and the constraints $\langle M_i, X \rangle \geq 0$ for $M_i \in \mathcal{M}_i$ get only more restrictive as $\epsilon \to 0$, we deduce

\[
S(\mathcal{M}) = S \left( \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} \right) = \mathbb{R}_+ I.
\]

We conclude $S(\mathcal{M}) = \mathbb{R}_+ I$ is not ROG. On the other hand, for any $\emptyset \neq \mathcal{M}' \subseteq \mathcal{M}$, we have $S(\mathcal{M}) \cap T(\mathcal{M}') = \{0\}$ (because $(M, I) \neq 0$ for any $M \in \mathcal{M}$) and is ROG.

**Lemma 5.** Let $\mathcal{M} \subseteq \mathbb{S}^n$ be finite. If $T(\mathcal{M}')$ is ROG for every $\mathcal{M}' \subseteq \mathcal{M}$, then $S(\mathcal{M})$ is ROG.

**Proof.** Let $\mathbb{R}_+ X$ be an extreme ray of $S(\mathcal{M})$. Define $\mathcal{M}' := \{ M \in \mathcal{M} : \langle M, X \rangle = 0 \}$. By Lemma 3 and the fact that any finite set is compact, we deduce that $\mathbb{R}_+ X$ is an extreme ray of $T(\mathcal{M}')$. We conclude that $\text{rank}(X) = 1$. \qed

**Remark 3.** The characterizations given in Lemmas 2, 4 and 5 and Corollary 1 are based on the facial structure of the sets $S(\mathcal{M})$ and $T(\mathcal{M})$ and in a sense are analogous to characterizations of integral polyhedra. \qed

**Remark 4.** The ROG property is not preserved under trivial liftings. When $\mathcal{M} = \{ M_1, \ldots, M_k \}$ is finite, one may attempt to replace all of the inequalities defining $S(\mathcal{M})$ with equalities by adding new slack variables. Specifically, let $\overline{M}_i \in \mathbb{S}^{n+k}$ be the following block matrix

\[
\overline{M}_i := \begin{pmatrix} M_i & e_i e_i^T \end{pmatrix}
\]

and let $\overline{M} := \{ \overline{M}_1, \ldots, \overline{M}_k \}$. It is straightforward to show that the ROG property is preserved under the projection of $\mathbb{S}^{n+k}$ onto $\mathbb{S}^n$. Thus, if $T(\overline{M})$ is ROG, then $S(\mathcal{M})$ is also ROG. Unfortunately the reverse implication is not true in general. We will give a counterexample in Section 4.4 (see Example 5). \qed

### 2.3 Simple operations preserving ROG property

We now present a few lemmas that are useful in reasoning about extreme rays of $S(\mathcal{M})$. The following lemma states that an extreme ray $\mathbb{R}_+ X$ “only cares about” constraints “in the range of $X$.”

**Lemma 6.** Let $\mathcal{M} \subseteq \mathbb{S}^n$ and let $\mathbb{R}_+ X$ be an extreme ray of $S(\mathcal{M})$. Let $W := \text{range}(X)$ and let $\mathcal{M}_W := \{ M_W : M \in \mathcal{M} \}$. Then $\mathbb{R}_+(X_W)$ is an extreme ray of $S(M_W)$. In particular, if $S(\{ M_W \})$ is ROG, then $\text{rank}(X) = \text{rank}(X_W) = 1$. 

Figure 1: A summary of Lemma 5 and Corollary 1
We next examine the ROG property of a set and its connection to the existence of nonzero solutions of quadratic systems of inequalities and/or equations.

2.4 The ROG property and solutions of quadratic systems

We next examine the ROG property of a set and its connection to the existence of nonzero solutions of quadratic systems of inequalities and/or equations.

Definition 3. Given \( M \subseteq \mathbb{R}^n \) and \( X \in S(M) \), we define

\[
\mathcal{E}(X, M) := \{ x \in \mathbb{R}^n : |x^T M x| \leq \langle M, X \rangle, \forall M \in \mathcal{M} \}.
\]

Lemma 9. \( S(M) \) is ROG if and only if for every nonzero \( X \in S(M) \) we have \( \text{range}(X) \cap \mathcal{E}(X, M) \neq \{0\} \).

Proof. \((\Rightarrow)\) Suppose \( X \in S(M) \) is nonzero. Because \( S(M) \) is ROG, we can write \( X = \sum_{i=1}^k x_i x_i^T \) using nonzero matrices \( x_i x_i^T \in S(M) \). As \( X \) is a nonzero matrix, we have \( k \geq 1 \) and thus \( \bar{x} := x_1 \) exists. Then, for every \( M \in \mathcal{M} \) and \( i \in [k] \), we have \( x_i^T M x_i \geq 0 \). In particular, \( 0 \leq \bar{x}^T M \bar{x} \leq \sum_{i=1}^k x_i^T M x_i = \langle M, X \rangle \). Furthermore, \( \bar{x} \in \text{range}(X) \). We conclude that \( \text{range}(X) \cap \mathcal{E}(X, M) \) contains the nonzero element \( \bar{x} \).

\((\Leftarrow)\) Let \( \mathbb{R}_+X \) be an extreme ray of \( S(M) \). By assumption, there exists a nonzero \( x \in \text{range}(X) \) such that

\[
|x^T M x| \leq \langle M, X \rangle, \forall M \in \mathcal{M}.
\]
By picking $\epsilon > 0$ small enough, we can simultaneously ensure that $X \pm \epsilon xx^\top \in \mathbb{S}_+^n$ and that

$$\langle M, X \pm \epsilon xx^\top \rangle \geq (1 - \epsilon) \langle M, X \rangle \geq 0, \forall M \in \mathcal{M}.$$ 

Hence, we conclude that the interval $[X - \epsilon xx^\top, X + \epsilon xx^\top]$ is contained in $\mathcal{S}(\mathcal{M})$. In particular, because $\mathbb{R}_+ X$ is an extreme ray of $\mathcal{S}(\mathcal{M})$, we deduce that $\epsilon xx^\top$ is a scalar multiple of $X$ and hence $\text{rank}(X) = 1$. □

When studying $\mathcal{T}(\mathcal{M})$, we can replace the set $\mathcal{E}(X, \mathcal{M})$ in Lemma 9 with a simpler set corresponding to solutions to a homogeneous system of quadratic equations.⁵

**Definition 4.** Given $\mathcal{M} \subseteq \mathbb{S}^n$, we define

$$\mathcal{N}(\mathcal{M}) := \{x \in \mathbb{R}^n : x^\top M x = 0, \forall M \in \mathcal{M}\}.$$ □

**Remark 5.** Note that for every $\mathcal{M} \subseteq \mathbb{S}^n$ and every $X \in \mathcal{S}(\mathcal{M})$, we have $\mathcal{N}(\mathcal{M}) \subseteq \mathcal{E}(X, \mathcal{M})$. □

**Corollary 2.** $\mathcal{T}(\mathcal{M})$ is ROG if and only if for every nonzero $X \in \mathcal{T}(\mathcal{M})$ we have $\text{range}(X) \cap \mathcal{N}(\mathcal{M}) \neq \{0\}$.

**Proof.** Note that $\mathcal{S}(-\mathcal{M} \cup \mathcal{M}) = \mathcal{T}(\mathcal{M})$ and apply Lemma 9. □

**Remark 6.** When applying Lemma 9, it suffices to check the right hand side only for matrices $X$ with rank at least two. Indeed if $X = xx^\top$, then $x \in \text{range}(X) \cap \mathcal{E}(X, \mathcal{M})$. The same is true for Corollary 2. □

### 2.5 Known ROG sets

In order to familiarize the reader with our notation and setup, we now recover three known results in our language. We begin with a result due to Sturm and Zhang [24] regarding spectrahedral cones defined by a single LMI.

**Lemma 10.** Consider any $M \in \mathbb{S}^n$, and let $\mathcal{M} = \{M\}$. Then $\mathcal{S}(\mathcal{M})$ is ROG.

**Proof.** By Lemma 4, $\mathcal{S}(\mathcal{M})$ is ROG if and only if $\mathcal{T}(\mathcal{M})$ is ROG. We will show that $\mathcal{T}(\mathcal{M})$ is ROG by appealing to Corollary 2.

Let $X \in \mathcal{T}(\mathcal{M})$ have rank at least two. Begin by performing a spectral decomposition $X = \sum_{i=1}^{r} \lambda_i x_i x_i^\top$, where $r = \text{rank}(X) \geq 2$, the $x_i$ are orthonormal eigenvectors of $X$, and $\lambda_i > 0$ for all $i \in [r]$.

If one of the eigenvectors $x_i$ is in $\mathcal{N}(\mathcal{M})$, then $\text{range}(X) \cap \mathcal{N}(\mathcal{M})$ contains $x_i$ and is clearly nontrivial.

Else, there exist distinct eigenvectors, without loss of generality $x_1$ and $x_2$, such that $\langle M, x_1 x_1^\top \rangle > 0 > \langle M, x_2 x_2^\top \rangle$. By continuity, there exists $x \in \mathbb{R}^n$ such that $\langle M, xx^\top \rangle = 0$. Note that $x$ is nonzero as $0 \notin \{x_1, x_2\}$ (this follows as $x_1$ and $x_2$ are orthonormal). Furthermore, $x \in \text{range}(X)$. This concludes the proof as we have constructed a nonzero $x \in \text{range}(X) \cap \mathcal{N}(\mathcal{M})$. □

Based on Lemmas 5 and 10 and Corollary 1, we have the following characterization of ROG sets defined by two inequalities.

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⁵Readers familiar with algebraic geometry will recognize this as the variety defined by $\mathcal{M}$. 

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Corollary 3. Suppose $|\mathcal{M}| = 2$, then $\mathcal{S} (\mathcal{M})$ is ROG if and only if $\mathcal{T} (\mathcal{M})$ is ROG.

The characterization given in Corollary 3 for the case of $|\mathcal{M}| = 2$ is, at the moment, unsatisfactory as we have yet to analyze when $\mathcal{T} (\mathcal{M})$ is itself ROG. Our developments in the remainder of this paper will make this implicit characterization much more explicit (see Section 4).

Next, we recover a result related to the S-lemma [13] and a convexity theorem due to Dines [12].

Lemma 11. Let $\mathcal{M} = \{M_1, M_2\}$ and suppose there exists $(\alpha_1, \alpha_2) \neq (0, 0)$ such that $\alpha_1 M_1 + \alpha_2 M_2 \in S_+^n$. Then $\mathcal{S} (\mathcal{M})$ is ROG.

Proof. By Corollary 3, it suffices to show that $\mathcal{T} (\mathcal{M})$ is ROG. Recall also that $\mathcal{T} (\mathcal{M})$ depends only on $\text{span}(\mathcal{M})$ (see Remark 2), thus we may without loss of generality suppose $M_1 \in S_+^n$. Let $W := \text{range}(M_1)$ and let $M_2 := (M_2)_W$. Then,

$$\mathcal{T} (\mathcal{M}) = 0_W \oplus \mathcal{T} (\overline{M}_2).$$

By Lemma 10 and Corollary 1, $\mathcal{T} (\overline{M}_2)$ is ROG. Then as $\mathcal{T} (\mathcal{M})$ is isomorphic via the rank-preserving map (2) to $\mathcal{T} (\overline{M}_2)$, we conclude that $\mathcal{T} (\mathcal{M})$ is ROG.

Example 2. Suppose $\mathcal{M} = \{M_1, M_2\} \subseteq S^3$ where $M_1 = \text{Diag}(1, 1, -1)$ and $M_2 = \text{Diag}(-1, 1, -1)$ so that

$$\mathcal{S} (\mathcal{M}) = \left\{ X \in S_+^3 : \begin{array}{l} X_{1,1} + X_{2,2} - X_{3,3} \geq 0 \\
-X_{1,1} + X_{2,2} - X_{3,3} \geq 0 \end{array} \right\}.$$ 

Then, noting that $M_1 - M_2 = \text{Diag}(2, 0, 0) \in S_+^3$, Lemma 11 implies that $\mathcal{S} (\mathcal{M})$ is ROG.

Intuitively, Lemma 11 states that if $M_1$ and $M_2$ “do not interact” on the interior of $S_+^n$ then $\mathcal{S} (\mathcal{M})$ is ROG (see the discussion after the statement of Theorem 4 for a more precise description of this geometric intuition). Indeed, in this example, the first constraint is redundant: if $X \in S_+^3$ satisfies the second constraint then it must also satisfy the first constraint as $X_{1,1} \geq 0$ is implied by $X \in S_+^3$.

3 Sufficient conditions

The following observation generalizes the key step in Lemma 11.

Observation 1. Let $\mathcal{M} \subseteq S^n$. Suppose there exists a nonzero $M \in \text{span}(\mathcal{M}) \cap S_+^n$. Let $W := \text{range}(M)$ and define $\mathcal{M}_W := \{M_W : M \in \mathcal{M}\}$. Then $\mathcal{T} (\mathcal{M})$ is isomorphic via a rank-preserving map to $\mathcal{T} (\mathcal{M}_W)$:

$$\mathcal{T} (\mathcal{M}) = 0_W \oplus \mathcal{T} (\mathcal{M}_W).$$

In particular, $\mathcal{T} (\mathcal{M})$ is ROG if and only if $\mathcal{T} (\mathcal{M}_W)$ is ROG.

Applying this observation repeatedly gives the following sufficient condition.

Proposition 1. Let $\mathcal{M} = \{M_1, \ldots, M_k\}$ for some $k \geq 2$. Suppose for all distinct indices $i, j \in [k]$, there exists $(\alpha, \beta) \neq (0, 0)$ such that $\alpha M_i + \beta M_j$ is positive semidefinite. Then $\mathcal{S} (\mathcal{M})$ is ROG.

Proof. By Lemma 5, it suffices to show that $\mathcal{T} (\mathcal{M}')$ is ROG for every $\mathcal{M}' \subseteq \mathcal{M}$. We will simply show that $\mathcal{T} (\mathcal{M})$ is ROG and note that every subset $\mathcal{M}' \subseteq \mathcal{M}$ of size at least two also has the assumed property.

Consider repeatedly applying Observation 1 to get a chain of subspaces $W_1 \subseteq W_2 \subseteq \ldots$ such that

$$\mathcal{T} (\mathcal{M}) = 0_{W_1} \oplus \mathcal{T} (\mathcal{M}_W) = 0_{W_2} \oplus \mathcal{T} (\mathcal{M}_W) = \ldots$$

We will repeat this process until $\text{span}(\mathcal{M}_W) \cap S_+^n = \{0\}$. This process necessarily terminates as the subspaces $W_i$ strictly increase in dimension. Let $\overline{M}_i := (M_i)_W$ and $\overline{M} := \{\overline{M}_1, \ldots, \overline{M}_k\}$.
We claim that \( \dim(\text{span}(\mathcal{M})) \leq 1 \). Suppose otherwise and let \( i, j \in [k] \) be distinct indices such that \( \mathcal{M}_i \) and \( \mathcal{M}_j \) are independent. By assumption, there exists \((\alpha, \beta) \neq (0, 0)\) such that \( \alpha M_i + \beta M_j \) is positive semidefinite. Then,

\[
\alpha \mathcal{M}_i + \beta \mathcal{M}_j = (\alpha M_i + \beta M_j)_W
\]

is positive semidefinite. Furthermore, this linear combination is nonzero by independence of \( \mathcal{M}_i \) and \( \mathcal{M}_j \). This contradicts the assumption that \( \text{span}(\mathcal{M}) \cap S^m_{W^+} = \{0\} \).

Let \( \mathcal{M}' \) denote a basis of \( \text{span}(\mathcal{M}) \). By the above discussion, \( |\mathcal{M}'| \leq 1 \). Thus, we have

\[
T(\mathcal{M}) = 0_W \oplus T(\mathcal{M}').
\]

By Lemma 10, \( T(\mathcal{M}') \) is ROG. Then as \( T(\mathcal{M}) \) is isomorphic via the rank-preserving map to \( T(\mathcal{M}') \), we conclude that \( T(\mathcal{M}) \) is ROG.

Next, we present a new sufficient condition for the ROG property suggested by Lemma 9 and Remark 5.

**Theorem 1.** Suppose \( \mathcal{M} = \{\text{Sym}(\mathbf{a}b^\top) : b \in \mathcal{B}\} \) for some \( \mathbf{a} \in \mathbb{R}^n \) and \( \mathcal{B} \subseteq \mathbb{R}^n \). Then, for every positive semidefinite \( X \) of rank at least two, we have \( \text{range}(X) \cap \mathcal{N}(\mathcal{M}) \neq \{0\} \). In particular, \( S(\mathcal{M}) \) is ROG.

**Proof.** For any \( v \in \mathbf{a}^\perp \), we have \( v^\top \text{Sym}(\mathbf{a}b^\top) v = v^\top \mathbf{a}b^\top v = 0 \). We deduce that \( \mathbf{a}^\perp \subseteq \mathcal{N}(\mathcal{M}) \), i.e., \( \mathcal{N}(\mathcal{M}) \) contains a vector space of codimension one.

Let \( X \) be a positive semidefinite matrix with rank at least two. As \( \dim(\text{range}(X)) = \text{rank}(X) \), we see that \( \text{range}(X) \cap \mathcal{N}(\mathcal{M}) \) must contain a vector space of dimension at least one. In particular, \( \text{range}(X) \cap \mathcal{E}(X, \mathcal{M}) \supseteq \text{range}(X) \cap \mathcal{N}(\mathcal{M}) \) and is nonempty. Lemma 9 then implies that \( S(\mathcal{M}) \) is ROG.

We list two immediate corollaries of Theorem 1.

**Corollary 4.** Let \( K \subseteq \mathbb{R}^n \) be any closed convex cone and consider an arbitrary vector \( c \in \mathbb{R}^n \). Then, the set \( \{X \in S^m_+ : Xc \in K\} \) is ROG.

**Proof.** Define \( \mathcal{M} := \{\text{Sym}(\mathbf{c}b^\top) : b \in K^*\} \) where \( K^* \) is the dual cone of \( K \). Then \( \{X \in S^m_+ : Xc \in K\} = S(\mathcal{M}) \), whence Theorem 1 implies the result.

**Corollary 5.** Let \( a, b, c \in \mathbb{R}^n \). Then the set \( \{X \in S^m_+ : a^\top Xc \geq 0, b^\top Xc \geq 0\} \) is ROG.

Let us demonstrate this new sufficient condition on an example.

**Example 3.** Consider \( \mathcal{M} = \{M_1, M_2\} \subseteq S^3 \) where \( M_1 = \text{Sym}(\mathbf{e}_1\mathbf{e}_1^\top) \) and \( M_2 = \text{Sym}(\mathbf{e}_1\mathbf{e}_2^\top) \). Then,

\[
S(\mathcal{M}) = \left\{ X \in S^3_+ : \begin{array}{c} X_{1,2} + X_{2,1} \geq 0 \\ X_{1,3} + X_{3,1} \geq 0 \end{array} \right\}.
\]

In contrast to Example 2, neither constraint in the definition of \( S(\mathcal{M}) \) is redundant. For example, the matrices

\[
\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

satisfy only the first inequality and only the second inequality, respectively. Nevertheless, \( S(\mathcal{M}) \) is ROG by Corollary 5.

By applying Lemma 5 once more, we next give a sufficient condition which is not covered by Theorem 1.

**Theorem 2.** Let \( a, b, c \in \mathbb{R}^n \). Then the set \( \{X \in S^m_+ : a^\top Xb \geq 0, b^\top Xc \geq 0, a^\top Xc \geq 0\} \) is ROG.
Proof. Let $\mathcal{M} = \{ \text{Sym}(ab^T), \text{Sym}(ac^T), \text{Sym}(bc^T) \}$. By Lemma 5 and Corollary 5, it suffices to show that $\mathcal{T}(\mathcal{M})$ is ROG.

We will show that $\mathcal{T}(\mathcal{M})$ is ROG by appealing to Corollary 2. Let $X \in \mathcal{T}(\mathcal{M})$ have rank at least two. Note that $\mathcal{N}(\text{Sym}(ab^T)) = a^\perp \cup b^\perp$. Hence,

$$\mathcal{N}(\mathcal{M}) = (a^\perp \cup b^\perp) \cap (a^\perp \cup c^\perp) \cap (b^\perp \cup c^\perp) = \{a, b\}^\perp \cup \{a, c\}^\perp \cup \{b, c\}^\perp.$$

If $Xa = Xb = Xc = 0$, then $\text{range}(X) \subseteq \{a, b, c\}^\perp$ and thus $\text{range}(X) \cap \mathcal{N}(\mathcal{M}) = \text{range}(X)$ is clearly nontrivial. Else, without loss of generality suppose $y = Xa \neq 0$. Because $X \in \mathcal{T}(\mathcal{M})$, we have $b^\top y = c^\top y = 0$, and thus $y \in \mathcal{N}(\mathcal{M})$. Noting that $y \neq 0$ and $y \in \text{range}(X)$, we have concluded $0 \neq y \in \text{range}(X) \cap \mathcal{N}(\mathcal{M})$ as desired. ■

Remark 7. By picking $n = 3$ and $\{a, b, c\} = \{e_1, e_2, e_3\}$ in Theorem 2, we recover the fact that the set of copositive matrices in $\mathbb{S}^3$ is ROG. □

Remark 8. A graph $G = (V, E)$ is chordal if every minimal cycle has at most 3 edges. It is well-known that the set of positive semidefinite matrices with a fixed chordal support is ROG $[1, 15, 21]$. Specifically, if $G = ([n], E)$ is a chordal graph containing all self-loops, then

$$\{X \in \mathbb{S}^n_+ : X_{i,j} = 0, \forall (i, j) \notin E\}$$

is ROG.

Unfortunately, the set in (3) does not necessarily remain ROG when the equality constraints are replaced with inequality constraints (see the example below). From this point of view, Theorem 2 and Remark 7 highlight a special chordal graph for which the inequality version of the set is also ROG.

Consider the path graph on four vertices with all self-loops. We will show that the following set is not ROG:

$$\mathcal{S} = \left\{ X \in \mathbb{S}^4_+ : \begin{array}{c} X_{1,2} \geq 0 \\ X_{2,3} \geq 0 \\ X_{3,4} \geq 0 \end{array} \right\}.$$

We will apply Lemma 9 to show that $\mathcal{S}$ is not ROG. Let $\mathcal{M} = \{ \text{Sym}(e_1e_2^T), \text{Sym}(e_2e_3^T), \text{Sym}(e_3e_4^T) \}$ so that $\mathcal{S} = \mathcal{S}(\mathcal{M})$. Let $x = (1, 0, 1, 1)^\top$ and $y = (0, 1, 1, -1)^\top$. Note that the following rank-two matrix

$$X := xx^\top + yy^\top = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{pmatrix}$$

satisfies $X \in \mathcal{S}$. We compute

$$\text{range}(X) \cap \mathcal{E}(X, \mathcal{M}) = \text{span} \{ x, y \} \cap \left\{ z \in \mathbb{R}^4 : \begin{array}{c} z_{1,2} = 0 \\ z_{2,3} \leq 1 \\ z_{3,4} = 0 \end{array} \right\}.$$

Let $z \in \text{range}(X) \cap \mathcal{E}(X, \mathcal{M})$. Then, writing $z = \alpha x + \beta y = (\alpha, \beta, \alpha + \beta, \alpha - \beta)^\top$, we deduce that $0 = z_{1,2} = \alpha \beta$ and $0 = z_{3,4} = \alpha^2 - \beta^2$ so that $\alpha = \beta = 0$. Thus, $\text{range}(X) \cap \mathcal{E}(X, \mathcal{M}) = \{0\}$. □

Finally, we show how our results can be used to recover a result due to Sturm and Zhang $[24]$; see also $[7$, Section 6.1]. Let $L^n \subseteq \mathbb{R}^n$ denote the second order cone (SOC)

$$L^n := \{ x = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \| y \|_2 \leq t \}.$$

Defining $L := \text{Diag}(-1, \ldots, -1, 1) \in \mathbb{S}^n$, we can write $L^n = \{ x \in \mathbb{R}^n : x^\top L x \geq 0, x_n \geq 0 \}$. 

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Lemma 12. Let \( c \in \mathbb{R}^n \) and define
\[
S = \left\{ X \in S^n_+ : Xc \in L^n, \langle L, X \rangle \geq 0 \right\}.
\]
Then \( S \) is ROG.

Proof. For the sake of contradiction suppose there exists an extreme ray \( \mathbb{R}_+ X \) of \( S \) with \( \text{rank}(X) \geq 2 \) and recall Lemma 3. If \( \langle L, X \rangle > 0 \) then \( \mathbb{R}_+ X \) is an extreme ray of \( \{ X \in S^n_+ : Xc \in L^n \} \), contradicting Corollary 4. If \( Xc \in \text{int}(L^n) \) then \( \mathbb{R}_+ X \) is an extreme ray of \( S(\{ L \}) \), contradicting Lemma 10. Finally, if \( Xc = 0 \) then \( \mathbb{R}_+(Xc^\top) \) is an extreme ray of \( S(\{ Lc^\top \}) \) by Lemma 6, contradicting Lemma 10. Then, we may assume that \( \langle L, X \rangle = 0 \) and \( y := Xc \) is a nonzero element in \( \text{bd}(L^n) \), i.e., \( y^\top Ly = 0 \).

Then for all \( \epsilon > 0 \) small enough, we have \( X \pm \epsilon yy^\top \succeq 0, \langle L, X \pm \epsilon yy^\top \rangle = \langle L, X \rangle = 0 \), and \( (X \pm \epsilon yy^\top)c = (1 \pm \epsilon y^\top c)y \in L^n \). This contradicts the assumption that \( \mathbb{R}_+ X \) is extreme. Thus all extreme rays \( \mathbb{R}_+ X \) of \( S \) have \( \text{rank}(X) \leq 1 \).

\( \blacksquare \)

4 Necessary conditions

In this section, we give a complete characterization of ROG cones defined by two LMIs.

Theorem 3. Let \( \mathcal{M} = \{ M_1, M_2 \} \). Then \( S(\mathcal{M}) \) is ROG if and only if one of the following holds:

(i) there exists \( (\alpha_1, \alpha_2) \neq (0, 0) \) such that \( \alpha_1 M_1 + \alpha_2 M_2 \in S^n_+ \), or

(ii) there exists \( a, b, c \in \mathbb{R}^n \) such that \( M_1 = \text{Sym}(ac^\top) \) and \( M_2 = \text{Sym}(bc^\top) \).

Note that the if direction of Theorem 3 is a direct consequence of the sufficient conditions identified in Proposition 1 and Corollary 5. Furthermore, recall from Corollary 3 that when \( |\mathcal{M}| = 2 \), the set \( S(\mathcal{M}) \) is ROG if and only if \( \mathcal{T}(\mathcal{M}) \) is ROG. Thus, Theorem 3 follows as a corollary to the following necessary condition.

Theorem 4. Let \( \mathcal{M} = \{ M_1, M_2 \} \). If \( \mathcal{T}(\mathcal{M}) \) is ROG, then one of the following holds:

(i) there exists \( (\alpha_1, \alpha_2) \neq (0, 0) \) such that \( \alpha_1 M_1 + \alpha_2 M_2 \in S^n_+ \), or

(ii) there exists \( a, b, c \in \mathbb{R}^n \) such that \( M_1 = \text{Sym}(ac^\top) \) and \( M_2 = \text{Sym}(bc^\top) \).

Conditions (i) and (ii) in Theorems 3 and 4 have simple geometric interpretations. We describe these interpretations for Theorem 4, i.e., in the case of two LMEs. Condition (i) covers the simple case where the two LMEs defining \( \mathcal{T}(\mathcal{M}) \) only interact with each other on a single (possibly empty) face of the positive semidefinite cone. Furthermore, on this face, the two LMEs impose the same (possibly trivial) constraint. Condition (ii) covers the important case when the two LMEs interact in a nontrivial manner inside \( S^n_+ \). Suppose for the sake of presentation that \( a = e_1, b = e_2, c = e_n \). Then, Corollary 5 implies that
\[
\mathcal{T}(\mathcal{M}) = \text{conv}\{ xx^\top : x_1 x_n = 0, x_2 x_n = 0 \}
= \text{conv} \left( \text{conv} \{ xx^\top : x_1 = x_2 = 0 \} \cup \text{conv} \{ xx^\top : x_n = 0 \} \right)
= \text{conv} \left( (0_2 \oplus S^{n-2}_+) \cup (S^{n-1}_+ \oplus 0_1) \right).
\]

In other words, condition (ii) covers the case where \( \mathcal{T}(\mathcal{M}) \) is the convex hull of the union of two faces of the positive semidefinite cone with a particular intersection structure. Theorem 4 states that these are the only ways for \( \mathcal{T}(\mathcal{M}) \) to be ROG when \( |\mathcal{M}| = 2 \).

The proof of Theorem 4 is nontrivial and will be the focus of the remainder of the section. Before completing this proof, let us first work out in detail a prototypical example. This example will highlight a number of the steps of our proof.
Example 4. Suppose $\mathcal{M} = \{M_1, M_2\}$ where $M_1 = \text{Diag}(1, -1, 0)$ and $M_2 = \text{Diag}(0, 1, -1)$ so that $\mathcal{T}(\mathcal{M}) = \{X \in S^3_+ : X_{1,1} = X_{2,2} = X_{3,3}\}$.

We first verify that neither condition (i) nor (ii) from Theorem 4 hold. Indeed, $\alpha_1 M_1 + \alpha_2 M_2 = \text{Diag}(\alpha_1, \alpha_2 - \alpha_1, -\alpha_2)$ is positive semidefinite if and only if $(\alpha_1, \alpha_2) = (0, 0)$ so that condition (i) is violated. Next, note that $2M_1 + M_2 = \text{Diag}(2, -1, 1)$ has rank three so that condition (ii) is also violated. We next demonstrate that $\mathcal{T}(\mathcal{M})$ is not ROG.

Let $w := (1, 1, \sqrt{2})^T$. We claim there exists a vector $z$ such that
\[
\begin{pmatrix}
    z^T M_1 z \\
    z^T M_2 z
\end{pmatrix} = - \begin{pmatrix}
    w^T M_1 w \\
    w^T M_2 w
\end{pmatrix}.
\]

Indeed for this example, $z = (-1, 1, 0)^T$ is such a vector. It is clear that $w$ and $z$ are linearly independent so that $X := ww^T + zz^T$ is a rank-two matrix contained in $\mathcal{T}(\mathcal{M})$. By Corollary 2, it suffices to show that $\text{range}(X) \cap \mathcal{N}(\mathcal{M}) = \{0\}$. We will write a generic element from $\text{range}(X)$ as $(\alpha - \beta, \alpha + \beta, \sqrt{2\alpha})^T$. Then
\[
\text{range}(X) \cap \mathcal{N}(\mathcal{M}) = \left\{ (\alpha - \beta, \alpha + \beta, \sqrt{2\alpha}) : (\alpha - \beta)^2 = (\alpha + \beta)^2 = 2\alpha^2 \right\}.
\]

The first equality implies $\alpha \beta = 0$. The second equality then implies that $\alpha = -\beta = 0$. We conclude $\text{range}(X) \cap \mathcal{N}(\mathcal{M}) = \{0\}$ and that $\mathcal{T}(\mathcal{M})$ is not ROG.

We now begin on the proof of Theorem 4. We first make a simplifying assumption that holds without loss of generality.

Lemma 13. Let $W := \text{span}(\bigcup_{M \in \mathcal{M}} \text{range}(M))$ have dimension $k \leq n$. For $M \in \mathcal{M}$, let $\overline{M} = M_W$ denote the restriction of $M$ to $W$. Let $\overline{\mathcal{M}} = \{\overline{M} : M \in \mathcal{M}\}$. Then, $\mathcal{T}(\mathcal{M})$ is ROG if and only $\mathcal{T}(\overline{\mathcal{M}})$ is ROG. Furthermore, if $\mathcal{M} = \{M_1, M_2\}$ and $\overline{\mathcal{M}} = \{\overline{M}_1, \overline{M}_2\}$, then each of conditions (i) and (ii) in Theorem 4 hold for $\mathcal{M}$ if and only if they hold for $\overline{\mathcal{M}}$.

Proof. We first prove that $\mathcal{T}(\mathcal{M})$ is ROG if and only if $\mathcal{T}(\overline{\mathcal{M}})$ is ROG.

(⇒) Note that $\mathcal{T}(\overline{\mathcal{M}})$ is isomorphic via a rank-preserving map to $0_{W^\perp} \oplus \mathcal{T}(\overline{\mathcal{M}})$. This latter set is a face of $\mathcal{T}(\mathcal{M})$ and thus ROG by Lemma 2.

(⇐) Let $X \in \mathcal{T}(\mathcal{M})$ be nonzero and let $\overline{X} := X_W$. First, suppose $\overline{X} = 0$, then $\text{range}(X) \subseteq W^\perp \subseteq \mathcal{N}(\mathcal{M})$. Thus, as $X \neq 0$, there exists a nonzero vector in $\text{range}(X) \cap \mathcal{N}(\mathcal{M})$. Next, suppose $\overline{X} \neq 0$. As $(\overline{M}, X) = \langle \overline{M}, \overline{X} \rangle$ for every $M \in \mathcal{M}$, we have that $\overline{X} \in \mathcal{T}(\overline{\mathcal{M}})$. Because $\mathcal{T}(\overline{\mathcal{M}})$ is ROG, by Corollary 2 there exists a nonzero $\overline{y} \in \text{range}(\overline{X}) \cap \mathcal{N}(\overline{\mathcal{M}})$. Let $\overline{z}$ such that $\overline{y} = \overline{X}\overline{z}$ and set
\[y := X(\overline{z} \oplus 0_{W^\perp}).\]

Note that $y_W = \overline{y}$ and hence $y$ is nonzero. Furthermore, $y^\top My = \overline{y}^\top \overline{M}\overline{y} = 0$ for all $M \in \mathcal{M}$, which implies $y \in \mathcal{N}(\mathcal{M})$. By construction we also have $y \in \mathcal{N}(\overline{\mathcal{M}})$. Then Corollary 2 implies that $\mathcal{T}(\mathcal{M})$ is ROG.

The last statement of the lemma follows from definition of $W$. \[\Box\]

We will henceforth assume that $\mathcal{M}$ spans $\mathbb{R}^n$ in the following sense.

Assumption 1. Assume that span $(\bigcup_{M \in \mathcal{M}} \text{range}(M)) = \mathbb{R}^n$. \[\Box\]

Proof of Theorem 4. By Lemma 13, we may without loss of generality assume that Assumption 1 holds. We will split the proof of Theorem 4 into a number of cases depending on the dimension $n$.

- The case $n = 1$ holds vacuously as we can set $(\alpha_1, \alpha_2)$ to either $(1, 0)$ or $(-1, 0)$ to satisfy (i).
• For $n = 2$, we will suppose condition (i) is not satisfied and explicitly construct an extreme ray of $\mathcal{T}(\mathcal{M})$ with rank two. The construction crucially uses the geometry of $\mathbb{R}^2$ (and $\mathbb{S}^2$). See Proposition 2.

• For $n = 3$, we will suppose that neither conditions (i) nor (ii) are satisfied and explicitly construct extreme rays of $\mathcal{T}(\mathcal{M})$ with rank two. The construction is based on understanding what the corresponding $\mathcal{N}(\mathcal{M})$ set looks like. This construction crucially use the geometry of $\mathbb{R}^3$. See Proposition 3.

• Finally, we will show how to reduce the case of $n \geq 4$ to the case of $n = 3$. Specifically, supposing that $\mathcal{T}(\mathcal{M})$ is a ROG cone, with $n \geq 4$, violating (i), we will construct $\tilde{\mathcal{M}}$ such that $\mathcal{T}(\tilde{\mathcal{M}})$ is a ROG cone, with $n = 3$, violating both (i) and (ii). See Proposition 4. ■

We will make use of the following theorem related to the convexity of the joint image of two quadratic maps. For completeness, a short proof of this result is included in Appendix A.

**Theorem 5** [Dines [12]]. Let $M_1, M_2 \in \mathbb{S}^n$ and suppose that for all $\langle \alpha_1, \alpha_2 \rangle \neq (0,0)$, we have $\alpha_1 M_1 + \alpha_2 M_2 \notin \mathbb{S}^n_+$. Then

$$\left\{ \begin{pmatrix} x^\top M_1 x \\ x^\top M_2 x \end{pmatrix} \in \mathbb{R}^2 : x \in \mathbb{R}^n \right\} = \mathbb{R}^2,$$

i.e., for every $y \in \mathbb{R}^2$, there exists an $x \in \mathbb{R}^n$ such that $x^\top M_1 x = y_1$ and $x^\top M_2 x = y_2$.

**Remark 9.** Both directions of Theorems 3 and 4 admit small certificates.

• Suppose $\mathcal{S}(\mathcal{M})$ is ROG. Then Theorem 3 implies that there exists either aggregation weights $(\alpha_1, \alpha_2) \neq (0,0)$ for which $\alpha_1 M_1 + \alpha_2 M_2 \in \mathbb{S}^n_+$ or vectors $a, b, c$ for which $M_1 = \text{Sym}(ac^\top)$ and $M_2 = \text{Sym}(bc^\top)$.

• Suppose $\mathcal{S}(\mathcal{M})$ is not ROG. Then by Theorem 3, it suffices to certify that neither conditions (i) nor (ii) hold. Let $\theta_1 = 0$, $\theta_2 = 2\pi/3$ and $\theta_3 = 4\pi/3$. By Theorem 5, there exists vectors $u_1, u_2, u_3$ such that $(u_i^\top M_1 u_i, u_i^\top M_2 u_i) = (\cos(\theta_i), \sin(\theta_i))$ for all $i \in \{3\}$. Then,

$$\{ (\alpha_1, \alpha_2) : \alpha_1 M_1 + \alpha_2 M_2 \in \mathbb{S}^n_+ \} \subseteq \left\{ (\alpha_1, \alpha_2) : \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{pmatrix} \right\} \geq 0, \forall i \right\} = \{ (0,0) \}.$$

Here, the first inclusion follows by considering the quadratic forms induced by the $u_i$. In words, the vectors $u_1, u_2, u_3$ certify that condition (i) does not hold. If either rank($M_1$) $\geq 3$ or rank($M_2$) $\geq 3$, then the spectral decomposition of the corresponding $M_i$ certifies that condition (ii) does not hold. Else, $M_1$ and $M_2$ both have rank two and we can write $M_1 = \eta_1 \text{Sym}(ab^\top)$ and $M_2 = \eta_2 \text{Sym}(cd^\top)$ where $\eta_i \in \mathbb{R}$, $a, b, c, d \in \mathbb{S}^{n-1}$. This decomposition is unique up to renaming $a$ and $b$ or $c$ and $d$. Then condition (ii) does not hold if and only if $a, b, c, d$ are distinct. In particular, this decomposition certifies that condition (ii) does not hold. ■

### 4.1 Dimension $n = 2$

We now prove Theorem 4 for the case $n = 2$.

**Proposition 2.** Let $\mathcal{M} = \{M_1, M_2\}$. Suppose Assumption 1 holds and $n = 2$. If $\mathcal{T}(\mathcal{M})$ is ROG then there exists $(\alpha_1, \alpha_2) \neq (0,0)$ such that $\alpha_1 M_1 + \alpha_2 M_2 \in \mathbb{S}^n_+$.

**Proof.** Suppose for all $(\alpha_1, \alpha_2) \neq (0,0)$, the linear combination $\alpha_1 M_1 + \alpha_2 M_2$ is not positive semidefinite. In particular, $M_1$ and $M_2$ are linearly independent in $\mathbb{S}^2$. As $\mathbb{S}^2$ has dimension three, the space orthogonal to both $M_1$ and $M_2$ has dimension one.

By Theorem 5, there exist $x, y \in \mathbb{R}^2$ such that

$$\begin{pmatrix} x^\top M_1 x \\ x^\top M_2 x \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y^\top M_1 y \\ y^\top M_2 y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Then, $X := xx^\top + yy^\top$ is contained in $\mathcal{T}(\mathcal{M})$. However, by the above discussion, $\mathcal{T}(\mathcal{M})$ is contained in a one-dimensional space so that $\mathcal{T}(\mathcal{M}) = \mathbb{R}_+ X$. Noting that $x$ and $y$ are linearly independent, we conclude that $xx^\top + yy^\top$ has rank two and that $\mathcal{T}(\mathcal{M})$ is not ROG. ■
4.2 Dimension $n = 3$

We will make use of the following lemma from Hildebrand [17, Lemma 3.13]. The lemma states that the Carathéodory number of an element $X$ of $\mathcal{T}(\mathcal{M})$ is equal to $\text{rank}(X)$ when $\mathcal{T}(\mathcal{M})$ is ROG.

**Lemma 14** ([17, Lemma 3.13]). Suppose $\mathcal{T}(\mathcal{M})$ is ROG. For every $X \in \mathcal{T}(\mathcal{M})$, we can write $X = \sum_{i=1}^{r} x_i x_i^\top$ where $x_i \in \mathcal{N}(\mathcal{M})$ for all $i \in [r]$ and $r = \text{rank}(X)$.

**Proof.** Let $X \in \mathcal{T}(\mathcal{M})$ be nonzero. By Corollary 2, there exists a nonzero $x \in \text{range}(X) \cap \mathcal{N}(\mathcal{M})$. We will write

$$X = \alpha xx^\top + (X - \alpha xx^\top)$$

for some $\alpha \geq 0$. As $x \in \text{range}(X)$ is nonzero, there exists a unique $\alpha \geq 0$ such that $X - \alpha xx^\top$ is positive semidefinite and $\text{rank}(X - \alpha xx^\top) = \text{rank}(X) - 1$. Furthermore, $\alpha xx^\top$ and $X - \alpha xx^\top$ are both elements of $\mathcal{T}(\mathcal{M})$. Repeating this procedure gives the desired decomposition. ■

The next lemma states that when neither conditions (i) nor (ii) hold, the set $\mathcal{N}(\mathcal{M})$ is extremely sparse in $\mathbb{R}^3$.

**Lemma 15.** Let $\mathcal{M} = \{M_1, M_2\}$. Suppose Assumption 1 holds and $n = 3$. If neither conditions (i) nor (ii) of Theorem 4 hold, then $\mathcal{N}(\mathcal{M})$ is the union of at most four one-dimensional subspaces of $\mathbb{R}^3$.

Readers familiar with algebraic geometry will recognize this as a consequence of Bézout’s theorem. We will give an alternate proof of this lemma using only elementary linear algebra. Unfortunately, the proof is technical and adds little insight to the discussion, and as such is deferred to Appendix B.

We are now ready to prove Theorem 4 for the case of $n = 3$. We will assume that neither conditions (i) nor (ii) hold and use Theorem 5 and Lemma 15 to construct a rank two matrix contained in $\mathcal{T}(\mathcal{M})$. We will then apply Lemma 14 to derive a contradiction.

**Proposition 3.** Let $\mathcal{M} = \{M_1, M_2\}$. Suppose Assumption 1 holds and $n = 3$. If $\mathcal{T}(\mathcal{M})$ is ROG, then one of conditions (i) or (ii) of Theorem 4 must hold.

**Proof.** Suppose $\mathcal{T}(\mathcal{M})$ is ROG but neither conditions (i) nor (ii) hold. Consider the subset of $\mathbb{R}^3$ given by

$$\mathcal{R} := \bigcup_{x,y \in \mathcal{N}(\mathcal{M})} \text{span}\{\{x, y\}\}.$$ 

By Lemma 15, we have that $\mathcal{R}$ is the union of a finite number of planes and lines in $\mathbb{R}^3$, and thus there exists $w \notin \mathcal{R}$. By Theorem 5, we can pick $z$ such that

$$\begin{pmatrix} z^\top M_1 z \\ z^\top M_2 z \end{pmatrix} = -\begin{pmatrix} w^\top M_1 w \\ w^\top M_2 w \end{pmatrix}. \tag{6}$$

Assuming that neither conditions (i) nor (ii) hold, the plane curves defined by $M_1$ and $M_2$ cannot share a common component. Then Bézout’s theorem implies that $\mathcal{N}(\mathcal{M})$ consists of at most four lines (or equivalently, four points in projective space).
As \( w \notin \mathcal{R} \), we deduce at least one of \( w^\top M_1 w \) and \( w^\top M_2 w \) is nonzero. Then, it is clear that \( w \) and \( z \) are linearly independent, and thus \( X := w w^\top + z z^\top \) is a rank two matrix contained in \( \mathcal{T}(\mathcal{M}) \).

As \( \mathcal{T}(\mathcal{M}) \) is ROG, we can apply Lemma 14. In particular, we can write \( X = x x^\top + y y^\top \) for some \( x, y \in \mathcal{N}(\mathcal{M}) \). Then, \( w \in \text{range}(X) = \text{span}(x, y) \subseteq \mathcal{R} \). This contradicts our choice of \( w \notin \mathcal{R} \).

\[ \blacksquare \]

### 4.3 Dimensions \( n \geq 4 \)

We will now reduce the case of \( n \geq 4 \) to \( n = 3 \). The proof will show that if \( \mathcal{M} \) violates condition (i) then there exists a three-dimensional subspace \( W \) for which the restriction of \( \mathcal{M} \) to \( W \) fails both conditions (i) and (ii).

We begin by showing that there exists a linear combination of \( M_1 \) and \( M_2 \) with rank at least three.

**Lemma 16.** Let \( \mathcal{M} = \{M_1, M_2\} \). Suppose Assumption 1 holds and \( n \geq 4 \). If condition (i) in Theorem 4 does not hold, then there exists \((\alpha_1, \alpha_2)\) such that \( \text{rank}(\alpha_1 M_1 + \alpha_2 M_2) \geq 3 \).

**Proof.** Suppose \( \text{rank}(\alpha_1 M_1 + \alpha_2 M_2) \leq 2 \) for all \((\alpha_1, \alpha_2)\). Then as condition (i) does not hold, we conclude that for all \((\alpha_1, \alpha_2) \neq (0, 0)\), the linear combination \( \alpha_1 M_1 + \alpha_2 M_2 \) has exactly one positive and one negative eigenvalue. Then, we can write \( M_1 = \text{Sym}(ab^\top) \) and \( M_2 = \text{Sym}(cd^\top) \). By Assumption 1, we have that \( a, b, c, d \) are linearly independent. By independence, there exists an \( x \) such that \( x^\top b = 1 \) and \( x^\top a = x^\top c = x^\top d = 0 \); we deduce that \( (M_1 + M_2)x = a \in \text{range}(M_1 + M_2) \). Similarly, \( b, c, d \in \text{range}(M_1 + M_2) \). Then \( \text{rank}(M_1 + M_2) = 4 \), a contradiction.

We are now ready to prove Theorem 4 for the case of \( n \geq 4 \).

**Proposition 4.** Let \( \mathcal{M} = \{M_1, M_2\} \). Suppose Assumption 1 holds and \( n \geq 4 \). If \( \mathcal{T}(\mathcal{M}) \) is ROG, then there exists \((\alpha_1, \alpha_2) \neq (0, 0)\) such that \( \alpha_1 M_1 + \alpha_2 M_2 \in \mathbb{S}^n_+ \).

**Proof.** Suppose for the sake of contradiction that \( \mathcal{T}(\mathcal{M}) \) is ROG but condition (i) in Theorem 4 does not hold. Then by Lemma 16, there exists \( M_\alpha := \alpha_1 M_1 + \alpha_2 M_2 \) with rank at least three. Let \( v_1, v_2, v_3 \in \mathbb{R}^n \) be orthonormal eigenvectors of \( M_\alpha \) with nonzero eigenvalues.

Let \( \theta_1 = 0 \), \( \theta_2 = 2\pi/3 \) and \( \theta_3 = 4\pi/3 \). Then, we may apply Theorem 5 to find three vectors \( u_1, u_2, u_3 \in \mathbb{R}^n \) satisfying

\[
\begin{pmatrix}
 u_1^\top M_1 u_1 \\
 u_2^\top M_2 u_2 \\
 u_3^\top M_2 u_3
\end{pmatrix} = \begin{pmatrix}
 \cos(\theta_1) \\
 \cos(\theta_2) \\
 \cos(\theta_3)
\end{pmatrix} \forall i \in [3].
\]  

(4)

Let \( x_i := (1 - \mu) u_i + \mu v_i \) for some \( \mu \in (0, 1] \) to be chosen later. We claim that for all \( \mu > 0 \) small enough, the set \( \{x_i\} \) are linearly independent. Indeed, consider

\[
\det \begin{pmatrix}
 v_1^\top x_1 & v_1^\top x_2 & v_1^\top x_3 \\
 v_2^\top x_1 & v_2^\top x_2 & v_2^\top x_3 \\
 v_3^\top x_1 & v_3^\top x_2 & v_3^\top x_3
\end{pmatrix}.
\]

This is a degree-3 polynomial in \( \mu \) which is not identically zero (taking \( \mu = 1 \) gives the determinant of the identity matrix), and thus \( \{x_i\} \) are linearly dependent for at most three choices of \( \mu \).

Let \( W \subseteq \mathbb{R}^n \) denote the three-dimensional span of the \( \{x_i\} \) identified with \( \mathbb{R}^3 \). Let \( \overline{\mathcal{M}}_1 := (M_1)_W \). Similarly define \( \overline{\mathcal{M}}_2 \) and \( \overline{\mathcal{M}}_{\alpha} \). Let \( \overline{\mathcal{M}} := \{\overline{\mathcal{M}}_1, \overline{\mathcal{M}}_2\} \).

We claim that for all \( \mu > 0 \) small enough, neither condition (i) nor (ii) in Theorem 4 hold for \( \overline{\mathcal{M}} \). Note

\[
\{(\alpha_1, \alpha_2) : \alpha_1 \overline{\mathcal{M}}_1 + \alpha_2 \overline{\mathcal{M}}_2 \geq 0 \} \subseteq \{(\alpha_1, \alpha_2) : x_i^\top (\alpha_1 M_1 + \alpha_2 M_2) x_i \geq 0, \forall i \in [3]\}
\]

\[
= \{(\alpha_1, \alpha_2) : \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} x_i^\top M_1 x_i \\ x_i^\top M_2 x_i \end{pmatrix} \geq 0, \forall i \in [3]\},
\]

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where the first relation follows from the definition of $\overline{M}_i$ and noting that $x_i \in W$. By continuity of the quadratic forms $x_i^\top M_i x_i$ and $x_i^\top M_2 x_i$ in the variable $\mu$, and the choice of the $u_i$ in Equation (4), the set on the second line above is the trivial set $\{0\}$ for all $\mu > 0$ small enough. Thus, $\overline{M}$ does not satisfy condition (i) for all $\mu > 0$ small enough. Next, consider $\overline{M}_\alpha$. Note that for all $\mu > 0$ small enough, $\overline{M}_\alpha$ is singular if and only if

$$\det \begin{pmatrix} x_1^\top M_\alpha x_1 & x_1^\top M_\alpha x_2 & x_1^\top M_\alpha x_3 \\ x_2^\top M_\alpha x_1 & x_2^\top M_\alpha x_2 & x_2^\top M_\alpha x_3 \\ x_3^\top M_\alpha x_1 & x_3^\top M_\alpha x_2 & x_3^\top M_\alpha x_3 \end{pmatrix} = 0.$$  

This is a degree-6 polynomial in $\mu$ (recall that $x_i$s are determined by $\mu$) that is not identically zero: for $\mu = 1$, this determinant evaluates to the product of three nonzero eigenvalues of $M_\alpha$. Then, for all $\mu > 0$ small enough, this polynomial is nonzero and hence $\det (\overline{M}_\alpha) = 0$. Thus, we deduce that $\overline{M}$ does not satisfy condition (ii) for all $\mu > 0$ small enough. We now fix $\mu$ such that $\overline{M}$ does not satisfy either condition (i) or (ii).

To complete the proof we will show that $T(\overline{M})$ is ROG. This will contradict Proposition 3. Let $Y \in T(\overline{M})$ be nonzero. Let $X = 0_{W^\perp} \oplus Y$. Then, $Y \in T(\overline{M})$ implies that $X \in T(M)$. Furthermore, $X$ is nonzero because $Y$ is nonzero. Then, as $X$ is nonzero and $T(M)$ is ROG, Corollary 2 implies there exists a nonzero $x \in \text{range}(X) \cap N(M)$. As $\text{range}(X) \subseteq W$, we deduce that, after identifying $W$ with $\mathbb{R}^{\dim(W)}$, in fact $x \in \text{range}(Y) \cap N(\overline{M})$. Applying Corollary 2 once more, we deduce that $T(\overline{M})$ is ROG, which is the desired contradiction. $\blacksquare$

This concludes the proof of Theorem 4.

4.4 Lifting LMIs into LMEs

In this section, we will show that a simple lifting of an LMI set $S$ into an LME set $T$ in a larger dimension may not preserve the ROG property.

Example 5. Consider the set

$$S := \left\{ X \in \mathbb{S}^3_+ : \begin{array}{l} X_{1,2} = 0 \\ X_{1,3} \geq 0 \end{array} \right\}.$$  

This set is ROG by Theorem 3 and Lemma 2. We can replace the LMIs defining $S$ with LMEs in a lifted space as follows: Let $\Pi : \mathbb{S}^4 \to \mathbb{S}^3$ denote the projection of a $4 \times 4$ matrix onto its top-left $3 \times 3$ principal submatrix. Then

$$S = \Pi \left( \left\{ X \in \mathbb{S}^4 : \begin{array}{l} X_{1,2} = 0 \\ X_{1,3} - X_{4,4} = 0 \end{array} \right\} \right) = \Pi (T(\{M'_1, M'_2\})).$$  

where

$$M'_1 := \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M'_2 := \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$  

Define $M' := \{M'_1, M'_2\}$. By Theorem 3, we see that $T(M')$ is not ROG. We conclude that the obvious lifting of LMIs into LMEs can take ROG sets $S(M)$ to non-ROG sets $T(M')$ (even when there is only a single inequality to lift). $\square$

5 Applications of ROG cones

5.1 Exactness of SDP relaxations of QCQPs

Recall that the general form of a QCQP and its SDP relaxation, given in Equation (1), contains exactly one inhomogeneous equality constraint. The following lemma relates the ROG property of a cone to the
ROG property of its affine slices. This will allow us to apply our main results on spectrahedra arising as the feasible domain of the SDP relaxations in Equation (1).

**Lemma 17.** Let $\mathcal{M} \subseteq \mathbb{S}^n$. If $\mathcal{S}(\mathcal{M})$ is ROG, then

$$\inf_{x \in \mathbb{R}^n} \left\{ x^\top M_0 x : \begin{array}{l} x^\top M x \geq 0, \forall M \in \mathcal{M} \\ x^\top B x = 1 \end{array} \right\} = \inf_{X \in \mathbb{S}^n} \left\{ \langle M_0, X \rangle : \begin{array}{l} \langle M, X \rangle \geq 0, \forall M \in \mathcal{M} \\ X \succeq 0 \end{array} \right\}$$

for all $B, M_0 \in \mathbb{S}^n$ for which the optimum SDP objective value is bounded from below. In particular, this equality holds whenever the SDP feasible domain is bounded.

**Proof.** Let $\mathcal{S} := \mathcal{S}(\mathcal{M})$.

$(\geq)$ This direction is immediate as the SDP gives a relaxation of the QCQP.

$(\leq)$ We may assume without loss of generality that the SDP is feasible. Let $X$ be a feasible SDP solution. As $X \in \mathcal{S}$ and $\mathcal{S}$ is an ROG cone, there exist $x_1, \ldots, x_r \in \mathbb{R}^n$ such that $x_i x_i^\top \in \mathcal{S}$ for all $i \in [r]$ and $X = \sum_{i=1}^r x_i x_i^\top$. That is, we have $x_i^\top M x_i \geq 0$ for all $M \in \mathcal{M}$ and $i \in [r]$. Without loss of generality, by re-arranging the indices in $[r]$ if needed, we may decompose

$$X = \hat{X} + \hat{X} := \left( \sum_{i=1}^k x_i x_i^\top \right) + \left( \sum_{i=k+1}^r x_i x_i^\top \right),$$

where $x_1^\top B x_1, \ldots, x_k^\top B x_k$ are positive scalars summing to one. Note that $\langle B, \hat{X} \rangle = \langle B, X \rangle - \langle B, \hat{X} \rangle = 0$. Moreover, because the optimum SDP objective value is bounded from below, we must have $\langle M_0, \hat{X} \rangle \geq 0$.

For $i \in [k]$, define $\mu_i := x_i^\top B x_i > 0$ and $\hat{x}_i := x_i/\sqrt{\mu_i}$. Then, $\hat{x}_i^\top B \hat{x}_i = 1$ and $\hat{x}_i^\top M \hat{x}_i \geq 0$ for all $M \in \mathcal{M}$. Finally, note that $1 = \sum_{i=1}^k x_i^\top B x_i = \sum_{i=1}^k \mu_i \hat{x}_i^\top M_0 \hat{x}_i$.

Using these facts, we deduce

$$\langle M_0, X \rangle \geq \langle M_0, \hat{X} \rangle = \sum_{i=1}^k x_i^\top M_0 x_i = \sum_{i=1}^k \mu_i \hat{x}_i^\top M_0 \hat{x}_i \geq \min_{i \in [k]} \hat{x}_i^\top M_0 \hat{x}_i \geq \inf_{x \in \mathbb{R}^n} \left\{ x^\top M_0 x : \begin{array}{l} x^\top M x \geq 0, \forall M \in \mathcal{M} \\ x^\top B x = 1 \end{array} \right\}. $$

The desired result follows by taking the infimum of this inequality over feasible solutions $X$ to the SDP. □

**Remark 10.** Lemma 17 extends [17, Lemma 1.2], which shows that the same statement holds in the case of finitely many LMEs. The proof we present is new and immediately shows how to construct a QCQP feasible solution achieving the SDP value (or a sequence approaching the SDP value).

Lemma 17 implies that equality holds in Equation (1) whenever $\mathcal{S}(\{M_1, \ldots, M_m\})$ is ROG and the SDP optimum value is bounded from below. It may be natural to ask whether the boundedness assumption can be dropped in the case where $B$ is specialized to $B = e_1 e_1^\top$. Indeed, this is the only case we need when analyzing Equation (1). The following example shows that this is not possible.

**Example 6.** Let $n = 2$ and $\mathcal{M} = \{ \text{Sym}(e_1 e_2^\top), -\text{Sym}(e_1 e_2^\top) \}$ so that

$$\mathcal{S}(\mathcal{M}) = \left\{ \begin{pmatrix} x_1^2 & 0 \\ 0 & x_2^2 \end{pmatrix} : x \in \mathbb{R}^2 \right\} = \text{conv} \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \end{pmatrix} : x_1 \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} : x_2 \in \mathbb{R} \right\}. $$

The representation on the right shows that $\mathcal{S}(\mathcal{M})$ is ROG. On the other hand, taking $B = e_1 e_1^\top$ and $M_0 = -e_2 e_2^\top$, we have

$$\inf_{x \in \mathbb{R}^2} \left\{ x^\top M_0 x : \begin{array}{l} xx^\top \in \mathcal{S}(\mathcal{M}) \\ x^\top B x = 1 \end{array} \right\} = \inf_{x \in \mathbb{R}^2} \left\{ -x_2^2 : \begin{array}{l} x_1 x_2 = 0 \\ x_1^2 = 1 \end{array} \right\} = 0,$$
which is not equal to
\[
\inf_{X \in \mathbb{S}^n} \left\{ \langle M_0, X \rangle : X, Y \in \mathcal{S} \right\} = \inf_{x \in \mathbb{R}^2} \{ -x_2^2 : x_1^2 = 1 \} = -\infty. \]

In a sense, Example 6 exhibits a particular worst-case behavior. Specifically, adding an arbitrary inhomogeneous constraint to a ROG cone produces a set that is rank-two generated.

**Lemma 18.** Let \( \mathcal{M} \subseteq \mathbb{S}^n \). If \( \mathcal{S}(\mathcal{M}) \) is ROG, then for all \( B \in \mathbb{S}^n \),
\[
\text{conv} \left( \left\{ X \in \mathbb{S}^n : \langle M, X \rangle \geq 0, \forall M \in \mathcal{M} \right\} \right) = \left\{ X \in \mathbb{S}^n : \langle B, X \rangle = 1, X \geq 0, \text{rank}(X) \leq 2 \right\}.
\]

In particular, when \( \mathcal{S}(\mathcal{M}) \) is ROG, for any \( M_0 \in \mathbb{S}^n \), there exists a sequence of rank-two solutions approaching the SDP optimum value in (1).

**Proof.** Let \( \mathcal{L} \) denote the inner set on the left hand side so that the left hand side is \( \text{conv}(\mathcal{L}) \) and let \( \mathcal{R} \) denote the right hand set.

(\( \subseteq \)) This follows upon noting that \( \mathcal{L} \subseteq \mathcal{R} \) and \( \mathcal{R} \) is convex.

(\( \supseteq \)) Let \( X \in \mathcal{R} \). As \( \mathcal{R} \subseteq \mathcal{S}(\mathcal{M}) \), we may decompose \( X = \sum_{i=1}^{r} x_i x_i^\top \) where \( x_i x_i^\top \in \mathcal{S}(\mathcal{M}) \) for all \( i \in [r] \). We may assume that \( r = \text{rank}(X) \) by Lemma 14. Let \( \beta_i := \langle B, x_i x_i^\top \rangle \).

If \( \beta_i > 0 \) for all \( i \in [r] \), then we are done. Else, without loss of generality \( \beta_1 > 0 \geq \beta_2 \). Consider the value of \( \mu := \alpha_1 \beta_1 + \alpha_2 \beta_2 \) as \((\alpha_1, \alpha_2)\) moves continuously on the line segments \((1, 0) \to (1, 1) \to (0, 1)\). Noting that \( \beta_1 > 0 \) and \( \beta_2 \leq 0 \), we may fix \((\alpha_1, \alpha_2)\) on this path such that \( \mu \in (0, 1) \). Then, we can decompose
\[
X = \mu \left( \frac{\alpha_1 x_1 x_1^\top + \alpha_2 x_2 x_2^\top}{\mu} \right) + (1 - \mu) \left( \frac{X - \alpha_1 x_1 x_1^\top - \alpha_2 x_2 x_2^\top}{1 - \mu} \right) =: \mu X_\ell + (1 - \mu) X_r.
\]

We have written \( X \) as a convex combination of two matrices \( X_\ell \) and \( X_r \). It can be verified easily that \( X_\ell \in \mathcal{L} \) and \( X_r \in \mathcal{R} \). As at least one of \( \alpha_1 \) or \( \alpha_2 \) takes the value 1, the element \( X_r \) has rank strictly less than \( r \). Iterating this procedure completes the proof.

**Remark 11.** A result similar to Lemma 18 in the case of a single homogeneous constraint is presented in [7, Lemma 5]. Specifically, it is shown that for an arbitrary closed convex cone \( \mathcal{S} \), the extreme rays of \( \mathcal{S} \) intersected with a hyperplane through the origin can be written as convex combinations of at most two extreme rays of \( \mathcal{S} \).

### 5.2 Convex hulls of bounded quadratically constrained sets

Consider a set
\[
\mathcal{Y} := \{ y \in \mathbb{R}^{n-1} : q_i(y) \geq 0, \forall i \in [m] \}
\]
where \( q_i \)s are quadratic functions of the form \( q_i(y) = y^\top A_i y + 2 b_i^\top y + c_i \). Let \( M_i := \begin{pmatrix} c_i & b_i^\top \\ b_i & A_i \end{pmatrix} \) and \( \mathcal{M} := \{ M_1, \ldots, M_m \} \).

The following lemma gives an explicit description of \( \text{conv}(\mathcal{Y}) \) under the assumption that \( \mathcal{S}(\mathcal{M}) \) is ROG and \( \mathcal{Y} \) satisfies a particular boundedness condition.

**Proposition 5.** Suppose there exists \( \lambda^* \in \mathbb{R}^m_+ \) such that \( \sum_{i=1}^{m} \lambda_i^* A_i \) is negative definite. If \( \mathcal{S}(\mathcal{M}) \) is ROG, then \( \text{conv}(\mathcal{Y}) \) is a semidefinite-representable set given by
\[
\text{conv}(\mathcal{Y}) = \left\{ y \in \mathbb{R}^{n-1} : \exists Y \succeq y y^\top : \langle A_i, Y \rangle + 2 \langle b_i, y \rangle + c_i \geq 0, \forall i \in [m] \right\}.
\]
Proof. Let $\mu \in \mathbb{R}$ to be fixed later. Then,
\[
\left\{ \begin{array}{l}
X \in \mathbb{S}^n : \\
(M_i, X) \geq 0, \forall i \in [m] \\
\langle e_1 e_1^T, X \rangle = 1 \\
X \succeq 0
\end{array} \right\} \subseteq \left\{ X \in \mathbb{S}^n : \\
\langle \mu e_1 e_1^T - \sum_{i=1}^m \lambda_i^* M_i, X \rangle \leq \mu, \forall i \in [m] \right\}.
\]
Note that
\[
\mu e_1 e_1^T - \sum_{i=1}^m \lambda_i^* M_i = \left( \mu - \frac{\sum_{i=1}^m \lambda_i^* c_i}{-\sum_{i=1}^m \lambda_i^* b_i} \right) - \sum_{i=1}^m \frac{\lambda_i^* b_i^T}{-\sum_{i=1}^m \lambda_i^* A_i}.
\]
The bottom right block of this matrix is positive definite by the assumption on $\lambda^*$. Thus, by picking $\mu$ large enough, we may ensure that this matrix is positive definite. Then, we deduce that the set on the right (and consequently the set on the left) of (5) is bounded.

Since $\mathcal{S}(\mathcal{M})$ is ROG, Lemma 17 then implies that for all $b_0 \in \mathbb{R}^{n-1}$ we have
\[
\inf_{y \in \mathbb{R}^{n-1}} \langle b_0, y \rangle = \inf_{X \in \mathbb{S}^n} \left\{ \langle \begin{array}{l}
0 \\
\begin{bmatrix} b_0 & 0 \\
0 & 0_{n-1}
\end{bmatrix}
\end{array}, X \rangle : \\
\langle M_i, X \rangle \geq 0, \forall i \in [m] \\
\langle e_1 e_1^T, X \rangle = 1 \\
X \succeq 0
\right\} = \inf_{y \in \mathbb{R}^{n-1}} \left\{ \langle b_0, y \rangle : \exists Y \succeq y y^T : \\
\langle A_i, Y \rangle + 2 \langle b_i, y \rangle + c_i \geq 0, \forall i \in [m] \right\}.
\]
Here both equalities follow by writing the variable $X$ on the right of the first line as $X = \begin{pmatrix} 1 & v^T \\ v & Y \end{pmatrix}$.

The proof then follows by noting that all sets in question are compact. \(\blacksquare\)

We next turn our attention to the closed convex hull of epigraph sets. Let $q_0$ be a quadratic function of the form $q_0(y) = y^T A_0 y + 2b_0^T y + c_0$ and define $M_0 = \begin{pmatrix} c_0 & b_0 \\ b_0 & A_0 \end{pmatrix}$.

**Proposition 6.** Suppose there exists $\lambda^* \in \mathbb{R}^m_+$ such that $A_0 - \sum_{i=1}^m \lambda_i^* A_i$ is positive definite. If $\mathcal{S}(\mathcal{M})$ is ROG, then the closed convex hull of
\[
\text{epi} := \left\{ (y, t) : y \in \mathbb{R}^{n-1} \times \mathbb{R} : \\
q_0(y) \leq t \right\}
\]
is a semidefinite-representable set given by
\[
\text{conv(epi)} = \left\{ (y, t) : y \in \mathbb{R}^{n-1} \times \mathbb{R} : \\
\exists Y \succeq y y^T : \\
\langle A_0, Y \rangle + 2 \langle b_0, y \rangle + c_0 \leq t \\
\langle A_i, Y \rangle + 2 \langle b_i, y \rangle + c_i \geq 0, \forall i \in [m] \right\}.
\]

*Proof.* Let $\mathcal{R}$ denote the set on the right. 

(\subseteq) By taking $Y = y y^T$, we have that epi $\subseteq \mathcal{R}$. It suffices to show that $\mathcal{R}$ is both convex and closed. As $\mathcal{R}$ is the projection of the SDP relaxation (a convex set) of epi, it is itself convex. Next consider a sequence $(y^{(i)}, t^{(i)}) \in \mathcal{R}$ converging to $(y, t)$. Let $Y^{(i)}$ denote a sequence of matrices certifying $(y^{(i)}, t^{(i)}) \in \mathcal{R}$. As there exists a $\lambda^* \in \mathbb{R}^m_+$ such that $A_0 - \sum_{i=1}^m \lambda_i^* A_i$ is positive definite, the sequence $Y^{(i)}$ is bounded and hence has a convergent subsequence with limit $Y$. By continuity, we deduce that $(y, t) \in \mathcal{R}$ and hence $\mathcal{R}$ is closed.

(\supseteq) Suppose $(y, t) \notin \text{conv(epi)}$. We will show that $(y, t) \notin \mathcal{R}$. By the strict hyperplane separation theorem, there exists $(\mu, \nu) \neq (0, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that
\[
\langle \mu, y \rangle + \nu t < \inf_{(y', t') \in \text{conv(epi)}} \langle \mu, y' \rangle + \nu t' = \inf_{(y', t') \in \text{epi}} \langle \mu, y' \rangle + \nu t'.
\]
We claim that we may assume $\nu > 0$ without loss of generality. First suppose $\mathcal{Y} = \emptyset$. In this case, epi $= \emptyset$ and any arbitrary $(\mu, \nu) \neq (0, 0)$ satisfies (6). On the other hand, if $\mathcal{Y}$ is nonempty then $e_n$ is a recessive
direction for epi. In particular, as the objective value of the program on the right is finite (by the bound on the left), we deduce that \( \nu \geq 0 \). Finally, as \( q_0(y) \) is bounded on \( \mathcal{Y} \) by compactness, we may increase \( \nu \) by some positive amount without affecting (6).

Then,
\[
\langle \mu, y \rangle + \nu t < \min_{y'} \left\{ \langle \mu, y' \rangle + \nu q_0(y') : y' \in \mathcal{Y} \right\}
\]
\[
= \min_{y'} \left\{ \langle \mu, y' \rangle + \nu \langle A_0, Y' \rangle + 2 \langle b_0, y' \rangle + c_0 : Y' \succeq y' y'^\top + A_i, Y' \rangle + 2 \langle b_i, y' \rangle + c_i \geq 0, \forall i \in [m] \right\}
\]
\[
\leq \min_{y'} \left\{ \langle \mu, y' \rangle + \nu \langle A_0, Y \rangle + 2 \langle b_0, y \rangle + c_0 : Y \succeq y y^\top + A_i, Y \rangle + 2 \langle b_i, y \rangle + c_i \geq 0, \forall i \in [m] \right\}.
\]

Here, the first line follows by substituting the optimal value of \( t' \) in (6), the second line follows from Lemma 17 (which we can apply as \( S(\mathcal{M}) \) is ROG and the SDP on the second line has finite objective value), and the third line follows by selecting \( y' = y \).

Subtracting \( \langle \mu, y \rangle \) from both sides and dividing by \( \nu > 0 \) leads to the desired conclusion that \( (y, t) \notin \mathcal{R} \) and completes the proof. \( \blacksquare \)

We next examine a classical example related to the “perspective reformulation/relaxation” trick \([9, 14, 16]\)
and demonstrate how this convex hull result can be recovered using our ROG toolsets. The nonconvex set in this example will involve both binary and continuous variables and complementarity constraints.

**Example 7.** Consider the quadratically constrained set
\[
\mathcal{Y} = \left\{ y \in \mathbb{R}^2 : \begin{array}{l}
(1 - y_1)y_1 = 0 \\
(1 - y_1)y_2 = 0
\end{array} \right\}.
\]

In words, \( y_1 \) is constrained to be a binary variable, \( y_2 \) is allowed to be arbitrary when \( y_1 = 1 \) is “on” and forced to be zero when \( y_1 = 0 \) is “off.” Letting \( M_1 := \text{Sym}((e_3 - e_1)e_1^\top) \) and \( M_2 := \text{Sym}((e_3 - e_1)e_2^\top) \), we have that
\[
\mathcal{Y} = \left\{ y \in \mathbb{R}^2 : \begin{array}{l}
\left( \begin{array}{c} y \\ 1 \end{array} \right)^\top M_1 \left( \begin{array}{c} y \\ 1 \end{array} \right) = 0 \\
\left( \begin{array}{c} y \\ 1 \end{array} \right)^\top M_2 \left( \begin{array}{c} y \\ 1 \end{array} \right) = 0
\end{array} \right\}.
\]

Let \( \mathcal{M} = \{M_1, M_2\} \). Note that \( T(\mathcal{M}) \) is ROG by Corollary 5. Let \( q_0(y) = y_2^2 \). It is easy to check that the assumptions of Proposition 6 hold. Then,
\[
\text{conv} \left\{ (y, t) \in \mathbb{R}^2 \times \mathbb{R} : \begin{array}{l}
y_2^2 \leq t \\
(1 - y_1)y_1 = 0 \\
(1 - y_1)y_2 = 0
\end{array} \right\} = \left\{ (y, t) \in \mathbb{R}^2 \times \mathbb{R} : \begin{array}{l}
y_1 \geq y_2^2 \\
y_1 - Y_{1,1} = 0 \\
y_2 - Y_{1,2} = 0
\end{array} \right\}
\]
\[
= \left\{ (y, t) \in \mathbb{R}^2 \times \mathbb{R} : \begin{array}{l}
y_1 \geq y_2^2 \\
t \geq y_2^2 \\
(y_1 - y_1^2)(t - y_2^2) \geq (y_2 - y_1y_2)^2
\end{array} \right\}.
\]

Note that the first constraint in the last formulation implies that \( y_1 \in [0, 1] \). By expanding and rearranging, we can write the last constraint as
\[
0 \leq (y_1 - y_1^2)(t - y_2^2) - (y_2 - y_1y_2)^2 = y_1t + y_1y_2^2 - y_1^2t - y_2^2 = (y_1 - y_2^2)(1 - y_1).
\]

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When \( y_1 \in [0, 1) \), this constraint is equivalent to \( y_1 t - y_2^2 \geq 0 \). On the other hand when \( y_1 = 1 \), the constraint \( y_1 t - y_2^2 \geq 0 \) is redundant. Hence, we deduce that

\[
\text{conv} \left\{ (y, t) \in \mathbb{R}^2 \times \mathbb{R} : \begin{array}{l}
y_2^2 \leq t \\
(1 - y_1)y_1 = 0 \\
(1 - y_1)y_2 = 0
\end{array} \right\} = \left\{ (y, t) \in \mathbb{R}^2 \times \mathbb{R} : \begin{array}{l}
y_1 \in [0, 1] \\
y_1 t \geq y_2^2
\end{array} \right\}.
\]

This gives the well-known perspective formulation of \( \text{conv}(\mathcal{Y}) \).

Remark 12. There are few known sufficient conditions guaranteeing that the convex hull of the epigraph of a QCQP is given by its SDP relaxation. The conditions presented by Wang and Kilınç-Karzan [25, Theorems 1 and 7] are among the most general in this direction. We claim that both [25, Theorems 1 and 7] are incomparable with Proposition 6. Note that [25, Theorem 1] cannot be applied directly to Example 7: the set of convex Lagrange multipliers (see [25, Section 2.1]) for this example is

\[
\Gamma := \{ \gamma \in \mathbb{R}^2 : \begin{array}{l}
0 \\
1 + \gamma_1 \begin{pmatrix} -1 \\
0 \end{pmatrix} + \gamma_2 \begin{pmatrix} 0 & -1/2 \\
-1/2 & 0 \end{pmatrix} \succeq 0
\end{array} \}
\]

which is not polyhedral. On the other hand, [25, Theorem 1] can be applied to QCQPs where the \( A_i \)s satisfy a “symmetry” condition. The following QCQP is such an example. Consider

\[
\inf_{y \in \mathbb{R}^4} \left\{ \|y\|^2 : \begin{array}{l}
y^\top \begin{pmatrix} 1 & 0 \\
0 & -1 \\
2 & 0 \end{pmatrix} y + 1 \geq 0 \\
y^\top \begin{pmatrix} 1 & -1 \\
-1 & -1 \\
2 & 1 \end{pmatrix} y + 1 \geq 0
\end{array} \right\}.
\]

The corresponding set \( \mathcal{M} \) for this example is \( \mathcal{M} = \{ \text{Diag}(1, 1, -1, -1, 1), \text{Diag}(-2, -2, 1, 1, 1) \} \). Theorem 3 implies that \( S(\mathcal{M}) \) is not ROG and thus Proposition 6 cannot be applied to this example. We conclude that [25, Theorem 1] and Proposition 6 are incomparable. Similar examples can be constructed to show that [25, Theorem 7] and Proposition 6 are incomparable.

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References


A Proof of Theorem 5

For completeness we restate Theorem 5.

Theorem 5 (Dines [12]). Let $M_1, M_2 \in S^n$ and suppose that for all $(\alpha_1, \alpha_2) \neq (0, 0)$, we have $\alpha_1 M_1 + \alpha_2 M_2 \notin S^n_+$. Then

\[
\left\{ \begin{pmatrix} x^T M_1 x \\
 x^T M_2 x \end{pmatrix} \in \mathbb{R}^2 : x \in \mathbb{R}^n \right\} = \mathbb{R}^2,
\]

i.e., for every $y \in \mathbb{R}^2$, there exists an $x \in \mathbb{R}^n$ such that $x^T M_1 x = y_1$ and $x^T M_2 x = y_2$.

Let $Q : \mathbb{R}^n \to \mathbb{R}^2$ denote the mapping $x \mapsto (x^T M_1 x, x^T M_2 x)^T$. We need to show that the assumption of Theorem 5 implies that $Q(\mathbb{R}^n) = \mathbb{R}^2$. By homogeneity, it is clear that $Q(\mathbb{R}^n)$ is conic. We next show that $Q(\mathbb{R}^n)$ is also convex.

Lemma 19. For any $M_1$ and $M_2$, the image $Q(\mathbb{R}^n)$ is convex.

Proof. Consider $Q(x), Q(y) \in \mathbb{R}^2$ corresponding to points $x, y \in \mathbb{R}^n$. We need to show that the interval $[Q(x), Q(y)]$ is contained in $Q(\mathbb{R}^n)$. If $Q(x), Q(y)$ are collinear then this interval is clearly contained in $Q(\mathbb{R}^n)$ as $Q(\mathbb{R}^n)$ is conic. Else there exists a vector $v \in \mathbb{R}^2$ such that $\langle v, Q(x) \rangle$ and $\langle v, Q(y) \rangle$ are both strictly positive. By negating $y$ if necessary (this does not change the image $Q(y)$), we may assume that

\[
\left\langle v, \begin{pmatrix} x^T M_1 y \\
 x^T M_2 y \end{pmatrix} \right\rangle > 0.
\]

Then, let $z(\mu) = (1 - \mu)x + \mu y$ for $\mu \in [0, 1]$. We compute

\[
\langle v, z(\mu) \rangle = \mu^2 \langle v, Q(x) \rangle + (1 - \mu)^2 \langle v, Q(y) \rangle + 2\mu(1 - \mu) \left\langle v, \begin{pmatrix} x^T M_1 y \\
 x^T M_2 y \end{pmatrix} \right\rangle \geq \mu^2 \langle v, Q(x) \rangle + (1 - \mu)^2 \langle v, Q(y) \rangle > 0.
\]

Thus there exists a continuous path $Q(z(\mu)) \subseteq Q(\mathbb{R}^n)$ parameterized by $\mu \in [0, 1]$ that connects $Q(x)$ to $Q(y)$. Furthermore, $\langle v, Q(z(\mu)) \rangle$ is strictly positive for all $\mu \in [0, 1]$. As $Q(\mathbb{R}^n)$ is conic, we conclude that the interval $[Q(x), Q(y)]$ is contained in $Q(\mathbb{R}^n)$. \hfill \blacksquare

Proof of Theorem 5. Suppose that $Q(\mathbb{R}^n) \neq \mathbb{R}^2$. By Lemma 19, $Q(\mathbb{R}^n)$ is a convex cone thus is contained in some half space $\{ q \in \mathbb{R}^2 : \alpha_1 q_1 + \alpha_2 q_2 \geq 0 \}$ where $(\alpha_1, \alpha_2) \neq (0, 0)$. But then for every $x \in \mathbb{R}^n$,

\[
0 \leq \alpha_1 x^T M_1 x + \alpha_2 x^T M_2 x = x^T (\alpha_1 M_1 + \alpha_2 M_2) x.
\]

We conclude that $\alpha_1 M_1 + \alpha_2 M_2 \notin S^n_+$, a contradiction. \hfill \blacksquare

B Proof of Lemma 15

For completeness we restate Lemma 15.

Lemma 15. Let $\mathcal{M} = \{M_1, M_2\}$. Suppose Assumption 1 holds and $n = 3$. If neither conditions (i) nor (ii) of Theorem 4 hold, then $\mathcal{N}(\mathcal{M})$ is the union of at most four one-dimensional subspaces of $\mathbb{R}^3$.

Proof. As $M_1, M_2 \notin S^3_+$ they must each have rank either two or three. We will break the proof into two cases.

Suppose first that $\text{rank}(M_1) = \text{rank}(M_2) = 2$. As $M_1, M_2 \notin S^3_+$, each $M_i$ has exactly one positive and one negative eigenvalue. We can then write $M_1 = \text{Sym}(ab^T)$ and $M_2 = \text{Sym}(cd^T)$. Then

\[
\mathcal{N}(\mathcal{M}) = \{ x : x^T(ab^T)x = x^T(cd^T)x = 0 \}
= (a^+ \cup b^+) \cap (c^+ \cup d^+)
= (a^+ \cap c^+) \cup (a^+ \cap d^+) \cup (b^+ \cap c^+) \cup (b^+ \cap d^+).
\]

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As condition (ii) does not hold, each of the four spaces on the final line have dimension one. Thus \( \mathcal{N}(\mathcal{M}) \) is the union of at most four distinct lines.

Next suppose without loss of generality that \( \text{rank}(M_1) = 3 \). As \( M_1 \notin \mathbb{S}^3_+ \), we may assume that it has two positive eigenvalues and one negative eigenvalue. Performing a change of basis, it suffices to consider when

\[
M_1 = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\quad \text{and} \quad
M_2 = \begin{pmatrix}
a & b & c \\
b & d & e \\
c & e & f
\end{pmatrix}.
\]

We will consider the intersection \( \mathcal{N}(\mathcal{M}) \setminus \{ x \in \mathbb{R}^3 : x_3 = 1 \} \). Note that if \( x \in \mathcal{N}(\mathcal{M}) \) has \( x_3 \) coordinate equal to zero, then \( x = 0 \). Thus, the number of distinct lines in \( \mathcal{N}(\mathcal{M}) \) is equal to the number of distinct points in

\[
\mathcal{P} := \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{array}{l}
x_1^2 + x_2^2 - 1 = 0 \\
(ax_1 + dx_2 + 2cx_1 + f) + x_2 (2bx_1 + 2e) = 0
\end{array} \right\}.
\]

Suppose that \( \mathcal{N}(\mathcal{M}) \) contains at least five lines so that \( \mathcal{P} \) contains at least five points. Without loss of generality, we may assume that the \( x_1 \) coordinates of these five points are distinct (else, perform an orthonormal change of basis on the first two dimensions). Let the \( x_1 \) coordinates of these five points be \( \xi_1, \xi_2, \ldots, \xi_5 \). For each \( \xi_i \), by the first constraint in the definition of \( \mathcal{P} \), we have that the corresponding \( x_2 \) coordinate must be either \( \sqrt{1 - \xi_i^2} \) or \( -\sqrt{1 - \xi_i^2} \). Hence,

\[
\left[(a\xi^2 + d(1 - \xi^2) + 2c\xi + f) + \sqrt{1 - \xi^2} (2b\xi + 2e)\right] \left[(a\xi^2 + d(1 - \xi^2) + 2c\xi + f) - \sqrt{1 - \xi^2} (2b\xi + 2e)\right]
= \left[(a - d)^2 + 4b^2\right] \xi^4 + [4(a - d)c + 8be] \xi^3 + \left[2(a - d)(d + f) + 4c^2 + 4e^2 - 4b^2\right] \xi^2 + [4c(d + f) - 8be] \xi + [(d + f)^2 - 4e^2]
\]

is a degree-4 polynomial in \( \xi \) which is zero on five distinct points \( \xi_1, \ldots, \xi_5 \). We conclude that this polynomial is identically zero. The coefficient of \( \xi^4 \) implies that \( a = d \) and \( b = 0 \). The coefficient of \( \xi^2 \) implies that \( c = e = 0 \). The constant term implies that \( f = -d \). We conclude that \( M_2 \) has the form

\[
M_2 = \begin{pmatrix}
a & a \\
-a & -a
\end{pmatrix}.
\]

This contradicts the assumption that there does not exist an \( (\alpha_1, \alpha_2) \neq (0, 0) \) such that \( \alpha_1 M_1 + \alpha_2 M_2 \in \mathbb{S}^n_+ \). □