An Exact Method for Bisubmodular Function Maximization

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Abstract

Bisubmodularity—a natural generalization of submodularity—applies to set functions with two arguments and appears in a broad range of applications, including coupled sensor placement in infrastructure, coupled feature selection in machine learning, and drug-drug interaction detection in healthcare. In this paper, we study maximization problems with bisubmodular objective functions. We propose valid linear inequalities, namely the bisubmodular inequalities, for the hypograph of any bisubmodular function. We show that maximizing a bisubmodular function is equivalent to solving a mixed-integer linear program with exponentially many bisubmodular inequalities. Using this representation in a delayed constraint generation framework, we design the first exact algorithm to solve general bisubmodular maximization problems. Our computational experiments on the coupled sensor placement problem demonstrate the efficacy of our algorithm in constrained nonlinear bisubmodular maximization problems for which no existing exact methods are available.

Keywords – bisubmodular maximization; cutting plane; coupled sensor placement

1 Introduction

Submodularity is an important concept in integer and combinatorial optimization. Several functions of great theoretical interest in combinatorial optimization are submodular, such as the set covering function and the graph cut function. Submodularity also arises in numerous practical applications, including the influence maximization problem (Kempe et al., 2015), the sensor placement problem (Krause et al., 2008), and the hub location problem (Contreras and Fernández, 2014). The unconstrained submodular minimization problem is polynomially solvable (Iwata et al., 2001; Lee et al., 2015; Orlin, 2009; Schrijver, 2000). However, submodular minimization with simple constraints, such as cardinality constraints, is NP-hard (Svitkina and Fleischer, 2011). Submodular maximization is known to be NP-hard even in the unconstrained case. Nemhauser et al. (1978) prove that the greedy method for maximizing a monotone submodular function subject to a cardinality constraint is a \((1 - 1/e)\)-approximation algorithm. Nemhauser and Wolsey (1988) also give an exact method for submodular maximization from a polyhedral perspective. They show that maximizing any
submodular function is equivalent to solving a mixed-integer linear program with exponentially many linear inequalities, referred to as the submodular inequalities. This approach has received renewed interest, both in terms of strengthening the submodular inequalities and extending their use to stochastic settings (Ahmed and Atamtürk, 2011; Wu and Küçükyavuz, 2018, 2019, 2020; Yu and Ahmed, 2017).

Bisubmodularity generalizes submodularity to functions with two arguments. This concept was first considered by Chandrasekaran and Kabadi (1988) and Qi (1988). Researchers have developed multiple methods to minimize bisubmodular functions. Qi (1988) proves an analogue of Lovász extension for bisubmodular functions. As a consequence of this result, unconstrained bisubmodular minimization can be solved in polynomial time using the ellipsoid method. Subsequently, Fujishige and Iwata (2005) propose a weakly polynomial-time algorithm for unconstrained bisubmodular minimization. McCormick and Fujishige (2010) improve this result and provide a strongly polynomial-time algorithm for unconstrained bisubmodular minimization. In contrast to these methods, Yu and Küçükyavuz (2020) apply a polyhedral approach and give a complete linear description of the convex hull of the epigraph of any bisubmodular function. In the same work, the authors propose a cutting plane algorithm that solves constrained bisubmodular minimization problems efficiently.

Bisubmodular maximization is more challenging than bisubmodular minimization. The bisubmodular maximization problem—a generalization of the NP-hard submodular maximization problem—is also NP-hard. Researchers have proposed various approximation algorithms to tackle bisubmodular maximization problems. Iwata et al. (2013) present a deterministic and a randomized greedy approximation algorithm. The deterministic algorithm achieves at least 1/3 of the optimal value, and the randomized algorithm achieves at least 1/2 of the optimal value in expectation. Singh et al. (2012) provide a constant-factor approximation algorithm for a class of bisubmodular functions that is non-negative, monotone, and a particular extension of another special class of bisubmodular functions. Ward and Živný (2016) also restrict their focus to bisubmodular functions under monotonicity assumptions, and their approximation algorithm is shown to achieve 1/2 approximation guarantee for such bisubmodular functions. Despite the strong interest and promising results in approximating the optimal solutions for bisubmodular maximization, there has been a paucity of research on the exact approaches. In this paper, we address this gap in our knowledge.

Before we outline our results, we provide a few examples from a wide range of applications of bisubmodular maximization.

1.1 Bisubmodular Maximization Applications

1.1.1 Coupled sensor placement

Sensor networks—enabled by internet of things (IoT) technology—provide real-time monitoring and control of systems to operate smart cities (Zanella et al., 2014), smart homes (Ghayvat et al., 2015), and smart grids (Abujubbeh et al., 2019). These applications often call for multiple types of sensors in the network. For example, in smart water distribution networks, multiple types of sensors are placed to measure different aspects of water quality in real time (Public Utilities Board Singapore, 2016).
Consider a smart building network with two types of sensors for different measurements, such as temperature and humidity, where we can place the sensors at a subset of a given set $N$ of $n$ potential locations. At most one sensor is allowed in each location. Every biset $(S_1, S_2) \in 3^N$ corresponds to a coupled sensor placement plan, in which the type-1 sensors are placed at the locations in $S_1$ and the type-2 sensors are placed at $S_2$. Due to a limited budget, we can place at most $B_1$ type-1 and $B_2$ type-2 sensors. Each sensor deployment plan is evaluated using entropy, which measures how much uncertainty in the environment the sensors can capture (Ohsaka and Yoshida, 2015). The entropy of a discrete random variable $X$ with support $\mathcal{X}$ is computed by

$$H(X) = -\sum_{x \in \mathcal{X}} P(X = x) \log P(X = x).$$

The entropy of $X$ is high if multiple outcomes occur with similar probabilities, making it difficult for us to predict what we may observe. For instance, it is harder for us to guess the outcome of throwing a fair dice correctly than that of a biased dice, so the entropy in the case of a fair dice is higher than the latter. In the context of coupled sensor placement, a discrete random variable $X_{S_1, S_2}$ captures the possible observations reported by sensors installed at $(S_1, S_2) \in 3^N$, and the set $\mathcal{X}_{S_1, S_2}$ contains all possible observations. The entropy of $X_{S_1, S_2}$ is

$$H(X_{S_1, S_2}) = -\sum_{x \in \mathcal{X}_{S_1, S_2}} P(X_{S_1, S_2} = x) \log P(X_{S_1, S_2} = x).$$

We refer the readers to Yu and Küçükşeyhavuz (2020) for a small numerical example. In an ideal coupled sensor placement plan, sensors are installed at locations where the corresponding observations are the most unpredictable. In other words, a placement $(S_1^*, S_2^*)$ is the best when $H(X_{S_1^*, S_2^*})$ is maximal among all feasible $(S_1, S_2) \in 3^N$. The function $f : 3^N \to \mathbb{R}$, defined by $f(S_1, S_2) = H(X_{S_1, S_2})$ for all $(S_1, S_2) \in 3^N$, is monotone and bisubmodular (Ohsaka and Yoshida, 2015). Thus the coupled sensor placement problem is a cardinality-constrained bisubmodular maximization problem with objective function $f$.

1.1.2 Coupled feature selection

Feature selection plays a key role in multiple fields of research including machine learning (Shalev-Shwartz and Ben-David, 2014), bioinformatics (Saeys et al., 2007), and data mining (Rokach and Maimon, 2008). This process improves analysis of large datasets by reducing the dimensionality of data. Bisubmodularity arises in a special class of feature selection problems, referred to as coupled feature selection in Singh et al. (2012). In this class of problems, there are two uncorrelated prediction variables, and their associated features are mixed in a pool. The task is not only to find the most informative features, but also to classify the features with respect to the prediction variables.

More formally, suppose that a Gaussian graphical model and a set of features $N$ are given. Let $C_1, C_2$ be two variables to be predicted. The goal is to partition the features in $N$ into two sets $S_1$ and $S_2$, such that $S_1$ is used to predict $C_1$, and $S_2$ is used to predict $C_2$. It is assumed that $S_1, S_2$ are mutually conditionally independent given $C = \{C_1, C_2\}$. The total number of selected features, $|S_1| + |S_2|$, is no more than a given number $k$. Next, we describe the score function that evaluates the mutual information obtained by a coupled feature selection. Suppose $X$ and $Y$ are discrete random variables, and $\mathcal{X}, \mathcal{Y}$ are the respective supports.
Then the conditional entropy of $X$ given $Y$ is

$$H(X|Y) = - \sum_{x \in X, y \in Y} \mathbb{P}(X = x, Y = y) \log \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}.$$ 

The biset mutual information is computed by

$$I(S_1, S_2; C) = H(S_1 \cup S_2) - \sum_{i \in S_1} H(i \mid C_1) - \sum_{j \in S_2} H(j \mid C_2),$$

where $H$ is the entropy function discussed in Section 1.1.1 and the conditional entropy is defined above. Intuitively, the features with higher mutual information are more informative about both prediction tasks. Let $f(S_1, S_2) = I(S_1, S_2; C)$. The function $f$ is monotone and bisubmodular (Singh et al., 2012), and the best features can be found by maximizing $f$.

### 1.1.3 Drug-drug interaction detection

Bisubmodularity can also be harnessed in a unique way to detect drug-drug interactions (DDIs) in the healthcare domain. DDIs are the reactions resulting from using multiple drugs concomitantly. DDIs are a major cause of morbidity and mortality (Lu et al., 2015)—adverse drug events cause 770,000 injuries and deaths every year, and as much as 30% of these adverse drug events are due to DDIs (Pirmohamed and Orme, 1998; Tatonetti et al., 2012). Therefore, identifying adverse DDIs is crucial to medication safety and patient care. Hu et al. (2019) gather DDI-related posts from online health forums and cull out all the co-occurring drugs and/or symptoms. The correlations among the combinations of drugs and associated symptoms are captured by a bisubmodular function, and the potential DDIs are determined by maximizing this bisubmodular function.

### 1.2 Our contributions

While there have been many developments in finding approximate solutions to bisubmodular maximization, there is no known exact method. To bridge this gap, we propose the first polyhedral approach to study bisubmodular function maximization. This approach allows us to develop an exact cutting-plane algorithm for constrained bisubmodular maximization problems. Our algorithm does not require a bisubmodular function to satisfy any restrictive assumptions, such as monotonicity. We demonstrate the effectiveness of our algorithm by experimenting on the coupled sensor placement problem, which has a highly nonlinear bisubmodular objective function.

### 1.3 Outline

The outline of this paper is as follows. In Section 2, we provide formal definitions and review known properties of bisubmodularity. In Section 3, we state and prove additional properties of bisubmodular functions that have not been studied in the literature. These properties will be used to establish our results. Next, we propose a class of valid linear inequalities which we call the *bisubmodular inequalities* for the hypograph of any bisubmodular function in Section 4. In particular, we show that maximizing a bisubmodular function is equivalent to solving a mixed-integer program with exponentially many bisubmodular inequalities. In Section 5, we give a cutting plane algorithm to solve the constrained maximization problems with bisubmodular
objective functions. We test the performance of our proposed algorithm on the coupled sensor placement problem in Section 6. Lastly, we conclude with a few remarks in Section 7.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be a non-empty finite set, and let $3^N = \{(S_1, S_2) \mid S_1, S_2 \subseteq N, S_1 \cap S_2 = \emptyset\}$ denote the collection of all pairs of disjoint subsets of $N$.

**Definition 2.1.** A function $f : 3^N \to \mathbb{R}$ is bisubmodular if for any $(X_1, X_2), (Y_1, Y_2) \in 3^N$,

$$f(X_1, X_2) + f(Y_1, Y_2) \geq f(X_1 \cap Y_1, X_2 \cap Y_2) + f((X_1 \cup Y_1) \setminus (X_2 \cup Y_2), (X_2 \cup Y_2) \setminus (X_1 \cup Y_1)).$$

We call $(S, T)$ a partition, or an orthant, of $N$, if $S \cup T = N$ and $S \cap T = \emptyset$.

**Definition 2.2.** A function $f : 3^N \to \mathbb{R}$ is submodular over a partition $(S, T)$ if

$$\hat{f}_{S,T}(X) := f(X \cap S, X \cap T)$$

is submodular over $X \subseteq N$.

**Theorem 2.3.** *(Ando et al., 1996, Ando Conditions)* A function $f : 3^N \to \mathbb{R}$ is bisubmodular if and only if

(A1) the function $f$ is bisubmodular over every partition of $N$, and

(A2) for any $(S_1, S_2) \in 3^N$ and $i \notin S_1 \cup S_2$, $f(S_1 \cup \{i\}, S_2) + f(S_1, S_2 \cup \{i\}) \geq 2f(S_1, S_2)$.

For $i \in N$ and $(X_1, X_2) \in 3^N$, we define $\rho_{1,i}(X_1, X_2) = f(X_1 \cup \{i\}, X_2) - f(X_1, X_2)$, and $\rho_{2,i}(X_1, X_2) = f(X_1, X_2 \cup \{i\}) - f(X_1, X_2)$. Intuitively, $\rho_{j,i}(X_1, X_2)$ represents the marginal contribution of adding $i \in N$ to $X_j$ where $j = 1, 2$. The following corollary is an immediate result from the Ando Condition (A1), and it is observed in McCormick and Fujishige (2010). This corollary captures the diminishing marginal return property of bisubmodular functions over every partition, which is analogous to the same characteristic of submodular functions.

**Corollary 2.4.** If $f$ is a bisubmodular function, then for any $(X_1, X_2)$ and $(Y_1, Y_2) \in 3^N$ satisfying $X_1 \subseteq Y_1 \subseteq N$, $X_2 \subseteq Y_2 \subseteq N$, $\rho_{j,i}(X_1, X_2) \geq \rho_{j,i}(Y_1, Y_2)$ for all $i \in N \setminus (Y_1 \cup Y_2)$ and $j = 1, 2$.

**Definition 2.5.** *(Ando et al., 1996)* A bisubmodular function $f$ over a ground set $N$ is monotone non-decreasing if for any $(X_1, X_2), (Y_1, Y_2) \in 3^N$ such that $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, $f(Y_1, Y_2) \geq f(X_1, X_2)$.

Equivalently, $f$ is monotone non-decreasing if for any $(S_1, S_2) \in 3^N$ and $i \in N \setminus (S_1 \cup S_2)$, $f(S_1 \cup \{i\}, S_2) \geq f(S_1, S_2)$ and $f(S_1, S_2 \cup \{i\}) \geq f(S_1, S_2)$ (Singh et al., 2012).

We call a monotone non-decreasing function simply a monotone function. Without loss of generality, we assume that $f(\emptyset, \emptyset) = 0$. By slightly abusing notation, we let $f(S_1, S_2) = f(x)$, where $x \in \{0, \pm 1\}^N$. More explicitly, $x_i = 1$ if $i \in S_1$, and $x_i = -1$ if $i \in S_2$, 0 otherwise. This is a unique one-to-one mapping between $3^N$ and $\{0, \pm 1\}^N$. The hypograph of $f$ is

$$\mathcal{T}_f = \{(x, \eta) \in \{0, \pm 1\}^N \times \mathbb{R} \mid \eta \leq f(x)\}.$$
In this study, we consider maximization problems with bisubmodular objective functions, namely

\[
\max_{(S_1, S_2) \in \mathcal{S}} f(S_1, S_2),
\]

where \( f \) is bisubmodular and \( \mathcal{S} \subseteq 3^N \) denotes the collection of feasible bisets. When the problem is unconstrained, \( \mathcal{S} \) is \( 3^N \). Let \( \mathcal{K} \) be the set of incidence vectors corresponding to the feasible bisets in \( \mathcal{S} \). Problem (2) can be rewritten as

\[
\max\{\eta \mid (x, \eta) \in \mathcal{T}_f, x \in \mathcal{K}\},
\]

In Section 4, we propose a set of valid linear inequalities for \( \mathcal{T}_f \). By using these inequalities in a cutting plane framework, we propose the first exact method to solve Problem (3) in Section 5. Before we do so, we first give additional properties of bisubmodular functions.

3 New Properties of Bisubmodular Functions

In this section, we establish a few properties of bisubmodular functions that are not previously discussed in the literature, to the best of our knowledge. These results are analogues of the results derived by Nemhauser and Wolsey (1988) for submodular functions. We then apply these properties in Section 4 to derive valid linear inequalities for \( \mathcal{T}_f \).

**Lemma 3.1.** Given a ground set \( N \), a biset function \( f : 3^N \to \mathbb{R} \) is bisubmodular and monotone if and only if

\[
\hat{f}_{S,T}(Y) \leq \hat{f}_{S,T}(X) + \sum_{i \in Y \setminus X} [\hat{f}_{S,T}(X \cup \{i\}) - \hat{f}_{S,T}(X)]
\]

for any \( X, Y \subseteq N \) over any partition \( (S, T) \) of \( N \), where \( \hat{f}_{S,T}(X) \) is defined in (1).

**Proof.** Nemhauser and Wolsey (1988) show that a function \( g \) is submodular and monotone if and only if

\[
g(T) \leq g(S) + \sum_{j \in T \setminus S} [g(S \cup \{j\}) - g(S)]
\]

for any \( S, T \subseteq N \).

Suppose \( f \) is bisubmodular and monotone. We first observe that, given any partition \( (S, T) \), \( \hat{f}_{S,T} \) is submodular by (A1). Next we consider any \( P \subseteq Q \subseteq N \). Let \( P_1 = P \cap S, P_2 = P \cap T, Q_1 = Q \cap S \), and \( Q_2 = Q \cap T \). Since \( P_1 \subseteq Q_1, P_2 \subseteq Q_2 \) and \( f \) is monotone, \( \hat{f}_{S,T}(P) = f(P_1, P_2) \leq f(Q_1, Q_2) = \hat{f}_{S,T}(Q) \). Thus \( \hat{f}_{S,T} \) is also monotone. It follows that

\[
\hat{f}_{S,T}(Y) \leq \hat{f}_{S,T}(X) + \sum_{i \in Y \setminus X} [\hat{f}_{S,T}(X \cup \{i\}) - \hat{f}_{S,T}(X)]
\]

holds for any \( X, Y \subseteq N \) over any partition \( (S, T) \) of \( N \).

Conversely, suppose (4) holds. Then \( \hat{f}_{S,T} \) is submodular and monotone over any partition \( (S, T) \), and (A1) immediately follows. For any \( (S_1, S_2) \in 3^N \) and \( i \notin S_1 \cup S_2 \), \( f(S_1 \cup \{i\}, S_2) = \hat{f}_{(N \setminus S_2), S_2}(S_1 \cup \{i\} \cup S_2) \geq \hat{f}_{(N \setminus S_2), S_2}(S_1 \cup S_2) = f(S_1, S_2) \). Similarly, \( f(S_1, S_2 \cup \{i\}) \geq f(S_1, S_2) \), so \( f \) is monotone. Moreover, \( f(S_1 \cup \{i\}, S_2) + f(S_1, S_2 \cup \{i\}) \geq 2f(S_1, S_2) \), so (A2) holds. We conclude that \( f \) is bisubmodular and monotone. \( \square \)
Given a non-monotone submodular function $g$ defined over a ground set $N$, Nemhauser and Wolsey (1988) show that $g^*(S) = g(S) - \sum_{i \in S} (f(N) - f(N \setminus \{i\}))$ is monotone and submodular. Lemma 3.2 generalizes this result to non-monotone bisubmodular functions.

**Lemma 3.2.** Let $f : 3^N \to \mathbb{R}$ be a bisubmodular function. For every $i \in N$, we let

$$
\xi_i^1 = \min_{L \cup Q = N, L \cap Q = \emptyset, i \in L} [f(L, Q) - f(L \setminus \{i\}, Q)],
$$

and

$$
\xi_i^2 = \min_{L \cup Q = N, L \cap Q = \emptyset, i \in Q} [f(L, Q) - f(L, Q \setminus \{i\})].
$$

The function

$$
f^*(X_1, X_2) = f(X_1, X_2) - \sum_{i \in X_1} \xi_i^1 - \sum_{j \in X_2} \xi_j^2
$$

is bisubmodular and monotone.

**Proof.** By Lemma 3.1, it suffices to show that $\hat{f}_{S,T}$ is submodular and monotone for any partition $(S, T)$. Consider any $X, Y \subseteq N$ and any partition $(S, T)$ of $N$. Let $(X_1, X_2)$ and $(Y_1, Y_2)$ be the corresponding bisets over the given partition. That is,

$$
X_1 = X \cap S, \quad X_2 = X \cap T,
$$

$$
Y_1 = Y \cap S, \quad Y_2 = Y \cap T.
$$

Then

$$
\hat{f}_{S,T}(X) + \sum_{i \in Y \setminus X} [\hat{f}_{S,T}(X \cup \{i\}) - \hat{f}_{S,T}(X)]
$$

is bisubmodular and monotone.

$$
f^*(X_1, X_2) + \sum_{i \in Y_1 \setminus X_1} [f^*(X_1 \cup \{i\}, X_2) - f^*(X_1, X_2)] + \sum_{i \in Y_2 \setminus X_2} [f^*(X_1, X_2 \cup \{i\}) - f^*(X_1, X_2)]
$$

is bisubmodular and monotone.
\[ f(Y_1, Y_2) + \sum_{i \in X_1 \setminus Y_1} [f(S, T) - f(S \setminus \{i\}, T)] + \sum_{j \in X_2 \setminus Y_2} [f(S, T) - f(S, T \setminus \{j\})] \geq f(Y_1, Y_2) - \sum_{i \in X_1 \setminus Y_1} \xi_i^1 - \sum_{j \in X_2 \setminus Y_2} \xi_j^2 \]

\[ \geq f(Y_1, Y_2) + \sum_{i \in X_1 \setminus Y_1} \xi_i^1 + \sum_{j \in X_2 \setminus Y_2} \xi_j^2 - \sum_{i \in X_1 \cup Y_1} \xi_i^1 - \sum_{j \in X_2 \cup Y_2} \xi_j^2 \]

\[ = f(Y_1, Y_2) - \sum_{i \in Y_1} \xi_i^1 - \sum_{j \in Y_2} \xi_j^2 \]

\[ = f^*(Y_1, Y_2) \]

\[ = \hat{f}^*_{S, T}(Y). \]

Equalities (5h)-(5e) rewrite \( f^* \) in terms of \( f \). Inequality (5f) is a consequence of Corollary 2.4 as we show next. Suppose we fix an order of elements in \( Y_1 \setminus X_1 \) and \( Y_2 \setminus X_2 \) to be respectively \((\alpha(1), \alpha(2), \ldots, \alpha(|Y_1 \setminus X_1|)) \) and \((\beta(1), \beta(2), \ldots, \beta(|Y_2 \setminus X_2|)) \). Then

\[ f(X_1 \cup Y_1, X_2 \cup Y_2) = f(X_1, X_2) + \sum_{i=1}^{|Y_1 \setminus X_1|} \rho_{1, \alpha(i)}(X_1 \cup \{\alpha(k)\}_{k=1}^{i-1}, X_2) \]

\[ + \sum_{j=1}^{|Y_2 \setminus X_2|} \rho_{2, \beta(j)}(X_1 \cup Y_1, X_2 \cup \{\beta(k)\}_{k=1}^{j-1}) \]

\[ \leq f(X_1, X_2) + \sum_{i=1}^{|Y_1 \setminus X_1|} \rho_{1, \alpha(i)}(X_1, X_2) + \sum_{j=1}^{|Y_2 \setminus X_2|} \rho_{2, \beta(j)}(X_1, X_2). \]

Similarly, inequality (5g) holds because

\[ \rho_{1, i}(X_1 \cup Y_1 \setminus \{i\}, X_2 \cup Y_2) \geq \rho_{1, i}(S \setminus \{i\}, T) \]

for any \( i \in X_1 \cup Y_1 \), and

\[ \rho_{2, j}(X_1 \cup Y_1, X_2 \cup Y_2 \setminus \{j\}) \geq \rho_{2, j}(S, T \setminus \{j\}) \]

for any \( j \in X_2 \cup Y_2 \). Inequality (5h) follows from the definitions of \( \xi_i^1 \) and \( \xi_j^2 \). Equations (5i) and (5k) follow from the definitions of \( f^* \) and \( \hat{f}^* \), respectively. \[ \square \]

**Lemma 3.3.** Let \( f \) be a monotone bisubmodular function. Given any \((X_1, X_2), (S_1, S_2) \in \mathcal{P}^N\),

\[ f(X_1, X_2) \leq f(S_1, S_2) + \sum_{i \in X_1 \setminus (S_1 \cup S_2)} \rho_{1,i}(S_1, S_2) + \sum_{i \in X_2 \setminus (S_1 \cup S_2)} \rho_{2,i}(S_1, S_2) \]

\[ + \sum_{i \in X_2 \cap S_1} \rho_{2,i}(\emptyset, \emptyset) + \sum_{i \in X_1 \cap S_2} \rho_{1,i}(\emptyset, \emptyset). \]

**Proof.** Let

\[ X_1 = J \cup L_1 \cup P_1, X_2 = K \cup L_2 \cup P_2, \]

and

\[ S_1 = K \cup L_1 \cup Q_1, S_2 = J \cup L_2 \cup Q_2, \]

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where $J,K,L_1,L_2,P_1,P_2,Q_1,Q_2$ are pairwise disjoint subsets of $N$.

\[
f(S_1,S_2) + \sum_{i \in X_1 \setminus (S_1 \cup S_2)} \rho_{1,i}(S_1,S_2) + \sum_{i \in X_2 \setminus (S_1 \cup S_2)} \rho_{2,i}(S_1,S_2) + \sum_{i \in S_1 \cap S_2} \rho_{2,i}(\emptyset,\emptyset) + \sum_{i \in K \setminus S_1} \rho_{1,i}(\emptyset,\emptyset) + \sum_{i \in J \setminus S_2} \rho_{1,i}(\emptyset,\emptyset)
\]

(6a)

\[
f(S_1,S_2) + \sum_{i \in P_1} \rho_{1,i}(S_1,S_2) + \sum_{i \in P_2} \rho_{2,i}(S_1,S_2) + \sum_{i \in K} \rho_{2,i}(\emptyset,\emptyset) + \sum_{i \in J} \rho_{1,i}(\emptyset,\emptyset)
\]

(6b)

\[
f(K \cup L_1 \cup Q_1 \cup P_1, J \cup L_2 \cup Q_2 \cup P_2) + f(J, K)
\]

(6c)

\[
f(L_1 \cup Q_1 \cup P_1, L_2 \cup Q_2 \cup P_2) + f(J, K)
\]

(6d)

\[
f(J \cup L_1 \cup Q_1 \cup P_1, K \cup L_2 \cup Q_2 \cup P_2)
\]

(6e)

\[
f(X_1, X_2)
\]

(6f)

Inequality (6c) follows from Corollary 2.4. Inequalities (6d) and (6f) are due to the monotonicity of $f$, and inequality (6e) holds because of the bisubmodularity of $f$. \hfill \Box

Lemma 3.3 applies to all monotone bisubmodular functions. By using the relationship between any general bisubmodular function $f$ and its monotone counterpart $f^*$ as stated in Lemma 3.2, we obtain the following result.

**Corollary 3.4.** Let $f$ be any bisubmodular function. Given any $(X_1, X_2), (S_1, S_2) \in 2^N$,

\[
f(X_1, X_2) \leq f(S_1, S_2) + \sum_{i \in X_1 \setminus (S_1 \cup S_2)} \rho_{1,i}(S_1,S_2) + \sum_{i \in X_2 \setminus (S_1 \cup S_2)} \rho_{2,i}(S_1,S_2) + \sum_{i \in S_1 \cap S_2} \rho_{2,i}(\emptyset,\emptyset) + \sum_{i \in K \setminus S_1} \rho_{1,i}(\emptyset,\emptyset) + \sum_{i \in J \setminus S_2} \rho_{1,i}(\emptyset,\emptyset) - \sum_{i \in S_1 \setminus X_1} \xi^1_i - \sum_{i \in S_2 \setminus X_2} \xi^2_i.
\]

With these properties of bisubmodular functions, we propose valid linear inequalities for the hypograph of any bisubmodular function in the next section.

### 4 Bisubmodular Inequalities

Let $f$ be any bisubmodular function defined over the ground set $N$. Let us consider the set

\[
\mathcal{T}_f = \{(x,t,y^1,y^2,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \mid \eta \leq f(x), x_i = y^1_i - y^2_i, t_i = y^1_i + y^2_i, y^1_i + y^2_i \leq 1, \forall i \in N\}.
\]

The set $\mathcal{T}_f$ is an extended representation of $\mathcal{T}_f$ with additional vector of variables $t$ that represents the absolute value of $x_i$. And $y^1, y^2$ that are vectors of binary indicator variables for the two set arguments. For any $i \in N$, $y^1_i = 1$ if and only if $i$ is chosen for the first set argument, and similarly $y^2_i = 1$ precisely when $i$ is included in the second set argument. Problem (3) is equivalent to

\[
\max \{\eta \mid (x, t, y^1, y^2, \eta) \in \mathcal{T}_f, x \in K\}.
\]

We propose two classes of valid linear inequalities for $\mathcal{T}_f$, depending on whether the function is monotone. We refer to these inequalities as the *bisubmodular inequalities*. 

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Proposition 4.1. Let $f$ be a monotone bisubmodular function. For a given $(S_1, S_2) \in 3^N$,

$$2\eta \leq 2f(S_1, S_2) + \sum_{i \notin S_1 \cup S_2} \{[\rho_{1,i}(S_1, S_2) - \rho_{2,i}(S_1, S_2)]x_i + [\rho_{1,i}(S_1, S_2) + \rho_{2,i}(S_1, S_2)]t_i\}$$

$$+ \sum_{i \in S_1} \rho_{2,i}(\emptyset, \emptyset)(t_i - x_i) + \sum_{i \in S_2} \rho_{1,i}(\emptyset, \emptyset)(t_i + x_i).$$

is valid for $T_f^t$.

Proof. Consider any $(x, t, y^1, y^2, \eta) \in T_f^t$. Let $(X_1, X_2) \in 3^N$ be the biset that corresponds to $x$. For any $(S_1, S_2) \in 3^N$,

$$2\eta \leq 2f(X_1, X_2) \leq 2f(S_1, S_2) + \sum_{i \in X_1 \setminus (S_1 \cup S_2)} 2\rho_{1,i}(S_1, S_2)$$

$$+ \sum_{i \in X_2 \cap S_1} 2\rho_{2,i}(\emptyset, \emptyset) + \sum_{i \in X_1 \cap S_2} 2\rho_{1,i}(\emptyset, \emptyset)$$

$$= 2f(S_1, S_2) + \sum_{i \in X_1 \cap (S_1 \cup S_2)} [\rho_{1,i}(S_1, S_2) - \rho_{2,i}(S_1, S_2)]$$

$$- \sum_{i \in X_2 \cap (S_1 \cup S_2)} [\rho_{1,i}(S_1, S_2) - \rho_{2,i}(S_1, S_2)]$$

$$+ \sum_{i \in X_1 \cup X_2 \setminus (S_1 \cup S_2)} [\rho_{1,i}(S_1, S_2) + \rho_{2,i}(S_1, S_2)]$$

$$+ \sum_{i \in X_2 \cap S_1} \rho_{2,i}(\emptyset, \emptyset) \cdot [1 - (-1)] + \sum_{i \in X_1 \cap S_2} \rho_{1,i}(\emptyset, \emptyset)(1 + 1)$$

$$= 2f(S_1, S_2) + \sum_{i \in S_1} \{[\rho_{1,i}(S_1, S_2) - \rho_{2,i}(S_1, S_2)]x_i + [\rho_{1,i}(S_1, S_2) + \rho_{2,i}(S_1, S_2)]t_i\}$$

$$+ \sum_{i \in S_2} \rho_{2,i}(\emptyset, \emptyset)(t_i - x_i) + \sum_{i \in S_2} \rho_{1,i}(\emptyset, \emptyset)(t_i + x_i).$$

Note that for any $i \in X_1$, $x_i = t_i = 1$. For any $i \in X_2$, $x_i = -1$ and $t_i = 1$. If $i \notin X_1 \cup X_2$, then $x_i = t_i = 0$. The sum $t_i + x_i = 2$ if and only if $i \in X_1$. Similarly, $t_i - x_i = 2$ if and only if $i \in X_2$. Inequality (8a) follows from the definition of $T_f^t$. Inequality (8b) is a consequence of Lemma 3.3, and equalities (8c)-(8d) rewrite (8b) in the form of the right-hand side of inequality (7).

Proof. Consider any $(x, t, y^1, y^2, \eta) \in T_f^t$. Let $(X_1, X_2) \in 3^N$ be the biset that corresponds to $x$. For any $i \in X_1$, $x_i = t_i = 1$, while for any $i \in X_2$, $x_i = -1$ and $t_i = 1$. If $i \notin X_1 \cup X_2$, then $x_i = t_i = 0$. For any
Inequality (10a) follows from the definition of \( t \) which is the right-hand side of inequality (9). (9) can be written equivalently with variables \( x, y, \eta \). We call both inequalities (7) and (9) bisubmodular inequalities associated with \((S_1, S_2)\). Inequalities (7) and (9) can be written equivalently with variables \( y^1, y^2 \) instead of \( x, t \). For any bisubmodular \( f \), (9) is equivalent to
\[
\eta \leq f(S_1, S_2) + \sum_{i \in S_1 \cup S_2} [\rho_{1,i}(S_1, S_2)y_i^1 + \rho_{2,i}(S_1, S_2)y_i^2] \\
+ \sum_{i \in S_1} \rho_{2,i}(\emptyset, \emptyset)(t_i - x_i) + \sum_{i \in S_2} \rho_{1,i}(\emptyset, \emptyset)(t_i + x_i) \\
- \sum_{i \in S_1} \xi^1_i(2 - x_i - t_i) - \sum_{i \in S_2} \xi^2_i(2 + x_i - t_i). 
\] (10c)

Inequality (10a) follows from the definition of \( T_f \). Inequality (10b) is a consequence of Corollary 3.4. Recall that \( t_i + x_i = 2 \) if and only if \( i \in X_1 \), and \( t_i - x_i = 2 \) if and only if \( i \in X_2 \), so (10b) is equivalent to (10c), which is the right-hand side of inequality (9).

We call both inequalities (7) and (9) bisubmodular inequalities associated with \((S_1, S_2)\). Inequalities (7) and (9) can be written equivalently with variables \( y^1, y^2 \) instead of \( x, t \). For any bisubmodular \( f \), (9) is equivalent to
\[
\eta \leq f(S_1, S_2) + \sum_{i \in S_1 \cup S_2} [\rho_{1,i}(S_1, S_2)y_i^1 + \rho_{2,i}(S_1, S_2)y_i^2] \\
+ \sum_{i \in S_1} \rho_{2,i}(\emptyset, \emptyset)y_i^2 + \sum_{i \in S_2} \rho_{1,i}(\emptyset, \emptyset)y_i^1 \\
- \sum_{i \in S_1} \xi^1_i(1 - y_i^1) - \sum_{i \in S_2} \xi^2_i(1 - y_i^2). 
\] (11)

For a monotone bisubmodular function \( f \), (7) is equivalent to
\[
\eta \leq f(S_1, S_2) + \sum_{i \in S_1 \cup S_2} [\rho_{1,i}(S_1, S_2)y_i^1 + \rho_{2,i}(S_1, S_2)y_i^2] \\
+ \sum_{i \in S_1} \rho_{2,i}(\emptyset, \emptyset)y_i^2 + \sum_{i \in S_2} \rho_{1,i}(\emptyset, \emptyset)y_i^1. 
\] (12)

Intuitively, the first summation term in the right-hand side of (12) represents the marginal contribution made by appending additional elements to \( S_1 \) or \( S_2 \). The second and the third summations are upper bounds for the change in functional value when some elements in \( S_1 \) are switched to \( S_2 \), or vice versa.

**Remark 4.3.** Notice that the submodular inequality proposed by Nemhauser and Wolsey (1981) for submodular functions is a special case of the bisubmodular inequality (11). Let \( g : 2^N \rightarrow \mathbb{R} \) be a submodular function over a ground set \( N \). We denote the hypograph of \( g \) by
\[
\{(y, \eta_y) \in \{0, 1\}^N \times \mathbb{R} \mid \eta_y \leq g(y)\}. 
\]
For any \( j \in N \) and \( S \subseteq N \), we let \( \gamma_j = g(N) - g(N \setminus \{ j \}) \) and \( \rho_j(S) = g(S \cup \{ j \}) - g(S) \). The submodular inequality for any given \( S \subseteq N \) is

\[
\eta_0 \leq g(S) + \sum_{i \in S} \rho_i(S)y_i - \sum_{j \in S} \gamma_j(1 - y_j).
\]

In the bisubmodular case, suppose we force one of the subsets of \( f \), say \( S_2 \) to always be empty, then \( f \) becomes a submodular function. Under this assumption, we can remove the second and the third summations in inequality (11) since we do not need to account for the exchanges of elements between \( S_1 \) and \( S_2 \). Observe that the resulting inequality is identical to the submodular inequality.

Now consider the polyhedron

\[
\mathcal{P}_f = \{(x, t, y^1, y^2, \eta) \in \mathbb{R}^{4n+1} \mid 2\eta \leq 2f(S_1, S_2) + \sum_{i \notin S_1 \cup S_2} \{[\rho_1,i(S_1, S_2) - \rho_2,i(S_1, S_2)]x_i + [\rho_1,i(S_1, S_2) + \rho_2,i(S_1, S_2)]t_i\}
+ \sum_{i \in S_1} \rho_2,i(\emptyset, \emptyset)(t_i - x_i) + \sum_{i \in S_2} \rho_1,i(\emptyset, \emptyset)(t_i + x_i)
- \sum_{i \in S_1} \xi^1_i(2 - x_i - t_i) - \sum_{i \in S_2} \xi^2_i(2 + x_i - t_i), \forall (S_1, S_2) \in 3^N, \]
\[
x_i = y^1_i - y^2_i,

t_i = y^1_i + y^2_i,
y^1_i + y^2_i \leq 1, \forall i \in N \}.
\]

**Theorem 4.4.** Given any general bisubmodular (not necessarily monotone) function \( f \), and any \((x, \eta) \in \{0, \pm 1\}^n \times \mathbb{R}\), we have \((x, |x|, y^1, y^2, \eta) \in \mathcal{P}_f\) if and only if \( \eta \leq f(X_1, X_2) \), where \((X_1, X_2)\) is the biset that corresponds to \( x \), and \( y^1, y^2 \) are binary characteristic vectors associated with \((X_1, X_2)\).

**Proof.** Suppose \((x, |x|, y^1, y^2, \eta) \in \mathcal{P}_f\). Then \( x_i = 1, t_i = 1 \) for all \( i \in X_1 \), and \( x_i = -1, t_i = 1 \) for all \( i \in X_2 \). For all \( i \notin X_1 \cup X_2, x_i = t_i = 0 \). It follows that

\[
2\eta \leq 2f(X_1, X_2) + \sum_{i \notin X_1 \cup X_2} \{[\rho_1,i(X_1, X_2) - \rho_2,i(X_1, X_2)] \cdot 0 + [\rho_1,i(X_1, X_2) + \rho_2,i(X_1, X_2)] \cdot 0\}
+ \sum_{i \in X_1} \rho_2,i(\emptyset, \emptyset) \cdot (1 - 1) + \sum_{i \in X_2} \rho_1,i(\emptyset, \emptyset) \cdot [1 + (-1)]
- \sum_{i \in X_1} \xi^1_i \cdot (2 - 1 - 1) - \sum_{i \in X_2} \xi^2_i \cdot (2 - 1 - 1)
= 2f(X_1, X_2).
\]

Conversely, suppose \( \eta \leq f(X_1, X_2) \). Let \( x \) be the characteristic vector of \((X_1, X_2)\). Consider \((x, |x|, y^1, y^2, \eta)\) where \( y^1, y^2 \) are binary characteristic vectors associated with \( x \). The last three sets of inequalities in the definition of \( \mathcal{P}_f \) trivially hold. For any \((S_1, S_2) \in 3^N,\)

\[
2\eta \leq 2f(X_1, X_2) \leq 2f(S_1, S_2) + \sum_{i \in X_1 \setminus (S_1 \cup S_2)} 2\rho_1,i(S_1, S_2) + \sum_{i \in X_2 \setminus (S_1 \cup S_2)} 2\rho_2,i(S_1, S_2)
+ \sum_{i \in X_2 \cap S_1} 2\rho_2,i(\emptyset, \emptyset) + \sum_{i \in X_1 \cap S_2} 2\rho_1,i(\emptyset, \emptyset)
\]

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\[
- \sum_{i \in S_1 \setminus X_1} 2\xi_1^i - \sum_{i \in S_2 \setminus X_2} 2\xi_2^i \\
= 2f(S_1, S_2) + \sum_{i \in S_1 \cup S_2} \{(\rho_{1,i}(S_1, S_2) - \rho_{2,i}(S_1, S_2))x_i + (\rho_{1,i}(S_1, S_2) + \rho_{2,i}(S_1, S_2))t_i\} \\
+ \sum_{i \in S_1} \rho_{2,i}(\emptyset, \emptyset)(t_i - x_i) + \sum_{i \in S_2} \rho_{1,i}(\emptyset, \emptyset)(t_i + x_i) \\
- \sum_{i \in S_1} \xi_1^i (2 - x_i - t_i) - \sum_{i \in S_2} \xi_2^i (2 + x_i - t_i)
\]

following the same argument as in the proof of Proposition 4.2. Thus \((x, |x|, y^1, y^2, \eta)\) satisfies the first set of inequalities in the definition of \(\mathcal{P}_f\). We conclude that \((x, |x|, y^1, y^2, \eta) \in \mathcal{P}_f\).

Corollary 4.5. Problem (3) is equivalent to
\[
\max \{\eta \mid (x, t, y^1, y^2, \eta) \in \mathcal{P}_f \cap \{0, \pm 1\}^n \times \mathbb{R}^{3n+1}, x \in \mathcal{K}\}.
\]

Proof. This result directly follows from Theorem 4.4.

Remark 4.6. It may be hard to compute \(\xi_1^i\) and \(\xi_2^i\) for non-monotone bisubmodular functions in practice. However, we do not require exact \(\xi_1^i\) and \(\xi_2^i\) values in the construction of the linear valid inequalities in our exact method. Proposition 4.2 still holds if we replace \(\xi_1^i\) and \(\xi_2^i\) by their lower bounds. One lower bound for each \(\xi_1^i\) and \(\xi_2^i\) is \(\zeta = f - \overline{f}\), where \(\overline{f}\) and \(\overline{f}\) are a lower and an upper bound of \(f\) respectively. This estimate can be improved depending on the problem context. Similarly, we can replace the \(\xi_1^i\) and \(\xi_2^i\) values in \(\mathcal{P}_f\) by their lower bounds that are cheaper to obtain. With the same proof, Theorem 4.4 and Corollary 4.5 hold for the modified \(\mathcal{P}_f\).

Theorem 4.4 and its corollary immediately suggest a cutting plane algorithm for bisubmodular maximization as we describe next.

5 A Cutting Plane Algorithm for Bisubmodular Maximization

In this section, we propose an algorithm to tackle any constrained bisubmodular maximization problem in the form of (2) or equivalently (3). Following the notation from Section 4, such a problem can be rewritten as
\[
\max \eta \quad (13a) \\
\text{s.t.} \quad (x, t, y^1, y^2, \eta) \in \mathcal{C} \quad (13b) \\
x \in \mathcal{K}. \quad (13c)
\]

Here, the polyhedral set \(\mathcal{C}\) is defined by the bisubmodular inequalities, which provide a piecewise linear representation of the objective function \(f\). The set \(\mathcal{K}\) contains the incidence vectors \(x\) associated with the feasible bisets in \(S\), and by abusing notation, \(\mathcal{K}\) here also includes the ternary restriction \(x \in \{0, \pm 1\}^n\).

We propose Algorithm 1 to solve Problem (13). In this algorithm, we start with a relaxed set \(\mathcal{C}\) and repeat the following subroutine until the optimality gap is within the given tolerance \(\epsilon\). We solve a relaxed version of Problem (13) to obtain \(\overline{\eta}\) and \(\overline{\pi}\). The current solution \(\overline{\eta}\) is an upper bound for the optimal objective, and
\( f(\bar{x}) \) serves as a lower bound. Let \((\bar{X}_1, \bar{X}_2)\) be the biset that corresponds to \(\bar{x}\). If \(\bar{\eta}\) overestimates \(f(\bar{x})\), then we restrict \(\mathcal{C}\) by adding the bisubmodular inequality (7) or (9) associated with \((\bar{X}_1, \bar{X}_2)\). We repeat the same procedure in the next iteration.

**Algorithm 1: Delayed Constraint Generation**

1. **Input** initial \(\mathcal{C}\), \(LB = -\infty\), \(UB = \infty\);
2. **while** \((UB - LB)/UB > \epsilon\) **do**
3.  Solve Problem (13) to get \((\bar{x}, \bar{\eta})\);
4.  **if** \(UB > \bar{\eta}\) **then**
5.      \(UB \leftarrow \bar{\eta}\);
6.  **end**
7.  compute \(f(\bar{x})\);
8.  **if** \(\bar{\eta} > f(\bar{x})\) **then**
9.      Add a bisubmodular inequality associated with \(\bar{x}\) to \(\mathcal{C}\);
10. **end**
11. **if** \(LB < f(\bar{x})\) **then**
12.      \(LB \leftarrow f(\bar{x})\);
13.      Update the incumbent solution to \(\bar{x}\);  
14. **end**
15. **end**
16. **Output** \(\bar{\eta}, \bar{x}\).

**Corollary 5.1.** Algorithm 1 converges to an optimal solution of Problem (13) in finitely many iterations.

*Proof.* This result follows from the fact that the number of feasible solutions is finite and from Theorem 4.4.

\[ \square \]

### 6 Numerical Study

In this numerical study, we demonstrate the effectiveness of our proposed Delayed Constraint Generation (DCG) Algorithm 1 by solving the coupled sensor placement problem—described in Section 1.1.1. To summarize, let a set \(N\) of \(n\) potential sensor deployment locations, and \(t\) pairs of observations at each location be given. Our goal is to determine a coupled sensor placement plan \((S_1, S_2) \in 3^N\), subject to cardinality constraints \(|S_1| \leq B_1\) and \(|S_2| \leq B_2\), such that the entropy is maximized. Since the entropy function is highly nonlinear, we cannot formulate the coupled sensor placement problem as a mixed-integer linear program.

Using the DCG approach, we formulate the coupled sensor placement problem as

\[
\max \ \eta \quad \tag{14a}
\]
The variables $x, t, y^1, y^2$ and $\eta$ are consistent with the notation in Problem (13). Constraint (14b) gives the piecewise linear representation of the entropy function by exploiting its bisubmodularity. The inequalities (14f)-(14g) ensure that the cardinality requirements are satisfied.

We create random problem instances using the Intel Berkeley research lab dataset (Bodik et al., 2004), which includes temperature and humidity sensor readings at 54 locations in the Intel Berkeley Research lab from February 28th to April 5th in 2004. We discretize the temperature data into three equal-width bins from the lowest temperature reading to the highest. Similarly, the humidity data is discretized into two equal-width bins. The experiments are executed on two threads of a Linux server with Intel Haswell E5-2680 processor at 2.5GHz and 128GB of RAM. Our algorithms are implemented in Python 3.6 and Gurobi Optimizer 7.5.1 with default settings and one-hour time limit for each instance.

First, we explore how the changes in the number of deployable locations, $n$, affect the computational performance of the DCG algorithm. We randomly select $n \in \{10, 20, 50\}$ out of the 54 locations in the dataset. At each of the $n$ locations, we randomly select $t \in \{10, 20, 50, 100\}$ pairs of temperature and humidity measurements. We set $B_1 = B_2 = \lfloor n/10 \rfloor$ so that the cardinality bounds for both types of sensors increase proportionally with $n$. The computational results are summarized in Table 1. The first two columns list the numbers of deployable sensor locations and the numbers of observations at each location. The last three columns present the relevant computational statistics, namely the running time in seconds, the number of bisubmodular inequalities added, and the number of branch-and-bound nodes visited.

In this set of experiments, DCG solves all the instances within the one hour time limit. For small $n$ values ($n = 10, 20$), all instances are solved in less than 3.6 seconds. The runtime, the number of branch-and-bound nodes as well as the number of bisubmodular inequalities added increase as $n$ increases. Variations in $t$ for small $n$ values do not significantly impact the computational statistics. When $n = 50$, all the statistics increase at a greater rate in response to increments in $t$ compared with the cases of $n = 10$ or 20.

Next, we explore the effects of the cardinality bounds $B_1, B_2$ on the computational performance of DCG. We consider all the placeable sensor locations; that is, $n = 54$. Again, at each location, we randomly select $t \in \{10, 20, 50\}$ pairs of temperature and humidity readings. We set $B_1 = B_2 = B$ where $B$ is an integer between 1 and 5. The computational results are summarized in Table 2. The first column shows the upper bounds on the number of each type of sensors. The second column lists the numbers of observations at each
Table 1: Computational performance of DCG in the coupled sensor placement problem. The statistics are averaged across 3 trials.

<table>
<thead>
<tr>
<th>n</th>
<th>time (s)</th>
<th># cuts</th>
<th># nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.11</td>
<td>10.67</td>
<td>17.33</td>
</tr>
<tr>
<td>20</td>
<td>0.19</td>
<td>12.00</td>
<td>26.67</td>
</tr>
<tr>
<td>50</td>
<td>0.28</td>
<td>24.33</td>
<td>62.33</td>
</tr>
<tr>
<td>100</td>
<td>0.75</td>
<td>18.33</td>
<td>43.00</td>
</tr>
<tr>
<td>20</td>
<td>0.24</td>
<td>23.33</td>
<td>460.67</td>
</tr>
<tr>
<td>20</td>
<td>0.53</td>
<td>67.00</td>
<td>758.00</td>
</tr>
<tr>
<td>50</td>
<td>1.59</td>
<td>121.67</td>
<td>1529.67</td>
</tr>
<tr>
<td>100</td>
<td>3.60</td>
<td>127.00</td>
<td>1896.67</td>
</tr>
<tr>
<td>50</td>
<td>25.00</td>
<td>197.67</td>
<td>196693.33</td>
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</tr>
<tr>
<td>50</td>
<td>681.99</td>
<td>944.67</td>
<td>775305.67</td>
</tr>
<tr>
<td>100</td>
<td>2477.59</td>
<td>1951.33</td>
<td>1660335.67</td>
</tr>
</tbody>
</table>

Based on Table 2, higher $B_1$ and $B_2$ values make the coupled sensor placement problem more challenging, with longer running time, more cuts added, and more branch-and-bound nodes visited. In particular, when $B_1 = B_2 \geq 4$, the computational statistics increase at a higher rate as $t$ increases. The decision space consisting of all the plausible deployment plans grows rapidly as more sensors are allowed. If, in addition, the number of observations is high, then each entropy evaluation becomes expensive, resulting in a significant increase in the running time. Nevertheless, even in the most challenging instances where $B_1 = B_2 = 4, 5$ and $t = 100$, DCG achieves low end gaps that are below 5% within the one hour time limit. DCG also solves the relatively easy cases efficiently. When the cardinality bounds are below three, DCG solves all the instances within five minutes.

7 Concluding Remarks

In this paper, we propose a polyhedral approach to solve maximization problems with bisubmodular objective functions. We propose valid linear inequalities, referred to as bisubmodular inequalities, for the hypograph of any bisubmodular function. This development leads us to construct the first exact method—a delayed constraint generation algorithm based on bisubmodular inequalities—to solve general bisubmodular maximization problems. Our numerical experiments on a highly nonlinear coupled sensor placement problem demonstrate the efficacy of the proposed delayed constraint generation algorithm when handling challenging bisubmodular maximization problems. For future work, we will extend the results in this paper...
Table 2: Computational performance of DCG in the coupled sensor placement problem. The statistics are averaged across 3 trials. The superscript $i_1$ means that out of the three trials, $i_1$ instances reach the time limit of one hour.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$t$</th>
<th>time (s)</th>
<th># cuts</th>
<th># nodes</th>
<th>end gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>0.47</td>
<td>17.00</td>
<td>64.33</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.14</td>
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<td>333.33</td>
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to $k$-submodular function maximization, where the functions have $k > 2$ arguments.

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**References**


