Robust Conic Satisficing

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Inspired by the principle of satisficing [Simon 1955], Long et al. [2021] propose an alternative framework for optimization under uncertainty, which we term as a robust satisficing model. Instead of sizing the uncertainty set in robust optimization, the robust satisficing model is specified by a target objective with the aim of delivering the solution that is least impacted by uncertainty in achieving the target. At the heart of this framework, we minimize the level of constraint violation under all possible realizations within the support set. Our framework is based on a constraint function that evaluates to the optimal objective value of a standard conic optimization problem, which can be used to model a wide range of constraint functions that are convex in the decision variables but can be either convex or concave in the uncertain parameters. We derive an exact semidefinite optimization formulation when the constraint is biconvex quadratic with quadratic penalty and the support set is ellipsoidal. We also show the equivalence between the more general robust satisficing problems and the classical adaptive robust linear optimization models with conic uncertainty sets, where the latter can be solved approximately using affine recourse adaptation. More importantly, under complete recourse, and reasonably chosen polyhedral support set and penalty function, we show that the exact reformulation and safe approximations do not lead to infeasible problems if the chosen target is above the optimum objective obtained when the nominal optimization problem is minimized. Finally, we extend our framework to the data-driven setting and showcase the modelling and the computational benefits of the robust satisficing framework over robust optimization with three numerical examples: portfolio selection, log-sum-exp optimization and adaptive lot-sizing problem.

Key words: robust optimization, robust satisficing, conic optimization, affine recourse adaptation
1. Introduction

“Of the impermanent there is no certainty” - Gita (2.16)

Uncertainty is an integral part of optimization problems without accounting for which, the deterministic optimal solution is fragile and does not provide meaningful insights (Ben-Tal et al. 2009). While the need to introduce uncertainty is well documented, the form in which it appears varies. In stochastic optimization problems, the uncertainty appears as random variables that are governed by an explicit probability distribution, which is assumed to be either available or else has to be estimated from historical data (Shapiro et al. 2014, Birge and Louveaux 2011). On the other hand, robust optimization only assumes that the uncertainty dwells in a restricted set, also known as uncertainty set, without any further statistical information (Soyster 1973, Ben-Tal and Nemirovski 1998, El Ghaoui et al. 1998). Robust optimization minimizes the worst-case cost while enforcing that the constraints are satisfied for every realization of the uncertainty within this set.

The selling point of robust optimization models is that their computational tractability is typically on par with their deterministic counterparts for many classes of constraints and characterizations of the uncertainty set (Bertsimas and Sim 2003, 2004, 2006). Ben-Tal et al. (2004) extend robust optimization to an adaptive optimization framework, where recourse decisions can adapt to the uncertain parameters that are realized. Not all recourse adaptations would lead to computationally tractable optimization problems, and Ben-Tal et al. (2004) propose a tractable safe approximation by restricting the recourse to an affine function of the uncertain parameters (see also Delage and Iancu 2013, Kuhn et al. 2011, Bertsimas et al. 2010, Iancu et al. 2013). Distributionally robust optimization, which generalizes robust optimization (Delage and Ye 2010, Wiesemann et al. 2014, Bertsimas et al. 2019), assumes that the probability distribution governing the uncertain parameters lies in an ambiguity set of distributions characterized by known properties of the unknown data-generating distribution. In the data-driven framework, distributionally robust optimization models with Wasserstein ambiguity sets (Mohajerin Esfahani and Kuhn 2018), or $\Phi$-divergence based ambiguity sets (Ben-Tal et al. 2013) could effectively overcome poor out-of-sample performance. Chen et al. (2020) provide a unified framework for modeling distributionally robust optimization problems including data-driven models, and develop an algebraic modeling toolbox “RSOME” for this purpose.

Long et al. (2021) recently proposed a new paradigm to model uncertainty called robustness optimization, which, in the stochastic-free version, corresponds to a case of the GRC-sum model of Ben-Tal et al. (2017). Unlike robust optimization methods, which only hedge against pre-defined uncertainty, robustness optimization offers full protection by giving nature a free-hand to vary over the entire uncertainty support. The robustness optimization model compensates for this increased
protection by allowing the constraints to be violated while simultaneously controlling the degree of infeasibility. The decision maker has the flexibility to choose the degree of sub-optimality relative to the nominal objective value, by specifying a target, unlike the robust optimization model, where the size of the uncertainty set needs to be known apriori. Simon (1955), who proposes the term satisficing, argues that target plays an important role in decision making, especially in complex situations involving uncertainty. To emphasize the role of the target, we prefer to use the term robust satisficing in place of robustness optimization proposed in Long et al. (2021). The same term has also been used in Schwartz et al. (2011) to describe a decision model that maximizes the robustness to uncertainty of achieving a satisfactory target. The decision criterion in our robust satisficing framework belongs to the family of satisficing decision criteria axiomatized by Brown and Sim (2009), which has an embedded preference for diversification that, serendipitously, also leads to computational tractability when used in convex optimization problems.

Since the inception of robust optimization, we now have an arsenal of tools to address and solve either exactly, or providing tractable safe approximations for various kinds of robust optimization models. We wish to highlight Bertsimas and de Ruiter (2016) for proposing the dualizing technique and applying affine dual recourse adaptation to address an adaptive robust linear optimization problem. When the uncertainty set is polyhedral, this approach can also be used to provide safe approximations for robust optimization models with biconvex constraint functions including those with recourse adaptation (de Ruiter et al. 2018, Roos et al. 2020). We use this approach to obtain solutions to our proposed robust conic satisficing models.

We summarize the contributions of this paper below:

1. We provide a unifying framework for conic optimization under uncertainty that is based on minimizing a linear objective function and a constraint function that evaluates to the optimal objective value of a standard conic optimization problem. We demonstrate that it can be used to model a wide range of robust optimization problems studied in the literature including biconvex linear or quadratic constraint functions (Ben-Tal and Nemirovski 1998), saddle constraint functions (Ben-Tal et al. 2017), non-linear biconvex constraint functions (Roos et al. 2020, Zhen et al. 2017), adaptive linear optimization model (Ben-Tal et al. 2004, Bertsimas and de Ruiter 2016) and adaptive convex optimization model (de Ruiter et al. 2018).

2. Based on the conic optimization framework, we demonstrate how we could solve the robust satisficing model, which has potentially infinite number of conic constraints, either exactly when possible, or safely approximated to a practicably solvable optimization problem. For a biconvex quadratic constraint with a quadratic penalty, we derive the exact reformulation in the form of tractable semidefinite optimization problem. Then, for a more general constraint with polyhedral support and penalty, we derive a tractable safe approximation using affine dual recourse
adaptation technique first proposed by Bertsimas and de Ruiter \(2016\). The key challenge is to show that under a condition of complete recourse, and reasonably chosen polyhedral support set and penalty function, the exact reformulation and safe approximations do not lead to infeasible problems as long as the chosen targets are above the optimum objective obtained when the nominal optimization problem is minimized.

3. We explore how the affine dual recourse adaptation can be used to provide safe approximations to two-stage adaptive conic optimization problems, including those in the data-driven settings explored in Long et al. \(2021\). For the important case of adaptive linear optimization, we show that the affine dual recourse adaptation provides a better approximation than the non-affine recourse adaptation proposed in Long et al. \(2021\).

4. We showcase the modelling and the computational benefits of the robust satisficing framework over the robust optimization counterpart with three numerical examples: portfolio selection, log-sum-exp optimization and adaptive lot-sizing problem. Using Monte-Carlo simulations, we show that the robust satisficing model obtains a family of solutions that have better statistical performance compared to the solutions generated by an equivalent robust optimization model. Additionally, we present computational results to show that the robust satisficing model has a remarkable improvement in computational time over the robust optimization model.

Notation: We use \(\mathbb{R}\) to denote the space of reals while \(\mathbb{R}_+\) and \(\mathbb{R}_{++}\) denote the sets of non-negative and strictly positive real numbers respectively. We use boldface small-case letters (e.g. \(x\)) to denote vectors, capitals (e.g. \(A\)) to denote matrices and capital caligraphic letters to denote sets (e.g. \(\mathbb{Z}\)) including cones (e.g. \(K\)). \(\mathcal{R}^{m,n}\) and \(\mathcal{L}^{m,n}\) are used to denote the set of all functions and its sub-class of affine functions, respectively, from \(\mathbb{R}^m\) to \(\mathbb{R}^n\). The transpose of a vector (matrix) is denoted by \(x^\top\) (\(A^\top\)). A random vector is denoted with a tilde sign (e.g. \(\tilde{z}\)), and \([n]\) is used to denote the running index set \(\{1,2,\ldots,n\}\). We use superscript indexing (e.g. \(w^i\)) to denote the \(i^{th}\) vector (matrix) among a countable set of vectors \(\{w^t\}\) (matrices) and subscript indexing (e.g. \(A_i\)) to denote the \(i^{th}\) row of a matrix \(A\). Finally, \(0\) (\(1\)) denotes the vector of all zeros (ones) and its dimension should be clear from the context, while the identity matrix of order \(n\) is denoted by \(I_n\).

2. Unifying framework for conic optimization under uncertainty

In this section, we propose a unifying conic optimization framework where the exact values of the model’s parameters are unknown, or unobservable, but proximal to some given nominal values. We first consider on a deterministic nominal conic optimization problem as follows:

\[
Z_0 = \text{minimize} \quad c^\top x \\
\text{subject to} \quad g(x, \tilde{z}) \leq 0 \\
x \in \mathcal{X},
\]

\(1\)
for a given function \( g : \mathcal{X} \times \mathcal{Z} \mapsto \mathbb{R} \) where the input to the second argument may be subjected to uncertainty and \( \hat{z} \in \mathcal{Z} \) is the nominal value. The model’s decision variable is denoted by the vector \( x \in \mathcal{X} \subseteq \mathbb{R}^n_x \) and the uncertain parameters are represented by the vector \( z \in \mathcal{Z} \subseteq \mathbb{R}^n_z \), where \( \mathcal{X} \) and \( \mathcal{Z} \) are respectively the feasible set and uncertainty support set. Central to our conic optimization model is how we define the function \( g \), which is designed to be as expressive as possible, yet allowing the conic optimization problem under uncertainty to be amendable to tractable reformulations or approximations. Specifically, the function \( g \) is a conic representable function of the form

\[
\begin{align*}
  g(x, z) &= \text{minimize} \quad d^\top y \\
  \text{subject to} \quad By &\preceq_K f(x) + F(x)z \\
  y &\in \mathbb{R}^n_y,
\end{align*}
\]

where \( f : \mathbb{R}^n_x \mapsto \mathbb{R}^n_f \), \( F : \mathbb{R}^n_x \mapsto \mathbb{R}^{n_f \times n_z} \) are affine mappings in \( x \), and \( K \) is a proper cone. Since the decision variable \( y \) in Problem (2) is made after observing \( x \) and \( z \), we will call \( y \) the recourse variable. Indeed, if the cone \( K \) is the non-negative orthant, then the \( g \) function would represent the second stage optimization problem of a standard two-stage stochastic optimization model. However, as we will reveal, the function \( g \) is sufficiently broad enough to represent many types of functions considered in the robust optimization literature, for instance a biconvex quadratic function (Ben-Tal and Nemirovski 1998), which are not necessarily associated with a two-stage optimization problem.

**Assumption 1 (Convexity and practicable solvability).** We assume that \( \mathcal{X} \) and \( \mathcal{Z} \) are compact and convex sets. Moreover, we assume that any convex optimization problem over \( x \in \mathcal{X} \), involving a modest number of additional decision variables, linear and \( K \)-conic inequalities are practicably solvable, i.e., it can be solved to optimality within reasonable time using current available solvers such as CPLEX, Gurobi, Mosek, SDPT3, among others.

The nominal problem is practicably solvable under Assumption 1. However, most optimization problems become much harder to solve when they are subject to uncertainty. Note that as \( x \) and \( z \) appear on the right-hand side of the conic constraint, \( g \) is a biconvex function, and we would expect the problem to be much harder to solve exactly if the second argument is subject to uncertainty. In the simplest case, where \( g \) is a biaffine function given by

\[
g(x, z) = x^\top A z + b^\top x + c^\top z + d,
\]

we can express

\[
g(x, z) = \min_{y \in \mathbb{R}} \{ y \mid y \geq (b^\top x + d) + (x^\top A + c^\top) z \}.
\]

In addition, the following example demonstrates how we can convert commonly used biconvex functions to the form of Problem (2).
**Example 1 (Common bi-convex function).** Consider a biconvex function of the following form

\[ g(x, z) = h(b(x) + A(x)z) \]

where \( h : \mathbb{R}^{n_a} \to \mathbb{R} \cup \{\infty\} \) is a convex map, \( b : \mathbb{R}^n \to \mathbb{R}^{n_a} \), \( A : \mathbb{R}^n \to \mathbb{R}^{n_a \times n_z} \) are affine mappings in \( x \). This is a bi-convex function, which is common in classical robust convex optimization models; see e.g. Roos et al. (2020). Observe that

\[
\begin{aligned}
g(x, z) &= \inf y \\
&\text{s.t.} \\
&\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} y \succeq_K \\
&\begin{pmatrix}
-b(x) \\
0 \\
-1
\end{pmatrix} + \begin{pmatrix}
-A(x) \\
0^T \\
0^T
\end{pmatrix} z,
\end{aligned}
\]

where the conic inequality involves a cone

\[
K = \text{cl} \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^{n_a} \times \mathbb{R} \times \mathbb{R}_+ \mid \gamma h(\alpha/\gamma) \leq \beta, \gamma > 0 \right\},
\]

which is proper and convex because a perspective function preserves convexity. For instance, a quadratic constraint

\[
\|A(x)z + a(x)\|_2^2 + b(x)^T z + c(x) \leq 0
\]

can be expressed as

\[
g(x, z) = \min_{y \in \mathbb{R}} \left\{ y \mid (A(x)z + a(x), y - b(x)^T z - c(x), 1) \in K \right\},
\]

where here \( K \) denotes the rotated second-order cone given by

\[
K = \left\{ (\alpha, \beta, \gamma) \in \mathbb{R}^{n_a} \times \mathbb{R}_+ \times \mathbb{R}_+ \mid \alpha^T \alpha \leq \beta \gamma \right\}.
\]

In this regard, the two-stage adaptive convex optimization framework proposed in de Ruiter et al. (2018) can also be represented in our unified conic optimization framework.

### 2.1. Robust optimization

To better protect the constraints against infeasibility, robust optimization fashions an uncertainty set of a given size \( \mathcal{U} \) around the nominal parameter \( z \) as follows:

\[
Z_r = \minimize_{x} \ c^T x \\
\text{subject to} \ g(x, z) \leq 0 \quad \forall z \in \mathcal{U}_r, \quad (3)
\]

\[ x \in \mathcal{X}, \]
where $\mathcal{U}_r$ is typically a distance-calibrated uncertainty set that contains $\tilde{z}$ and is given by

$$\mathcal{U}_r = \{ z \in \mathcal{Z} \mid p(z - \tilde{z}) \leq r \},$$

and $p(\zeta) : \mathbb{R}^{nz} \mapsto \mathbb{R}_+$ is a convex penalty function that penalizes deviations of $\zeta$ from the origin such that $p(\zeta) = 0$ if and only if $\zeta = 0$. Hence, $\mathcal{U}_0 = \{ \tilde{z} \}$ and $\mathcal{U}_r \subseteq \mathcal{U}_{r'}$ for all $0 \leq r \leq r'$.

It has well been known that the tractability of the robust optimization depends on the constraint function $g$ along with the characterization of the uncertainty set, and there are few examples beyond linear constraint that would yield tractable reformulations. Notably, if $\mathcal{U}_r$ is a compact convex set, and the function associated with the robust constraint $h(x, z) \leq 0$, $\forall z \in \mathcal{U}_r$ is a saddle function, i.e., $h(x, z)$ is convex in $x$ for a given $z$ and concave in $z$ for a given $x$, then in many interesting examples, we would be able to use standard robust optimization techniques (see, e.g., Ben-Tal et al. 2017) to tractably address the robust constraint through a reformulation. As we will show, although such a saddle function may not necessarily be represented as a $g$ function of Problem (2), we can always transform the constraint to an equivalent one where the constraint function can be expressed as the $g$ function.

**Example 2 (Saddle Function).** Consider a saddle function $h(x, z) : \mathcal{X} \times \mathbb{R}^{nz} \mapsto \mathbb{R} \cup \{-\infty\}$ on an extended real number line. Specifically, for a given $x \in \mathcal{X}$, $h(x, z)$ is upper-semicontinuous and concave in $z \in \mathbb{R}^{nz}$, and for a given $z \in \mathbb{R}^{nz}$, the function is convex in $x \in \mathcal{X}$. Hence, due the biconjugate property, for a given $(x, z) \in \mathcal{X} \times \mathbb{R}^{nz}$, the function can be rewritten as

$$h(x, z) = \inf_{v \in \mathbb{R}^{nz}} \left\{ f(x, v) - v^\top z \right\}$$

where $f$ is the convex conjugate of $-h$ with respect to the second argument as follows

$$f(x, v) = \sup_{y \in \mathbb{R}^{nz}} \left\{ v^\top y + h(x, y) \right\},$$

which is a jointly convex function, since it can be expressed as a maximum of functions that are convex in $(x, v)$. Therefore, we can express

$$h(x, z) = \inf_{(u, v) : (u, v, x) \in \mathcal{Y}} \left\{ u - v^\top z \right\}$$

(4)

where

$$\mathcal{Y} = \{(u, v, x) \in \mathbb{R} \times \mathbb{R}^{nz} \times \mathcal{X} \mid u \geq f(x, v)\}$$

is an epigraph of a jointly convex function and thus a convex set. However, it should be noted that Problem (4) is not in the same form expressed in Problem (2) because the recourse matrix $B$ in the latter does not depend on $z$. Note that the robust optimization constraint

$$h(x, z) \leq 0 \quad \forall z \in \mathcal{U}_r$$
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is equivalent to

$$\max_{z \in \mathcal{U}} \inf_{(u,v)(u,v,x) \in \mathcal{Y}} \{ u - v^\top z \} \leq 0.$$ 

Since $\mathcal{U}$ is a convex compact set, we can use Sion (1958) minimax result to obtain an equivalent representation,

$$\max_{z \in \mathcal{U}} \inf_{(u,v)(u,v,x) \in \mathcal{Y}} \{ u - v^\top z \} \leq 0 \iff \inf_{(u,v)(u,v,x) \in \mathcal{Y}} \max_{z \in \mathcal{U}} \{ u - v^\top z \} \leq 0.$$ 

Hence, we can express the robust optimization constraint as

$$g(u,v,z) \leq 0 \ \forall z \in \mathcal{U}_r,$$

where $g(u,v,z) = \min_y \{ y \mid y \geq u - v^\top z \}$ for some first-stage variables $(u,v,x) \in \mathcal{Y}$. Therefore, we can always transform a robust optimization problem with saddle constraint function to an equivalent robust optimization problem with a biaffine constraint function.

2.2. Robust satisficing

Long et al. (2021) has recently proposed a target-oriented robustness optimization framework for data-driven optimization, which we term as robust satisficing to emphasize the role of the target in the model specification. We focus on the stochastic-free model, where the robust satisficing model also corresponds to the GRC-sum model of Ben-Tal et al. (2017) as follows:

$$\begin{align*}
\text{minimize} \quad & k \\
\text{subject to} \quad & g(x,z) \leq kp(z - \bar{z}), \quad \forall z \in \mathcal{Z} \\
& c^\top x \leq \tau \\
& x \in \mathcal{X}, \quad k \in \mathbb{R}_+.
\end{align*}$$

In contrast to the robust formulation in Problem (3), where the protection offered is only against the subset $\mathcal{U}_r \subseteq \mathcal{Z}$ of all possible realizations of $z$, the robust satisficing model allows the uncertainty to range over the entire support $\mathcal{Z}$, but controls the level of constraint violations as much as possible whenever $z$ deviates from its nominal value, $\bar{z}$. Additionally, as a trade-off for the model’s greater ability to withstand uncertainty, an acceptable loss in optimality is specified by a target $\tau \geq \tau_0$. As also observed in Long et al. (2021), the robust satisficing model is almost in the same complexity class as its robust optimization counterpart. We also note that if the constraint function $h(x,z)$ is a saddle function, then the function $h(x,z) - kp(z - \bar{z})$ is also a saddle function on domains $(x,k) \in \mathcal{X} \times \mathbb{R}_+$ and $z \in \mathcal{Z}$, which, as illustrated in Example 2, can be transformed to a robust constraint in which the constraint function can be represented as a $g$ function of Problem (3).
Depending on the nominal problem, there are different variants of the robust optimization and satisficing models. If the nominal problem has $g$ appearing at the objective function, then we can introduce artificial variables so that it can be framed as Problem (1). For instance

\[ Z_0 = \min \ c^\top x + g(x, z) \quad \iff \quad Z_0 = \min x_0 \]

s.t. $x \in \mathcal{X}$

\[ g((x_0, x), z) \leq 0 \]

$x \in \mathcal{X}, x_0 \in \mathbb{R}$

where $\bar{g}((x_0, x), z) = \min_{y \in \mathbb{R}} \{g(x, z) + y | y \geq c^\top x - x_0 \}$. Hence, the corresponding robust optimization problem becomes

\[ Z_r = \min x_0 \]

s.t. $\bar{g}((x_0, x), z) \leq 0 \ \forall z \in \mathcal{U}_r$

$x \in \mathcal{X}, x_0 \in \mathbb{R}$

and the robust satisficing problem would be

\[ \min k \]

s.t. $\bar{g}((x_0, x), z) \leq kp(z - \hat{z}) \ \forall z \in \mathcal{Z}$

$x_0 \leq \tau$

$x \in \mathcal{X}, x_0 \in \mathbb{R}$

We can also extend these frameworks to consider an arbitrary number of constraints, say $m \in \mathbb{N}$ as follows,

<table>
<thead>
<tr>
<th>Nominal</th>
<th>Robust optimization</th>
<th>Robust satisficing ($\tau \geq Z_0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_0 = \min c^\top x$</td>
<td>$\min c^\top x$</td>
<td>$\min w^\top k$</td>
</tr>
<tr>
<td>s.t. $g_i(x, z) \leq 0$</td>
<td>s.t. $g_i(x, z) \leq 0$</td>
<td>s.t. $g_i(x, z) \leq kp(z - \hat{z})$</td>
</tr>
<tr>
<td>$\forall i \in [m]$</td>
<td>$\forall z \in \mathcal{U}(r_i), i \in [m]$</td>
<td>$\forall z \in \mathcal{Z}, i \in [m]$</td>
</tr>
<tr>
<td>$x \in \mathcal{X}$</td>
<td>$x \in \mathcal{X}$</td>
<td>$c^\top x \leq \tau$</td>
</tr>
</tbody>
</table>

where $\{r_i\}_{i \in [m]}$ is a collection of the non-negative radii of the uncertainty set, and $\{w_i\}_{i \in [m]}$ is a collection of non-negative weights.

### 3. Tractable reformulations and safe approximations

In this section, we demonstrate how we could solve the robust satisficing of Problem (6) that has infinite number of conic constraints, either exactly when possible, or safely approximated to a practically solvable optimization problem under Assumption 1. For a biconvex quadratic function with
a quadratic penalty, we derive the exact reformulation in the form of a tractable semidefinite optimization problem. Then, for a more general function with polyhedral support and penalty, we focus on obtaining a tractable safe approximation using affine dual recourse adaptation technique. The key challenge is to provide the conditions under which the exact reformulation and safe approximations would not lead to infeasible problems for any reasonably chosen target above the optimum objective obtained when the nominal optimization problem is minimized, i.e., \( \tau > Z_0 \).

### 3.1. Biconvex quadratic constraint with quadratic penalty

This problem is motivated from the classical robust optimization proposed in Ben-Tal and Nemirovski (1998) involving a quadratic constraint and an ellipsoidal uncertainty set

\[
\begin{align*}
Z_r &= \text{minimize} \quad c^\top x \\
\text{subject to} \quad &\|A(x)z + a(x)\|^2_2 + b(x)^\top z + c(x) \leq 0 \quad \forall z \in \mathcal{E}_r \\
x \in \mathcal{X},
\end{align*}
\]

where \( \mathcal{E}_r = \{ z \in \mathbb{R}^{n_z} | z^\top z \leq r \} \), \( r \geq 0 \), is a non-empty ellipsoidal uncertainty set and \( a : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_z} \), \( A : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_a \times n_z} \), \( b : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_z} \), \( c : \mathbb{R}^{n_x} \mapsto \mathbb{R} \) are affine mappings of the decision variables \( x \).

Note that here the nominal value is \( \hat{z} = 0 \), which can be assumed without any loss of generality. Biconvex robust optimization problems are typically difficult to deal with; however, as shown below, Problem (8) is a notable exception. Besides, we assume that the nominal problem is strictly feasible.

**Theorem 1.** (Ben-Tal and Nemirovski 1998, Theorem 3.2). For any \( r > 0 \), the constraints

\[
\|A(x)z + a(x)\|^2_2 + b(x)^\top z + c(x) \leq 0, \quad \forall z \in \mathcal{E}_r
\]

is equivalent to following the positive semidefinite constraint

\[
\exists \lambda \geq 0 : \begin{bmatrix}
I_{n_a} & a(x) & A(x) \\
A(x)^\top & -c(x) - \lambda r - \frac{1}{2} b(x)^\top & -\frac{1}{2} b(x) \\
A(x)^\top & -\frac{1}{2} b(x) & \lambda I_{n_z}
\end{bmatrix} \succeq 0.
\]

**Proof.** To avoid clutter, we first drop the dependency of \( A, b, c \) on \( x \), and then we expand the squared Euclidean norm in the robust constraint \( \|A(x)z + a(x)\|^2_2 + b(x)^\top z + c(x) \leq 0, \forall z \in \mathcal{E}_r \) to equivalently express it as

\[
\begin{bmatrix}
1 \\
z
\end{bmatrix}^\top \begin{bmatrix}
1 - c - a^\top a & -(A^\top a + \frac{1}{2} b)^\top \\
-(A^\top a + \frac{1}{2} b) & -A^\top A
\end{bmatrix} \begin{bmatrix}
1 \\
z
\end{bmatrix} \succeq 0, \forall z : \begin{bmatrix}
1 \\
z
\end{bmatrix}^\top \begin{bmatrix}
r & 0^\top \\
0 & -I_{n_z}
\end{bmatrix} \begin{bmatrix}
1 \\
z
\end{bmatrix} \succeq 0.
\]
Applying $S$-lemma leads to another equivalent representation of our quadratic constraint which is

$$
\exists \lambda \geq 0 : \begin{bmatrix}
-c - a^\top a & -\left(A^\top a + \frac{1}{2}b\right)^\top \\
\left(A^\top a + \frac{1}{2}b\right) & -A^\top A
\end{bmatrix} - \lambda \begin{bmatrix}
r & 0^\top \\
0 & -I_{n_a}
\end{bmatrix} \succeq 0
\quad \Leftrightarrow \quad \exists \lambda \geq 0 : \begin{bmatrix}
-c - \lambda r - \frac{1}{2}b^\top \\
-\frac{1}{2}b & \lambda I_{n_a}
\end{bmatrix} - \begin{bmatrix}
a^\top \\
A^\top
\end{bmatrix} I_{n_a} \begin{bmatrix}
a^\top \\
A^\top
\end{bmatrix}^\top \succeq 0.
$$

Finally, using Schur's complement and recovering the dependency of $A, b, c$ on $x$ yields the desired result.

Analogously, we now consider a robust quadratic satisficing model by choosing the penalty function to be the squared Euclidean norm $p(z) = \|z\|^2 = z^\top z$. Suppose we consider the ellipsoidal support set $\mathcal{Z} = \mathcal{E}_r$, $r \geq r$, which contains the earlier uncertainty set $\mathcal{E}_r$. The arising robust satisficing problem can then be written down as

$$
\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad \|A(x)z + a(x)\|^2 + b(x)^\top z + c(x) \leq kz^\top z, \quad \forall z \in \mathcal{Z}, \\
& \quad c^\top x \leq \tau \\
& \quad x \in \mathcal{X}, \quad k \in \mathbb{R}_+,
\end{align*}
$$

(9)

where $\tau > Z_0$ is the prescribed target.

**Theorem 2.** For any $\bar{r} > 0$, the constraints

$$
\|A(x)z + a(x)\|^2 + b(x)^\top z + c(x) \leq kz^\top z, \quad \forall z \in \mathcal{Z}
$$

is equivalent to the following positive semidefinite constraint

$$
\exists \lambda \geq 0 : \begin{bmatrix}
I_{n_a} & a(x) & A(x) \\
a(x)^\top & -c(x) - \lambda \bar{r} - \frac{1}{2}b(x)^\top \\
A(x)^\top & -\frac{1}{2}b(x) & (k + \lambda)I_{n_a}
\end{bmatrix} \succeq 0.
$$

Under Assumption [1] and that the nominal problem is strictly feasible, then Problem (9) is feasible for any chosen target, $\tau > Z_0$.

**Proof.** By relocating $kz^\top z$ to the left-hand side, the constraint function is still quadratic in $x$ and in $z$. Using similar arguments of the $S$-lemma and Schur’s complement from the proof of Theorem [1] thus completes the first half of the theorem.

For the second half of the proof, we denote by $x^s \in \mathcal{X}$ any Slater’s point of the nominal problem. Let $\hat{x}$ be the optimum solution to the nominal problem so that $Z_0 = c^\top \hat{x}$. Since $\mathcal{X}$ is a convex set, it follows that there is a point $\hat{x}^s \in \mathcal{X}$ on the line segment connecting $\hat{x}$ and $x^s$ such that

$$
c^\top \hat{x}^s \leq \tau \quad \text{and} \quad \|a(\hat{x}^s)\|^2 + c(\hat{x}^s) < 0.
$$
It thus suffices to show that there exists a sufficiently large positive $\hat{k}$ such that $(x, k, \lambda) = (\hat{x}^s, \hat{k}, 0)$ is feasible in Problem (6). To this end, we first note from the above quadratic inequality that

$$\begin{bmatrix} I_{na} & a(\hat{x}^s) \\ a(\hat{x}^s)^\top & -c(\hat{x}^s) \end{bmatrix} \succ 0.$$  

Thus, there exists $\hat{k} > 0$ such that

$$\begin{bmatrix} I_{na} & a(\hat{x}^s) \\ a(\hat{x}^s)^\top & -c(\hat{x}^s) \end{bmatrix} \succ \begin{bmatrix} A(\hat{x}^s) \\ -\frac{1}{2}b(\hat{x}^s)^\top \end{bmatrix} \begin{bmatrix} A(\hat{x}^s) \\ -\frac{1}{2}b(\hat{x}^s)^\top \end{bmatrix}^\top.$$  

Therefore, by the virtue of Schur’s complement, $(x, k, \lambda) = (\hat{x}^s, \hat{k}, 0)$ satisfies the semidefinite constraint presented in the theorem. The proof is now complete. \hfill \square

Theorems 1 and 2 show that both the robust quadratic optimization and satisficing problems can be reformulated as tractable semidefinite optimization problems that can be solved exactly in polynomial time by, for instance, the interior point algorithm. These results heavily rely on the nature of the problem: $g$ is a biconvex quadratic function, the uncertainty set and the support are ellipsoidal, and the penalty function is quadratic in the form of squared Euclidean norm. For a more general robust conic satisficing problem of the form of Problem (3), exact tractable reformulations may not exist, and thus approximations would be needed. We consider next the case of polyhedral support sets, and we use polyhedral penalty function to penalize the deviation of the uncertain parameter $z$ from its nominal value $\hat{z}$.

### 3.2. Tractable safe approximation with affine dual recourse adaptation

Assuming again without loss of generality that $\hat{z} = 0$, we denote the optimal solution of the nominal optimization problem (3) by $\hat{x}$ and by $\hat{y}$ the corresponding optimal recourse, that is,

$$\hat{y} \in \arg\min_y \{ d^\top y : By \succeq \kappa f(\hat{x}) \}.$$  

As a consequence of our notation, we have $Z_0 = c^\top \hat{x}$. Problem (9) can be expressed in the following explicit formulation,

$$\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad d^\top y(z) \leq kp(z) \quad \forall z \in Z \\
& \quad By(z) \succeq \kappa f(x) + F(x)z \quad \forall z \in Z \\
& \quad c^\top x \leq \tau \\
& \quad x \in \mathcal{X}, \; y \in \mathbb{R}^{nz \times n_y}, \; k \in \mathbb{R}_+,
\end{align*}$$  

\[ \tag{11} \]
where $\mathcal{R}^{m,n}$ denotes the family of all functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, i.e.,

$$\mathcal{R}^{m,n} = \{y : y : \mathbb{R}^m \mapsto \mathbb{R}^n\}.$$ 

The problem remains intractable even if we restrict the recourse $y$ to a static function that does not depend on $z$. As we will reveal, we overcome this challenge by using a technique of dualizing twice similar to that in Roos et al. (2020), first over the recourse variables $y$ and then over the uncertain parameters $z$ to absorb the conic nature of the original problem into a new uncertainty set while the polyhedral support and penalty show up as linear constraints of the resultant formulation. We can thus use familiar approximation methods such as affine recourse adaptation for this resulting problem. Although feasibility is not guaranteed with such approximations even under assumptions of complete recourse (see, e.g., Bertsimas et al. 2019), our results however show that whenever the target $\tau \geq Z_0$, the affine dual recourse adaptation would always yield a feasible solution under the stated assumptions as follows.

**Assumption 2.** We assume the following:

(i) **Solvable:** The optimal nominal solutions $\hat{x}$ and $\hat{y}$ are solvable.

(ii) **Complete and bounded recourse:** For any $v \in \mathbb{R}^{n_f}$, there exists a $y \in \mathbb{R}^{n_y}$ such that $By \succeq \chi v$. Moreover, there does not exist $y \in \mathbb{R}^{n_y}$ such that $By \succeq 0$ and $d^Ty < 0$. These conditions ensure that Problem (2) is always finite and strictly feasible.

(iii) **Polyhedral support:** The uncertainty set $Z$ is a polytope

$$Z = \{z \in \mathbb{R}^{n_z} \mid Hz \leq h\},$$

for some $H \in \mathbb{R}^{n_h \times n_z}$ and $h \in \mathbb{R}^{n_h}$.

(iv) **Polyhedral penalty:** The polyhedral penalty function $p(\zeta) : \mathbb{R}^{n_z} \mapsto \mathbb{R}_+$ can be expressed as

$$p(\zeta) = \max_{(\lambda, \eta) \in \mathcal{V}} \{\lambda^T \zeta - \eta\},$$

where $\mathcal{V}$ is a polyhedral

$$\mathcal{V} = \{(\lambda, \eta) \in \mathbb{R}^{n_z} \times \mathbb{R}_+ \mid \exists \mu \in \mathbb{R}^{n_\mu} : M\lambda + N\mu + s\eta \leq t\},$$

for some $M \in \mathbb{R}^{n_m \times n_z}$, $N \in \mathbb{R}^{n_m \times n_\mu}$, and $s, t \in \mathbb{R}^{n_m}$. In addition, (i) $\mathcal{V}$ contains the origin so that $p(\zeta) \geq 0$ and $p(0) = 0$, (ii) there exists $\hat{\mu} \in \mathbb{R}^{n_\mu}$ such that $N\hat{\mu} < t$, which ensures $p(\zeta) > 0$ if $\zeta \neq 0$, and (iii) $\mathcal{V}$ is bounded, which ensures $p(\zeta) < \infty$.

We remark that the term complete recourse is an extension of the same term used in stochastic linear optimization (see, e.g., Birge and Louveaux 2011) to our model where the second stage problem is a conic optimization problem. It ensures that Problem (2) is strictly feasible. To see the strict
feasibility, let $v \in \mathbb{R}^{n_f}$ such that $v \succeq_{K} 0$, and under compete recourse, there exists a $y \in \mathbb{R}^{n_y}$ such that $By \succeq_{K} f(x) + F(x)z + v \succeq_{K} f(x) + F(x)z$. The bounded recourse condition ensures that the $g$ function of Problem (2) is always finite.

It is common to choose a polyhedral norm as the penalty function, which has a similar representation.

**Proposition 1 (Polyhedral norm).** Under Assumption 2, if the penalty function $p$ is a norm, then it has the representation

$$p(\zeta) = \max_{\lambda \in \mathbb{R}^{n_z}, \mu \in \mathbb{R}^{n_\mu}} \{\lambda^T \zeta | M\lambda + N\mu \leq t\},$$

for which

$$\{\lambda | M\lambda + N\mu \leq t\} = \{\lambda | -M\lambda + N\mu \leq t\}.$$ 

Its dual norm is given by

$$p^*(\zeta) = \min_{\mu \in \mathbb{R}^{n_\mu}, \delta \in \mathbb{R}_+} \{\delta | M\zeta + N\mu \leq \delta t\}.$$ 

**Proof.** The proof is relegated to Appendix A.

**Example 3 (Budgeted norm).** We illustrate the modeling potential of the norm-based penalty defined in Proposition 1 with a budgeted norm which computes the sum of the $n_z$ largest absolute components of an $n_z$-dimensional vector, i.e.,

$$p_r(\zeta) = \max_{S \subseteq \{1, \ldots, n_z\} : |S| = r} \sum_{i \in S} |\zeta_i|$$

so that $p_1(\zeta) = \|\zeta\|_\infty$ and $p_{n_z}(\zeta) = \|\zeta\|_1$. This can be represented as the following linear optimization problem

$$p_r(\zeta) = \max_{\lambda \in \mathbb{R}^{n_z}} \left\{ \lambda^T \zeta \mid \sum_{i \in \{1, \ldots, n_z\}} |\lambda_i| \leq \Gamma, |\lambda_i| \leq 1, \forall i \in \{1, \ldots, n_z\} \right\}$$

$$= \max_{\lambda, \mu \in \mathbb{R}^{n_z}} \left\{ \lambda^T \zeta \mid \begin{array}{l} \sum_{i \in \{1, \ldots, n_z\}} \mu_i \leq \Gamma \\
\lambda_i - \mu_i \leq 0, \forall i \in \{1, \ldots, n_z\} \\
-\lambda_i - \mu_i \leq 0, \forall i \in \{1, \ldots, n_z\} \\
\mu_i \leq 1, \forall i \in \{1, \ldots, n_z\} \end{array} \right\},$$

which satisfies the properties of the polyhedral penalty in Assumption 2.

We also remark that since second-order conic constraints can be approximated accurately via a modest sized polyhedron (Ben-Tal and Nemirovski 2001), the representation of polyhedral penalty is quite broad and can be used to approximate many different types of convex nonlinear penalty functions such as, *inter alia*, convex polynomials and $\ell_p$-norms, for $p \geq 1$. 
Under Assumption 2, we will show that the robust satisficing problem (3) with a conic constraint and a polyhedral uncertainty set admits an equivalent reformulation as a problem of a similar nature but with linear constraints and a conic uncertainty set. The following result is a precursor for obtaining the alternate formulation of the robust satisficing problem, which would enable us to obtain a safe approximation via affine dual recourse adaptation.

**Proposition 2.** Under Assumption 2, for any \( \mathbf{a} \in \mathbb{R}^n \) and \( k \geq 0 \), we have

\[
\max_{z \in \mathcal{Z}} \{ \mathbf{a}^\top z - k p(z) \} = \min_{\eta \in \mathbb{R}_+, \beta \in \mathbb{R}_+^n} \{ \beta^\top \mathbf{h} + \eta \mid (\mathbf{a} - H^\top \beta, \eta, k) \in \tilde{\mathcal{V}} \}
\]

where \( \tilde{\mathcal{V}} \) is the perspective cone of \( \mathcal{V} \) given by

\[
\tilde{\mathcal{V}} = \{ (\lambda, \eta, k) \in \mathbb{R}^n \times \mathbb{R}^2 \mid \exists \mu \in \mathbb{R}^{n_\mu} : M \lambda + N \mu + s \eta \leq tk \}.
\]

If the penalty function \( p \) is a norm, then we have

\[
\max_{z \in \mathcal{Z}} \{ \mathbf{a}^\top z - k p(z) \} = \min_{\beta \in \mathbb{R}_+^n} \{ \beta^\top \mathbf{h} \mid p^*(\mathbf{a} - H^\top \beta) \leq k \}.
\]

**Proof.** The proof is relegated to Appendix A. \( \square \)

**Theorem 3.** Under Assumption 2, for any \( \mathbf{x} \in \mathbb{R}^n \) and \( k \geq 0 \), the robust satisficing constraint

\[
g(\mathbf{x}, z) \leq k p(z), \forall z \in \mathcal{Z}
\]

is equivalent to

\[
\forall \rho \in \mathcal{P}, \exists \beta \in \mathbb{R}^n, \mu \in \mathbb{R}^{n_\mu}, \eta \in \mathbb{R} : \begin{cases} 
\rho^\top f(\mathbf{x}) + \beta^\top \mathbf{h} + \eta \leq 0 \\
M(F(\mathbf{x})^\top \rho) - H^\top \beta + N \mu + s \eta \leq tk \\
\beta \geq 0, \eta \geq 0,
\end{cases}
\]

where \( \mathcal{P} = \{ \rho \in \mathcal{K}^* \mid B^\top \rho = d \} \).

**Proof.** First, it follows from the specificity of \( g \) in (2) that

\[
g(\mathbf{x}, z) \leq k p(z), \forall z \in \mathcal{Z} \iff \max_{z \in \mathcal{Z}} \min_y \{ d^\top y - k p(z) \mid By \succeq \kappa f(\mathbf{x}) + F(\mathbf{x}) z \} \leq 0.
\]

Under complete and bounded recourse, observe that the inner minimization (over \( y \)) is strictly feasible and its objective is finite. Thus, we can transform it into a maximization problem via conic duality, that is,

\[
g(\mathbf{x}, z) \leq k p(z), \forall z \in \mathcal{Z} \iff \max_{z \in \mathcal{Z}} \max_{\rho \in \mathcal{P}} \{ \rho^\top (f(\mathbf{x}) + F(\mathbf{x}) z) - k p(z) \} \leq 0
\]

\[
\iff \max_{\rho \in \mathcal{P}} \{ \rho^\top f(\mathbf{x}) + \max_{z \in \mathcal{Z}} \{ \rho^\top F(\mathbf{x}) z - k p(z) \} \} \leq 0.
\]

Invoking Proposition 2 to transform the inner maximization (over \( z \)) to a minimization problem (over \( \beta, \mu \) and \( \eta \)) completes the proof. \( \square \)
Robust Conic Satisficing

Theorem 3 allows us to reformulate Problem (5), which is the same as Problem (11), as a classical adaptive robust linear optimization model with a conic uncertainty set as follows,

$$\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad \rho^\top f(x) + \beta(\rho)^\top h + \eta(\rho) \leq 0 \quad \forall \rho \in \mathcal{P} \\
& \quad M(F(x)^\top \rho - H^\top \beta(\rho)) + N\mu(\rho) + s\eta(\rho) \leq tk \quad \forall \rho \in \mathcal{P} \\
& \quad \beta(\rho) \geq 0, \eta(\rho) \geq 0 \quad \forall \rho \in \mathcal{P} \\
& \quad c^\top x \leq \tau \\
& \quad x \in \mathcal{X}, k \in \mathbb{R}_+, \beta \in \mathcal{R}^{nf,nh}, \mu \in \mathcal{R}^{nf,n\mu}, \eta \in \mathcal{R}^{nf,1},
\end{align*}$$

(12)

where $\beta, \mu, \eta$ replaces $y$ as the recourse variables and $\mathcal{P}$ represents the (dual) uncertainty set defined in Theorem 3. To distinguish the two different recourse variables, we may refer $y$ as the primal recourse and $(\beta, \mu, \eta)$ as the dual recourse. Comparing Problems (11) and (12), we find that, even though Problem (11) is always feasible (thanks to Assumption 2), it is not necessarily easy to construct a feasible solution, let alone computing the optimal solution. In contrast, we will argue below that Problem (12) admits an almost trivial feasible dual recourse solution. The existence of this feasible solution will then be used to support the appropriateness of approximately solving Problem (12) using affine dual recourse adaption as follows:

$$\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad \rho^\top f(x) + \beta(\rho)^\top h + \eta(\rho) \leq 0 \quad \forall \rho \in \mathcal{P} \\
& \quad M(F(x)^\top \rho - H^\top \beta(\rho)) + N\mu(\rho) + s\eta(\rho) \leq tk \quad \forall \rho \in \mathcal{P} \\
& \quad \beta(\rho) \geq 0, \eta(\rho) \geq 0 \quad \forall \rho \in \mathcal{P} \\
& \quad c^\top x \leq \tau \\
& \quad x \in \mathcal{X}, k \in \mathbb{R}_+, \beta \in \mathcal{L}^{nf,nh}, \mu \in \mathcal{L}^{nf,n\mu}, \eta \in \mathcal{L}^{nf,1},
\end{align*}$$

(13)

where $\mathcal{L}^{m,n}$ denotes the sub-class of functions in $\mathcal{R}^{m,n}$ that are affinely dependent on the inputs as follows:

$$\mathcal{L}^{m,n} = \left\{ y \in \mathcal{R}^{m,n} \mid \exists \pi \in \mathcal{R}^n, \Pi \in \mathcal{R}^{n \times m} : y(z) = \pi + \Pi z \right\}.$$

**Theorem 4.** Under Assumption 4, there exists a feasible solution for Problem (13) whenever the target $\tau \geq Z_0$. Moreover, Problem (13) is practically solvable under Assumption 4.

**Proof.** We first show that the solution $x = \hat{x}$, $\beta(\rho) = 0$, and $\eta(\rho) = 0$ robustly satisfies the first and third constraints of Problem (13). To achieve this, it suffices to derive the maximum value that the left-hand side of the first constraint could take, i.e.,

$$\max_{\rho \in \mathcal{P}} \rho^\top f(\hat{x}) = \min_y \{ d^\top y \mid By \succeq_k f(\hat{x}) \} = d^\top \hat{y} = g(\hat{x}, 0) \leq 0,$$
where the first equality is due to the strong duality (as the minimization problem is strictly feasible) and the second equality holds because of the optimality of the recourse variable $\hat{y}$ given the decision $\hat{x}$ in (10).

We next show that the remaining constraints can be satisfied because there exists $\hat{k} > 0$ such that

$$MF(\hat{x})^\top \rho + N\hat{\mu}\hat{k} \leq t\hat{k} \quad \forall \rho \in \mathcal{P}. \tag{14}$$

Indeed, since $N\hat{\mu} < t$, it implies that the set $\{\lambda \mid M\lambda + N\hat{\mu} \leq t\}$ must contain the origin in its interior. Hence, there exists a norm $\| \cdot \|$ such that

$$\{\lambda \mid \|\lambda\| \leq 1\} \subseteq \{\lambda \mid M\lambda + N\hat{\mu} \leq t\}.$$ 

Therefore, it suffices to show that there exists a finite $\hat{k} > 0$ such that

$$\max_{\rho \in \mathcal{P}} \|F(\hat{x})^\top \rho\| \leq \hat{k}.$$ 

Suppose that the dual uncertainty set $\mathcal{P}$ is unbounded for the sake of a contradiction. Then, there exists a vector $v \in \mathbb{R}^{n_f}$ such that $\max_{\rho \in \mathcal{P}} \rho^\top v$ is unbounded, and therefore its corresponding dual

$$\min_y \left\{ d^\top y \mid By \succeq_K v \right\}$$ 

must be infeasible, which contradicts with Assumption 2. Hence, $\mathcal{P}$ is a bounded set, and a finite and positive $\hat{k}$ exists. Hence, the solution $x = \hat{x}$, $\beta(\rho) = 0$, $\mu(\rho) = \hat{\mu}\hat{k}$, $k = \hat{k}$ and $\eta(\rho) = 0$ robustly satisfies the second constraint of Problem (13).

To show that Problem (13) can be expressed as a modest sized conic optimization problem, observe that since the recourse variables are restricted to affine functions, we can express Problem (13) more compactly as

$$\begin{aligned}
\text{minimize} & \quad k \\
\text{subject to} & \quad \gamma(x, k, L) + \Gamma(x, k, L)\rho \leq 0 \quad \forall \rho \in \mathcal{P} \\
& \quad x \in X, \ k \in \mathbb{R}_+, \ L \in \mathbb{R}^{(n_h+n\mu+1)\times(n_f+1)},
\end{aligned}$$

where $L$ gathers the affine dual recourse adaption coefficients of $(\beta, \mu, \eta)$ and $\gamma, \Gamma$ are appropriate affine mappings. We next show that under Assumption 2, for any $\gamma \in \mathbb{R}^{n\gamma}$ and $\Gamma \in \mathbb{R}^{n\gamma\times n_f}$, the robust counterpart of $\gamma + \Gamma\rho \leq 0$, $\forall \rho \in \mathcal{P}$ is given by the following linear conic constraint

$$\exists V \in \mathbb{R}^{n\gamma \times n\gamma} : \begin{cases}
\gamma + Vd \leq 0 \\
B(V_i)^\top \succeq_K \Gamma_i^\top \forall i \in [n_\gamma].
\end{cases}$$

Indeed, observe that the given robust constraint can be written down as

$$\max_{\rho \in \mathcal{P}} \gamma_i + \Gamma_i\rho \leq 0 \quad \forall i \in [n_\gamma] \iff \min_{v^i \in \mathbb{R}^{n\gamma}} \left\{ \gamma_i + d^\top v^i \mid Bv^i \succeq_K \Gamma_i^\top \right\} \leq 0 \quad \forall i \in [n_\gamma],$$

where the equivalence holds because the minimization problem is convex and strictly feasible. We then denote $[v^1, \ldots, v^{n\gamma}]^\top$ by $V$. Hence, the problem is practicably solvable under Assumption 1, which completes the proof. □
This result is computationally significant since, despite the difficulty to solve it exactly, if \(\tau \geq Z_0\), we can still obtain a feasible solution of Problem (13) by solving a modest sized conic optimization problem.

4. Two-stage adaptive optimization

In this section, we explore how the affine dual recourse adaptation can be used to provide safe approximations to two-stage adaptive optimization problems. We investigate a two-stage adaptive linear optimization problem under complete recourse focusing on the \(\ell_1\)-norm penalty function, \(p(\zeta) = \|\zeta\|_1\), which has been previously tackled by Long et al. (2021) in the data-driven setting. For simplicity, we first focus on the stochastic-free setting with the nominal value being \(\hat{z} = 0\). With \(K = \mathbb{R}_{+}^{n_f}\), the two-stage nominal problem can be written as:

\[
Z_0 = \minimize \ c^\top x + d^\top y \\
\text{subject to} \quad By \geq f(x) \\
\quad x \in \mathcal{X}, \ y \in \mathbb{R}^{n_y},
\]

which has the same format as Problem (1). We analogously denote by \((\hat{x}, \hat{y})\) the optimal nominal solution. Assuming further that \(\hat{z} = 0\), the robust satisficing model, which is based on Problem (7), becomes the following standard two-stage robust linear program with \((x, k)\) being the first-stage decisions and \(y\) being the recourse or second-stage decisions:

\[
\minimize \ k \\
\text{subject to} \quad c^\top x + d^\top y(z) \leq \tau + k\|z\|_1 \quad \forall z \in \mathcal{Z} \\
By(z) \geq f(x) + F(x)z \quad \forall z \in \mathcal{Z} \\
x \in \mathcal{X}, \ y \in \mathcal{L}^{n_x, n_y}, \ k \in \mathbb{R}_+.
\]

Generally, two-stage robust linear programs are not tractable and are typically solved approximately via affine primal recourse adaptation, that is,

\[
\minimize \ k \\
\text{subject to} \quad c^\top x + d^\top y(z) \leq \tau + k\|z\|_1 \quad \forall z \in \mathcal{Z} \\
By(z) \geq f(x) + F(x)z \quad \forall z \in \mathcal{Z} \\
x \in \mathcal{X}, \ y \in \mathcal{L}^{n_x, n_y}, \ k \in \mathbb{R}_+.
\]

Note that, while Problem (16) is always feasible under Assumption 2, it is not necessarily the case for the restricted problem (17) (Long et al. 2021). As a result, we consider a more flexible, non-affine primal recourse adaptation extension with extra coefficients \(q^\dagger \in \mathbb{R}^{n_y}\):

\[
y(z) = q + Qz + q^\dagger\|z\|_1.
\]
Observe that when such a non-linear primal recourse adaptation is substituted for \( y(z) \) in Problem (16), the resultant optimization problem can be written down as

\[
\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad c^\top x + d^\top (q + Qz + q^\top \| z \|_1) \leq \tau + k \| z \|_1 \quad \forall z \in Z \\
& \quad B(q + Qz + q^\top \| z \|_1) \geq f(x) + F(x)z \quad \forall z \in Z \\
& \quad k - d^\top q^\top \geq 0, \quad Bq^\top \geq 0 \\
& \quad x \in \mathcal{X}, \ q \in \mathbb{R}^{n_y}, \ Q \in \mathbb{R}^{n_y \times n_z}, \ q^\top \in \mathbb{R}^{n_y}, \ k \in \mathbb{R}_+,
\end{align*}
\]

(19)

where the two linear constraints \( k - d^\top q^\top \geq 0 \) and \( Bq^\top \geq 0 \) are added to ensure that the two robust constraints are concave in the uncertain \( z \) and consequently tractability of Problem (19). Observe that with \( q^\top = 0 \), Problem (19) is equivalent to Problem (17), implying that the former problem is a less conservative approximation.

**Theorem 5.** (Long et al. 2021). Under complete recourse, there is always a feasible solution to Problem (19) whenever the target \( \tau \geq Z_0 \).

**Proof** The proof is relegated to Appendix A. \( \square \)

Next, we derive the robust counterpart of Problem (19).

**Proposition 3.** Problem (19) is equivalent to

\[
\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad c^\top x + d^\top q + h^\top \omega_0 \leq \tau & \\
& \quad -(k - d^\top q^\top) 1 \leq H^\top \omega_0 - Q^\top d \leq (k - d^\top q^\top) 1 & \\
& \quad f_i(x) + h^\top \omega_i \leq B_iq_i^\top \quad \forall i \in [n_f] \\
& \quad -B_i q_i^\top 1 \leq H^\top \omega_i - F_i^\top(x) + Q^\top B_i^\top \leq B_i q_i^\top 1 \quad \forall i \in [n_f] \\
& \quad x \in \mathcal{X}, \ q \in \mathbb{R}^{n_y}, \ Q \in \mathbb{R}^{n_y \times n_z}, \ q^\top \in \mathbb{R}^{n_y}, \ \omega_i \in \mathbb{R}^{n_y}, \ \omega_0, \ldots, \omega_{nf} \in \mathbb{R}_+^{n_y}, \ k \in \mathbb{R}_+.
\end{align*}
\]

(20)

**Proof** The proof is relegated to Appendix A. \( \square \)

Next, we will similarly analyze the dual robust satisficing problem (12) when \( p(\zeta) = \| \zeta \|_1 \) and \( \mathcal{K} = \mathbb{R}_+^{n_f} \). From Proposition 2 with \( p^*(\zeta) = \| \zeta \|_\infty \), Problem (12) reduces to

\[
\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad c^\top x + \rho^\top f(x) + \beta(\rho)^\top h \leq \tau \quad \forall \rho \in \mathcal{P} \\
& \quad -k 1 \leq H^\top \beta(\rho) - F(x)^\top \rho \leq k 1 \quad \forall \rho \in \mathcal{P} & \\
& \quad \beta(\rho) \geq 0 \quad \forall \rho \in \mathcal{P} \\
& \quad x \in \mathcal{X}, \ k \in \mathbb{R}_+, \ \beta \in \mathbb{R}_+^{n_f}, \ \rho \in \mathcal{P},
\end{align*}
\]

(21)
where the dual uncertainty set $\mathcal{P}$ is $\{\rho \in \mathbb{R}_+^n | B^T \rho = d\}$. We note that Problem (21) and the primal robust satisficing model (16) are equivalent. However, unlike Problem (16) which may not admit a feasible affine primal recourse $y$, Problem (21) always admits a feasible affine dual recourse for $\beta$. Hence, for this problem, we are not required to come up with an ingenious idea of how to construct a non-affine recourse adaptation that can ensure feasibility. Our objective here is therefore to show that, despite being simpler, the affine dual recourse adaptation of Problem (21) is closer to the original Problem (16) compared to the previously discussed non-affine primal recourse adaptation of Problem (16) itself.

To begin with, we write down the affine recourse approximation of Problem (21) as

$$\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad c^T x + \rho^T f(x) + (\pi + \Pi \rho)^T h \leq \tau & \forall \rho \in \mathcal{P} \\
& \quad -k1 \leq H^T (\pi + \Pi \rho) - F(x)^\top \rho \leq k1 & \forall \rho \in \mathcal{P} \\
& \quad \pi + \Pi \rho \geq 0 & \forall \rho \in \mathcal{P} \\
& \quad x \in \mathcal{X}, \, \pi \in \mathbb{R}^{n_h}, \, \Pi \in \mathbb{R}^{n_h \times n_f}, \, k \in \mathbb{R}_+ \tag{22}
\end{align*}$$

where the dual recourse $\beta$ is restricted to an affine function $\pi + \Pi \rho$ of $\rho$.

**Theorem 6.** Under complete recourse, the affine dual recourse adaptation in Problem (22) is a lower bound of Problem (19).

**Proof.** First of all, we compare the variants of Problems (19) and (22) when $x$ is fixed to $x'$. We can then abbreviate $f(x')$ and $F(x')$ to $f'$ and $F'$, respectively. Besides, we let $\tau'$ denote the value of $\tau - c^\top x'$. It suffices to show that the optimal objective value of

$$\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad \rho^T f' + (\pi + \Pi \rho)^T h \leq \tau' & \forall \rho \in \mathcal{P} \\
& \quad -k1 \leq H^T (\pi + \Pi \rho) - (F')^\top \rho \leq k1 & \forall \rho \in \mathcal{P} \\
& \quad \pi + \Pi \rho \geq 0 & \forall \rho \in \mathcal{P} \\
& \quad \pi \in \mathbb{R}^{n_h}, \, \Pi \in \mathbb{R}^{n_h \times n_f}, \, k \in \mathbb{R}_+ \tag{23}
\end{align*}$$

is smaller than or equal to that of

$$\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad d^T (q + Q z + q^\top \|z\|_1) \leq k\|z\|_1 + \tau' & \forall z \in \mathbb{Z} \\
& \quad B(q + Q z + q^\top \|z\|_1) \geq f' + F' z & \forall z \in \mathbb{Z} \\
& \quad k - d^T q^\top \geq 0, \, Bq^\top \geq 0 \\
& \quad q \in \mathbb{R}^{n_y}, \, Q \in \mathbb{R}^{n_y \times n_z}, \, q^\top \in \mathbb{R}^{n_y}, \, k \in \mathbb{R}_+. \tag{24}
\end{align*}$$
If $\tau' < \min \{ d^T q \mid Bq \geq f' \} = \max_{\rho \in \mathcal{P}} \{ \rho^T f' \}$, then Problems (23) and (24) are infeasible because their first respective robust constraint cannot be satisfied (recall that $0 \in \mathcal{Z}$ and $h \geq 0$). Otherwise if $\tau' \geq \min \{ d^T q : Bq \geq f' \}$, they must be feasible thanks to Theorems 3 and 4, which hold when $\mathcal{X} = \{ x' \}$. Henceforth, we assume that $\tau'$ is sufficiently large to avoid trivialities.

It suffices to show that, for any $(q, Q, q', k)$ feasible in Problem (24), we can construct $\pi$ and $\Pi$ such that $(\pi, \Pi, k)$ is feasible in Problem (23). To achieve this, we specifically consider $\pi = w^0 \in \mathbb{R}^n_+$ and $\Pi = [w^1, \ldots, w^m] \in \mathbb{R}^n_+ \times n_f$, where $\{ w^i \}^m_{i=0}$ satisfy the following conditions.

\begin{align}
  d^T q + h^T w^0 & \leq \tau' \quad (25a) \\
  -(k - d^T q^i)1 & \leq H^T w^0 - Q^T d \leq (k - d^T q^i)1 \\
  f' + h^T w^i & \leq B_i q \quad \forall i \in [n_f] \quad (25b) \\
  -B_i q^1 & \leq H^T w^i - (F'_i)^T + Q^T B_i^T \leq B_i q^1 \quad \forall i \in [n_f] \quad (25c)
\end{align}

Note that the existence $\{ w^i \}^m_{i=0}$ is guaranteed by Proposition 3 and Theorem 8.

We will now show that this choice of $(\pi, \Pi, k)$ satisfies all constraints in Problem (23). By a usual duality argument, the first robust constraint of this problem is satisfied if and only if there exists a vector $\nu \in \mathbb{R}^{n_y}$ such that

\[ \pi^T h + \nu^T d \leq \tau' \quad \text{and} \quad B\nu \geq f' + \Pi^T h. \]

Thanks to the inequalities (25a) and (25c), $\nu$ can be simply chosen as $q$.

Similarly, the next two constraints of Problem (23), namely $-k1 \leq H^T (\pi + \Pi \rho) - (F')^T \rho \leq k1$, are robustly satisfied for all $\rho \in \mathcal{P}$ if and only if there exist matrices $\Phi, \Psi \in \mathbb{R}^{n_y \times n_z}$ such that

\[ \begin{cases} 
  H^T \pi + \Phi^T d \leq k1 \\
  B\Phi \geq \Pi^T H - F'
\end{cases} \quad \text{and} \quad \begin{cases} 
  -H^T \pi + \Psi^T d \leq k1 \\
  B\Psi \geq F' - \Pi^T H.
\end{cases} \]

In this case, we can choose, $\Phi = q^1 1^T_{n_z} - Q$ and $\Psi = q^1 1^T_{n_z} + Q$. Observe that

\[ H^T \pi + \Phi^T d = H^T w^0 + (q^i)^T d - Q^T d \leq k1 \]

\[ B\Phi = \Pi^T H + F' = [B_i q^1 1^T_{n_z} - B_i Q - (w^i)^T H + F'_i]_{i=1}^{n_f} \geq 0, \]

where the inequalities holds due to the inequalities (25b) and (25d), as desired. Similarly,

\[ -H^T \pi + \Psi^T d = -H^T w^0 + (q^i)^T d + Q^T d \leq k1 \]

\[ B\Psi = \Pi^T H - F' = [B_i q^1 1^T_{n_z} + B_i Q + (w^i)^T H - F'_i]_{i=1}^{n_f} \geq 0. \]

Finally, the third constraint of (23) is trivially satisfied because of the non-negativity of $\pi$, $\Pi$ and $\rho$. Hence, Problem (23) is a lower bound of Problem (24), and the theorem follows. \[\square\]
4.1. Data-driven adaptive conic optimization

We can generalize our earlier results to the data-driven scheme of Long et al. (2021) where the data comprises $\Omega$ samples, and $\hat{z}^\omega$, $\omega \in [\Omega]$, denotes each known realization of $z$. Consistently with the classical data-driven stochastic optimization problem framework, we consider the nominal problem of the form that minimizes the total first stage and the average second stage costs over the $\Omega$ realizations as follows:

$$ Z_0 = \text{minimize } c^T x + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} d^T y^\omega $$
subject to $By \succeq_K f(x) + F(x)\hat{z}^\omega \quad \forall \omega \in [\Omega]$  
$x \in \mathcal{X}$, $y_1, \ldots, y^\Omega \in \mathbb{R}^{n_y}$.  

We assume that the optimal solution of Problem (26) exists, and we denote it by $\hat{x}, \hat{y}_1, \ldots, \hat{y}^\Omega$. Thus, $Z_0 = c^T \hat{x} + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} d^T \hat{y}^\omega$.

For consistency with the previous adaptive optimization framework in which the nominal uncertainty value is at the origin, for each $\omega \in [\Omega]$, we define the shifted support set

$$ Z_\omega = \{\zeta \in \mathbb{R}^{n_z} | H \zeta \leq h^\omega\} $$
where $h^\omega = h - H \hat{z}^\omega$, and the function $g^\omega : \mathcal{X} \times Z_\omega \rightarrow \mathbb{R}$,

$$ g^\omega(x, z) = \text{minimize } d^T y \quad \text{subject to } By \succeq_K f^\omega(x) + F(x)z $$
where $f^\omega(x) = f(x) + F(x)\hat{z}^\omega$. We also define the joint shifted support set as

$$ \hat{Z} = Z_1 \times \cdots \times Z_\Omega $$
and the function $\tilde{g} : \mathcal{X} \times \hat{Z}$,

$$ \tilde{g}(x, (z^1, \ldots, z^\Omega)) = \frac{1}{\Omega} \sum_{\omega \in [\Omega]} g^\omega(x, z^\omega) $$
so that

$$ Z_0 = \text{minimize } c^T x + \tilde{g}(x, (0, \ldots, 0)) $$
subject to $x \in \mathcal{X}$.  

We provide a different perspective from Long et al. (2021) to derive the robust satisficing model without the need of introducing an ambiguity set of uncertain probability distributions. In the data-driven setting, the penalty function $\tilde{p} : \mathbb{R}^{n_z \times \Omega} \rightarrow \mathbb{R}$ is also defined over the sample averages as follows

$$ \tilde{p}(\zeta^1, \ldots, \zeta^\Omega) = \frac{1}{\Omega} \sum_{\omega \in [\Omega]} p(\zeta^\omega). $$
Hence, for a chosen target $\tau \geq Z_0$, the robust satisficing model on the lines of Problem (3) can be expressed as

$$\begin{align*}
\text{minimize} \quad & k \\
\text{subject to} \quad & c^\top x + \bar{g}(x, (z^1, \ldots, z^\Omega)) \leq \tau + k\bar{p}(z^1, \ldots, z^\Omega), \quad \forall (z^1, \ldots, z^\Omega) \in \bar{Z} \\
& x \in \mathcal{X}, \quad k \in \mathbb{R}_+,
\end{align*}$$

or equivalently

$$\begin{align*}
\text{minimize} \quad & k \\
\text{subject to} \quad & c^\top x + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} (g^\omega(x, z^\omega) - kp(z^\omega)) \leq \tau \quad \forall (z^1, \ldots, z^\Omega) \in \bar{Z} \\
& x \in \mathcal{X}, \quad k \in \mathbb{R}_+,
\end{align*}$$

which recovers the data-driven robust satisficing model of Long et al. (2021). Our result is however more general than that in Long et al. (2021) since it covers polyhedral penalty functions beyond $\ell_1$-norm, and accommodates second-stage conic optimization problems beyond linear optimization.

**Theorem 7.** Under Assumption 2, the approximation of Problem (30) via affine dual recourse adaptation is given as:

$$\begin{align*}
\text{min} \quad & k \\
\text{s.t.} \quad & c^\top x + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} v_\omega \leq \tau \\
& \rho^\top f^\omega(x) + (h^\omega)^\top \beta^\omega(\rho) + \eta^\omega(\rho) \leq v_\omega \quad \forall \rho \in \mathcal{P}, \omega \in [\Omega] \\
& M (F(x)^\top \rho - H^\top \beta^\omega(\rho)) + N \mu^\omega(\rho) + s\eta^\omega(\rho) \leq tk \quad \forall \rho \in \mathcal{P}, \omega \in [\Omega] \\
& \beta^\omega(\rho) \geq 0, \eta^\omega(\rho) \geq 0 \quad \forall \rho \in \mathcal{P}, \omega \in [\Omega] \\
& x \in \mathcal{X}, \quad k \in \mathbb{R}_+, \quad v \in \mathbb{R}^\Omega, \quad \beta^1, \ldots, \beta^\Omega \in \mathcal{L}^{n_f, n_h}, \quad \mu^1, \ldots, \mu^\Omega \in \mathcal{L}^{n_f, n_h}, \quad \eta^1, \ldots, \eta^\Omega \in \mathcal{L}^{n_f, 1}
\end{align*}$$

where $\mathcal{P} = \{ \rho \in \mathcal{K}^* \mid B^\top \rho = d \}$. Additionally, the problem is feasible whenever the target $\tau \geq Z_0$ and is practically solvable under Assumption 4.

**Proof** The proof is relegated to Appendix A. □

We now consider a special, but an important case where each function $g^\omega$, $\omega \in [\Omega]$ evaluates the maximum of $n_f$ biaffine functions and can be expressed as

$$\begin{align*}
g^\omega(x, z) &= \text{minimize} \quad y \\
\text{subject to} \quad & 1y \geq f^\omega(x) + F(x)z \\
& y \in \mathbb{R}.
\end{align*}$$
Theorem 8. Suppose each function $g^\omega$, $\omega \in [\Omega]$ is represented as Problem (32), then Problems (30) and (31) are equivalent.

Proof. Observe that the dual uncertainty set is now a simplex, $\mathcal{P} = \{\rho \in \mathbb{R}^n_+ | 1^\top \rho = 1\}$. It has well been shown (see, e.g., Zhen et al. 2018) that if the uncertainty set is a simplex, then approximation via affine recourse adaption is exact.

5. Applications and computational studies

In this section, we illustrate the improved performance of the robust satisficing model over the classical robust model with an example for each of the three variants of the problems considered in Sections 3 and 4, i.e., the quadratic problem with exact reformulation, and the biconvex as well as two-stage linear optimization problems with affine dual adaptation. Our results in Sections 5.1 and 5.2 were obtained using Mosek 9.2.38 together with YALMIP modeling language (Löfberg 2004), whereas the results in Section 5.3 were obtained using Gurobi 9.1.1 with RSOME (Robust Stochastic Optimization Made Easy) modeling language (Chen et al. 2020) and Python 3.7.7. All experiments were conducted on an Intel Core i7 2.7GHz MacBook with 16GB of RAM.

5.1. Growth-optimal portfolios

When the constraint is quadratic and the uncertainty set is ellipsoidal, we demonstrate with a growth-optimal portfolio example that the exact semidefinite reformulation of the robust satisficing model numerically performs better than that of the robust model. Consider an investor who aims to accumulate wealth from trading asset by maximizing a logarithmic utility function. We let $x_i$, $i \in [n]$, denote the proportion of capital allocated to the $i^{th}$ asset, and we impose the budget constraint $1^\top x = 1$ and the non-negativity constraint $x \geq 0$ to disallow short-selling. We henceforth denote by $\mathcal{X}$ our (simplex) feasible set of portfolios and write down the utility maximization problem as

$$\text{maximize } \mathbb{E}_\mathcal{P} \left[ \log(1 + x^\top \tilde{r}) \right]$$

subject to $x \in \mathcal{X}$,

where $\tilde{r} \sim \mathcal{P}$ denotes a random vector of asset returns. Assuming that the asset returns are serially independent and identically distributed, the logarithmic utility function is of a particular interest to long-term investors because, if $\mathbb{E}_\mathcal{P} \left[ \log(1 + x^\top \tilde{r}) \right] > 0$, then (with probability one) a fixed-mix strategy that is underlied by $x$ achieves infinite wealth in the long run. On the other hand, if the expected utility is strictly negative, then the same fixed-mix strategy will eventually lead investors to ruin.
Any optimal solution of the above optimization problem is known as the growth-optimal portfolio. Following Rujeerapaiboon et al. (2016), we approximate the logarithmic function using a second-order Taylor expansion around one, resulting in

$$\begin{align*}
\text{maximize} & \quad x^\top \mu - \frac{1}{2} x^\top (\Sigma + \mu \mu^\top) x \\
\text{subject to} & \quad x \in \mathcal{X},
\end{align*}$$

where $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{S}_+^n$ denote the mean vector and the covariance matrix of the asset return distribution $\mathbb{P}$, respectively. As portfolio optimization problems are very sensitive to the estimation errors in $\mu$, we may robustify Problem (33) by seeking a portfolio $x$ that is robustly optimal in

$$\begin{align*}
\text{minimize} & \quad \max_{z \in \mathcal{E}_r} h(x, z) \\
\text{subject to} & \quad x \in \mathcal{X},
\end{align*}$$

where $z \in \mathbb{R}^n$ represents an uncertain vector belonging to the ellipsoidal uncertainty set $\mathcal{E}_r$, $r \geq 0$, which perturbs $\mu$ and $h(x, z) = \frac{1}{2} x^\top (\Sigma + (\mu + z)(\mu + z)^\top) x - x^\top (\mu + z)$. The corresponding robust satisficing investment problem can be formulated as

$$\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad h(x, z) \leq \tau + kz^\top z \quad \forall z \in \mathbb{R}^n \\
& \quad x \in \mathcal{X}, \quad k \in \mathbb{R}_+.
\end{align*}$$

To facilitate the comparisons between (34) and (35), we derive their respective robust counterparts in the following propositions.

**Proposition 4.** For any $r > 0$, Problem (34) is equivalent to

$$\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^\top \Sigma x + \frac{x_0}{2} \\
\text{subject to} & \quad \begin{bmatrix}
\lambda I_n & x & x \\
 x^\top & x_0 + 2x^\top \mu - \lambda r & x^\top \mu \\
 x^\top & x^\top \mu & 1
\end{bmatrix} \succeq 0 \\
& \quad x \in \mathcal{X}, \quad x_0 \in \mathbb{R}, \quad \lambda \in \mathbb{R}_+.
\end{align*}$$

**Proof** The proof is relegated to Appendix A.
**Proposition 5.** Problem (35) is equivalent to

\[
\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad \begin{bmatrix} 2kI_n & x & x \\ x^\top & x_0 + 2x^\top \mu & x^\top \mu \\ x^\top \mu & 1 \end{bmatrix} \succeq 0 \\
& \quad \frac{1}{2} x^\top \Sigma x + x_0 \leq \tau \\
& \quad x \in \mathcal{X}, \; x_0 \in \mathbb{R}, \; k \in \mathbb{R}_+.
\end{align*}
\tag{36}
\]

**Proof.** The proof is relegated to Appendix A.

Furthermore, it can be shown that the equally weighted portfolio \( x = \frac{1}{n}1 \) is increasingly close to being optimal in (36) as the target \( \tau \) increases (i.e., as the investor becomes increasingly risk-averse).

A similar observation in a data-driven setting can be found in Mohajerin Esfahani and Kuhn (2018) and Long et al. (2021), among others. We refer to DeMiguel et al. (2007) for the thorough statistical comparison between the equally weighted portfolio and other investment strategies.

**Theorem 9.** Denoting by \( k^*(\tau) \) the optimal objective value of Problem (36) for a given target \( \tau \) and by \( \bar{k}^*(\tau) \) the optimal objective value of the same problem restricted with \( x = \frac{1}{n}1 \), we have that

\[
\lim_{\tau \to \infty} \bar{k}^*(\tau) - k^*(\tau) = 0.
\]

**Proof.** The proof is relegated to Appendix A.

**Evaluation and discussion of results:** Our asset universe consists of \( n = 8 \) assets with the following means and variances

\[
\mu = [0.12, 0.16, 0.14, 0.13, 0.15, 0.12, 0.14, 0.15]^\top,
\]

\[
\text{diag}(\Sigma) = [0.18^2, 0.22^2, 0.20^2, 0.16^2, 0.14^2, 0.10^2, 0.14^2, 0.19^2]^\top.
\]

We suppose further that the first four assets (and the last four) are from the same industrial sector and are thus positively correlated with all pairwise correlations equal to 0.1 and that correlations between assets from different sectors are –0.1. We are now ready to compare the robust and the robust satisficing investment problems. To this end, for any fixed \( r > 0 \), which characterizes the radius of the uncertainty set in Problem (34), we solve the robust counterpart from Proposition 4. We denote the optimal objective value and the optimal solution by \( Z_r \) and \( x_r^{rb} \), respectively. Intuitively, we can interpret \( Z_r \) as the optimal (minimal) worst-case risk. We then solve the robust satisficing problem (36) by setting the target risk \( \tau = Z_r \) and denote the optimal solution by \( x_r^{st} \). Then, compute the expected shortfall of both the robust and the robust satisficing solution from

\[
\mathbb{E} \left[ h(x_r^{rb}, \tilde{z}) - Z_r \mid h(x_r^{rb}, \tilde{z}) > Z_r \right] \quad \text{and} \quad \mathbb{E} \left[ h(x_r^{st}, \tilde{z}) - Z_r \mid h(x_r^{st}, \tilde{z}) > Z_r \right]
\]
as well as their probability of ruin

\[ \mathbb{P} \left[ \mathbb{E} \left[ \log (1 + \tilde{r}^\top x^{rb}) \mid \tilde{r} \sim \mathcal{N}(\mu + \tilde{z}, \Sigma) \right] < 0 \right] \quad \text{and} \quad \mathbb{P} \left[ \mathbb{E} \left[ \log (1 + \tilde{r}^\top x^{st}) \mid \tilde{r} \sim \mathcal{N}(\mu + \tilde{z}, \Sigma) \right] < 0 \right] \]

via simulation from 10^5 independent realizations of \( \tilde{z} \) generated from \( \mathcal{N}(0, sI_n) \), for some \( s > 0 \) (and 10^5 independent realizations of \( \tilde{r} \) for each realization of \( \tilde{z} \)). Obtained results are shown in Figure 1, and they are clearly in favour of the robust satisficing solutions. Similar observations can be made for different choices of \( \mu \) and \( \Sigma \).

![Figure 1](image-url)  
Figure 1  Expected shortfalls and probabilities of ruin of the robust optimization (\( \text{rob opt} \)) and the robust satisficing (\( \text{rob sat} \)) portfolios.

5.2. Log-sum-exp optimization

Next, we consider robust optimization and satisficing problems involving biconvex constraints (cf. Example 1). Particularly, we consider a robust optimization problem

\[ Z_r = \text{minimize} \quad 1^\top x \]
\[ \text{subject to} \quad g_i(x, z) \leq 0 \quad \forall z \in \mathcal{U}_i, \; i \in [m] \]
\[ x \in \mathbb{R}^{n_x}, \]

where \( g_i(x, z) = \log(\exp((-1 + R_i^\top z)^\top x) + \exp((-1 + S^i z)^\top x)) \) for some matrices \( R_i, S^i \in \mathbb{R}^{n_x \times n_z} \).

Since \( g_i \)'s are convex in the uncertain \( z \), it is typically not possible to derive the exact robust counterpart by the standard duality argument (Ben-Tal and Nemirovski 1998) and instead one might
need to resort to an approximation scheme, see e.g. Roos et al. (2020). For the robust satisficing formulation, we choose the penalty function \( p(\zeta) = \|\zeta\|_1 \) and hence consider

\[
\begin{align*}
\text{minimize} & \quad 1^\top k \\
\text{subject to} & \quad g_i(x, z) \leq k_i \|z\|_1 \quad \forall z \in \mathcal{Z}, \ i \in [m] \\
& \quad 1^\top x \leq \tau \\
& \quad x \in \mathbb{R}^n, \ k \in \mathbb{R}_+^m,
\end{align*}
\]

where \( \tau \geq Z_0 \) is the prescribed target objective. As discussed in Section 3, this problem can be solved approximately using affine dual recourse adaptation. In the following, we assume that \( \hat{z} = 0 \) so that the nominal problem is bounded, the support set is given by \( \mathcal{Z} = \{z \in \mathbb{R}^{n_z} : \|z\|_\infty \leq 1\} \), and the uncertainty set is \( \mathcal{U}_r = \{z \in \mathbb{R}^{n_z} : \|z\|_\infty \leq r\}, \ r \in [0, 1] \).

We can cast the constraint functions \( g_i \)'s in our conic framework as

\[
\begin{align*}
g_i(x, z) = \text{minimize} & \quad y \\
\text{subject to} & \quad \begin{bmatrix} -1 & 1^\top x \end{bmatrix} y \preceq_K \begin{bmatrix} 1^\top x \end{bmatrix} + \begin{bmatrix} -x^\top R^i \end{bmatrix} z \\
& \quad \begin{bmatrix} 0^\top & -1 \end{bmatrix} y \preceq \begin{bmatrix} 0^\top 
\end{bmatrix},
\end{align*}
\]

where the cone \( K \) is given by \( \text{cl}(\{v \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^3 : v_3 \log(\exp(v_1/v_3) + \exp(v_2/v_3)) \leq 0\}) \), which is representable as an intersection of multiple exponential cones.

**Evaluation and discussion of results:** We choose \( n_x = m = 20 \) and \( n_z = 5 \). We randomly generate \( R^i \) and \( S^i \) as matrices with sparse density 0.1 whose non-zero elements are independently and uniformly picked from the unit interval. To facilitate the comparison between the robust and the robust satisficing models, for a fixed radius \( r \in [0, 1] \) of the robust uncertainty set, we solve the robust log-sum-exp problem *exactly* by enumerating all vertices of the uncertainty set, i.e., we solve

\[
Z_r = \text{minimize} \quad 1^\top x \\
\text{subject to} & \quad g_i(x, z) \leq 0 \quad \forall z \in \{-r, +r\}^{n_z} \\
& \quad x \in \mathbb{R}^{n_x}
\]

and denote the resulting robustly optimal solution by \( x^{\text{rb}}_r \). Subsequently, we solve the robust satisficing problem only *approximately* using affine dual recourse adaptation while setting the target objective \( \tau \) to \( Z_r \) and denote the optimal solution by \( x^{\text{st}}_r \). Finally, we generate \( 10^5 \) independent realizations of \( \hat{z} \) (whose components are independent and drawn uniformly from \([0, 1]\)) to compute the *normalized probability of constraint violation* of \( x^{\text{rb}}_r \) and \( x^{\text{st}}_r \):

\[
\frac{1}{m} \sum_{i=1}^{m} \mathbb{P} [g_i(x^{\text{rb}}_r, \hat{z}) > 0] \quad \text{and} \quad \frac{1}{m} \sum_{i=1}^{m} \mathbb{P} [g_i(x^{\text{st}}_r, \hat{z}) > 0]
\]
as well as their total expected shortfall:

\[
\sum_{i=1}^{m} \mathbb{E}[g_i(x_{ir}, \tilde{z}) | g_i(x_{ir}, \tilde{z}) > 0] \quad \text{and} \quad \sum_{i=1}^{m} \mathbb{E}[g_i(x_{is}, \tilde{z}) | g_i(x_{is}, \tilde{z}) > 0].
\]

Results from a hundred realizations of \(\{(R^i, S^i)\}_{i=1}^{m}\) are reported in Figure 3, where we see the robust satisficing solutions stochastically dominate the robust solutions in both performance metrics.

![Figure 2](image-url)  
**Figure 2** Normalized probabilities of constraint violation and total expected shortfalls of the robust optimization (rob opt) and the robust satisficing (rob sat) log-sum-exp solutions. Bold lines report mean values, whereas the shaded areas refer to the 10%-90% percentile ranges.

### 5.3. Adaptive network lot-sizing

Next, we present a network lot-sizing example similar to Bertsimas and de Ruiter (2016) and de Ruiter et al. (2018). Suppose that there are \(n\) nodes, each of which faces a random demand \(z_i, i \in [n]\). Throughout, the support set of \(z\) is assumed to be a hyperrectangle, \(i.e., \mathcal{Z} = \{z \in \mathbb{R}_+^n \mid z \leq \bar{z}\}\). The initial stock \(x_i \geq 0\) at each node is to be determined prior to the realization of the random demands. Similarly, we impose that \(\mathcal{X} = \{x \in \mathbb{R}_+^n \mid x \leq \bar{x}\}\). After observing the demand, we can transport stock \(y_{ij} \geq 0\) from node \(i\) to node \(j\). To ensure that the demands can always be fulfilled, we allow for an emergency order \(w_i \geq 0\) to be made at each node. Initial and emergency orders are purchased at the unit costs \(c_i \geq 0\) and \(\ell_i \geq c_i\), respectively, while the unit transportation costs are denoted by \(t_{ij} \geq 0\). It is assumed that \(t_{ij}\) is equal to the distance between the two nodes, and hence \(t_{ij} = t_{ji}\). 


First, we present the robust variant of the lot-sizing problem

\[
Z_r = \min \quad c^\top x + x_0 \\
\text{subject to} \quad x + Y(z)^\top 1 - Y(z)1 + w(z) \geq z \quad \forall z \in \mathcal{U}_r \\
\langle T, Y(z) \rangle + \ell^\top w(z) \leq x_0 \quad \forall z \in \mathcal{U}_r \\
Y(z) \geq 0, w(z) \geq 0 \quad \forall z \in \mathcal{U}_r \\
x \in \mathcal{X}, x_0 \in \mathbb{R}, Y \in \mathcal{R}^{n \times n}, w \in \mathcal{R}^{n, n},
\]

where \( \mathcal{U}_r = \{z \in \mathbb{R}^n_+ \mid z \leq \bar{z}, 1^\top z \leq r\} \) denotes the uncertainty set and the epigraph variable \( x_0 \) captures the worst-case transportation and emergency purchase costs. Similar to the proof of Theorem 3, we can dualize Problem (37) twice (the first time over the second-stage decisions \( Y, w \) and the second time over the uncertain demands \( z \)) to obtain an alternative formulation.

**Proposition 6.** Problem (37) is equivalent to

\[
Z_r = \min \quad c^\top x + x_0 \\
\text{subject to} \quad \beta(\rho)^\top \bar{z} + \hat{\beta}(\rho)r \leq \rho^\top x + x_0 \quad \forall \rho \in \mathcal{P} \\
\beta(\rho) + \hat{\beta}(\rho)1 \geq \rho \quad \forall \rho \in \mathcal{P} \\
\beta(\rho) \geq 0, \hat{\beta}(\rho) \geq 0 \quad \forall \rho \in \mathcal{P} \\
x \in \mathcal{X}, x_0 \in \mathbb{R}, \beta \in \mathcal{R}^{n, n}, \hat{\beta} \in \mathcal{R}^{n, 1},
\]

where the dual uncertainty set \( \mathcal{P} \) is given by \( \{\rho \in \mathbb{R}^n_+ \mid \rho \leq \ell, \rho 1^\top - 1 \rho^\top \leq T\} \).

**Proof.** We relegate the proof to Appendix A. □

Due to the linearity of both formulations, we can solve Problems (37) and (38) approximately using affine adaptations on the primal and dual recourse, respectively. These approximations however turn out to be equivalent.

**Proposition 7.** Problems (37) and (38) attain the same objective value when solved approximately using affine recourse adaptation.

**Proof.** We relegate the proof to Appendix A. □

For the robust satisficing variant of the problem, we consider

\[
\begin{align*}
\text{minimize} & \quad k \\
\text{subject to} & \quad x + Y(z)^\top 1 - Y(z)1 + w(z) \geq z \quad \forall z \in \mathcal{Z} \\
& \quad c^\top x + \langle T, Y(z) \rangle + \ell^\top w(z) \leq \tau + k\|z\|_1 \quad \forall z \in \mathcal{Z} \\
& \quad Y(z) \geq 0, w(z) \geq 0 \quad \forall z \in \mathcal{Z} \\
& \quad x \in \mathcal{X}, k \in \mathbb{R}_+, Y \in \mathcal{R}^{n \times n}, w \in \mathcal{R}^{n, n},
\end{align*}
\]
where $\tau \geq Z_0 = 0$ is the prescribed target objective. The dualized formulation \cite{12} corresponding to this problem is explicitly given below.

**Proposition 8.** Problem (39) is equivalent to

$$
\begin{align*}
\text{minimize} & \quad k \\
\text{subjectto} & \quad (c - \rho)^\top x + \beta(\rho)^\top \bar{z} \leq \tau \quad \forall \rho \in \mathcal{P} \\
& \quad \beta(\rho) + k1 \geq \rho \quad \forall \rho \in \mathcal{P} \\
& \quad \beta(\rho) \geq 0 \quad \forall \rho \in \mathcal{P} \\
& \quad x \in \mathcal{X}, k \in \mathbb{R}_+, \beta \in \mathcal{R}^{n,n},
\end{align*}
$$

where the dual uncertainty set $\mathcal{P}$ is given by $\{\rho \in \mathbb{R}^n_r : \rho \leq \ell, \rho 1^\top - 1\rho^\top \leq T\}$.

**Proof.** We relegate the proof to Appendix A.

**Proposition 9.** Problems (39) and (40) attain the same objective value when solved approximately using affine recourse adaptation.

**Proof.** We relegate the proof to Appendix A.

**Evaluation and discussion of results:** We consider a network of $n = 20$ nodes with $\mathcal{X} = \mathcal{Z} = [0, 20]^n$. To ensure suitable variability among the nodes, the components of the initial ordering cost $c$ and emergency cost $\ell$ are generated uniformly from $[8, 10]$ and $[18, 20]$ respectively. We then select the node locations randomly from $[0, 10]^2$ and accordingly compute the (Euclidean) distance matrix $T$. For the evaluation of the robust lot-sizing models, we vary $r \geq 0$ and approximate $x_r^{\text{rb}}$ by solving either solving (37) or (38) using affine recourse adaptation. Similarly for the robust satisficing models, we vary $\tau \geq 0$ and determine approximate robust satisficing solutions $x_r^{\text{st}}$ from either (39) or (40). We then compare their respective first-stage costs (i.e., $c^\top x_r^{\text{rb}}$ and $c^\top x_r^{\text{st}}$) with the expected total costs:

$$
\mathbb{E} \left[ c^\top x_r^{\text{rb}} + \min_{Y \geq 0, w \geq 0} \left\{ \langle Y, T \rangle + \ell^\top w \mid x_r^{\text{rb}} + Y 1 - Y 1 + w \geq \bar{z} \right\} \right], \quad \text{and}
$$

$$
\mathbb{E} \left[ c^\top x_r^{\text{st}} + \min_{Y \geq 0, w \geq 0} \left\{ \langle Y, T \rangle + \ell^\top w \mid x_r^{\text{st}} + Y 1 - Y 1 + w \geq \bar{z} \right\} \right],
$$

from a hundred independent realizations of $\bar{z} = \bar{z}^{\text{tot}} \bar{z}' / (1_n^\top \bar{z}')$, where $\bar{z}^{\text{tot}}$ representing the total demand is drawn uniformly from $[20\sqrt{n}, 40\sqrt{n}]$ while each component $\bar{z}'_i$ is independently and uniformly drawn from $[0, 1]$. This is primarily to enforce possible correlations between the demands.

Figure 3 (left) shows the relative out-of-sample performance of the robust and robust satisficing models by comparing the expected total cost for the same first-stage cost. The experiment is carried out multiple times, each with different $(c, \ell, T)$. We arbitrarily highlight the results of a single run in bold curves and show those of another four runs in dashed curves. The expected total cost appears
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Figure 3 Costs and computational times (in seconds) of the robust optimization (rob opt) and the robust satisficing (rob sat) lot-sizing solutions.

to be minimized when the first-stage cost is around 1,150. When the manager is less committed to making purchases in the first stage, she has to exorbitantly compensate in the second stage via transshipment or emergency orders. On the other hand, if the manager is overly committed in the first stage, the total cost could still be high because of the excessive advance purchases. Figure 3 (right) compares the computational times required by Gurobi & RSOME in logarithmic scale of the primal robust problem (37), dual robust problem (38), primal robust satisficing problem (39) and dual robust satisficing problem (40) by varying the number of nodes n \in \{10, 15, \ldots, 100\}. The primal models (37) and (39) cannot be solved within a time limit of one hour when the number of nodes exceed 80. When n = 100, the dual robust model (38) takes about 15 minutes, whereas the dual robust satisficing model (40) takes only about 30 seconds. The better efficiency of the dual approaches was first observed and explained in Bertsimas and de Ruiter (2016).

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References


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A. Proofs of Results

Proof of Proposition Suppose that \( p \) is a norm function and define
\[
J = \bigcup_{\zeta \in \mathbb{R}^n_+} \arg \max \{ \lambda^\top \zeta - \eta \}.
\]
We will first show that \( (\lambda^*, \eta^*) \in J \) implies \( \eta^* = 0 \), that is, \( \eta \) is superfluous in the optimization problem underlying the definition of \( p \). Suppose otherwise for the sake of a contradiction that there exists \( (\zeta^*, \lambda^*, \eta^*) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \) such that \( (\lambda^*, \eta^*) \in \arg \max_{(\lambda, \eta) \in V} \{ \lambda^\top \zeta^* - \eta \} \). It follows that 
\[
p(\zeta^*) = (\lambda^*)^\top \zeta^* - \eta^* \quad \text{and, as } p \text{ is a norm, } p(2\zeta^*) = 2(\lambda^*)^\top \zeta^* - 2\eta^*.
\]
Note also that
\[
p(2\zeta^*) = \max_{(\lambda, \eta) \in V} \{ \lambda^\top (2\zeta^*) - \eta \} \geq (\lambda^*)^\top (2\zeta^*) - \eta^*.
\]
By comparing \( p(2\zeta^*) \) with its lower bound, we find \( \eta^* \leq 0 \), reaching hence a contradiction. In conclusion, \( \eta \) always vanish at optimality, and we can assume that \( s = 0 \) without any loss of generality:
\[
p(\zeta) = \max_{\lambda \in \mathbb{R}^n_+, \mu \in \mathbb{R}^n_+} \{ \lambda^\top \zeta \mid M\lambda + N\mu \leq t \}.
\]
Since \( p(\zeta) = p(-\zeta) \) for all \( \zeta \in \mathbb{R}^n_+ \), the polytope \( Q_1 = \{ \lambda \mid M\lambda + N\mu \leq t \} \) must be identical to the polytope \( Q_2 = \{ \lambda \mid -M\lambda + N\mu \leq t \} \). Otherwise, suppose \( \lambda^* \in Q_1 \) but \( \lambda^* \notin Q_2 \), then by a separating hyperplane argument there would exist a vector \( \zeta^* \in \mathbb{R}^n_+ \) such that
\[
p(\zeta^*) = \max_{\lambda \in Q_1} \{ \lambda^\top \zeta^* \} \geq \lambda^{\text{star}}^\top \zeta^* > \max_{\lambda \in Q_2} \{ \lambda^\top \zeta^* \} = p(-\zeta^*),
\]
which is a contradiction. Likewise, similar contradiction can be established if \( \lambda^* \in Q_2 \) but \( \lambda^* \notin Q_1 \). Hence, \( Q_1 = Q_2 \). Next, we derive the dual norm \( p^* \):
\[
p^*(\zeta) = \max_{\omega \in \mathbb{R}^n_+} \{ \omega^\top \zeta \mid p(\omega) \leq 1 \}
= \max_{\omega \in \mathbb{R}^n_+} \{ \omega^\top \zeta \mid \lambda^\top \omega \leq 1, \forall (\lambda, \mu) : M\lambda + N\mu \leq t \}
= \max_{\omega \in \mathbb{R}^n_+, \alpha \in \mathbb{R}^n_+} \{ \omega^\top \zeta \mid \alpha^\top t \leq 1, M^\top \alpha = \omega, N^\top \alpha = 0 \}
= \max_{\alpha \in \mathbb{R}^n_+} \{ \alpha^\top M\zeta \mid \alpha^\top t \leq 1 \}
= \min_{\mu \in \mathbb{R}^n_+, \delta \in \mathbb{R}_+} \{ \delta \mid M\zeta + N\mu \leq \delta t \},
\]
where the third maximization problem constitutes a robust counterpart of the second and the fifth equation holds due to the standard linear optimization duality argument.

Proof of Proposition We first consider \( k > 0 \). From the definition of \( p \) in Assumption, we find
\[
\max_{z \in \mathcal{Z}} \{ a^\top z - kp(z) \} = \max_{z \in \mathcal{Z}} \min_{(\lambda, \eta) \in V} \{ (a - k\lambda)^\top z + k\eta \}.
\]
Since this is linear optimization problem, by the standard linear optimization duality argument, this latter maximization problem can be expressed as

\[
\max_{x \in \mathcal{Z}} \min_{(\lambda, \eta) \in \mathcal{V}} \left\{ (a - k\lambda)^\top z + k\eta \right\}
\]

\[
= \min_{(\lambda, \eta) \in \mathcal{V}} \max_{z \in \mathcal{Z}} \left\{ (a - k\lambda)^\top z + k\eta \right\}
\]

\[
= \min_{(\lambda, \eta) \in \mathcal{V}, \beta} \left\{ \beta^\top h + k\eta \mid \beta \geq 0, H^\top \beta = a - k\lambda \right\}
\]

\[
= \min_{\beta \geq 0, \eta \geq 0} \left\{ \beta^\top h + k\eta \mid (a - H^\top \beta, k\eta, k) \in \tilde{\mathcal{V}} \right\}
\]

\[
= \min_{\beta \geq 0, \eta \geq 0} \left\{ \beta^\top h + \eta \mid (a - H^\top \beta, \eta, k) \in \tilde{\mathcal{V}} \right\},
\]

where the first interchange between minimization and maximization is justified because \(\mathcal{Z}\) is compact, and this concludes the desired equivalence. Next, we consider the case when \(k = 0\). Since \(\mathcal{V}\) is bounded, we must have \(\{(\lambda, \eta) \mid (\lambda, \eta, 0) \in \mathcal{V}\} = \{(0, 0)\}\). Hence, \((a - H^\top \beta, \eta, 0) \in \mathcal{V}\) is equivalent to \(\eta = 0\) and \(H^\top \beta = a\). By noting that \(\max_{z \in \mathcal{Z}} a^\top z = \min_{\beta \in \mathbb{R}^n_k} \{\beta^\top h \mid H^\top \beta = a\}\), the first half of the proposition follows.

If the penalty function \(p\) is a norm, from the derivation of the dual norm in Proposition 1, we have

\[
\min_{\beta \in \mathbb{R}^n_k} \left\{ \beta^\top h \mid p^*(a - H^\top \beta) \leq k \right\}
\]

\[
= \min_{\beta \in \mathbb{R}^n_k, \mu \in \mathbb{R}^n, \delta \in \mathbb{R}_+} \left\{ \beta^\top h \mid \delta \leq k, M(a - H^\top \beta) + N\mu \leq \delta t \right\}
\]

\[
= \min_{\beta \in \mathbb{R}^n_k, \mu \in \mathbb{R}^n} \left\{ \beta^\top h \mid M(a - H^\top \beta) + N\mu \leq kt \right\},
\]

where the second equality follows because, to ensure that \((0, 0) \in \mathcal{V}, t\) must be non-negative making the constraint \(\delta \leq k\) binding at optimality. Noting from the previous half of this proposition and the proof of Proposition 3 that, as \(s\) vanishes when the penalty function is a norm, the latter minimization problem is equivalent to \(\max_{z \in \mathcal{Z}} \{a^\top z - kp(z)\}\) completes the proof.

**Proof of Theorem 3** Consider the following solution to Problem (19): \(x = \hat{x}, q = \hat{y}, Q = 0\) and \(k = d^\top \hat{q^\dagger}\) where \(\hat{q^\dagger} \in \mathbb{R}^n_y\) satisfies \(B\hat{q}^\dagger \geq 0\) and \(B\hat{q}^\dagger \geq \max_{i \in [n_f], j \in [n_z]} |F_{ij}(\hat{x})| \cdot 1\). Note that \(\hat{q}^\dagger\) always exists because of complete recourse. The suggested solution robustly satisfies first constraint of Problem (19) because

\[
c^\top \hat{x} + d^\top (\hat{y} + q^\dagger \|z\|_1) = c^\top \hat{x} + d^\top \hat{y} + k\|z\|_1 = Z_0 + k\|z\|_1 \leq \tau + k\|z\|_1.
\]

Moreover, the second constraint is also robustly satisfied as

\[
B(\hat{y} + q^\dagger \|z\|_1) \geq f(\hat{x}) + Bq^\dagger \|z\|_1
\]

\[
\geq f(\hat{x}) + \|z\|_1 \left\{ \max_{i \in [n_f], j \in [n_z]} |F_{ij}(\hat{x})| \right\} \cdot 1 \geq f(\hat{x}) + F(\hat{x})z, \quad \forall z \in \mathcal{Z}
\]
where the first two inequalities follow from the feasibility of \((\hat{x}, \hat{y})\) in the nominal problem \((15)\) and the construction of \(q^1\), respectively. Consequently, it readily follows that the constructed solution is feasible in Problem \((19)\). □

Proof of Proposition 2 The first robust constraint of Problem \((19)\) can be expressed as
\[
\max_z \{d^T Qz - (k - d^T q^1)\|z\|_1 \mid Hz \leq h\} \leq \tau - d^T q - c^\top x.
\]
By Proposition 1, we can replace the maximization problem on the left-hand side of by a minimization problem:
\[
\min_{w^0} \{h^\top w^0 \mid \|H^\top w^0 - Q^\top d\|_\infty \leq k - d^T q^1, w^0 \geq 0\}.
\]
Similarly, the second robust constraint of Problem \((19)\) can be written down as
\[
\max_x \{(F_i(x) - B_i Q)z - B_i q^1 \|z\|_1 : H z \leq h\} \leq B_i q - f_i(x) \quad \forall i \in [n_f],
\]
whose left-hand side maximization problem can be replaced by
\[
\min_{w^i} \{h^\top w^i : \|H^\top w^i - F_i^\top (x) + Q^\top B_i^\top\|_\infty \leq B_i q^1, w^i \geq 0\} \quad \forall i \in [n_f],
\]
Finally, the two deterministic linear constraints (namely, \(k - d^T q^1 \geq 0\) and \(B q^1 \geq 0\)) of \((19)\) are redundant in view of problem \((20)\) and can therefore be safely omitted. The proof is thus completed. □

Proof of Theorem 1 First, it follows from the specificity of \(g^\omega\) in Problem \((27)\) that the robust constraint in Problem \((30)\) is equivalent to
\[
c^\top x + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} \left( \max_{z \in z_\omega} \{d^\top y - k p(z) : B y \succeq_k f^\omega(x) + F(x) z\} \right) \leq \tau.
\]
Observe that, for each \(\omega \in [\Omega]\), the inner minimization (over \(y\)) is strictly feasible and thus we can transform it into a maximization problem via conic duality, that is,
\[
c^\top x + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} \left( \max_{\rho \in \mathcal{P}} \left\{ \rho^\top f^\omega(x) + \max_{z \in z_\omega} \{ (\rho^\top F(x) z - k p(z) \} \right\} \right) \leq \tau.
\]
Next, similarly to the proof of Theorem 1, we will make use of Proposition 2 to transform the inner maximization (over \(z^\omega\) for each \(\omega \in [\Omega]\)) to a minimization problem (over the variables \(\beta^\omega, \mu^\omega\) and \(\eta^\omega\) for each \(\omega \in [\Omega]\)). As a result, we can express Problem \((30)\) as
\[
\min_k \quad k
\]
s.t. \(c^\top x + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} v_\omega \leq \tau\)
\[
\rho^\top f^\omega(x) + (h^\omega)^\top \beta^\omega(\rho) + \eta^\omega(\rho) \leq v_\omega \quad \forall \rho \in \mathcal{P}, \omega \in [\Omega]\)
\[
M (F(x)^\top \rho - H^\top \beta^\omega(\rho)) + N \mu^\omega(\rho) + s \eta^\omega(\rho) \leq tk \quad \forall \rho \in \mathcal{P}, \omega \in [\Omega]\)
\[
\beta^\omega(\rho) \geq 0, \eta^\omega(\rho) \geq 0 \quad \forall \rho \in \mathcal{P}, \omega \in [\Omega]\)
\[
x \in \mathcal{X}, k \in \mathbb{R}_+, v \in \mathbb{R}^\Omega, \beta^1, \ldots, \beta^\Omega \in \mathcal{R}^{n_f, n_h}, \mu^1, \ldots, \mu^\Omega \in \mathcal{R}^{n_f, n_p}, \eta^1, \ldots, \eta^\Omega \in \mathcal{R}^{n_f, 1}.\)
Approximating all recourse variables using affine adaptation results in Problem (31), which completes the first half of the proof.

Assume now that \( \tau \geq Z_0 \) and consider the solution \( x = \hat{x}, \beta^\omega(\rho) = 0, \eta^\omega(\rho) = 0 \) and \( v_\omega = \max_{\rho \in P} \left\{ \rho^T (f^\omega(\hat{x})) \right\} \) (with \( \mu^\omega \) and \( k \) to be chosen later), for all \( \omega \in [\Omega] \), which robustly satisfies the first constraint of Problem (31) since the left-hand side of this constraint evaluates to:

\[
\begin{align*}
    c^T \hat{x} + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} \max_{\rho \in P} \left\{ \rho^T (f^\omega(\hat{x})) \right\} & = c^T \hat{x} + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} \max_{\rho \in P} \left\{ \rho^T (f(\hat{x}) + F(\hat{x})\hat{x}^\omega) \right\} \\
& = c^T \hat{x} + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} \min_y \left\{ d^T y : By \succeq_K f(\hat{x}) + F(\hat{x})\hat{x}^\omega \right\} \\
& = c^T \hat{x} + \frac{1}{\Omega} \sum_{\omega \in [\Omega]} d^T \hat{x}^\omega \leq \tau,
\end{align*}
\]

where the last equality follows from the optimality of \((\hat{x}, \hat{y}^1, \ldots, \hat{y}^\Omega)\) in Problem (26). Besides, the proof of Theorem 4 reveals that there exists \( \hat{\mu} \) and \( \hat{k} > 0 \) such that the constraint (14) holds. Hence, by completing the suggested solution with \( \mu^\omega(\rho) = \hat{\mu} \hat{k} \), for all \( \omega \in [\Omega] \), and \( k = \hat{k} \), the remaining constraints of Problem (31) are satisfied and the feasibility argument is completed.

**Proof of Proposition 4.** By introducing an epigraph variable \( x_0 \in \mathbb{R} \) to denote the uncertain part of the objective function of Problem (34), we obtain the following equivalent reformulation

\[
\begin{align*}
    \min_{x_0} & \quad \frac{1}{2} x^T \Sigma x + x_0 \\
\text{subject to} & \quad x_0 \geq x^T (\mu + z)(\mu + z)^T x - 2x^T (\mu + z) \quad \forall z \in \mathcal{E}(r) \\
    & \quad x \in \mathcal{X}, \ x_0 \in \mathbb{R}.
\end{align*}
\]

Next, we rewrite the arising robust constraint in an explicit quadratic form

\[
\begin{align*}
    \begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} -xx^T & x - x\mu^T x \\ x^T - x^T \mu x^T & x_0 + 2x^T \mu - x^T \mu \mu^T x \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0 \quad \forall z : \begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} -I_n & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} \geq 0
\end{align*}
\]

\[
\begin{align*}
    \iff \begin{bmatrix} \lambda I_n & x \\ x^T & x_0 + 2x^T \mu - \lambda r \end{bmatrix} \succeq \begin{bmatrix} x \\ x^T \mu \end{bmatrix}^T, \end{align*}
\]

where the equivalence is due to \( S \)-lemma, which holds whenever \( r > 0 \). Finally, invoking the Schur’s complement to linearize the above positive semidefinite constraint gives the desired equivalence.

**Proof of Proposition 5.** The proof widely parallels to that of Proposition 4 and is therefore omitted.

**Proof of Theorem 3.** Observe that the positive semidefinite constraint of (36) implies that

\[
\begin{align*}
    \begin{bmatrix} 2kI_n & x \\ x^T & 1 \end{bmatrix} \succeq 0 \iff 2kI_n \succeq xx^T \iff 2k1^T I_n 1 \geq (1^T x)^2 = 1,
\end{align*}
\]
where the equality follows because \( \mathbf{x} \in \mathcal{X} \). Therefore, any feasible \( k \) must satisfy \( k \geq \frac{1}{2n} \).

Next, for any \( \delta > 0 \), we introduce a matrix \( \Xi(\delta) \in \mathbb{S}^{n+1} \) and a scalar \( \xi(\delta) \):

\[
\Xi(\delta) = \begin{bmatrix}
(1+\delta)\mathbf{I}_n & 1 \\
1^\top & n
\end{bmatrix} \quad \text{and} \quad \xi(\delta) = \sup_{\mathbf{v}} \left\{ \left( \mathbf{v}^\top \begin{bmatrix} 1 & 1^\top \mu \end{bmatrix} \right)^2 \mathbf{v}^\top \Xi(\delta) \mathbf{v}^{-1} \mid \|\mathbf{v}\|_2 = 1 \right\}.
\]

Observe that for any nonzero vector \( (\mathbf{w}, w_{n+1}) \in \mathbb{R}^{n+1} \),

\[
\begin{bmatrix}
\mathbf{w} \\
w_{n+1}
\end{bmatrix}^\top \Xi(\delta) \begin{bmatrix}
\mathbf{w} \\
w_{n+1}
\end{bmatrix} = \delta \mathbf{w}^\top \mathbf{w} + (\mathbf{w} + 1w_{n+1})^\top (\mathbf{w} + 1w_{n+1}) > 0.
\]

Hence, \( \Xi(\delta) \) is strictly positive definite, and thus \( \xi(\delta) \) is positive and finite because \( \{\mathbf{v} \in \mathbb{R}^{n+1} \mid \|\mathbf{v}\|_2 = 1\} \) is compact. Therefore, we have

\[
\xi(\delta) \mathbf{v}^\top \Xi(\delta) \mathbf{v} \geq \mathbf{v}^\top \begin{bmatrix} 1 & 1^\top \mu \end{bmatrix} \begin{bmatrix} 1 & 1^\top \mu \end{bmatrix} \mathbf{v}
\]

for all \( \mathbf{v} \in \mathbb{R}^{n+1} \), or equivalently,

\[
\xi(\delta) \Xi(\delta) \geq \begin{bmatrix} 1 \\
1^\top \mu 
\end{bmatrix} \begin{bmatrix} 1 \\
1^\top \mu 
\end{bmatrix}.
\]

For a fixed \( \delta > 0 \), consider now a target objective \( \tau \geq \frac{1}{2n^2} \mathbf{1}^\top \Xi \mathbf{1} + \frac{\xi(\delta)}{2n} - \frac{1^\top \mu}{n} \). As \( \xi(\delta) \) is decreasing in \( \delta > 0 \), this lower bound on \( \tau \) is increasing as \( \delta \) decreases. Our next step is to show that \( (\mathbf{x}, x_0, k) = \left( \frac{1}{n}, \frac{\xi(\delta)-2\mathbf{1}^\top \mu}{\xi(\delta)}, \frac{1+\delta}{2n} \right) \) is feasible in Problem (36). Observe that

\[
\xi(\delta) \Xi(\delta) \geq \begin{bmatrix} 1 \\
1^\top \mu 
\end{bmatrix} \begin{bmatrix} 1 \\
1^\top \mu 
\end{bmatrix} \iff \begin{bmatrix}
(1+\delta)I_n & 1 & 1 \\
1^\top & n & 1^\top \mu \\
1^\top & 1^\top \mu & \xi(\delta)
\end{bmatrix} \succeq 0
\]

\[
\iff \begin{bmatrix}
2k\mathbf{I}_n & nx & nx \\
nx^\top & n & nx^\top \mu \\
nx^\top & nx^\top \mu & nx_0 + 2nx^\top \mu
\end{bmatrix} \succeq 0
\]

where the first equivalence follows from Schur’s complement and the definition of \( \Xi(\delta) \) and the second follows from the suggested value of \( (\mathbf{x}, x_0, k) \). Dividing both sides of the above inequality by \( n \) shows that \( (\mathbf{x}, x_0, k) \) satisfies the positive semidefinite constraint in Problem (36). Besides, the remaining constraints in Problem (36) are trivially satisfied. Therefore, \( \hat{k}^*(\tau) \) cannot exceed \( \frac{1+\delta}{2n} \). This result together with our earlier observation (42) implies \( \frac{1}{2n} \leq k^*(\tau) \leq \frac{1+\delta}{2n} \). By taking a limit as \( \delta \) approaches zero from above and considering \( \tau \) that exceeds the prescribed lower bound (which itself increases as \( \delta \) decreases), the theorem follows. □
Proof of Proposition 8. We could express the robust constraints of Problem (37) as
\[
\max_{z \in \mathcal{U}, \gamma \geq 0, w \geq 0} \left\{ \langle T, Y \rangle + \ell^T w \mid x + w + Y^T 1 \geq z \right\} \leq x_0
\]
\[\iff \max_{z \in \mathcal{U}, \gamma \geq 0, w \geq 0} \min_{\rho \geq 0} \left\{ \langle T, Y \rangle + \ell^T w + \rho^T (z - x - w + Y 1 - Y 1 - Y 1) \right\} \leq x_0\]
\[\iff \max_{z \in \mathcal{U}, \rho \geq 0} \max_{\gamma \geq 0, w \geq 0} \left\{ \rho^T (z - x) + \min_{\gamma \geq 0, w \geq 0} \left\{ \langle Y, T + \rho 1^T - 1 \rho^T \rangle + (\ell - \rho)^T w \right\} \leq x_0\right\}
\]
\[\iff \max_{z \in \mathcal{U}, \rho \geq 0} \min_{\beta \geq 0, \beta \geq 0} \left\{ \beta^T z + \hat{\beta} r \mid \beta + \beta 1 \geq \rho \right\} \leq x_0 + \rho^T x.\]

Treating $\rho$ as uncertainty as well as $\beta$ and $\hat{\beta}$ as dual recourse variables finally completes the proof. □

Proof of Proposition 9. In their Theorem 1, Bertsimas and de Ruiter (2016) show an equivalent reformulation of Problem (37) which is
\[
\text{minimize } c^T x + x_0
\]
subject to \[
\beta(\rho, \hat{\rho})^T z + \hat{\beta}(\rho, \hat{\rho}) r \leq \rho^T x + \hat{\rho} x_0 \quad \forall (\rho, \hat{\rho}) \in \hat{\mathcal{P}}
\]
\[
\beta(\rho, \hat{\rho}) + \hat{\beta}(\rho, \hat{\rho}) 1 \geq \rho \quad \forall (\rho, \hat{\rho}) \in \hat{\mathcal{P}}
\]
\[
\beta(\rho, \hat{\rho}) \geq 0, \hat{\beta}(\rho, \hat{\rho}) \geq 0 \quad \forall (\rho, \hat{\rho}) \in \hat{\mathcal{P}}
\]
\[x \in \mathcal{X}, x_0 \in \mathbb{R}, \beta \in \mathbb{R}^{n+1,n}, \hat{\beta} \in \mathbb{R}^{n+1,1},\]
where \[\hat{\mathcal{P}} = \{ (\rho, \hat{\rho}) \in \mathbb{R}_+^n \times \mathbb{R}_+ \mid \rho \leq \hat{\rho}, \rho 1^T - 1 \rho^T \leq \hat{\rho} T, \hat{\rho} + \rho = 1 \}.\] They also argue in their Theorem 2 that the respective affine recourse approximations of Problems (37) and (43) are equivalent. As a result, it suffices to show that the affine recourse approximations of Problems (38) and (43) are equivalent.

First, we write down the affine recourse approximation of Problem (38) by restricting $\beta(\rho)$ and $\hat{\beta}(\rho)$ to $\beta^i + \beta^s \rho$ and $\hat{\beta}^i + \hat{\beta}^s \rho$ (where ‘i’ and ‘s’ indicate the intercept and slope of the affine decision rules), respectively, and obtain
\[
\text{minimize } c^T x + x_0
\]
subject to \[
\beta^i + \beta^s \rho \leq \rho^T x + x_0 \quad \forall \rho \in \mathcal{P}
\]
\[
\beta^i + 1 \beta^s + (\beta^s + 1 \beta^s) \rho \geq \rho \quad \forall \rho \in \mathcal{P}
\]
\[
\beta^i + \beta^s \rho \geq 0, \hat{\beta}^i + \hat{\beta}^s \rho \geq 0 \quad \forall \rho \in \mathcal{P}
\]
\[x \in \mathcal{X}, x_0 \in \mathbb{R}, \beta^i \in \mathbb{R}^n, \beta^s \in \mathbb{R}^{n \times n}, \hat{\beta}^i \in \mathbb{R}, \hat{\beta}^s \in \mathbb{R}^{1 \times n}.\]

Next, for Problem (43), we first observe that the uncertainty set $\hat{\mathcal{P}}$ requires $\hat{\rho}$ to be linearly dependent on $\rho$. Hence, we can simply ignore the additional uncertain parameter $\hat{\rho}$ and work with the projection of $\hat{\mathcal{P}}$ on $\rho$. By a slight abuse of notation, we will denote this projection by $\hat{\mathcal{P}}$ and note that
\[\hat{\mathcal{P}} = \{ \rho \in \mathbb{R}_+^n \mid \rho \leq (1 - 1^T \rho) \ell, \rho 1^T - 1 \rho^T \leq (1 - 1^T \rho) T \}.\]
Note that as \( \hat{P} \subset \mathbb{R}^n_+ \) and as \( \ell \geq 0 \), it is a necessity that \( 1^T \rho < 1 \) for all \( \rho \in \hat{P} \). We are now ready to present the explicit affine recourse approximation of Problem (43), which is

\[
\begin{align*}
\text{minimize} & \quad c^T x + x_0 \\
\text{subject to} & \quad z^T \beta' + r \beta' + (z^T \beta^* + r \beta^*) \rho \leq \rho^T x + (1 - 1^T \rho)x_0 \quad \forall \rho \in \hat{P} \\
& \quad \beta^* + 1 \beta' + (\beta^s + 1 \beta') \rho \geq \rho \quad \forall \rho \in \hat{P} \\
& \quad \beta^* + \beta' \rho \geq 0 \quad \forall \rho \in \hat{P} \\
& \quad x \in \mathcal{X}, x_0 \in \mathbb{R}, \beta' \in \mathbb{R}^n, \beta^* \in \mathbb{R}^{n \times n}, \beta' \in \mathbb{R}, \beta^* \in \mathbb{R}^{1 \times n}.
\end{align*}
\]

It remains to show that Problems (44) and (45) are equivalent. First, we will show that Problem (45) is a relaxation of Problem (44). To this end, for any feasible solution \( X = (x, x_0, \beta', \beta^*, \hat{\beta'}, \hat{\beta}^*) \) of Problem (44), we will show that \( X' = (x, x_0, \beta', \beta^* - \beta' \mathbf{1}^T, \hat{\beta'}, \hat{\beta}^* - \hat{\beta}' \mathbf{1}^T) \) is feasible in Problem (45).

Note that, as both solutions share the same \( x \) and the same \( x_0 \), they attain the same objective value in their respective problem.

For any \( \rho \in \hat{P} \), it is readily seen that \( \rho/(1 - 1^T \rho) \in \mathcal{P} \). As a result, the feasibility of \( X \) in view of Problem (44) implies, for all \( \rho \in \hat{P} \), that

\[
\begin{align*}
(1 - 1^T \rho)(z^T \beta' + r \beta') + (z^T \beta^* + r \beta^*) \rho \leq \rho^T x + (1 - 1^T \rho)x_0, \\
(1 - 1^T \rho)(\beta' + 1 \beta') + (\beta^* + 1 \beta') \rho \geq \rho, \\
(1 - 1^T \rho)\beta^* + \beta' \rho \geq 0, \\
(1 - 1^T \rho)\beta' + \beta^* \rho \geq 0.
\end{align*}
\]

Rearranging terms in the above four inequalities yields

\[
\begin{align*}
z^T \beta' + r \beta' + (z^T (\beta^* - \beta') \mathbf{1}^T + r (\hat{\beta}' - \hat{\beta}^* \mathbf{1}^T)) \rho \leq \rho^T x + (1 - 1^T \rho)x_0, \\
\beta' + 1 \beta' + ((\beta^* - \beta') \mathbf{1}^T + 1 (\hat{\beta}' - \hat{\beta}^* \mathbf{1}^T)) \rho \geq \rho, \\
\beta^* + (\beta^* - \beta' \mathbf{1}^T) \rho \geq 0, \\
\beta' + (\beta^* - \beta^* \mathbf{1}^T) \rho \geq 0,
\end{align*}
\]

for all \( \rho \in \hat{P} \), which in turn implies that \( X' \) is indeed feasible in Problem (45).

Conversely, for any feasible solution \( X = (x, x_0, \beta', \beta^*, \hat{\beta'}, \hat{\beta}^*) \) of Problem (45), one can similarly show that \( X' = (x, x_0, \beta', \beta^* + \beta' \mathbf{1}^T, \hat{\beta'}, \hat{\beta}^* + \hat{\beta}' \mathbf{1}^T) \) is feasible in Problem (44) to conclude that
Problem (44) is a relaxation of Problem (45). To see this, observe that \( \rho/(1 + 1^T \rho) \in \hat{\mathcal{P}} \) for any \( \rho \in \mathcal{P} \). As a result, the feasibility of \( X \) in view of Problem (45) implies, for all \( \rho \in \mathcal{P} \), that

\[
(1 + 1^T \rho)(\bar{z}^T \hat{\beta}^i + r_i^\hat{\beta}) + (\bar{z}^T \hat{\beta}^s + r_i^\hat{\beta}) \rho \leq \rho^T x + x_0,
\]

\[
(1 + 1^T \rho)(\beta^i + 1^\hat{\beta}^i) + (\beta^s + 1^\hat{\beta}^s) \rho \geq \rho,
\]

\[
(1 + 1^T \rho)\beta^i + \beta^s \rho \geq 0,
\]

\[
(1 + 1^T \rho)\hat{\beta}^i + \hat{\beta}^s \rho \geq 0.
\]

Rearranging the terms in the above inequalities shows that \( X' \) is indeed feasible in Problem (44) as desired. Therefore, the optimal objective value of Problem (44) constitutes both a lower and an upper bound of that of Problem (45). The proof is hence completed. \( \square \)

**Proof of Proposition 8** The proof widely parallels to that of Proposition 6 and is thus omitted. \( \square \)

**Proof of Proposition 9** First, we can use Bertsimas and de Ruiter (2016, Theorem 1) to argue that Problem (39) is equivalent to

\[
\begin{align*}
&\text{minimize } k \\
&\text{subject to } \rho(c - \rho)^T x + \bar{z}^T \beta(\rho, \hat{\rho}) \leq \hat{\rho}_\tau & \forall (\rho, \hat{\rho}) \in \hat{\mathcal{P}} \\
&\beta(\rho, \hat{\rho}) + k \hat{\rho} 1 \geq \rho & \forall (\rho, \hat{\rho}) \in \hat{\mathcal{P}} \\
&\beta(\rho, \hat{\rho}) \geq 0 & \forall (\rho, \hat{\rho}) \in \hat{\mathcal{P}} \\
&x \in \mathcal{X}, k \in \mathbb{R}_+, \beta \in \mathcal{R}^{n+1,n},
\end{align*}
\]

where \( \hat{\mathcal{P}} \) is the same as that in the proof of Proposition 7. The proof largely follows that of Proposition 7, i.e., one can show that the affine adaptation approximation of the above problem is equivalent to that of (40), which in turn implies that the affine adaptation approximations of (39) and (40) are equivalent (Bertsimas and de Ruiter 2016, Theorem 2). Details are omitted for brevity. \( \square \)