Linear Convergence of Quasi-Newton Methods for Solving Constrained Generalized Equations

R. Andreani † R. M. Carvalho † L. D. Secchin § G. N. Silva ¶

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Abstract

In this paper, we focus on quasi-Newton methods to solve constrained generalized equations. As is well-known, this problem was firstly studied by Robinson and Josephy in the 70’s. Since then, it has been extensively studied by many other researchers, specially Dontchev and Rockafellar. Here, we propose two Broyden-type quasi-Newton approaches to dealing with constrained generalized equations, one that requires the exact resolution of the subproblems, and other that allows inexactness, which is closer to numerical reality. In both cases, projections onto the feasible set are also inexact. The local convergence of general quasi-Newton approaches is established under a limited deterioration of the update matrix and Lipschitz continuity hypotheses. In particular, we prove that our Broyden-type schemes converge linearly to the solution under suitable assumptions. Some numerical examples illustrate the applicability of the proposed methods.

1 Introduction

All of our study and contributions are focused on the problem known as the Constrained Generalized Equation. Basically, it consists of finding \( x \in \mathbb{R}^n \) such that

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1 Introduction

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\[
x \in C, \quad 0 \in f(x) + F(x),
\]
where $f : \Omega \to \mathbb{R}^m$ is a continuously differentiable function, $\Omega \subseteq \mathbb{R}^n$ is an open set, $C \subset \Omega$ is a nonempty closed convex set, and $F : \Omega \rightrightarrows \mathbb{R}^m$ is a multifunction with a closed nonempty graph.

Generalized equations were first proposed by Robinson [29]. In that work, the author deals with the unconstrained problem, which aims to find $x \in \mathbb{R}^n$ such that

$$0 \in f(x) + F(x),$$

where $f$ and $F$ are basically the same as in problem (1). This problem differs from (1) by the absence of the constraint $x \in C$. Josephy [25] shows that several problems can be rewritten as in (2), namely, the general nonlinear optimization, the variational inequality and the equilibrium problems. In the last ten years, many researchers have devoted their efforts to studying the application of Newton’s method and its variants to solve (2), see for instance [25, 5, 6, 9, 16, 19, 20, 21, 22, 23, 28, 30, 31]. In particular, we highlight the important contributions of Dontchev [5, 6, 19], Adly [2, 4], Bonnans [9], Ferreira [21, 23, 13] and their collaborators.

The problem addressed in this paper appeared in a recent work by Oliveira et al [13], where Newton’s method for solving (1) was considered. The presence of the constrained set $C$ allows us to write, in addition to the problems already mentioned previously, others in the form (1). For instance, the Constrained Variational Inequality Problem (CVIP) find $x \in U \cap V$ such that $\langle f(x), y - x \rangle \geq 0$ for all $y \in U$,

$$U, V \subset \mathbb{R}^n$$ closed convex sets, can be stated as

$$\text{find } x \in U \cap V \text{ such that } 0 \in f(x) + N_U(x),$$

where $N_U$ is the normal cone associated to $U$. Problem (3) has been extensively studied over the past ten years, see for instance [10, 24]. Another important equivalence to constrained generalized equations is the Split Variational Inequality Problem (SVIP), stated as follows: let $U, V \subset \mathbb{R}^m$ be nonempty, closed convex sets, and $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^m$ be functions. Then SVIP consists of

$$\text{find } x_* \in U \text{ such that } \langle f(x_*), x - x_* \rangle \geq 0, \text{ for all } x \in U,$$

and such that $y_* = Ax_* \in V$ satisfies

$$\langle g(y_*), y - y_* \rangle \geq 0 \text{ for all } y \in V.$$

Taking $D := U \times V$ and $V := \{w = (x, y) \in \mathbb{R}^n \times \mathbb{R}^m | Ax = y\}$, SVIP is equivalent to the following CVIP [10, Lemma 5.1]:

$$\text{find } w_* \in D \cap V \text{ such that } \langle h(w_*), w - w_* \rangle \geq 0 \text{ for all } w \in D,$$
where \( w = (x,y) \) and \( h(x,y) := (f(x), g(y)) \). In turn, this CVIP is equivalent to the following constrained generalized equation:

\[
\begin{align*}
\text{find} & \quad w_\ast \in D \cap V \quad \text{such that} \quad 0 \in h(w_\ast) + N_D(w_\ast).
\end{align*}
\]

It is known that SVIP includes several optimization problems, for instance, *Split Minimization Problem* and *Common Solutions to Variational Inequalities Problem*. For more details about these problems, see [10, 24, 1, 11, 27].

Artacho et al [5] studied a quasi-Newton method for the unconstrained problem (2). The authors considered the following iterative scheme:

\[
\begin{align*}
f(x_k) + B_k(x_{k+1} - x_k) + F(x_{k+1}) \ni 0, \quad k = 0, 1, \cdots, (4)
\end{align*}
\]

where \( \{B_k\} \) is a sequence of bounded linear mappings between Banach spaces \( X \) and \( Y \) satisfying the classical Broyden update rule. They proved that if the multifunction \( f + F \) is metrically regular at \( x_\ast \) for 0 and the derivative mapping \( Df \) is Lipschitz continuous, then the sequence \( \{x_k\} \) generated by (4) is linearly convergent to \( x_\ast \), see [5, Theorem 4.3]. More generally, Adly and Huynh [3] introduced quasi-Newton schemes like (4) for solving (2), allowing \( f \) possibly not differentiable. In this case, the authors assume the regularity metric condition with respect to a kind of semismooth regularization of \( f + F \). They proved that if \( B_k \) satisfies a suitable modified Broyden update, the sequence \( \{x_k\} \) generated by (4) is linearly convergent to a solution \( x_\ast \) of (2) [3, Theorem 4.3]. Similar approach was employed in [7, 26].

In this paper, we propose two quasi-Newton schemes to solve the constrained generalized equation (1). The first is are based on the following idea: given \( x_0 \in C \) and the initial \( B_0 \), near to \( f'(x_\ast) \), we compute, in each iteration, an intermediate point \( y_k \) such that

\[
\begin{align*}
f(x_k) + B_k(y_k - x_k) + F(y_k) \ni 0, \quad k = 0, 1, \cdots, (5)
\end{align*}
\]

Since \( y_k \) can be infeasible, that is, \( y_k \notin C \), we project it onto \( C \) by an inexact projection procedure, obtaining a new iterate \( x_{k+1} \) almost feasible. Thus we prove that under suitable assumptions the main sequence \( \{x_k\} \) converges linearly to a solution of (1). The second method follows an analogous idea. The difference is that (5) is allowed to be solved inexactly. Specifically, \( y_k \) only must be in an open ball centred at \( x_k \) that intersects the set \( \{y \mid f(x_k) + B_k(y - x_k) + F(y_k) \} \). This strategy is more suitable to be implemented, since solving (5) exactly can be practically impossible even in simple problems. Naturally, allowing inexact solutions leads to a more complicated convergence theory, which is addressed in Section 4.

This paper is divided into two parts. In the first one, we use the quasi-Newton approach (5) to find a solution of problem (1) with \( B_k \) satisfying the classical Broyden update rule. By assuming the regularity metric of the multifunction \( f + F \) at \( x_\ast \) to 0, we show that the sequence \( \{x_k\} \) generated by (5) is linearly convergent to \( x_\ast \). Firstly, we suppose
that \( B_k \) satisfies a limited deterioration condition to obtain a general convergence result. As a particular case, we show that the Broyden update satisfies this limited deterioration. It is worth mentioning that the used limited deterioration condition on \( B_k \) was previously considered in [5, 3]. Furthermore, we use the fact that Lipschitz properties of the multifunction \( F^{-1} \) are inherited by the multifunction \((f + F)^{-1}\), as demonstrated by Dontchev and Hager in [17]. In the second part, we address the problem (1) using an inexact approach and similar ideas proposed by Dontchev and Rockafellar in [19]. The proposed inexact quasi-Newton method is described by

\[
(f(x_k) + B_k(y_k - x_k) + F(y_k)) \cap R_k(x_k, y_k) \neq \emptyset, \quad k = 0, 1, \ldots, \quad (6)
\]

where \( R_k : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is a sequence of multifunctions with closed graphs representing the inexactness. It is not difficult to see that if \( F \equiv \{0\} \) and \( R_k(x_k, y_k) = \mathcal{B}_{\eta_k} f(x_k)(0), \eta_k > 0 \), then the iterative scheme (6) reduces to

\[
\|f(x_k) + B_k(y_k - x_k)\| \leq \eta_k \|f(x_k)\|, \quad k = 0, 1, \ldots,
\]

which can be seen as an inexact quasi-Newton method for solving \( f(x) = 0, x \in C \). Then, assuming the multifunction \( f + F \) metrically regular at \( x^* \) for \( 0 \), \( R_k \) partially Aubin continuous, and \( d(0, R_k(u, x_*)) \) fulfilling a suitable boundedness property, we show that the sequence \( \{x_k\} \) generated by (6) is linearly convergent to \( x_* \), with \( B_k \) satisfying the Broyden update. It should be mentioned that inexact quasi-Newton methods for solving the unconstrained problem (2) are considered in [12].

This work is organized as follows. In Section 2, we present the notations and basic necessary concepts. In Section 3, we present the first quasi-Newton algorithm, where subproblems are solved exactly, and its local convergence analysis. Section 4 is devoted to the inexact quasi-Newton algorithm and its convergence. In Section 5, we discuss the important particular case of Broyden-type methods. Numerical experiments are presented in Section 6, illustrating the theory. Finally, Section 7 brings our conclusions.

## 2 Preliminaries

In this section, we briefly present the basic concepts that we will use throughout the work. A detailed presentation can be found in [20].

Firstly we establish some notations. A generic norm will be denoted by \( \| \cdot \| \). The sets

\[
\mathcal{B}_\delta := \{y \in \mathbb{R}^n \mid \|x - y\| < \delta\}, \quad \mathcal{B}_\delta[x] := \{y \in \mathbb{R}^n \mid \|x - y\| \leq \delta\}
\]

will denote the *open* and *closed* balls of radius \( \delta > 0 \), centered at \( x \), respectively. The *vector space of all continuous linear mappings* \( A : \mathbb{R}^n \to \mathbb{R}^m \) will be denoted by \( \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \), and the *norm of* \( A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \) is defined by \( \|A\| := \sup \{\|Ax\| \mid \|x\| \leq 1\} \). Let \( f : \Omega \to \mathbb{R}^m \) be differentiable in an open set \( \Omega \subseteq \mathbb{R}^n \). The *derivative* of \( f \) at \( x \) will be
denoted by \( f'(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m \). Given a multifunction \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \), its graph is the set \( \text{gph} \ F := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid u \in F(x)\} \). The multifunction \( F^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \) defined by \( F^{-1}(u) := \{x \in \mathbb{R}^n \mid u \in F(x)\} \) denotes the inverse of \( F \). Given \( C, D \subset \mathbb{R}^n \),

\[
d(x, D) := \inf_{y \in D} \|x - y\|, \quad e(C, D) := \sup_{x \in C} d(x, D) \tag{7}
\]

are the distance from \( x \) to \( D \) and the excess of \( C \) beyond \( D \), respectively. The following conventions are adopted: \( d(x, \emptyset) = +\infty \), \( e(\emptyset, D) = 0 \) when \( D \neq \emptyset \), and \( e(\emptyset, \emptyset) = +\infty \).

In the context of generalized equations, it is common to consider some regularity condition over \( F \). Here, we will use the following notion:

**Definition 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be an open and nonempty set. We say that the multifunction \( G : \Omega \rightrightarrows \mathbb{R}^m \) is metrically regular at \( \bar{x} \in \Omega \) for \( \bar{u} \in \mathbb{R}^m \) with modulus \( \lambda \geq 0 \) when \( \bar{u} \in G(\bar{x}) \), gph \( G \) is locally closed at \((\bar{x}, \bar{u})\), and there exist \( a > 0 \) and \( b > 0 \) such that \( \mathbb{B}_a[\bar{x}] \subset \Omega \) and

\[
d(x, G^{-1}(u)) \leq \lambda d(u, G(x)) \quad \text{for all} \quad x \in \mathbb{B}_a[\bar{x}], \ u \in \mathbb{B}_b[\bar{u}].
\]

**Remark 2.2.** It is known that a multifunction \( \Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is metrically regular at \( \bar{x} \in \mathbb{R}^n \) for \( \bar{y} \in \mathbb{R}^m \) with modulus \( \lambda > 0 \) if and only if \( \Gamma^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \) has the Aubin property at \( \bar{y} \) for \( \bar{x} \) with the same constant \( \lambda \), i.e.,

\[
e(\Gamma^{-1}(y) \cap X, \Gamma^{-1}(y')) \leq \lambda \|y - y'\| \quad \text{for all} \quad y, y' \in Y,
\]

where \( X \) and \( Y \) are neighborhoods of \( \bar{x} \) and \( \bar{y} \), respectively. See [20, Theorem 5A.3, p. 255].

The next result establishes a connection between the metric regularity of \( f + F \) and the Aubin property of an associated map, which proof is analogous to that presented in [19].

**Proposition 2.3.** Let \( \zeta > 0 \) and assume that the multifunction \( f + F \) is metrically regular at \( \bar{x} \) for \( 0 \) with modulus \( \lambda \), where \( \lambda \zeta < 1 \). Consider the multifunction

\[
G_u(x) = f(u) + B_u(x - u) + F(x), \tag{8}
\]

where the operator \( B_u \) is such that \( \|B_u - f'(\bar{x})\| \leq \zeta \). Then, for every \( \kappa > \lambda/(1 - \lambda \zeta) \), there exist positive numbers \( a \) and \( b \) such that

\[
e\left(G_u^{-1}(y) \cap \mathbb{B}_a[\bar{x}], G_u^{-1}(y')\right) \leq \kappa \|y - y'\| \quad \text{for all} \quad u \in \mathbb{B}_a[\bar{x}], \ y, y' \in \mathbb{B}_b[0]. \tag{9}
\]

Other important result is a generalization of the contraction mapping principle for set-valued mappings, stated below. It will be useful to prove the convergence of the quasi-Newton method in the next section. Its proof can be found in [20, Theorem 5E.2, p. 313].

**Theorem 2.4.** Let \( \Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) be a multifunction and \( \bar{x} \in \mathbb{R}^n \). Suppose that there exist scalars \( \rho > 0 \) and \( \mu \in (0, 1) \) such that the following conditions hold:
(i) the set \( \text{gph} \Phi \cap (B_{\rho}[\bar{x}] \times B_{\rho}[\bar{x}]) \) is closed;

(ii) \( d(\bar{x}, \Phi(\bar{x})) \leq \rho(1 - \mu) \);

(iii) \( e(\Phi(p) \cap B_{\rho}[\bar{x}], \Phi(q)) \leq \mu \|p - q\| \) for all \( p, q \in B_{\rho}[\bar{x}] \).

Then, there exists \( y \in B_{\rho}[\bar{x}] \) such that \( y \in \Phi(y) \).

In the sequel, we present the feasible inexact projection used in our proposed algorithms, as well as some of their properties of interest.

**Definition 2.5.** Let \( \theta \geq 0, C \subset \mathbb{R}^n \) be a closed convex set and \( x \in C \). The feasible inexact projection mapping relative to \( x \) with error tolerance \( \theta \), denoted by \( P_C(\cdot, x, \theta) : \mathbb{R}^n \Rightarrow C \), is the multifunction

\[
P_C(y, x, \theta) := \{ z \in C \mid \langle y - z, u - z \rangle \leq \theta \|y - x\|^2, \forall u \in C \}. \tag{10}
\]

We say that \( w \in P_C(y, x, \theta) \) is a feasible inexact projection of \( y \) onto \( C \) with respect to \( x \) and with error tolerance \( \theta \).

**Remark 2.6.** It follows from [8, Proposition 2.1.3, p. 201] that, for each \( y \in \mathbb{R}^n \), the exact projection \( P_C(y) \) is a vector in \( P_C(y, x, \theta) \). Hence, \( P_C(y, x, \theta) \) is nonempty for all \( y \in \mathbb{R}^n \) and \( x \in C \).

**Remark 2.7.** \( P_C(y, x, 0) := \{ P_C(y) \} \) for all \( y \in \mathbb{R}^n \) and \( x \in C \). From now on, we write \( P_C(y, x, 0) = P_C(y) \).

**Remark 2.8.** [13, Lemma 1] If \( y, \bar{y} \in \mathbb{R}^n, x, \bar{x} \in C, \) and \( \theta \geq 0 \), then we have

\[
\|w - P_C(\bar{y}, \bar{x}, 0)\| \leq \|y - \bar{y}\| + \sqrt{2\theta}\|y - x\|
\]

for any \( w \in P_C(y, x, \theta) \).

Finally, we enunciate an useful version of the Aubin property suitable for multifunctions with two blocks of variables. We say that a multifunction \( T : V \times W \Rightarrow S \) is partially Aubin continuous at \((\bar{v}, \bar{w}) \in V \times W\) with respect to \( w \) uniformly in \( v \) for \( \bar{s} \in S \) with modulus \( \lambda \) [20] (or simply, partially Aubin continuous at \((\bar{v}, \bar{w})\) w.r.t. \( w \) for \( \bar{s} \) with modulus \( \lambda \)) if \( \bar{s} \in T(\bar{v}, \bar{w}) \) and there are neighborhoods \( \mathcal{V} \) of \( \bar{v} \), \( \mathcal{W} \) of \( \bar{w} \) and \( \mathcal{S} \) of \( \bar{s} \) such that

\[
e(T(v, w) \cap \mathcal{S}, T(v, w')) \leq \lambda \|w - w'\| \quad \text{for all} \quad v \in \mathcal{V}, \quad w, w' \in \mathcal{W}.
\]
Algorithm 1: Quasi-Newton with Inexact Projections (QN-Inexp)

**Step 0.** Let $x^0 \in C$, $B_0$ and $\{\theta_k\} \subset [0, +\infty)$. Set $k \leftarrow 0$.

**Step 1.** If $0 \in f(x_k) + F(x_k)$ then **stop** returning $x_k$ as solution. Otherwise, compute $y_k \in \mathbb{R}^n$ such that

$$0 \in f(x_k) + B_k(y_k - x_k) + F(y_k).$$

**Step 2.** If $y_k \in C$, set $x_{k+1} = y_k$. Otherwise, take any $x_{k+1} \in C$ satisfying

$$x_{k+1} \in P_C(y_k, x_k, \theta_k).$$

**Step 3.** Compute $B_{k+1}$, set $k \leftarrow k + 1$ and go to **Step 1**.

3 The quasi-Newton method and its local convergence analysis

In this section, we propose the first quasi-Newton method for solving (1). Here, it is required that the auxiliary iterate $y_k$ be an exact solution of the correspondent unconstrained subproblem $0 \in f(x_k) + B_k(y - x_k) + F(y)$. As we already mentioned, $y_k$ may be infeasible. So, an inexact projection onto $C$ is employed to achieve feasibility at the limit, in the spirit of Definition 2.5. Algorithm 1 below formalizes this idea.

**Remark 3.1.** The projection in **Step 3** can be computed as an approximate feasible solution of the problem $\min_{z \in C}\{\|z - y_k\|^2/2\}$ satisfying $\langle y_k - x_{k+1}, z - x_{k+1}\rangle \leq \theta_k\|y_k - x_k\|^2$ for all $z \in C$. In order to maintain feasibility, we employ an interior point method in our numerical tests. Also, the choice of $\theta_k$ will be detailed in Section 6.

Next, we state the local convergence of the QN-Inexp method. This is the main result of this section. The point $x_*$ will always refer to a solution of (1).

**Theorem 3.2.** As in (1), let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \rightarrow \mathbb{R}^m$ be a function, $F : \Omega \rightrightarrows \mathbb{R}^m$ be a multifunction with closed graph and $C \subset \Omega$ be a nonempty closed convex set. Furthermore, let $x_*$ such that $0 \in f(x_*) + F(x_*)$, $x_* \in C$. Suppose the following conditions hold:

(i) $f + F$ is metrically regular at $x_*$ for 0 with modulus $\lambda > 0$;

(ii) there exist $\epsilon > 0$ and a neighborhood $\mathcal{X}$ of $x_*$ such that

$$\|f(u) - f(v) - f'(x_*)(u - v)\| \leq \epsilon\|u - v\| \quad \text{for all } u, v \in \mathcal{X}; \quad (11)$$
(iii) \( B_0 \) is chosen so that
\[
\|B_0 - f'(x_*)\| < \frac{1}{2\lambda}; \tag{12}
\]

(iv) \( \theta_k \geq 0 \) for all \( k \geq 0 \) and \( \hat{\theta} := \sup \theta_k < \frac{1}{2} \);

(v) There exists a constant \( c > 0 \) such that, for each \( k \geq 0 \), \( B_{k+1} \) satisfies the limited deterioration condition
\[
\|B_{k+1} - f'(x_*)\| \leq \|B_k - f'(x_*)\| + c(\|x_k - x_*\| + \|y_k - x_*\|) . \tag{13}
\]

Then there exists a neighborhood \( \mathcal{U} \) of \( x_* \) such that, starting from any \( x_0 \in C \cap \mathcal{U}\backslash\{x_*\} \), there is a sequence \( \{x_k\} \subset C \cap \mathcal{U} \) generated by the QN-INEXP method that converges linearly to \( x_* \).

**Proof.** Let us consider the radii \( a > 0 \) and \( b > 0 \) associated with the metric regularity of \( f + F \) (see Definition 2.1). Taking \( \lambda' > \lambda \), we can assume without loss of generality that \( a \) is small enough to \( B_a(x_*) \subset X \), \( \lambda(\delta + 2ca) < 1 \) and \( \lambda'(\epsilon + \delta + 2ca) < 1 \).

We define the constant \( 
\gamma := \max \left\{ \hat{\gamma}, (1 + \sqrt{2\theta_0}) \frac{\lambda'(\epsilon + \delta + 2ca)}{1 - \lambda(\delta + 2ca)} + \sqrt{2\theta_0} \right\}. \tag{15}\)

It is immediate from (14) that \( 0 < \gamma < 1 \). In the sequel, we use induction to prove that, starting from any \( x_0 \) close enough to \( x_* \), is possible to generate a sequence \( \{x_k\} \) linearly convergent to \( x_* \).

Take \( x_0 \in C \cap \mathcal{B}_{r_*}(x_*)\backslash\{x_*\} \), where
\[
r_* := \min \left\{ a, \frac{b}{\epsilon + 2\hat{\delta}}, \frac{b(1 - \hat{\gamma})}{2(1 - \hat{\gamma})(\epsilon + \hat{\delta}) + 2ca} \right\}. \tag{16}
\]

To construct the next iterate \( x_1 \), let us verify the conditions in Theorem 2.4. Defining the auxiliary multifunction
\[
\Phi_{x_0}(z) := G_{x_*}^{-1}(f(x_* - f(x_0) - B_0(z - x_0) + f'(x_*)(z - x_*)),
\]

8
Thus, we have
\[ z, \quad e \quad x \quad f(x) - f(x_0) - B_0(x - x_0). \]

By using (11) and the definition of \( r_* \), we get that
\[ \| f(x_*) - f(x_0) - B_0(x - x_0) \|
\leq \| f(x_*) - f(x_0) - f'(x_*)(x_* - x_0) \|
+ \| f'(x_*) - B_0(x - x_0) \|
\leq (\varepsilon + \delta)\| x_* - x_0 \| \leq b. \]  
(16)

Then, since \( 0 \in G_{x_*}(x_*) \) (see (8)), we obtain from (16) and Definition 2.1
\[ d(x_*, \Phi_{x_0}(x_*)) \leq \lambda' d(f(x_*) - f(x_0) - B_0(x_* - x_0), G_{x_*}(x_*)) \]
\[ \leq \lambda' \| f(x_*) - f(x_0) - B_0(x_* - x_0) \| \leq \lambda'(\varepsilon + \delta)\| x_* - x_0 \|. \]

Thus, we have \( d(x_*, \Phi_{x_0}(x_*)) \leq \rho(1 - \lambda\delta) \) for
\[ \rho := \frac{\lambda'(\varepsilon + \delta)\| x_* - x_0 \|}{1 - \lambda\delta}. \]

Now, let \( p, q \in \mathbb{B}_\rho[x_*] \). Taking into account (14) and \( x_0 \in \mathbb{B}_{r_*}(x_*) \{ x_* \} \), we can verify that \( \rho < r_* \). Therefore, for \( s = p \) or \( s = q \) we obtain that
\[ \| f(x_*) - f(x_0) - B_0(s - x_0) + f'(x_*)(s - x_*) \|
\leq \| f(x_*) - f(x_0) - f'(x_*)(x_* - x_0) \|
+ \| f'(x_*)(x_* - x_0) - f'(x_*)(x_* - s) - B_0(s - x_0) \|
\leq (\varepsilon + 2\delta)\| x_* - x_0 \| \leq b, \]
where the second inequality holds since \( p, q \in \mathbb{B}_\rho[x_*] \) and, by (14), \( \rho < \| x_0 - x_* \| \). As \( e(\emptyset, \Phi_{x_0}(q)) = 0 \), we can assume that \( \Phi_{x_0}(p) \cap \mathbb{B}_\delta[x_*] \neq \emptyset \) for all \( p \in \mathbb{B}_\rho[x_*] \). Let \( z \in \Phi_{x_0}(p) \cap \mathbb{B}_\delta[x_*] \). From Definition 2.1 with \( \bar{x} = x_* \), \( \bar{a} = 0 \), \( x = z \), \( u = f(x_*) - f(x_0) - B_0(q - x_0) + f'(x_*)(q - x_*) \) and \( G = \Phi_{x_0}(q) \), we have
\[ d(z, \Phi_{x_0}(q)) \leq \lambda d(f(x_*) - f(x_0) - B_0(q - x_0) + f'(x_*)(q - x_*), G_{x_*}(z)). \]  

Since \( z \in \Phi_{x_0}(p) \) implies \( f(x_*) - f(x_0) - B_0(p - x_0) + f'(x_*)(p - x_*) \in G_{x_*}(z) \), the definition of distance given in (7) ensures that
\[ d(f(x_*) - f(x_0) - B_0(q - x_0) + f'(x_*)(q - x_*), \quad G_{x_*}(z)) \leq \| B_0 - f'(x_*) \|\| p - q \|. \]
So, \( d(z, \Phi_{x_0}(q)) \leq \lambda \| B_0 - f'(x_*) \|\| p - q \| \), which implies \( e(\Phi_{x_0}(p) \cap \mathbb{B}_\rho[x_*], \Phi_{x_0}(q)) \leq \lambda \| B_0 - f'(x_*) \|\| p - q \|. \)
Futhermore, since \( \rho < r_* \leq a \), we have \( e(\Phi_{x_0}(p) \cap \mathbb{B}_\rho[x_*], \Phi_{x_0}(q)) \leq e(\Phi_{x_0}(p) \cap \mathbb{B}_\rho[x_*], \Phi_{x_0}(q)). \) Hence, using (12), we obtain
\[ e(\Phi_{x_0}(p) \cap \mathbb{B}_\rho[x_*], \Phi_{x_0}(q)) \leq \lambda \| B_0 - f'(x_*) \|\| p - q \| \leq \lambda\delta\| p - q \|. \]
As $\lambda \delta < 1$ (see (12)), we can apply Theorem 2.4 with $\Phi = \Phi_{x_0}$, $\bar{x} = x_*$ and $\mu = \lambda \delta$ to conclude that there exists $y_0 \in \Phi_{x_0}(y_0)$ such that

$$
\|y_0 - x_*\| \leq \frac{\lambda'(\epsilon + \delta)}{1 - \lambda \delta}\|x_0 - x_*\|.
$$

At this point, we have constructed $y_0$. The next iterate $x_1$ is obtained according to \textbf{Step 2}, that is, $x_1 := y_0 \in C \cap B_r[x_*]$ if $y_0 \in C$ and $x_1 \in P_C(y_0, x_0, \theta_0)$ otherwise. Also, it follows from Remark 2.8 with $w = x_1$, $y = y_0$, $x = x_0$, $\bar{y} = x_*$, $\bar{x} = x_*$ and $\theta = \theta_0$ that

$$
\|x_1 - P_C(x_*, x_*, 0)\| \leq \left(1 + \sqrt{2\theta_0}\right)\|x_* - y_0\| + \sqrt{2\theta_0}\|x_* - x_0\|
\leq \left[1 + \frac{\lambda'(\epsilon + \delta)}{1 - \lambda \delta} + \sqrt{\theta_0}\right]\|x_* - x_0\| < \bar{\gamma}\|x_0 - x_*\|.
$$

Hence, remembering that $P_C(x_*, x_*, 0) = x_*$ we conclude that $\|x_1 - x_*\| \leq \bar{\gamma}\|x_0 - x_*\|$. Since $\bar{\gamma} < 1$, this yields $x_1 \in C \cap B_r[x_*]$.

Now, suppose that for $k > 0$ there are $y_0, \ldots, y_{k-1}, x_1, \ldots, x_k$ such that $y_j \in B_r[x_*]$ for all $0 \leq j \leq k-1$, $x_j \in C \cap B_r[x_*]$ for all $1 \leq j \leq k$, and

$$
\|y_{j-1} - x_*\| \leq \bar{\gamma}\|y_{j-1} - x_*\| \quad \text{for} \quad j = 1, \ldots, k, \quad (17)
$$

$$
\|x_j - x_*\| \leq \bar{\gamma}\|x_{j-1} - x_*\| \quad \text{for} \quad j = 1, \ldots, k. \quad (18)
$$

To complete the induction process we proceed analogously to the first step. As in (16), we need to show that $\|f(x_*) - f(x_k) - B_k(x_*) - x_k\| \leq b$. Indeed, since $x_k \in C \cap B_r[x_*]$ we have

$$
\|f(x_*) - f(x_k) - B_k(x_*) - x_k\| \leq \|f(x_*) - f(x_k) - f'(x_*)(x_* - x_k)\|
+ \|f'(x_*) - B_k(x_*) - x_k\| \leq (\epsilon + \|f'(x_*) - B_k\|)\|x_* - x_k\|. \quad (19)
$$

On the other hand, from (13) we obtain that

$$
\|f'(x_*) - B_k\| \leq \|f'(x_*) - B_{k-1}\| + c(\|x_{k-1} - x_*\| + \|y_{k-1} - x_*\|) \\
\leq \|f'(x_*) - B_{k-2}\| + c(\|x_{k-2} - x_*\| + \|y_{k-2} - x_*\|) \\
+ c(\|x_{k-1} - x_*\| + \|y_{k-1} - x_*\|) \\
\leq \cdots \leq \|f'(x_*) - B_0\| + c \left(\sum_{j=0}^{k-1} \|x_j - x_*\| + \sum_{j=1}^{k} \|y_{j-1} - x_*\|\right).
$$
By (17) and (18), the last inequality becomes
\[
\|f'(x_*) - B_k\| \leq \|f'(x_*) - B_0\| + c \left( \|x_0 - x_*\| + \sum_{j=1}^{k-1} \|x_j - x_*\| + \sum_{j=1}^{k} \|y_{j-1} - x_*\| \right)
\]
\[
\leq \delta + c\|x_0 - x_*\| + 2c \sum_{j=1}^{k} \tilde{\gamma}^j\|x_0 - x_*\|
\]
\[
\leq \delta + 2c \sum_{j=0}^{k} \tilde{\gamma}^j\|x_0 - x_*\| \leq \delta + \frac{2ca}{1-\tilde{\gamma}}. \tag{20}
\]

Hence, from (19), (20) and the condition on \(r_*\), we have
\[
\|f(x_*) - f(x_k) - B_k(x_* - x_k)\| \leq \left( \epsilon + \delta + \frac{2ca}{1-\tilde{\gamma}} \right) r_* \leq b.
\]

By Definition 2.1 and taking into account that \(0 \in G_{x_*}(x_*)\), we obtain
\[
d(x_*, \Phi_{x_k}(x_*)) \leq \lambda'\|f(x_*) - f(x_k) - f'(x_*)(x_* - x_k)\|
\]
\[
+ \lambda'\|f'(x_*) - B_k\|\|x_* - x_k\|
\]
\[
\leq \lambda' \left( \epsilon + \delta + \frac{2ca}{1-\tilde{\gamma}} \right) \|x_* - x_k\| = \tilde{\rho} \left[ 1 - \lambda \left( \delta + \frac{2ca}{1-\tilde{\gamma}} \right) \right],
\]
where
\[
\tilde{\rho} = \frac{\lambda' \left( \epsilon + \delta + \frac{2ca}{1-\tilde{\gamma}} \right)}{1 - \lambda \left( \delta + \frac{2ca}{1-\tilde{\gamma}} \right)} \|x_* - x_k\|.
\]

Now, let \(p, q \in B_{\tilde{\rho}}[x_*]\). From (15) we have \(\tilde{\rho} < r_*\). Thus, it is not difficult to see that
\[
\|f(x_*) - f(x_k) - B_k(v - x_k) + f'(x_*)(v - x_*)\| < b \quad \text{for} \quad v = p \text{ or } v = q.
\]
Let \(y \in \Phi_{x_k}(p) \cap B_{\tilde{\rho}}[x_*]\). From Definition 2.1 with \(\bar{x} = x_*, \bar{u} = 0, x_k = y, u = f(x_*) - f(x_k) - B_k(q - x_k) + f'(x_*)(q - x_*)\) and \(G = \Phi_{x_k}(q)\), we obtain
\[
d(y, \Phi_{x_k}(q)) \leq \lambda d\left( f(x_*) - f(x_k) - B_k(q - x_k) + f'(x_*)(q - x_*), G_{x_*}(y) \right).
\]

Since \(y \in \Phi_{x_k}(p)\) implies \(f(x_*) - f(x_k) - B_k(p - x_k) + f'(x_*)(p - x_*) \in G_{x_*}(y)\), the definition of distance given in (7) gives
\[
d(f(x_*) - f(x_k) - B_k(q - x_k) + f'(x_*)(q - x_*), G_{x_*}(y)) \leq \|B_k - f'(x_*)\||p - q||.
\]

Combining the two last inequalities we conclude that
\[
d(y, \Phi_{x_k}(q)) \leq \lambda \|B_k - f'(x_*)\||p - q||.
\]

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Taking the supremum with respect to \( z \in \Phi_{x_k}(p) \cap \mathcal{B}_a[x_\ast] \) in the last inequality and using the definition of excess given in (7), we have

\[
e(\Phi_{x_k}(p) \cap B_a[x_\ast], \Phi_{x_k}(q)) \leq \lambda \| B_k - f'(x_\ast) \| \| p - q \|.
\]

As \( \bar{\rho} < r_\ast \leq a \), we have \( e(\Phi_{x_k}(p) \cap \mathcal{B}_\rho[x_\ast], \Phi_{x_k}(q)) \leq e(\Phi_{x_k}(p) \cap \mathcal{B}_a[x_\ast], \Phi_{x_k}(q)) \). Hence, from the last inequality and the properties of the norm, we obtain

\[
e(\Phi_{x_k}(p) \cap B_\rho[x_\ast], \Phi_{x_k}(q)) \leq \lambda \| B_k - f'(x_\ast) \| \| p - q \| \leq \lambda \left( \delta + \frac{2ca}{1 - \gamma} \right) \| p - q \|.
\]

By (14) we have \( \lambda \left( \delta + \frac{2ca}{1 - \gamma} \right) < 1 \), so we can apply Theorem 2.4 with \( \Phi = \Phi_{x_k}, \bar{x} = x_\ast \) and \( \mu = \lambda' \left( \delta + \frac{2ca}{1 - \gamma} \right) \) to conclude that there exists \( y_k \in \Phi_{x_k}(y_k) \) such that

\[
\| y_k - x_\ast \| \leq \frac{\lambda' \left( \epsilon + \delta + \frac{2ca}{1 - \gamma} \right)}{1 - \lambda \left( \delta + \frac{2ca}{1 - \gamma} \right)} \| x_\ast - x_k \|.
\]

As before, we take \( x_{k+1} := y_k \in C \cap \mathcal{B}_{\bar{\rho}}[x_\ast] \) if \( y_k \in C \), and \( x_{k+1} \in P_C(y_k, x_\ast, \theta_k) \) otherwise. From Remarks 2.7 and 2.8 with \( w = x_{k+1}, y = y_k, x = x_k, \bar{y} = x_\ast, \bar{x} = x_\ast, \theta = \theta_k \), we get

\[
\| x_\ast - x_{k+1} \| \leq \left[ (1 + \sqrt{2\theta}) \frac{\lambda' \left( \epsilon + \delta + \frac{2ca}{1 - \gamma} \right)}{1 - \lambda \left( \delta + \frac{2ca}{1 - \gamma} \right)} + \sqrt{2\theta} \right] \| x_\ast - x_k \| < \gamma \| x_\ast - x_k \|.
\]

Therefore, we conclude the induction process. \( \square \)

### 4 The inexact quasi-Newton approach

In this section, we propose a version of QN-INEXP where subproblems need not to be solved exactly. Specifically, the they become

\[
(f(x_k) + B_k(y_k - x_k) + F(y_k)) \cap R_k(x_k, y_k) \neq \emptyset, \quad k = 0, 1, \ldots, (21)
\]

where \( \{B_k\} \) is a sequence of matrices. To illustrate the flexibility of condition (21), we observe that when \( F \equiv \{0\} \) and \( R_k(x_k, x_{k+1}) := \mathcal{B}_{\eta_k\|f(x_k)\|}(0), \eta_k > 0 \), then we recover the inexact quasi-Newton method developed in [14] for nonlinear systems of equations. Also, if \( R_k(x_k, x_{k+1}) = \{-r_k(x_k)\} \) where \( \{r_k\} \) is a sequence of functions representing the inexactness, our method reduces to an instance of the inexact quasi-Newton method considered in [12]. Similarly to the previous section, we formally state our inexact quasi-Newton scheme with inexact projections in Algorithm 2.
Algorithm 2: Inexact Quasi-Newton with Inexact Projections (IQN-INEXP)

Step 0. Let $x_0 \in C$, $B_0$ and $\{\theta_k\} \subset [0, +\infty)$. Set $k \leftarrow 0$.

Step 1. If $0 \in f(x_k) + F(x_k)$ then stop returning $x_k$ as solution. Otherwise, compute $y_k \in \mathbb{R}^n$ such that

$$\left(f(x_k) + B_k(y_k - x_k) + F(y_k)\right) \cap R_k(x_k, y_k) \neq \emptyset.$$

Step 2. If $y_k \in C$, set $x_{k+1} = y_k$. Otherwise, take any $x_{k+1} \in C$ satisfying

$$x_{k+1} \in P_C(y_k, x_k, \theta_k).$$

Step 3. Compute $B_{k+1}$, set $k \leftarrow k + 1$ and go to Step 1.

As we made for QN-INEXP, we will establish the local convergence of IQN-INEXP. One may think this can be done simply by adapting the proof of Theorem 3.2 by introducing inexactness, but this is not totally true. Here, differently from the previous theorem that uses the principle of contractions, the Coincidence Theorem will serve as support [19, Theorem 1].

Theorem 4.1 (Coincidence Theorem). Let $X$ and $Y$ be two metric spaces and consider the multifunctions $\Phi : X \rightrightarrows Y$ and $\Gamma : Y \rightrightarrows X$. Let $\hat{x} \in X$ and $\hat{y} \in Y$. Also, let $\eta$, $\kappa$, and $\mu$ be positive scalars such that $\mu \kappa < 1$. Suppose that one of the sets

$$\text{gph } \Phi \cap (B_{\eta}[\hat{x}] \times B_{\eta/\mu}[\hat{y}]) \quad \text{or} \quad \text{gph } \Gamma \cap (B_{\eta/\mu}[\hat{y}] \times B_{\eta}[\hat{x}])$$

is closed while the other is complete, or that both sets

$$\text{gph } (\Phi \circ \Gamma) \cap (B_{\eta}[\hat{x}] \times B_{\eta}[\hat{x}]) \quad \text{and} \quad \text{gph } (\Gamma \circ \Phi) \cap (B_{\eta/\mu}[\hat{y}] \times B_{\eta/\mu}[\hat{y}])$$

are complete. Also, suppose the following conditions hold:

(i) $d(\hat{y}, \Phi(\hat{x})) < \eta(1 - \kappa \mu)/(2 \mu)$;

(ii) $d(\hat{x}, \Gamma(\hat{y})) < \eta(1 - \kappa \mu)/2$;

(iii) $e (\Phi(p) \cap B_{\eta/\mu}[\hat{y}], \Phi(q)) \leq \mu \rho(p, q)$ for all $p, q \in B_{\eta}[\hat{x}]$ such that $\rho(p, q) \leq \eta(1 - \kappa \mu)/\mu$;

(iv) $e (\Gamma(p) \cap B_{\eta}[\hat{x}], \Gamma(q)) \leq \kappa \rho(p, q)$ for all $p, q \in B_{\eta/\mu}[\hat{y}]$ such that $\rho(p, q) \leq \eta(1 - \kappa \mu)$.

Then there exist $\hat{x} \in B_{\eta}[\hat{x}]$ and $\hat{y} \in B_{\eta/\mu}[\hat{y}]$ such that $\hat{y} \in \Phi(\hat{x})$ and $\hat{x} \in \Gamma(\hat{y})$. If the mappings $B_{\eta}[\hat{x}] \ni x \mapsto \Phi(x) \cap B_{\eta/\mu}[\hat{y}]$ and $B_{\eta/\mu}[\hat{y}] \ni y \mapsto \Gamma(y) \cap B_{\eta}[\hat{x}]$ are single-valued, then the points $\hat{x}$ and $\hat{y}$ are unique in $B_{\eta}[\hat{x}]$ and $B_{\eta/\mu}[\hat{y}]$, respectively.
Now, we apply Theorem 4.1 to obtain the desired convergence result for Algorithm 2.

**Theorem 4.2.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( f : \Omega \to \mathbb{R}^m \) be a function, \( F : \Omega \rightrightarrows \mathbb{R}^m \) be a multifunction with closed graph and \( C \subset \Omega \) be a nonempty closed convex set. Furthermore, let \( x_* \) such that \( 0 \in f(x_*) + F(x_*) \), \( x_* \in C \).

Suppose valid conditions (i) to (v) of Theorem 3.2. Also, suppose that the following additional conditions hold:

(vi) for each \( k \geq 0 \), the mapping \( (u, x) \mapsto R_k(u, x) \) is partially Aubin continuous at \( (x_*, x_*) \) w.r.t. \( x \) for \( 0 \) with modulus \( \mu > 0 \);

(vii) there are scalars \( t \in \left( 0, \frac{1 - \sqrt{2\hat{\theta}}}{1 + \sqrt{2\hat{\theta}}} \right) \) with \( 0 < \gamma < t(1 - \lambda \mu)/2\mu \) and \( \beta > 0 \) such that

\[
d(0, R_k(u, x_*)) \leq \gamma \| u - x_* \| \quad \text{for all} \quad u \in B_{\beta}(x_*), \quad k \geq 0. \tag{22}\]

Then there exists a neighborhood \( U \) of \( x_* \) such that, starting from any \( x_0 \in C \cap U \setminus \{x_*\} \), there is a sequence \( \{x_k\} \subset C \cap U \) generated by the IQN-IEXP method that converges linearly to \( x_* \).

**Proof.** From item (vi), there exist positive constants \( a \) and \( b \) satisfying

\[
e(R_k(u, x) \cap B_0, R_k(u, x')) \leq \mu \| x - x' \| \quad \text{for all} \quad u, x, x' \in B_a[x_*]. \tag{23}\]

By (11), we can assume \( a \) small enough so that

\[
\| f(x) - f(x_*) - f'(x_*)(x - x_*) \| \leq \epsilon \| x - x_* \| \quad \text{for all} \quad x \in B_a[x_*], \tag{24}\]

and

\[
\left( \delta + \frac{2ca}{1 - \tilde{t}_0} \right) \lambda < 1, \tag{25}\]

where \( c \) is the constant in (13) and \( \tilde{t}_k := \left[ (1 + \sqrt{2\theta_k})t + \sqrt{2\theta_k} \right] \), for all \( k \geq 0 \). Note that \( \tilde{t}_k < 1 \) from item (vii) and the fact that \( \theta_k \leq \hat{\theta} \). Also, we choose a constant \( \kappa \) such that \( \kappa > \lambda \), \( \kappa \mu < 1 \), \( \gamma < t(1 - \kappa \mu)/(2\mu) \),

\[
\kappa > \omega := \frac{\lambda}{1 - \lambda \left( \delta + \frac{2ca}{1 - \tilde{t}_0} \right)}
\]

and

\[
\epsilon + \delta + \frac{2ca}{1 - \tilde{t}_0} < \frac{t(1 - \kappa \mu)}{2\kappa}. \tag{26}\]

In order to apply Proposition 2.3, we define the following positive constant:

\[
r_* := \min \left\{ a, \frac{b}{\left( \epsilon + \delta + \frac{2ca}{1 - \tilde{t}_0} \right)}, \beta , b\mu \right\}. \tag{27}\]
Choose any $x_0 \in \mathbb{B}_r[x_0] \setminus \{x_*\}$. Since $r_* \leq a$ and $(\epsilon + \delta)r_* \leq b$ we obtain
\[
\|f(x_*) - f(x_0) - B_0(x_* - x_0)\| \\
\leq \|f(x_*) - f(x_0) - f'(x_*)(x_* - x_0)\| + \|f'(x_*) - B_0(x_* - x_0)\| \\
\leq (\epsilon + \delta)r_* \leq b.
\]
Combining the above inequality with the fact that $0 \in f(x_*) + F(x_*)$ we have
\[
-f(x_*) + f(x_0) + B_0(x_* - x_0) \in G_{x_0}(x_*) \cap \mathbb{B}_0[0]. \tag{28}
\]
Then, since $\kappa > \varpi$ and $\lambda \delta < 1$, we can apply Proposition 2.3 with $u = x_0$, $\bar{x} = x_*$, $\zeta = \delta$, $y = -f(x_*) + f(x_0) + B_0(x_* - x_0)$, $y' = 0$ and $a = r_*$ to conclude that
\[
e \left( G_{x_0}^{-1}(-f(x_*) + f(x_0) + B_0(x_* - x_0)) \cap \mathbb{B}_r[x_*], G_{x_0}^{-1}(0) \right) \\
\leq \kappa \| -f(x_*) + f(x_0) + B_0(x_* - x_0) \|.
\]
Now, we use the definition of excess in (7) and (28) to obtain
\[
d(x_*, G_{x_0}^{-1}(0)) \leq \kappa \| -f(x_*) + f(x_0) + B_0(x_* - x_0) \|.
\]
After simple manipulations and using (26), item (vii) and (24), the above inequality implies that
\[
d(x_*, G_{x_0}^{-1}(0)) \leq \kappa(\epsilon + \delta)\|x_* - x_0\| < \frac{t(1 - \kappa \mu)}{2} \|x_* - x_0\| < \frac{\eta(1 - \kappa \mu)}{2},
\]
where the second inequality follows by (26) with $\eta := t\|x_0 - x_*\|$. On the other hand, as $\|x_0 - x_*\| \leq r_* \leq \beta$, $\kappa \mu < 1$ and $\gamma < t(1 - \kappa \mu)/(2\mu)$, we obtain from (22) that
\[
d(0, R_0(x_0, x_*)) \leq \gamma \|x_0 - x_*\| < \frac{t(1 - \kappa \mu)}{2\mu} \|x_0 - x_*\| = \frac{\eta(1 - \kappa \mu)}{2\mu}. \tag{29}
\]
Hence, the conditions (i) and (ii) in Theorem 4.1 are satisfied with $\Phi(x) = R_0(x_0, x_*)$, $\Gamma = G_{x_0}^{-1}$, $\kappa$, $\mu$, $\bar{x} = x_*$, $\bar{y} = 0$ and $\eta = t\|x_0 - x_*\|$.

Now, since $x_0 \in \mathbb{B}_r[x]$ and $t < 1$, we obtain that $\eta \leq r_* \leq a$ and $\eta/\mu \leq b$. Thus, $\mathbb{B}_{\eta/\mu}[0] \subset \mathbb{B}_b[0]$ and $\mathbb{B}_{\eta}[x_*] \subset \mathbb{B}_a[x_*]$. Hence, for any $x, x' \in \mathbb{B}_{\eta}[x_*]$, condition (23) implies
\[
e(R_0(x_0, x) \cap B_{\eta/\mu}[0], R_0(x_0, x')) \leq e(R_0(x_0, x) \cap B_0[0], R_0(x_0, x')) \leq \mu\|x - x'\|
\]
that is, condition (iii) in Theorem 4.1 holds. Furthermore, from (9) we conclude that
\[
e \left( G_{x_0}^{-1}(x) \cap \mathbb{B}_{\eta}[x_*], G_{x_0}^{-1}(x') \right) \leq e \left( G_{x_0}^{-1}(x) \cap \mathbb{B}_a[x_*], G_{x_0}^{-1}(x') \right) \leq \kappa\|x - x'\|
\]
for all $x, x' \in \mathbb{B}_{\eta/\mu}[0] \subset \mathbb{B}_b[0]$. So, condition (iv) in Theorem 4.1 also holds. Therefore, we can apply Theorem 4.1 to to ensure the existence of $y_0 \in \mathbb{B}_\eta[x_*]$ and $v_1 \in \mathbb{B}_{\eta/\mu}[0]$ such that $y_0 \in G_{x_0}^{-1}(v_1)$ and $v_1 \in R_0(x_0, y_0)$; that is,
\[
v_1 \in G_{x_0}(y_0) \cap R_0(x_0, y_0) = (f(x_0) + B_0(y_0 - x_0) + F(y_0)) \cap R_0(x_0, y_0). \tag{30}
\]
Moreover, Theorem 4.1 implies \( \|y_0 - x_1\| \leq t \|x_0 - x_1\| \). The inclusion (30) yields that \( y_0 \) satisfies (21) for \( k = 0 \). In particular, since \( t < 1 \), \( y_0 \in \mathbb{B}_{r_0}[x_1] \). If \( y_0 \in C \cap \mathbb{B}_\eta[x_1] \) then take \( x_1 := y_0 \). Otherwise, take \( x_1 \in P_C(y_0, x_0, \theta_0) \). By using similar arguments as in Theorem 3.2 (see also Remark 2.8) we obtain

\[
\|x_1 - x_*\| \leq (1 + \sqrt{2\theta_0})\|x_* - y_0\| + \sqrt{2\theta_0}\|x_* - x_0\|
= [(1 + \sqrt{2\theta_0})t + \sqrt{2\theta_0}]\|x_* - x_0\| = \tilde{t}_0\|x_0 - x_*\|.
\]

Since \( \tilde{t}_0 < 1 \), we have \( x_1 \in C \cap \mathbb{B}_\eta[x_1] \subset C \cap \mathbb{B}_{r_0}[x_1] \). By induction, we suppose that there exist an integer \( k > 1 \) and points \( x_1, x_2, \ldots, x_k \in C \cap \mathbb{B}_{r_*}[x_*] \) and \( y_1, y_2, \ldots, y_k \in \mathbb{B}_{r_*}[x_*] \), satisfying

\[
\|y_j - x_*\| \leq t\|x_{j-1} - x_*\|, \forall j = 1, 2, \ldots, k. \tag{31}
\]

\[
\|x_j - x_*\| \leq \tilde{t}_0\|x_{j-1} - x_*\|, \forall j = 1, 2, \ldots, k. \tag{32}
\]

Without loss of generality, we assume that \( y_j - x_{j-1} \) and \( x_* \) are distinct from each other. Note that, as \( x_k \in \mathbb{B}_{r_*}[x_*] \) and \( r_* \leq \beta \), we can repeat the same argument from (29) with \( x_0 \) replaced by \( x_k \), obtaining

\[
d(0, R_k(x_k, x_*)) \leq \frac{\eta(1 - \kappa \mu)}{2\mu}.
\]

To apply Proposition 2.3 in the induction step, firstly we need to show that \( \|B_k - f'(x_*)\| \leq \zeta \) for some positive scalar \( \zeta \) such that \( \zeta \lambda < 1 \). By combining (13), (31), (32) and using the fact that \( x_0 \in \mathbb{B}_{r_*}[x_*] \), we have

\[
\|f'(x_*) - B_k\| \leq \|f'(x_*) - B_{k-1}\| + c(\|x_{k-1} - x_*\| + \|y_{k-1} - x_*\|)
\leq \|f'(x_*) - B_0\| + c \sum_{j=0}^{k} (\|x_j - x_*\| + t\|x_j - x_*\|)
\leq \delta + (1 + t)c\|x_0 - x_*\| + (1 + t)c \sum_{j=1}^{k} \|x_j - x_*\|
\leq \delta + (1 + t)c \sum_{j=0}^{k} \|x_j - x_*\| \leq \delta + 2c \sum_{j=0}^{\infty} \tilde{t}_0^j\|x_0 - x_*\| \leq \delta + \frac{2ca}{1 - \tilde{t}_0}.
\]

That is, for all \( k \geq 1 \),

\[
\|f'(\bar{x}) - B_k\| \leq \delta + \frac{2ca}{1 - \tilde{t}_0}. \tag{33}
\]

The above inequality combined with (24) implies

\[
\| - f(x_*) + f(x_k) + B_k(x_k - x_*)\|
\leq \| - f(x_*) + f(x_k) + f'(x_*)(x_* - x_k)\| + \|f'(x_*) - B_k)(x_* - x_k)\|
\leq c\|x_* - x_k\| + (\delta + \frac{2ca}{1 - \tilde{t}_0})\|x_* - x_k\| \leq \left(\epsilon + \delta + \frac{2ca}{1 - \tilde{t}_0}\right) r_* \leq b.
\]

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Thus, \(-f(x^*) + f(x_k) + B_k(x_k - x^*) \in B_0[0] \cap G_{x_k}(x^*)\). In particular, \(x_k \in G_{x_k}^{-1}(-f(x^*) + f(x_k) + B_k(x_k - x^*))\). Then, using (25), (26), (33) and the last inequality we can apply Proposition 2.3 to obtain
\[
d(x^*, G_{x_k}^{-1}(0)) \leq \kappa \left( \epsilon + \frac{2ca}{1-\theta} \right) \|x^* - x_k\|
\]
\[
\leq \frac{t(1 - \kappa \mu)}{2} \|x^* - x_k\| = \frac{\eta(1 - \kappa \mu)}{2}.
\]
Thus the conditions (vi) and (vii) in Theorem 4.1 are satisfied with \(u = x_k\). Now, since \(x_k \in B_{r_0}[x^*]\) and \(\eta = t\|x_k - x^*\|\), condition (27) ensures that \(\eta \leq r_0 \leq a\) and \(\eta/\mu \leq b\). So, (23) implies condition (iii) of Theorem 4.1. Furthermore, from (9) we conclude that condition (iv) in Theorem 4.1 holds for \(\Gamma = G_{x_k}^{-1}\) with \(j = k\). Thus, the assumptions of Theorem 4.1 are satisfied with \(\eta = t\|x_k - x^*\|\), and hence there exists \(y_k \in B_{\eta}[x^*]\) and \(v_{k+1} \in B_{\eta/\mu}[0]\) such that \(y_k \in G_{x_k}^{-1}(v_{k+1})\) and \(v_{k+1} \in R_k(x_k, y_k)\); that is,
\[
v_{k+1} \in G_{x_k}(y_k) \cap R_k(x_k, y_k) = (f(x_k) + B_k(y_k - x_k) + F(y_k)) \cap R_k(x_k, y_k).
\]
Moreover, Theorem 4.1 implies that \(\|y_k - x^*\| \leq t\|x_k - x^*\|\). The inclusion (34) also ensures that \(y_k\) satisfies (21) for all \(k \geq 0\). Since \(t < 1\) we have \(y_k \in B_{r_0}[x^*]\). Again, if \(y_k \in C \cap B_{\eta}[x^*]\) then take \(x_{k+1} := y_k\) and \(x_{k+1} \in P_{G \cap B_{\eta}[x^*]}(y_k, x_k, \theta_k)\) otherwise. By using similar arguments employed in Theorem 3.2, we finally get
\[
\|x_{k+1} - x^*\| \leq \left(1 + \sqrt{2\theta_k}t + \sqrt{2\theta_k}\right) \|x^* - x_k\| = \tilde{t}_k\|x_k - x^*\|.
\]
As \(\tilde{t}_k < 1\), we obtain \(x_{k+1} \in B_{r_0}[x^*]\), concluding the proof.

5 Convergence of the Broyden-type quasi-Newton methods

In this section, we consider both QN-INEXP and IQN-INEXP methods when \(B_{k+1}\) in **Step 3** is computed by the classical Broyden update scheme. Let us denote by \(\langle \cdot, \cdot \rangle\) scalar products in \(\mathbb{R}^n\). The Broyden update rule is defined as
\[
B_{k+1} := B_k + \frac{(z_k - B_k s_k) - B_k s_k \langle s_k, \cdot \rangle}{\|s_k\|^2},
\]
where \(z_k := f(y_k) - f(x_k)\) and \(s_k := y_k - x_k\). Despite of (12), a practical usual choice for \(B_0\) is \(B_0 = f'(x_0)\).

The aim of this section is to show that the Broyden update rule satisfies the limited deterioration property (13), a crucial condition in Theorems 3.2 and 4.2. Therefore, (35) is one practical choice that fulfills the general framework of our quasi-Newton methods.
Lemma 5.1. Let $A \in \mathcal{L}(X,Y)$, where $X$ and $Y$ are Hilbert spaces. If $x \in X \setminus \{0\}$, then
\[
\left\| A - \frac{\langle x, \cdot \rangle Ax}{\|x\|^2} \right\| = \begin{cases} 
0, & \text{if } \dim X = 1; \\
\|A\|, & \text{if } \dim X > 1.
\end{cases}
\]

The proof of this Lemma can be found in [5].

Proposition 5.2. Suppose that the Fréchet derivative mapping $f$ is Lipschitz continuous with constant $L$ in a convex neighborhood $\mathcal{X}$ of a point $x_*$. Given $B_k \in \mathcal{L}(X,Y)$ and $x_k, y_k \in \mathcal{X}$, $y_k \neq x_k$, the operator $B_{k+1}$ defined as in (35) satisfies
\[
\|B_{k+1} - f'(x_*)\| \leq \|B_k - f'(x_*)\| + \frac{L}{2}(\|y_k - x_*\| + \|x_* - x_k\|).
\]

Proof. Let $y_k, x_k \in \mathcal{X}$, $x_k \neq y_k$ and let $B_{k+1}$ be defined as in (35). We have
\[
B_{k+1}f'(x_*) = B_k - f'(x_*) + \frac{(z_k - B_k s_k)\langle s_k, \cdot \rangle}{\|s_k\|^2}
\]
\[
= B_k - f'(x_*) - \frac{(B_k - f'(x_*) s_k)\langle s_k, \cdot \rangle}{\|s_k\|^2} + \frac{(z_k - f'(x_*) s_k)\langle s_k, \cdot \rangle}{\|s_k\|^2}.
\]
Thus,
\[
\|B_{k+1} - f'(x_*)\| \leq \left| B_k - f'(x_*) - \frac{(B_k - f'(x_*) s_k)\langle s_k, \cdot \rangle}{\|s_k\|^2} \right| + \frac{\|z_k - f'(x_*) s_k\|}{\|s_k\|}.
\]

It follows from Lemma 5.1 that
\[
\left| B_k - f'(x_*) - \frac{(B_k - f'(x_*) s_k)\langle s_k, \cdot \rangle}{\|s_k\|^2} \right| \leq \|B_k - f'(x_*)\|.
\]
Finally, by the mean value theorem we obtain
\[
\|z_k - f'(x_*) s_k\| = \|f(y_k) - f(x_k) - f'(x_*) s_k\|
\]
\[
= \left\| \int_0^1 [f'(x_k + t(y_k - x_k))(y_k - x_k) - f'(x_*) s_k] dt \right\|
\]
\[
\leq \|s_k\| \int_0^1 \|f'(x_k + t(y_k - x_k)) - f'(x_*)\| dt
\]
\[
\leq L \|s_k\| \int_0^1 ((1 - t)\|x_k - x_*\| + t\|y_k - x_*\|) dt
\]
\[
= \frac{L}{2} \|s_k\| (\|y_k - x_*\| + \|x_k - x_*\|)
\]
which, in view of (36), concludes the proof. \qed
Using Proposition 5.2, the local convergence of the Broyden variants of QN-INEXP and IQN-INEXP follows directly from Theorems 3.2 and 4.2, respectively. Next we state such results.

**Corollary 5.3.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}^m$ be a function, $F : \Omega \rightrightarrows \mathbb{R}^m$ be a multifunction with closed graph and $C \subset \Omega$ be a nonempty closed convex set. Furthermore, let $x_*$ such that $0 \in f(x_*) + F(x_*)$, $x_* \in C$.

Suppose $f + F$ is metrically regular at $x_*$ for $0$ with modulus $\lambda > 0$, and that $f$ has Fréchet derivative Lipschitz continuous locally around $x_*$. Consider the QN-INEXP method with $\theta_k \geq 0$ for all $k$ and assume that

$$\hat{\theta} := \sup \theta_k < \frac{1}{2}, \quad \delta := \|B_0 - f'(x_*)\| < \frac{1}{2\lambda}.$$

Then, if we choose $B_{k+1}$ as in (35), there exists a neighborhood $\mathcal{U}$ of $x_*$ such that, starting from any $x_0 \in C \cap \mathcal{U} \setminus \{x_*\}$, there is a sequence $\{x_k\} \subset C \cap \mathcal{U}$ generated by the QN-INEXP method that converges linearly to $x_*$. If, in addition, items (vi) and (vii) of Theorem 4.2 are fulfilled, then the same conclusion is valid for the IQN-INEXP method with $B_{k+1}$ as in (35).

### 6 Numerical experiments

In this section we consider the quasi-Newton scheme with matrices $B_k$ following (35). For simplicity, we suppose $\Omega = \mathbb{R}^n$. In order to write the inclusion $\bar{z} \in F(x)$ in a tractable way, we assume that the set $F(x)$ can be described by equality and inequality constraints, that is,

$$F(x) = \{z \in \mathbb{R}^m \mid h(x, z) = 0, \ g(x, z) \leq 0\},$$

where $h$ and $g$ are continuously differentiable functions. Thus

$$\bar{z} \in F(x) \Leftrightarrow h(x, \bar{z}) = 0, \ g(x, \bar{z}) \leq 0.$$

Therefore, determining a solution $y_k$ of the subproblem $0 \in f(x_k) + B_k(y - x_k) + F(y)$ is equivalent to find $y_k \in \mathbb{R}^n$ and $z_k \in \mathbb{R}^m$ such that

$$z_k = -[f(x_k) + B_k(y_k - x_k)], \quad h(y_k, z_k) = 0, \quad g(y_k, z_k) \leq 0.$$

In turn, a sufficient condition for the above expressions to be satisfied is $(y_k, z_k)$ to be an optimal solution of

$$\min_{y, z} \frac{1}{2} \|z + [f(x_k) + B_k(y - x_k)]\|^2_2$$

subject to $h(y, z) = 0$, $g(y, z) \leq 0$ (38a)
with null objective value. This problem can be solved by standard nonlinear optimization methods. So, we have a practical way to test whether \( \bar{z} \in F(x) \) or not, at least approximately since (38) is considered solved when an optimality accuracy is achieved. This is in agreement with the inexactness allowed in the IQN-INEXP variant (Algorithm 2).

Taking into account (37), the stopping criterion \( 0 \in f(x_k) + F(x_k) \) is equivalent to

\[
    h(x_k, -f(x_k)) = 0, \quad g(x_k, -f(x_k)) \leq 0.
\]

Thus, the following (approximate) criterion is natural to declare convergence:

\[
    e_k := \max \{ \| h(x_k, -f(x_k)) \|, \| g(x_k, -f(x_k)) \| \} \leq 10^{-6}.
\]

Note that \( x_k \in C \) by construction in Algorithms 1 and 2, since the approximate projection in the sense of Definition 2.5 maintains feasibility.

The iterate \( x_{k+1} \in P_C(y_k, x_k, \theta_k) \) in must satisfy condition (10). However, as \( P_C(y_k) = P_C(y_k, x_k, 0) \in P_C(y_k, x_k, \theta_k) \), the solution of the standard projection problem

\[
    \min \frac{1}{2} \| x - y_k \|_2^2 \quad \text{subject to} \quad x \in C \tag{39}
\]

fulfils such condition (see Remark 3.1). Dealing with this convex problem is more convenient than try to handle (10) directly, since (39) can be solved by standard algorithms at global optimality. Nevertheless, iterative algorithms generally stop only at an approximate solution of (39). Thus, the inexact projection (10) encompasses in some sense this behaviour. Again, because Step 2 of algorithms require \( x_{k+1} \in C \), it is reasonable solving problem (39) by an interior point method since it maintains feasibility along its iterations.

In the next sections, we present three illustrative numerical examples of our theory. Our implementation is made in Matlab© R2016b using double precision. All tests were run on GNU/Linux Ubuntu 20.04. To solve problem (38) and (39) we use the interior point method implemented in the fmincon routine with maximum number of iterations equals to 1000. For (38), we set the optimality tolerance at \( 10^{-6} \), while for (39) this parameter is equal to \( \theta_k^2 \), where \( \theta_0 = 10^{-2} \) and \( \theta_k = \max \{ 0.9 \theta_{k-1}, 10^{-8} \}, k \geq 1 \). This choice is purely empiric, and tries to reflect the inexact computation allowed by theory (note that even \( \theta_k \to 0 \) is not required in Theorems 3.2 and 4.2). A more accurate choice may be supported by the inequality in Remark 2.8, but more research is necessary to connect such inequality to the accuracy of the solver applied on (39). The gradients of \( h_i \) and \( g_j \) are given in each case. Finally, since it is required that \( x_0 \in C \), we project the given initial point onto \( C \) if necessary.
6.1 Problem 1

Similarly to [5], let us consider $f: \mathbb{R}^2 \to \mathbb{R}^2$, $F: \mathbb{R}^2 \mapsto \mathbb{R}^2$ and $C$ given by

$$f(x_1, x_2) = (3x_1^3 - 2x_1^2, 0),$$
$$F(x_1, x_2) = \begin{cases} 
-x_1, x_1 \times \{0\}, & x_1 \geq 0 \\
\emptyset \times \{0\}, & x_1 < 0 
\end{cases},$$
$$C = \mathbb{R} \times [1/2, 1].$$

The solutions of the generalized equation $f(x) + F(x) \ni 0$ subject to $x \in C$ are $(0, x_2)$ and $(1, x_2)$, $x_2 \in [1/2, 1]$.

There are different ways to write $F(x)$ as (37). Each of them affects the numerical resolution of the problem by a previously selected algorithm (in our case, the interior point method of Matlab©). For this example, we choose $F(x) = \{z \in \mathbb{R}^2 \mid z_1^2 - x_1 = 0\}$.

Table 1 shows the execution starting from $x_0 = (0.7, 0)$. Columns “iter” and “conv. rate” stands for “iteration” and $\|x_k - x_\ast\|_\infty/\|x_{k-1} - x_\ast\|_\infty$, respectively, where $x_\ast = (0, 0.5)$ is the point to that algorithm approximates. From the third column, we can observe the linear convergence rate of $\{x_k\}$ to $x_\ast$ related in the theory (e.g. Theorem 3.2). We run the algorithm from several distinct initial points, and in all cases it converges to $(0, \min\{1, \max\{0.5, (x_2)_0\}\})$. That is, the sequence $\{(x_2)_k\}$ converges to the extreme of the interval $[0.5, 1]$ if the initial value is not in this interval, while otherwise it is constant. This behaviour is expected and reflects the projection onto $C$. The algorithm has a preference for solutions with $x_1 = 0$. This probably occurs by the fact that the term $3x_1^3$ of $f$ tends to be minimized in the objective function of (38). Evidently, the algorithm reaches a valid solution of the problem.

<table>
<thead>
<tr>
<th>iter</th>
<th>$e_k$</th>
<th>conv. rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.975990e-01</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3.075904e-01</td>
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</tr>
<tr>
<td>3</td>
<td>4.694257e-02</td>
<td>1.472804e-01</td>
</tr>
<tr>
<td>4</td>
<td>1.808056e-04</td>
<td>3.850256e-03</td>
</tr>
<tr>
<td>5*</td>
<td>2.367476e-10*</td>
<td>1.309404e-06</td>
</tr>
</tbody>
</table>

Table 1: Resolution of Problem 1 starting from $(0.7, 0)$.  

21
6.2 Problem 2

The next problem is a modification of Problem 1 that imposes a stricter relationship between the variables \(x_1\) and \(x_2\):

\[
f(x_1, x_2) = (3x_1^3 - 2x_1^2, -x_1x_2),
\]

\[
F(x_1, x_2) = \begin{cases}
\{-x_1, x_1\} \times \{1 - x_1\}, & x_1 \geq 0 \\
\emptyset \times \{0\}, & x_1 < 0
\end{cases},
\]

\[
C = \mathbb{R} \times [0, 2].
\]

The solutions are \((0, x_2), x_2 \in [0, 2]\), and \((1, 0)\). Here, \(F(x) = \{z \in \mathbb{R}^2 \mid z_1^2 - x_1 = 0, \ z_2 = 1 - x_2\}\).

The algorithm is attracted to \((0, 1)\) for different initial point, including the solution \((0, 0)\). Evidently this depends on the implementation, and can be justified by the way that we initialize \(z\) when solving problem (38). Table 2 shows the execution for \(x_0 = (0.9, 0.5)\).

As in Problem 1, we can observe a linear rate of convergence to \((0, 1)\).

<table>
<thead>
<tr>
<th>iter</th>
<th>(e_k)</th>
<th>conv. rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.785110e−01</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5.344404e−01</td>
<td>6.067537e−01</td>
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<td>3</td>
<td>2.963137e−01</td>
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<td>4</td>
<td>1.049330e−01</td>
<td>3.465348e−01</td>
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<td>5</td>
<td>3.895669e−02</td>
<td>4.236108e−01</td>
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<tr>
<td>6</td>
<td>6.432412e−04</td>
<td>1.409197e−02</td>
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<td>7</td>
<td>6.269612e−06</td>
<td>9.883571e−03</td>
</tr>
<tr>
<td>8*</td>
<td>2.101206e−09*</td>
<td>3.351656e−04</td>
</tr>
</tbody>
</table>

Table 2: Resolution of Problem 2 starting from \((0.9, 0.5)\).

6.3 Problem 3

Here we consider the bilevel problem

\[
\min_{x,y} \phi(x, y)
\]

subject to \((x, y) \in D\)

\[
y \in \arg \min_y q(x, y)
\]

subject to \(Yy \leq c - Xx,\)

where \(x \in \mathbb{R}^n, y \in \mathbb{R}^m\) and \(c \in \mathbb{R}^p\). We assume that \(D \neq \emptyset\) is closed and convex, and that \(q(x, \cdot)\) is a continuously differentiable convex function for all \(x\).
It is well known that computing a feasible point \((x, y)\) of the above problem is NP-hard in general, since \(y\) must be a global minimizer of an optimization problem. This difficulty is increased if the upper level constraints \((x, y) \in D\) contain lower level variables \(y_i\)'s. For details on bilevel problems, see [15]. One way to deal with bilevel problems is rewriting the lower level problem under their Karush-Kuhn-Tucker (KKT) conditions:

\[
\nabla_y q(x, y) + Y^t \lambda = 0 \\
\lambda \geq 0 \\
(Xx + Yy - c)^t \lambda = 0 \\
Yy \leq c - Xx.
\]

Since \(q(x, \cdot)\) is convex, these conditions are sufficient to optimality of the lower level problem. Defining

\[
F(x, y, \lambda) = \{0\}^2 \times N_{\mathbb{R}}(\lambda_1) \times \cdots \times N_{\mathbb{R}}(\lambda_p)
\]

\[
f(x, y, \lambda) = \left( \nabla_y q(x, y) - Y^t \lambda , Xx + Yy - c \right)
\]

\[
C = (D \cap \{(x, y) \mid Xx + Yy \leq c\}) \times \mathbb{R}^p,
\]

a point \((x, y)\) is feasible for the bilevel problem if, and only if, there is a \(\lambda\) such that

\[
0 \in f(x, y, \lambda) + F(x, y, \lambda), \quad (x, y, \lambda) \in C.
\]

Note that the above generalized equation does not involve inequalities. They are totally encapsulated by the projection step.

To illustrate the functionality of our algorithm, let us consider the particular instance

\[
q(x, y) = \frac{1}{2}(\|x\|^2 + \|y\|^2)
\]

\[
X = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -1 \end{bmatrix},
\]

\[
D = \{(x, y) \in \mathbb{R}^4 \mid x_1 + x_2 + y_1 + y_2 \leq 1\}
\]

In general, our implementation converges to different points when we vary the initial point. This is in agreement with the fact that, for each \(x\), the lower level problem possibly admits a different minimizer. Starting from \((x_0, y_0, \lambda_0) = (0.5, 0, 0, 1, 0.1, 0)\), the algorithm reaches

\[
x_* \approx (-0.4264, -0.2793), \quad y_* \approx (0.0661, 0.0661), \quad \lambda_* \approx (0.0521, 0.014),
\]

with \(e_* \approx 2.82 \cdot 10^{-7}\) (in this case, \(e_k\) encompasses the KKT residue of the lower level problem). The execution is presented in Table 3.
Table 3: Resolution of Problem 3.

<table>
<thead>
<tr>
<th>iter</th>
<th>$e_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$5.285722e-01$</td>
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<tr>
<td>2</td>
<td>$7.819067e-02$</td>
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<td>$3.158594e-02$</td>
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<tr>
<td>4</td>
<td>$5.256745e-03$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>$6.993861e-04$</td>
</tr>
<tr>
<td>20</td>
<td>$6.970235e-04$</td>
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<td>$6.993202e-04$</td>
</tr>
<tr>
<td>22*</td>
<td>$3.195855e-08*$</td>
</tr>
</tbody>
</table>

7 Conclusions

In this paper, we deal with constrained generalized equations. This is a very general class of problems, encompassing several other contexts, such as standard nonlinear optimization, variational inequalities and equilibrium problems. We presented two general quasi-Newton frameworks for solving constrained generalized equations that employ an inexact projection step. Firstly, we discuss a quasi-Newton scheme where subproblems are solved exactly. Its local convergence is provided under a limited deterioration condition on the update quasi-Newton operator/matrix. Secondly, we extend the proposed method allowing subproblems to be solved inexactly. The resulting (inexact quasi-Newton) method is closer to the numerical practice, where inexactness is naturally present. We also proved that the inexact scheme converges locally under mild assumptions.

We analysed the particular case when the classical Broyden update rule is employed. For both exact and inexact quasi-Newton methods, we show that, in this case, the deterioration condition is satisfied directly. Illustrative numerical experiments with the Broyden variant were made to align theory with practice.

As future works, we intend to answer whether the condition of limited deterioration can be replaced by something weaker or whether, with additional assumptions, we could obtain another update rule besides Broyden’s. Another line of research is to study the
variational inequality problem  
\[ 0 \in f(x) + N_C(x), \]
by considering the inexact quasi-Newton method  
\[ (f(x_k) + (B_k + \bar{B})(x_{k+1} - x_k) + N_C(x_{k+1})) \cap \mathbb{B}_{\eta_k \psi(x_k)}(0) \neq \emptyset, \quad k = 0, 1, \ldots, \quad (40) \]
with \( B_k \) satisfying the Broyden update, \( \bar{B} \) an \( n \times n \) matrix, \( \{\eta_k\} \) a sequence of positive numbers converging to zero and \( \psi \) a Lipschitz function. As in [18], we intend to consider the Newton-Kantorovich theorem on (40). Moreover, we propose to apply the quasi-Newton iteration  
\[ \nabla h(x_k) + B_k(x_{k+1} - x_k) + N_C(x_{k+1}) \ni 0 \quad (41) \]
to the first-order necessary optimality condition \( \nabla h(x) + N_C(x) \ni 0 \) of the optimization problem  
\[
\text{minimize } h(x) \quad \text{subject to } x \in C,
\]
where \( h \) is a twice continuously differentiable function and \( C \) is a closed and convex set. We propose to update \( B_k \) defined in (41) firstly by the Broyden update (35) with  
\[ z_k = \nabla h(x_{k+1}) - \nabla h(x_k) \quad \text{and} \quad s_k = x_{k+1} - x_k, \]
and secondly using the BFGS method, that is,  
\[ B_{k+1} := B_k + \frac{y_k y_k^T}{y_k^T s_k} - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k}, \]
where  
\[ y_k := \nabla h(x_{k+1}) - \nabla h(x_k) \quad \text{and} \quad s_k := x_{k+1} - x_k. \]
It is worth noting that, as reported in [18], the scheme (41) can be used for solving control-constrained optimal control problems. So, we also propose to do numerical experiments to the quasi-Newton method (41) and to compare the results with the ones obtained in [18].

References


