ABSTRACT. In this work, we study Bishop-Phelps cones (briefly, BP cones) given by an equation in Banach spaces. Due to the special form, these cones enjoy interesting properties. We show that nontrivial BP cones given by an equation form a “large family” in some sense in any Banach space and they can be used to characterize the strict convexity of the space. We obtain conditions for a convex cone to be included in or to be itself a BP cone given by an equation. Further, we give an affirmative answer to the open question whether these cones may possess a nonempty interior in infinite dimensional spaces and introduce Lorentz cones in some classical Banach spaces of sequences as illustrations. We also obtain explicit representations for all BP cones given by an equation in some classical Banach spaces. Finally, we present some short applications in optimal control and approximation.

1. Introduction

In infinite dimensional optimization as well as in conic programming, inequality constraints are defined using a convex cone. Moreover, the minimization of a vector-valued or set-valued objective is based on concepts where a convex cone plays a central role. Very often such a convex cone belongs to the well-known class of so called Bishop-Phelps cones (briefly, BP cones) introduced in [4] (compare [27]). These cones have applications in nonlinear analysis, optimization and mathematical economics. In particular, they are useful tools in obtaining optimality conditions, approximation and scalarization techniques in vector optimization, see for instance [3, 13, 14, 15, 18, 19, 22, 23, 24, 25, 32]. Special BP cones known under the name Lorentz cones play an important role in second order cone programming [1, 11]. Due to the important role of BP cones, various works have been dedicated to their properties (representation of the interior and the dual cone, conditions for a cone to be included in or to be a BP cone, relations to other kinds of cones), see for instance [14, 15, 18, 19, 25, 32].

Roughly speaking, a BP cone is a convex cone consisting of points, which satisfy an inequality involving two parameters, a continuous linear functional and a positive coefficient.
For a given BP cone, either at least one point of the cone satisfies a strict inequality or all its points satisfy an equality. We will call this cone “a BP cone given by an inequation” and “a BP cone given by an equation”, respectively (see Section 2 for the precise definition).

**BP cones given by an equation in Banach spaces are the main object of our study and it will be shown that these cones possess interesting properties due to their special representation.** Our attention is mostly concentrated on *infinite dimensional Banach spaces*, because a research on this topic in finite dimensional spaces and Hilbert spaces has already been carried out in [12], see also [19].

It is known that nonnegative orthants in some classical Banach spaces are BP cones given by an equation, see for instance [12, 19]. Let us note that there are significant differences between the mentioned two types of BP cones. In our sense, a BP cone given by an inequation always is nontrivial (i.e., it contains nonzero vectors), reproducing (i.e., its linear span coincides with the space) and has a nonempty interior, which is the set of points satisfying a strict inequality [15, 18]. In contrast, this nontriviality is not always observed among BP cones given by an equation in nonreflexive Banach spaces and there exist BP cones given by an equation, which are not reproducing, see Section 2. Yet it remains an open question whether a BP cone given by an equation in an infinite dimensional Banach space may possess a nonempty interior [12, p. 4]. This question is of interest in relation to the Slater type constraint qualification when a BP cone is involved in the constraint set of an optimization problem and it also is not trivial from the analysis point of view as it concerns the interior of a set given by an equation.

**Our main results** are the following. First, we show that nontrivial BP cones given by an equation form a large family in any Banach space (the meaning will be clarified in Section 2). Next, we use these cones to characterize the strict convexity of the space and we obtain conditions for a convex cone to be included in or to be itself a BP cone given by an equation. In particular, we prove the existence of convex cones with special properties (we call them sets of additivity of the norm) in any non-strictly convex Banach space, which are the only candidates for being included in a BP cone given by an equation and become BP cones given by an equation under an additional condition. Further, we give an affirmative answer to the open question mentioned above and we introduce Lorentz cones in some Banach spaces of sequences as illustrative examples. We also obtain explicit representations for all or for some BP cones given by an equation in several classical Banach spaces and study the emptiness of the interior of these cones. Finally, we present some short applications to optimal control and approximation.

The paper is organized as follows. Section 2 concerns with the nontriviality of BP cones given by an equation in Banach spaces. In Section 3, these cones are studied in context of
strict convexity of the space. Section 4 is devoted to the open question about nonemptiness of their interior in infinite dimensional spaces. Explicit forms for BP cones given by an equation in some classical Banach spaces are obtained in Section 5. Section 6 is devoted to applications.

2. Nontrivial Bishop-Phelps cones given by an equation in Banach spaces

In this section, we recall the concept of Bishop-Phelps cone (BP cone for short) and provide examples of BP cones given by an equation. In particular, we show that the family of nontrivial BP cones given by an equation form a large set in any Banach space.

Throughout the paper \((X,\|\cdot\|)\) denotes a real Banach space, which is shortly written by \(X\), and its dual space is denoted by \(X^*\). For a nonempty subset \(U\) of \(X\), the interior, closure, convex hull of \(U\) and the cone generated by \(U\) are denoted by \(\text{int} U\), \(\text{cl} U\), \(\text{conv} U\) and \(\text{cone} U\), respectively. Denote by \(B(x,\epsilon)\) the closed ball in \(X\) with center \(x\) and radius \(\epsilon\).

The following definition for a Bishop-Phelps cone is used in this paper.

**Definition 2.1.** For an arbitrary continuous linear functional \(\ell\) on \(X\) the convex cone
\[ C(\ell) := \{x \in X : \ell(x) \geq \|x\|\} \]
is called Bishop-Phelps cone (BP cone for short). A BP cone \(C(\ell)\) is said to be nontrivial iff it contains nonzero vectors, i.e., \(C(\ell)\backslash\{0\} \neq \emptyset\).

**Remark 2.1.** In the original definition of Bishop-Phelps [4], a BP cone is introduced in the following form
\[ C(\ell, \alpha) := \{x \in X : \ell(x) \geq \alpha \|x\|\}, \]
where \(\ell \in X^*\) is a functional with \(\|\ell\| = 1\) and \(\alpha \in (0,1]\) is some positive scalar. Nowadays, several authors do not use the constant \(\alpha\) and let the functional \(\ell \in X^*\) to have a norm varying in \([1,\infty[,\) see [12, 15, 23, 24, 32], and we follow this line. The authors of [19] consider BP cones \(C(\ell, \alpha)\) with arbitrary \(\ell \in X^*\) and \(\alpha \in ]0,\infty[\).

Let us state a relation between the representation of a BP cone \(C(\ell)\) and the norm of \(\ell\).

**Lemma 2.1.** A BP cone \(C(\ell)\) has the form
\[ C(\ell) = \{x \in X : \ell(x) = \|x\|\}. \]
if and only if \(\|\ell\| = 1\).

**Proof.** Recall that by the definition, the norm of the functional \(\ell\) satisfies
\[ \|\ell\| = \sup\{\ell(x)/\|x\| : x \in X \backslash \{0\}\}. \]
The “if part” follows from the fact that if \( \|\ell\| = 1 \), then for any \( x \in C(\ell) \) we have \( \|x\| \leq \ell(x) \leq \|x\| \) and hence, \( \ell(x) = \|x\| \). Next, assume that \( \|\ell\| > 1 \). Then there exists \( x \in X\setminus\{0\} \) such that \( \ell(x)/\|x\| > 1 \) or \( \ell(x) > \|x\| \). It follows that \( x \in C(\ell) \). Hence \( C(\ell) \) cannot be of the form \( \{x \in X : \ell(x) = \|x\|\} \). The “only if part” follows. □

From Lemma 2.1 and its proof, we see that for a given BP cone \( C(\ell) \), there are only two possibilities: either at least one point of the cone satisfies a strict inequality or all its points satisfy an equality, depending on whether the norm of \( \ell \) is strictly greater than 1 or equals 1. We will call this cone “a BP cone given by an inequation” and “a BP cone given by an equation”, respectively. We would like to remark that in this sense, the case “given by an inequation” does not include the case “given by an equation” as its special one. To emphasize the involved functional \( \ell \), we will use in parallel the name “a BP cone \( C(\ell) \) with \( \|\ell\| > 1 \)” and “a BP cone \( C(\ell) \) with \( \|\ell\| = 1 \)”, respectively. It is clear that a BP cone \( C(\ell, \alpha) \) from [19] is a BP cone given by an equation iff \( \|\ell\| = \alpha \).

Let us state a property of BP cones given by an equation, which will be used later.

**Corollary 2.1.** Let \( C(\ell) \) be a BP cone with \( \|\ell\| = 1 \). Then

\[
\|x + y\| = \|x\| + \|y\| \quad \text{for all} \quad x, y \in C(\ell).
\]

*Proof.* It follows from the equalities \( \|x + y\| = \ell(x + y) = \ell(x) + \ell(y) = \|x\| + \|y\| \). □

**Example 2.1.** For an illustration of these special BP cones consider \( X = \mathbb{R}^2 \) with the \( l_1 \)-norm \( \|\cdot\|_1 \) or \( l_2 \)-norm \( \|\cdot\|_2 \).

Fig. 1 and 2 illustrate a BP cone \( C(\ell) \) with \( \|\ell\| = 1 \) in \( \mathbb{R}^2 \) equipped with the \( l_1 \)-norm and with the \( l_2 \)-norm, respectively (the part colored in blue). Note that the BP cone \( C(\ell) \) in Fig. 1 has interior points whereas the BP cone \( C(\ell) \) in Fig. 2 has an empty interior. So, the norm plays an important role for topological properties of these special BP cones.
We will study the family of nontrivial BP cones given by an equation in an arbitrary Banach space. Observe first that such a cone exists in every Banach space. Indeed, an extended version of the Hahn-Banach theorem implies that for every nonzero \( x \in X \) there exists some \( \ell \in X^* \) with \( \| \ell \| = 1 \) such that \( \ell(x) = \| x \| \), i.e. \( x \in C(\ell) \). It is also known that a Banach space \( X \) is reflexive if and only if every \( \ell \in X^* \) with \( \| \ell \| = 1 \) attains its supremum on the unit sphere of \( X \) [17, Theorem 5]. This fact implies, in particular, the nontriviality of any BP cone \( C(\ell) \) with \( \| \ell \| = 1 \) in reflexive spaces.

**Proposition 2.1.** [12, Theorem 2.2] Let \( X \) be a Banach space. Then \( X \) is reflexive if and only if for every \( \ell \in X^* \) with \( \| \ell \| = 1 \), the BP cone \( C(\ell) \) contains a nonzero element.

If \( X \) is not reflexive, Proposition 2.1 implies that for some \( \ell \in X^* \) with \( \| \ell \| = 1 \), the corresponding BP cone \( C(\ell) \) is trivial. For instance, the cone \( C(\ell) \) in \( l_1 \) with \( \ell = (1, 1/2, 1/3, ... \) belongs to \( l_\infty \) is trivial, see the proof of this fact in forthcoming Proposition 5.1.

However, in any Banach space, the family of nontrivial BP cones given by an equation is “large” in the following sense: the family of functionals \( \ell \in X^* \) with unit norm such that the corresponding BP cones \( C(\ell) \) contain nonzero elements is norm dense in the unit sphere of \( X^* \). This follows from a result due to Bishop and Phelps [4], which states that every Banach space is subreflexive. Recall that a Banach space \( X \) is subreflexive if and only if those functionals which attain their supremum on the unit sphere of \( X \) are norm dense in \( X^* \), i.e., for each \( x^* \in X^* \) and each \( \epsilon > 0 \), there exist \( \ell \in X^* \) and \( x \in X \), \( \| x \| = 1 \) such that \( \ell(x) = \| x \| \) and \( \| x^* - \ell \| \leq \epsilon \). Namely, we have the following.

**Theorem 2.1.** For every \( x^* \in X^* \) with \( \| x^* \| = 1 \) and every \( \epsilon > 0 \), there exists some \( \tilde{\ell} \in X^* \) with \( \| \tilde{\ell} \| = 1 \) such that \( \| x^* - \tilde{\ell} \| \leq \epsilon \) and the BP cone \( C(\tilde{\ell}) \) is nontrivial.

**Proof.** Applying the result established by Bishop and Phelps to \( x^* \) and \( \epsilon \), we find some \( \ell \in X^* \) and \( x \in X \) with \( \| x \| = 1 \) such that \( \ell(x) = \| \ell \| \) and \( \| x^* - \ell \| \leq \epsilon/2 \). Set \( \tilde{\ell} := \ell/\| \ell \| \). It is clear that \( \| \tilde{\ell} \| = 1 \). Next, the equalities \( \ell(x) = \| \ell \| \) and \( \| x \| = 1 \) give \( \tilde{\ell}(x) = \| x \| \) and hence \( x \in C(\tilde{\ell}) \). It remains to show that \( \| x^* - \tilde{\ell} \| \leq \epsilon \). Observe that the triangle inequality implies \( \| x^* \| - \| x^* - \ell \| - \| \ell - \tilde{\ell} \| \leq \| x^* - \ell \| \), and we have

\[
\| x^* - \tilde{\ell} \| \leq \| x^* - \ell \| + \| \ell - \tilde{\ell} \| \| \ell \| \| = \| x^* - \ell \| + \| \ell \| \| 1 - 1/\| \ell \| \| = \| x^* - \ell \| + \| \ell \| - 1 \|
\]

as it was to be shown. \( \square \)

Recall that when \( X \) is a Hilbert space, Proposition 2.5 in [12] states that any nontrivial BP cone \( C \) given by an equation has the form \( C = \{ \lambda \ell : \lambda \in [0, \infty[ \} \) = cone \{ \ell \} \) with some \( \ell \in X \) satisfying \( \| \ell \| = 1 \).
Example 2.2. Below are some nontrivial BP cones given by an equation in infinite dimensional non-reflexive Banach spaces.

(a) The nonnegative orthant in the Banach space \( l_1 \) is a BP cone \( C(\ell) \) with \( \|\ell\| = 1 \), where for any \( x = (x_i)_{i \in \mathbb{N}} \in l_1, \ell(x) := \sum_{i=1}^{\infty} x_i \).

(b) The nonnegative orthant in the Banach space \( L^1_{[0,1]} \) is a BP cone \( C(\ell) \) with \( \|\ell\| = 1 \), where for any \( x \in L^1_{[0,1]}, \ell(x) := \int_0^1 x(t) \, dt \).

(c) Let \( X \) be the Banach space \( C_{[0,1]} \) and let \( \ell \in X^* \) with \( \ell(x(\cdot)) := x(1) \) be given. Then \( \|\ell\| = 1 \) and \( C(\ell) \) is the set

\[
C(\ell) = \{ x \in C_{[0,1]} : \|x\| = x(1) \}.
\]

Note that the closed convex cone \( K_0 \) of nonnegative non-decreasing continuous functions defined on \([0,1]\) is included in \( C(\ell) \).

(d) Let \( X \) be the Banach space \( l_1 \). For each \( i = 1, 2, ..., \) denote

\[
e_i := (0, 0, \ldots, 0, 1, 0, 0, \ldots),
\]

i.e. \( e_i \) is a vector with 1 being at the \( i \)th place and zeros at the other places. Consider the convex cones \( K_i := \text{conv cone}\{e_1, \ldots, e_{i+1}\} \), or equivalently,

\[
K_i = \{ x = (x_i)_{i \in \mathbb{N}} \in l_1 : x_1 \geq 0, \ldots, x_{i+1} \geq 0, x_j = 0 \text{ for } j \geq i+2 \}.
\]

For each \( i = 1, 2, ..., K_i \) is a BP cone \( C(\ell) \) with \( \ell = e_i^* \) and \( \|\ell\| = 1 \), where \( e_i^* \in l_\infty \) is a vector, the first \( i \) coordinates of which are 1 and the other ones are zero.

For the following comments recall that a convex cone \( K \) is said to be reproducing iff its linear span coincides with the space, i.e. \( K - K = X \) [20, p.17].

Remark 2.2. (i) It has been proved in [19, Examples 5.5-5.7] that the cones in Example 2.2, (a)-(c) are (in our terminology) BP cones given by an equation. This fact has been mentioned without proof in [12, Example 2.7]. It is known that the nonnegative orthants in \( l_p, L^p_{[0,1]} (1 \leq p < \infty) \) have an empty interior, see for instance [20, p. 18]. Let \( K \) be one of cones in Example 2.2, (a)-(b) or the cone \( K_0 \) in Example 2.2, (c) and \( \ell \) be the corresponding functional associated to this cone. It is known that \( \ell \) is uniformly positive on \( K \) (uniform positivity of \( \ell \) on \( K \) means \( \ell(x) \geq \alpha \|x\| \) for all \( x \in K \) and some \( \alpha > 0 \) [20, p. 31]), or more, precisely, \( K = \{ x \in K : \ell(x) = \|x\| \} \), see for instance [20, p. 31] and [2, p. 45-46]. In other words, the cone \( K \) has shown to be included in the corresponding BP cone \( C(\ell) \).

(ii) It has been established in [15, Corollary 2.7] that every BP cone given by an inequation is reproducing. In contrast, while the BP cones given by an equation considered in Example 2.2, (a)-(b) are reproducing [2, 20], the ones in Example 2.2, (d) are not.
3. BP cones given by an equation and the strict convexity of Banach spaces

In this section, we obtain necessary and/or sufficient conditions for a convex cone to be included in or to be itself a BP cone given by an equation. Establishing that a BP cone given by an equation in any strictly convex space is just a ray, we concentrate our attention to the case when the Banach space $X$ is not strictly convex. It turns out that there exist in such a space some convex cones with special properties (we call them sets of additivity of the norm), which are the only candidates for being included in a BP cone given by an equation and become BP cones given by an equation under an additional condition. We also characterize the strict convexity of the space by the existence of a BP cone given by an equation, which contains at least two linearly independent vectors.

To our knowledge, the concept of a strictly convex normed space may have been considered for the first time in the paper [7, p. 404] by Clarkson in 1936 and we recall it in the following definition.

**Definition 3.1.** [7, p. 404] A normed space $X$ is strictly convex iff $x, y$ in $X \setminus \{0\}$ and $\|x + y\| = \|x\| + \|y\|$ imply $x = cy$ for some scalar $c > 0$.

Geometrically, a normed linear space is strictly convex if and only if for any two given distinct vectors in the closed unit sphere, the midpoint of the line segment joining them must not lie in the closed unit sphere.

**Remark 3.1.** All Hilbert spaces, the Banach spaces $l^p$ and $L^p_{[0,1]} (1 < p < \infty)$ are strictly convex; but the Banach spaces $l^p$ and $L^p_{[0,1]} (p = 1$ or $p = \infty)$, $C_{[0,1]}$ are not strictly convex. Klee (compare [9]) made the conjecture that every reflexive Banach space is isomorphic to a strictly convex space. Clarkson [7, Theorem 9] showed that in every separable Banach space a new norm being equivalent to the original norm may be given, with respect to which the space is strictly convex. There are many papers investigating and characterizing strictly convex normed spaces (e.g., see [8, 16, 28, 30, 33, 34]).

We begin with the form of a BP cone given by an equation and its dual in a strictly convex Banach space.

**Proposition 3.1.** Suppose that $X$ is a strictly convex Banach space and $C(\ell) := \{x \in X : \ell(x) = \|x\|\}$ (with $\|\ell\| = 1$) is a nontrivial BP cone given by an equation. Then $C(\ell)$ has the form

$$C(\ell) = \{\lambda \bar{x} : \lambda \in [0, \infty)\} = \text{cone} \{\bar{x}\},$$

for some $\bar{x} \in X$ satisfying $\|\bar{x}\| = \ell(\bar{x}) = 1$. The dual cone of $C(\ell)$ is given by

$$(C(\ell))^* = \{v \in X^* : v(\bar{x}) \geq 0\}$$
with the interior
\[ \text{int } (C(\ell))^* = \{ v \in X^* : v(\bar{x}) > 0 \}. \]

Proof. Since \( C(\ell) \) is nontrivial, there exists a nonzero vector \( v \in C(\ell) \). Denote \( \bar{x} := v/\|v\| \).

Suppose that there exists a vector \( u \in C(\ell) \) such that \( u \) and \( \bar{x} \) are linearly independent. By Corollary 2.1, we have \( \|\bar{x} + u\| = \|\bar{x}\| + \|u\| \). On the other hand, since the space \( X \) is strictly convex, from [10, p. 458] we get \( \|\bar{x} + u\| < \|\bar{x}\| + \|u\| \), a contradiction. Hence, \( C(\ell) = \text{cone}\{\bar{x}\} \). The forms of the dual cone and its interior follow from [15, Theorems 2.5 and 2.13]. \( \square \)

Remark 3.2. (a) Figure 2 illustrates the result of Proposition 3.1.

(b) Proposition 3.1 is an extension of Proposition 2.5 and Theorem 3.4 (iii) of [12] from Hilbert spaces to strictly convex Banach spaces.

In the remaining part of this section, assume that the space \( X \) is not strictly convex. By the definition (see also [33], [16, Definition 4.4, p. 274]), there exist two linearly independent vectors of \( X \), say \( x_1 \) and \( x_2 \), satisfying \( \|x_1 + x_2\| = \|x_1\| + \|x_2\| \). Moreover, we have.

Lemma 3.1. [34, Lemma, p. 544] If the Banach space \( X \) is not strictly convex, then there exist linearly independent vectors \( x_1, x_2 \in X \) such that
\[ \|\alpha x_1 + (1 - \alpha)x_2\| = 1 \quad \text{for all } \alpha \in [0, 1]. \] (1)

We will need the following concept.

Definition 3.2. Let \( K \) be a convex cone in a Banach space \( X \). We say that \( K \) is a set of additivity of the norm, if it contains at least two linearly independent vectors and for all \( k_1, k_2 \in K \) one has
\[ \|k_1 + k_2\| = \|k_1\| + \|k_2\|. \]

The following result states that sets of additivity of the norm exist in every Banach space, which is not strictly convex.

Proposition 3.2. Assume that the Banach space \( X \) is not strictly convex. Then it contains a set of additivity of the norm.

Proof. Let \( x_1, x_2 \in X \) be linearly independent vectors as in Lemma 3.1 such that the relation (1) is satisfied. Let \( K \) be the closed convex cone generated by \( x_1, x_2 \), i.e.
\[ K := \{ \alpha x_1 + \beta x_2 : \alpha, \beta \in [0, +\infty]\}. \]

We show that \( K \) is a set of additivity of the norm. Since \( x_1 \) and \( x_2 \) are linear independent, each \( k \in K \) is uniquely represented in the form \( \alpha x_1 + \beta x_2 \) with nonnegative scalars \( \alpha \) and
\(\beta\). Observe that the equality (1) implies
\[
\|\alpha x_1 + \beta x_2\| = (\alpha + \beta) \left(\|\frac{\alpha}{\alpha + \beta} x_1\| + \frac{\beta}{\alpha + \beta} x_2\|\right) = \alpha + \beta.
\]
(2)

To complete the proof, we have to show that \(\|k_1 + k_2\| = \|k_1\| + \|k_2\|\) for all \(k_1, k_2 \in K\).

Assume that \(k_1 = \alpha_1 x_1 + \beta_1 x_2\) and \(k_2 = \alpha_2 x_1 + \beta_2 x_2\) are arbitrarily chosen, where \(\alpha_1, \alpha_2, \beta_1\) and \(\beta_2\) are nonnegative scalars. Then \(k_1 + k_2 = (\alpha_1 + \alpha_2)x_1 + (\beta_1 + \beta_2)x_2\) and applying the equality (2) we get
\[
\|k_1 + k_2\| = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) = \|k_1\| + \|k_2\|,
\]
as it was to be shown.

**Example 3.1.** Corollary 2.1 implies that the BP cones presented in Example 2.2 are sets of additivity of the norm. Below are examples of sets of additivity of the norm containing linearly independent vectors satisfying the equation (1).

(i) The closed convex cone
\[
K := \{ x \in C_{[0,1]} : \exists \alpha \geq 0, \beta \geq 0 \text{ such that } x(t) = \alpha t^2 + \beta t \text{ for all } t \in [0,1] \}
\]
(with \(x_1(t) := t^2\) and \(x_2(t) := t\) for all \(t \in [0,1]\)).

(ii) Let \(X = l_1\). The closed convex cone
\[
K := \{ k = (k_i)_{i \in \mathbb{N}} \in l_1 : k_1 \geq 0, k_2 \geq 0, k_j = 0 \text{ for } j \neq 1, 2 \}
\]
(with \(x_1 := e_1\) and \(x_2 := e_2\)).

(iii) The convex cone \(K \subset l_\infty\) defined by
\[
K := \{ k = (k_i)_{i \in \mathbb{N}} \in l_\infty : 0 \leq k_i \leq k_j < \|k\| \text{ for any } i \leq j \} \cup \{0\}
\]
(with \(x_1 = (1, \frac{3}{2}, ..., 2 - \frac{1}{n}, ...,)\) and \(x_2 = (0, 1, ..., 2 - \frac{2}{n}, ...)\)).

(iv) The convex cone \(K \subset L_{[0,1]}^\infty\) defined by
\[
K := \text{conv cone } \{u_1, u_2, ...\},
\]
where for each \(i \in \mathbb{N}\), \(u_i \in L_{[0,1]}^\infty\) is the function defined by \(u_i(t) = t^i\) for all \(t \in [0,1]\).

Any \(x \in K\) has the representation \(x = \sum_{j \in J(x)} \lambda_j u_j\), where the index set \(J(x)\) is a finite subset of \(\mathbb{N}\) and \(\lambda_j \geq 0\). One can check that \(\|x\| = \sum_{j \in J(x)} \lambda_j\). Note that all vectors \(u_1, u_2, ...\) are linearly independent.

We are interested in finding conditions for a convex cone \(K\) to be included in or to be a BP cone \(C(\ell)\) given by an equation. Note that results for the general case (\(\ell\) does not need to have a norm equal 1) may be divided into two types, roughly speaking, “existence” type and “representation” type. Namely, given a convex cone \(K\) with a bounded base, it has been obtained the existence either of a BP cone \(C\) so that \(K \subset C\) [14, Lemma 2.2] or of a BP
cone \( C \) in a renormed space so that \( K = C \) [25, Theorem 3.2]. Note that no explicit form of \( C \) is given. In contrast, under the assumption that \( K \) satisfies a condition involving some a priori known \( \ell \) and \( \alpha \), it has been established that \( K \) has the representation \( C(\ell, \alpha) \), i.e. \( K = C(\ell, \alpha) \) [19, Theorem 4.5]. Our results are of both types. Let us start with a result of the “existence” type.

**Theorem 3.1.** Assume that the Banach space \( X \) is not strictly convex and \( K \subset X \) is a convex cone containing at least two linearly independent vectors. Then the following assertions hold.

(i) A necessary and sufficient condition for \( K \) to be included in some BP cone given by an equation is that \( K \) is a set of additivity of the norm.

(ii) A sufficient condition for \( K \) to be a BP cone given by an equation is that \( K \) is a set of additivity of the norm and \( K \) is reproducing.

**Proof.** Our aim is to prove the existence of a functional \( \ell \in X^* \) with \( \|\ell\| = 1 \) such that \( K \subset C(\ell) \) or \( K = C(\ell) \), where \( C(\ell) = \{ x \in X : \ell(x) = \|x\| \} \) is the BP cone generated by \( \ell \). Our arguments are based on the Hahn-Banach theorem.

(i) The “only if part”: Since the BP cone containing \( K \) is a set of additivity of the norm by Corollary 2.1, so is \( K \).

The “if part”: Since \( K \) is a convex cone, the affine hull (or the linear span) of \( K \) is the set \( L := K - K \). Observe that \( -K \subset L \). We define a functional \( \varphi \) on \( L \) as follows:

\[
\varphi(x) = \|k_1\| - \|k_2\| \quad \text{if} \quad x = k_1 - k_2 \quad \text{and} \quad k_1, k_2 \in K.
\]

Note that by the definition we have \( \varphi(x) = \|x\| \) if \( x \in K \) and \( \varphi(x) = -\|x\| \) if \( x \in -K \). We show that the value of \( \varphi \) does not depend on the presentation of \( x \). Assume that \( x = k_1 - k_2 = k_1' - k_2' \) with \( k_1, k_2, k_1', k_2' \in K \). We have to check that \( \varphi(k_1 - k_2) = \varphi(k_1' - k_2') \). Indeed, since the norm is additive on \( K \) and \( K \) is a convex cone, it follows that \( k_1 + k_2, k_1' + k_2' \in K \) and taking into account the equality \( k_1 + k_2 = k_1' + k_2' \), we get

\[
\|k_1\| + \|k_2\| = \|k_1 + k_2\| = \|k_1' + k_2'\| = \|k_1'\| + \|k_2'\|.
\]

Then \( \varphi(k_1 - k_2) = \|k_1\| - \|k_2\| = \|k_1'\| - \|k_2'\| = \varphi(k_1' - k_2') \).

Next, we show that \( \varphi \) is linear on \( L \), i.e.,

\[
\varphi(\alpha x) = \alpha \varphi(x) \quad \text{for all} \ \alpha \in \mathbb{R}, \ x \in L,
\]

\[
\varphi(x + x') = \varphi(x + x') \quad \text{for all} \ x, x' \in L.
\]

First, we check the homogeneity. Let \( x = k_1 - k_2, k_1, k_2 \in K \) and \( \alpha \in \mathbb{R} \) be arbitrarily chosen. Assume first that \( \alpha \geq 0 \). Then we have

\[
\varphi(\alpha x) = \varphi(\alpha k_1 - \alpha k_2) = \|\alpha k_1\| - \|\alpha k_2\| = \alpha (\|k_1\| - \|k_2\|) = \alpha \varphi(k_1 - k_2) = \alpha \varphi(x).
\]
Now, if $\alpha < 0$, then $-\alpha > 0$ and we have

$$\varphi(\alpha x) = \varphi((-\alpha)(-x)) = (-\alpha)\varphi(-x) = (-\alpha)(\|k_2\| - \|k_1\|) = \alpha(\|k_1\| - \|k_2\|) = \alpha\varphi(x),$$

as it was to be shown. Next, let us check the additivity. Let $x_1, x_2 \in L$ be given. We show that $\varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2)$. Assume that $x = k_1 - k_2$ and $x' = k'_1 - k'_2$, where $k_1, k_2, k'_1, k'_2 \in K$. Since $K$ is a convex cone and the norm is linear on $K$, it follows that $k_1 + k'_1, k_2 + k'_2 \in K$ and

$$\varphi(x + x') = \varphi((k_1 + k'_1) - (k_2 + k'_2)) = \|k_1 + k'_1\| - \|k_2 + k'_2\| = (\|k_1\| + \|k'_1\|) - (\|k_2\| + \|k'_2\|) = \|k_1\| - \|k_2\| + (\|k'_1\| - \|k'_2\|) = \varphi(x) + \varphi(x'),$$

as it was to be shown.

Further, it follows from the definition of the functional $\varphi$ that for every $x \in L$ with a presentation $x = k_1 - k_2, k_1, k_2 \in K$ one has

$$\varphi(x) = \|k_1\| - \|k_2\| \leq \|k_1 - k_2\| = \|x\|.$$

If $L \subset X$, the Hahn-Banach dominated extension theorem [29, Theorem 3.2] implies the existence of a linear extension $\ell : X \to \mathbb{R}$ of $\varphi$ to the whole space $X$ that is dominated by the norm. This means that

$$\ell(x) = \varphi(x) \quad \text{for all } x \in L,$$

$$|\ell(x)| \leq \|x\| \quad \text{for all } x \in X.$$

So, the linear functional $\ell$ is continuous. In particular, we have $\ell(k) = \|k\|$ for $k \in K$ and $\|\ell\| = 1$. This means that $K \subset C(\ell)$ and the assertion (i) is proved.

(ii) Let $L$ and $\varphi$ be as in the proof of the "if part" of the first assertion. If $L = X$, i.e. $K$ is reproducing, then we can take $\ell = \varphi$. It is easy to check that $\ell \in X^*$, $\|\ell\| = 1$ and $C(\ell) = K$.

\begin{remark}

(i) All the convex cones considered in Example 3.1 are included in BP cones given by an equation. The nonnegative orthants in the Banach spaces $l_1$ and $L_{[0,1]}^1$ being sets of additivity of the norm and reproducing cones, are BP cones given by an equation.

(ii) Let $X = C_{[0,1]}$, $K := \{x \in C_{[0,1]} : \exists \alpha \geq 0, \beta \geq 0 \text{ such that } x(t) = \alpha t^2 + \beta t \text{ for all } t \in [0,1]\}$, see Example 3.1,(i) and $C(\ell) = \{x \in C_{[0,1]} : \|x\| = x(1)\}$, see Example 2.2,(c). One can check that $K$ is contained in the BP cone $C(\ell)$. Observe that the cone $K$ is not reproducing. It will be shown later in Proposition 3.3 that $K$ cannot be itself a BP cone given by an equation. It then follows that the assumption on the reproducing property of the cone $K$ in Theorem 3.1, (ii) cannot be dropped.

(iii) The Hahn-Banach theorem used in the proof of Theorem 3.1 does not give an explicit cone $C(\ell)$, and since various extensions of $\varphi$ are possible [26], $K$ may be contained in different

\end{remark}
BP cones $C(\ell)$. Indeed, for each $i \in \mathbb{N}$ the cone $K_i$ in Example 2.2, (d) is a BP cone given by an equation and is a proper subset of $K_{i+1}$.

(iv) Recall that a BP cone given by an equation is not necessarily reproducing, see Remark 2.2, (ii). Hence, the condition in Theorem 3.1, (ii) is not always satisfied.

Next, let us establish a result of “representation” type, which is inspired by Theorem 4.5 in [19]. This theorem states, roughly speaking, that a convex cone $K$ is representable as a BP cone $C(\ell, \alpha) := \{ x \in X : \ell(x) \geq \alpha \|x\| \}$, i.e., $K = C(\ell, \alpha)$, if and only if the equality
\[ \text{cl conv}\{k \in K : \|k\| = 1\} = \{ x \in X : \ell(x) = \|x\| = 1 \} \] (3)
holds (the “only if part” holds under an additional assumption that the space is reflexive).

**Theorem 3.2.** A necessary and sufficient condition for a closed convex cone $K$ in a Banach space $X$ to be a BP cone $C(\ell) := \{ x \in X : \ell(x) = \|x\| \}$ with $\|\ell\| = 1$ is that the equality
\[ \text{cl conv}\{k \in K : \|k\| = 1\} = \{ x \in X : \ell(x) = \|x\| = 1 \} \] (4)
is satisfied.

**Proof.** The sufficient condition follows from [19, Theorem 4.5] because (4) is a specialization of (3) to our case with $\|\ell\| = \alpha = 1$.

Necessary condition. Since $K = C(\ell) = \{ x \in X : \ell(x) = \|x\| \}$, we get
\[ \text{cl conv}\{k \in K : \|k\| = 1\} = \text{cl conv}\{ x \in X : \ell(x) = \|x\| = 1 \}. \]

It remains to show that the set $U := \{ x \in X : \ell(x) = \|x\| = 1 \}$ is closed and convex. Let $x_1, x_2 \in U$, i.e. $\ell(x_1) = \|x_1\| = 1$ and $\ell(x_2) = \|x_2\| = 1$. For any $\lambda \in [0, 1]$, we have
\[ \ell(\lambda x_1 + (1 - \lambda) x_2) = \lambda \ell(x_1) + (1 - \lambda) \ell(x_2) = \lambda + (1 - \lambda) = 1. \]

Since $C(\ell)$ is a closed convex cone and $x_1, x_2 \in C(\ell)$, we have $x_1 + x_2 \in C(\ell)$ and hence, $\ell(\lambda x_1 + (1 - \lambda) x_2) = \|\lambda x_1 + (1 - \lambda) x_2\|$. Then, $\ell(\lambda x_1 + (1 - \lambda) x_2) = \|\lambda x_1 + (1 - \lambda) x_2\| = 1$. The convexity of the set $U$ follows. The proof of the closedness of the set $U$ can be proved similarly taking account of the closedness of the cone $C(\ell)$. □

**Remark 3.4.** In contrast to [19, Theorem 4.5], the necessary condition in Theorem 3.2 holds true without assuming that $X$ is reflexive.

Let us state a new characterization of a strictly convex Banach space.

**Theorem 3.3.** A Banach space is strictly convex if and only if there does not exist a BP cone given by an equation, which contains at least two linearly independent vectors.
Proof. Since the “only if” part has been established in Proposition 3.1, it remains to prove the “if” part. Assume by contradiction that the Banach space is not strictly convex. By Proposition 3.2, there exists a set $K$ of additivity of the norm. Theorem 3.1,(i) implies that $K$ is contained in a BP cone given by an equation. Since $K$ contains at least two linearly independent vectors, so does this BP cone, a contradiction. □

It should be noted that there is also another possibility to prove Theorem 3.3 by using a known characterization of a strictly convex Banach space given in [10, Exercise 7, p. 458].

We conclude the section by proving that the cone $K$ in Remark 3.3, (ii) cannot be itself a BP cone given by an equation. Let us start with an auxiliary result.

**Lemma 3.2.** Assume that the Banach space $X$ is not strictly convex and $K$ is a BP cone given by an equation. Then $K$ satisfies the following Condition (C): $x \in X, k \in K, x+k \in K$ and $\|x+k\| = \|x\| + \|k\|$ together imply $x \in K$.

**Proof.** Suppose that $K$ is the BP cone $C(\ell)$ with $\|\ell\| = 1$. Let $x \in X$ and $k \in C(\ell)$ be such that $x+k \in C(\ell)$ and $\|x\| + \|k\| = \|x+k\|$. One has

$$\|x\| + \|k\| = \|x+k\| = \ell(x+k) = \ell(x) + \ell(k) = \ell(x) + \|k\|,$$

which implies that $\ell(x) = \|x\|$ and hence $x \in C(\ell)$. Thus, $K$ satisfies Condition (C). □

We do not know whether Condition (C) is sufficient for a set of additivity of the norm to be a BP cone given by an equation or not.

**Proposition 3.3.** Let $K := \{x \in C_{[0,1]} \mid \exists \alpha \geq 0, \beta \geq 0 \text{ such that } x(t) = \alpha t^2 + \beta t \text{ for all } t \in [0,1]\}$. Then $K$ cannot be itself a BP cone given by an equation.

**Proof.** By Lemma 3.2, it suffices to show that the set $K$ does not satisfy Condition (C). Let $x \in C_{[0,1]}$ be the function defined by $x(t) = t^2 - (\sqrt{2} - 1)t$ for all $t \in [0,1]$. A simple calculation gives $\|x\| = 2 - \sqrt{2}$. Let $k \in K$ be the function with $k(t) = (\sqrt{2} - 1)t$ for all $t \in [0,1]$. It is clear that $x+k \in K$ because $(x+k)(t) = t^2$ for all $t \in [0,1]$. We have $\|k\| = \sqrt{2} - 1$ and $\|x+k\| = 1$. Therefore,

$$\|x+k\| = 1 = 2 - \sqrt{2} + (\sqrt{2} - 1) = \|x\| + \|k\|.$$ 

However, we have $x \notin K$. Thus, the set $K$ does not satisfy Condition (C). □

4. **Nonempty interior of BP cones given by an equation in infinite dimensional Banach spaces**

This section is devoted to the open question about the existence of BP cones given by an equation, which have a nonempty interior, in infinite dimensional Banach spaces. First,
we prove that every Banach space admits an equivalent norm, in which such cones exist. Next, we introduce and study Lorentz cones in some classical Banach spaces of sequences. We show that these Lorentz cones are BP cones given by an equation either in the original spaces or in the renormed ones and provide representations of their nonempty interior.

BP cones given by an equation may have a nonempty interior in finite dimensional Banach spaces [12] (see Fig. 1) but the known ones in infinite dimensional spaces, see for instance Example 2.2, (a)-(c), have an empty interior. In strictly convex Banach spaces, Proposition 3.1 implies the following (compare Fig. 2).

**Proposition 4.1.** If the Banach space $X$ is strictly convex, then every nontrivial BP cone given by an equation has an empty interior.

To answer the open question, we need a result from [12]. Recall that the Minkowski functional $\mu_F : X \to \mathbb{R}$ associated to any nonempty convex balanced absorbing set $F$ in a real normed space $X$ is given by

$$\mu_F(x) := \inf\{t > 0 : x \in tF\} \quad \text{for all } x \in X. \quad (5)$$

Let $K \subset X$ be a closed convex cone and $B \subset K$ be a nonempty convex set. We say that $B$ is a **base** of $K$ if $0 \not\in \text{cl } B$ and $K = \text{cone } B$, i.e., for every $k \in K \setminus \{0\}$ there exist $\lambda > 0$ and $b \in B$ such that $k = \lambda b$. When the base $B$ of $K$ is a bounded set, we say that $K$ has a **bounded base**.

For a nontrivial BP cone $C(\ell)$ it is obvious that the set

$$B = \{x \in C(\ell) : \ell(x) = 1\}$$

is a base and this base is bounded because for $x \in B$ we have $\|x\| \leq \ell(x) = 1$. Recall that a functional $\ell \in X^*$ is said to be strictly positive on $K$ if $\ell(k) > 0$ for all $k \in K \setminus \{0\}$. The following result is an important tool in this section.

**Theorem 4.1.** [12, Theorem 4.4, Remark 4.5] Let $X$ be a real normed space, and let $K \subset X$ be a closed convex cone with a nonempty interior and a closed bounded base $B$, which is defined by a strictly positive linear functional $\ell \in X^*$ with

$$B := \{x \in K : \ell(x) = 1\}. \quad (6)$$

Let $\mu_F$ be the Minkowski functional associated to the set

$$F := \text{cl conv}(-B \cup B)$$

given by (5).

Then $\|\cdot\|' := \mu_F(\cdot)$ is an equivalent norm in $X$, $F$ is the closed unit ball of this new norm,

$$B = \{x \in X : \|x\|' = \ell(x) = 1\} \quad (7)$$
and $K$ is a BP cone with
\[ K = \{ x \in X : \ell(x) = \|x\|' \}. \]

We show that in fact the cone $K$ in Theorem 4.1 is a BP cone given by an equation in the space $X$ equipped with the new norm $\| \cdot \|'$. For the sake of convenience, we make a \textit{convention} that the norm of $\ell$ considered as a functional defined on $(X, \| \cdot \|')$ is denoted by $\| \ell \|'$.

\textbf{Lemma 4.1.} Let the assumptions of Theorem 4.1 be satisfied. Then the cone $K$ in the space $X$ equipped with the new norm $\| \cdot \|'$ is a BP cone $C(\ell)$ with $\| \ell \|' = 1$.

\textit{Proof.} It is easy to see from (6) that $|\ell(x)| \leq 1$ on $F$ and from (7) we get $\ell(x) = 1$ on the set $B \subset F$. Since $F$ is the unit ball of the norm $\| \cdot \|'$, we deduce that $\| \ell \|' = 1$.\hfill $\Box$

The answer to the open question reads as follows.

\textbf{Theorem 4.2.} Every infinite dimensional Banach space possesses an equivalent norm, in which a BP cone given by an equation with a nonempty interior exists.

\textit{Proof.} Let $X$ be an arbitrary infinite dimensional Banach space. Take an arbitrary $\ell \in X^*$ with $\| \ell \| > 1$ and let $C(\ell)$ be the BP cone generated by $\ell$. By [15, Theorem 2.5], the interior of $C(\ell)$ is nonempty. Further, Theorem 4.1 and Lemma 4.1 imply that one can construct an equivalent norm $\| \cdot \|'$ on $X$ such that $C(\ell) = \{ x \in X : \ell(x) = \|x\|' \}$ and $\| \ell \|' = 1$. Thus, $C(\ell)$ is a BP cone given by an equation in $(X, \| \cdot \|')$. As the new norm is equivalent to the original one of $X$, $C(\ell)$ has a nonempty interior in the topology induced by this new norm.\hfill $\Box$

To illustrate Theorem 4.2 and to provide examples of BP cones given by an equation with a nonempty interior, we extend the concept of Lorentz cones to the Banach spaces of sequences $l_p$ ($1 \leq p \leq \infty$), $c_0$ and $c$ (recall that $c_0$ is the Banach space of all sequences converging to zero and $c$ is the Banach space of all convergent sequences). Note that $c_0$ and $c$ are subspaces of the Banach space $l_\infty$ and their dual spaces coincide with $l_1$.

Recall [15] that the \textit{extended Lorentz cone} (also called second-order cone or briefly, Lorentz cone) in the $n$-dimensional space $\mathbb{R}^n$ equipped with the $l_p$-norm is given by
\[ C_p := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \|(x_2, \ldots, x_n)\|_p \leq x_1 \}. \]

Here, $\|(x_2, \ldots, x_n)\|_p := (\sum_{i=2}^n |x_i|^p)^{1/p}$ means the $l_p$-norm of $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. These cones play an important role in complementarity problems.

A Lorentz cone in an infinite dimensional Hilbert space $\mathcal{H}$ (see [5, 35]), here denoted by $C_\mathcal{H}$, can be represented in the form
\[ C_\mathcal{H} := \{ x \in \mathcal{H} : \langle \sqrt{2} e, x \rangle \geq \|x\| \}, \]
where $e \in \mathcal{H}$ is a fixed unit vector, i.e., it is a BP cone $C(\ell)$ with $\ell = \sqrt{2}e$ and $\|\ell\| = \sqrt{2} > 1$.

We extend the concept of Lorentz cone to the Banach spaces of sequences as follows.

**Definition 4.1.** Let $X$ be one of the Banach spaces of sequences $l_p$ $(1 \leq p \leq \infty)$, $c_0$ and $c$ with the norm $\| \cdot \|_X$. The Lorentz cone in $X$, denoted by $C_X$, is the set

$$C_X := \{ x = (x_i)_{i \in \mathbb{N}} \in X : \|(x_2, x_3, \ldots)\|_X \leq x_1 \}. \quad (8)$$

We establish some properties of the Lorentz cone in concrete spaces. Namely, we show that the Lorentz cone is a BP cone given by an inequality when $X$ is the Banach space $l_p$ ($1 \leq p < \infty$) and it is a BP cone given by an equation when $X$ is one of the Banach spaces $l_\infty$, $c_0$ and $c$. Moreover, the Lorentz cone always has a nonempty interior.

From now on, we make a convention that $e_1$ is, as usually, the vector $(1, 0, 0, \ldots)$ and $x = (x_i)_{i \in \mathbb{N}}$ is a vector in the considered space of sequences.

**Proposition 4.2.** The Lorentz cone in $l_p$ $(1 \leq p < \infty)$, here denoted by $C_p$, is a BP cone $C(\ell)$ generated by the functional $\ell = \sqrt{2}e_1$ (with $\|\ell\| = \sqrt{2}$)

$$C_p = \{ x \in l_p : \|x\|_p \leq \sqrt{2}x_1 \} \quad (9)$$

and it has a nonempty interior

$$\text{int } C_p = \{ x \in l_p : \|x\|_p < \sqrt{2}x_1 \} = \{ x \in l_p : \|(x_2, x_3, \ldots)\|_p < x_1 \}. \quad (10)$$

**Proof.** Note that $\ell(x) = \sqrt{2}x_1$. The equality $C_p = C(\ell)$ with $\ell = \sqrt{2}e_1$ follows from the equivalences

$$\|x\|_p \leq \ell(x) \iff (\sum_{i=1}^\infty |x_i|^p)^{1/p} \leq \sqrt{2}x_1 \text{ and } x_1 \geq 0$$

$$\iff \sum_{i=1}^\infty |x_i|^p \leq 2x_1^p \text{ and } x_1 \geq 0$$

$$\iff \sum_{i=2}^\infty |x_i|^p \leq x_1^p \text{ and } x_1 \geq 0$$

$$\iff (\sum_{i=2}^\infty |x_i|^p)^{1/p} \leq x_1 \text{ and } x_1 \geq 0$$

$$\iff \|(x_2, x_3, \ldots)\|_p \leq x_1.$$

Since $\|\ell\| = \sqrt{2} > 1$, Theorem 2.5 in [15] implies that the interior of $C(\ell)$ is nonempty and the first equality in (10) holds. The second equality in (10) can be proved in the way similar to the one used in the proof of the equality $C_p = C(\ell)$. \qed

**Proposition 4.3.** Let $X$ be one of the Banach spaces $l_\infty$, $c_0$ and $c$ with the norm $\| \cdot \|_X$. The Lorentz cone $C_X$ in $X$ is a BP cone $C(\ell)$ generated by the functional $\ell = e_1$ (with $\|\ell\| = 1$)

$$C_X = \{ x \in X : \|x\|_X = x_1 \} = \{ x \in X : |x_i| \leq x_1 \text{ for all } i \in \mathbb{N} \}$$

and it has a nonempty interior

$$\text{int } C_X = \{ x \in X : \|(x_2, x_3, \ldots)\|_X < x_1 \}.\quad (11)$$
Proof. We provide the proof for the case $X = l_{\infty}$ as the other cases can be considered analogously due to the fact that the norms in all these spaces coincide. Denote by $C_{\infty}$ the Lorentz cone in $l_{\infty}$ defined as in (8). To prove the equality $C_{\infty} = C(\ell)$ with $\ell = e_{1}$, observe that $\ell(x) = x_{1}$ and the following equivalences hold

$$
\|x\|_{\infty} \leq \ell(x) \iff \sup\{|x_{i}| : i = 1, 2, \ldots\} \leq x_{1} \text{ and } x_{1} \geq 0
$$

$$
\iff \sup\{|x_{i}| : i = 2, 3, \ldots\} \leq x_{1} \text{ and } x_{1} \geq 0
$$

$$
\iff \|(x_{2}, x_{3}, \ldots)\|_{\infty} \leq x_{1}.
$$

Since $\|e_{1}\|_{(l_{\infty})^*} = 1$, we have $C(\ell) = \{x \in l_{\infty} : \|x\|_{\infty} = x_{1}\}$. Therefore, $C_{\infty} = \{x \in l_{\infty} : \|x\|_{\infty} = x_{1}\}$.

Further, let us prove the representation of the interior of $C_{\infty}$. It is easy to see that $e_{1} \in U := \{x \in l_{\infty} : \|(x_{2}, x_{3}, \ldots)\|_{\infty} < x_{1}\}$. Let $x \in U$ be an arbitrary vector. We claim that $B(x, \epsilon) \subset C_{\infty}$, where $\epsilon := \frac{1}{2}(x_{1} - \|(x_{2}, x_{3}, \ldots)\|_{\infty})$. Let $u = (u_{i})_{i \in \mathbb{N}} \in l_{\infty}$ such that $\|u\|_{\infty} \leq \epsilon$. We show that $x + u \in C_{\infty}$. Observe that

$$
x_{1} + u_{1} \geq x_{1} - \|u\|_{\infty} \geq x_{1} - \epsilon = x_{1} - \frac{1}{2}(x_{1} - \|(x_{2}, x_{3}, \ldots)\|_{\infty}) = \frac{1}{2}(x_{1} + \|(x_{2}, x_{3}, \ldots)\|_{\infty})
$$

and for $i = 2, 3, \ldots$, one has

$$
|x_{i} + u_{i}| \leq |x_{i}| + |u_{i}| \leq \|(x_{2}, x_{3}, \ldots)\|_{\infty} + \|u\|_{\infty} \leq \|(x_{2}, x_{3}, \ldots)\|_{\infty} + \epsilon
$$

$$
= \|(x_{2}, x_{3}, \ldots)\|_{\infty} + \frac{1}{2}(x_{1} - \|(x_{2}, x_{3}, \ldots)\|_{\infty}) = \frac{1}{2}(x_{1} + \|(x_{2}, x_{3}, \ldots)\|_{\infty}).
$$

Hence $\|(x_{2} + u_{2}, x_{3} + u_{3}, \ldots)\|_{\infty} \leq x_{1} + u_{1}$, which means that $x + u \in C_{\infty}$. Thus, $U \subset \text{int } C_{\infty}$. It remains to check that any $x \in C_{\infty} \setminus U$ does not belong to the interior of $C_{\infty}$. Since $x \in C_{\infty} \setminus U$, we have $\|(x_{2}, x_{3}, \ldots)\|_{p} = x_{1}$. Let $u^{j} = (u_{1}^{j}, u_{2}^{j}, \ldots) := x - \frac{1}{j}e_{1} = (x_{1} - \frac{1}{j}, x_{2}, \ldots)$. It is clear that $\lim_{j \to \infty} u^{j} = x$ and $u^{j} \notin C_{\infty}$ because for all $j = 1, 2, \ldots$ one has

$$
\|(u_{2}^{j}, u_{3}^{j}, \ldots)\|_{\infty} = \|(x_{2}, x_{3}, \ldots)\|_{\infty} = x_{1} > x_{1} - \frac{1}{j} = u_{1}^{j}.
$$

Thus $x$ cannot belong to the interior of $C_{\infty}$. \qed

Remark 4.1. Proposition 4.2 is motivated by Lemma 2.4 and Corollary 2.6 in [15], in which similar facts have been established for Lorentz cones in $\mathbb{R}^{n}$ equipped with the $l_{p}$-norm. Note that although the set $\{x \in X : \|x\|_{X} < x_{1}\}$ is empty when $X$ is one of the spaces $l_{\infty}$, $c_{0}$ and $c$, we still have a representation for the interior of the cone $C_{X}$.

In contrast to the case $p = \infty$, the Lorentz cone $C_{p}$ with $p \in [1, \infty[$ becomes a BP cone given by an equation only when $l_{p}$ is renormed with an equivalent norm following the procedure presented in Theorem 4.2. We define a new norm $\|\cdot\|_{p}'$ in the space $l_{p}$, $1 \leq p < \infty$ as follows: For $x = (x_{1}, x_{2}, \ldots) \in l_{p}$$
$$
\|x\|_{p}' = \max\{\|(x_{2}, x_{3}, \ldots)\|_{p}, |x_{1}|\} = \max\{\left(\sum_{i=2}^{\infty} |x_{i}|^{p}\right)^{1/p}, |x_{1}|\}.
$$

(11)
Note that the space equipped with the new norm is not strictly convex.

**Lemma 4.2.** The norms $\| \cdot \|_p$ and $\| \cdot \|'_p$ are equivalent. More precisely, one has

$$\frac{1}{\sqrt{2}} \| x \|_p \leq \| x \|'_p \leq \| x \|_p \text{ for all } x \in l_p.$$  \hfill (12)

**Proof.** We start with the first inequality in (12) and consider two possible cases:

(i) Assume that $\| x \|'_p = |x_1|$ and $\| (x_2, x_3, \ldots) \|_p \leq |x_1|$. Then $\sum_{i=2}^{\infty} |x_i|^p \leq |x_1|^p$ and we get $\sum_{i=1}^{\infty} |x_i|^p \leq 2 |x_1|^p$. Hence,

$$\frac{1}{\sqrt{2}} \| x \|_p \leq |x_1| = \| x \|'_p.$$  

(ii) Assume that $\| x \|'_p = \| (x_2, x_3, \ldots) \|_p$ and $\| (x_2, x_3, \ldots) \|_p > |x_1|$. Then $|x_1|^p < \sum_{i=2}^{\infty} |x_i|^p$ and $\sum_{i=1}^{\infty} |x_i|^p < 2 \sum_{i=2}^{\infty} |x_i|^p$. Hence,

$$\frac{1}{\sqrt{2}} \| x \|_p \leq \sum_{i=2}^{\infty} |x_i|^p = \| x \|'_p.$$  

Thus, the first inequality in (12) holds in both cases.

Finally, the second inequality in (12) follows from the equality $\| x \|'_p = \max\{\| (x_2, x_3, \ldots) \|_p, |x_1|\}$ and the inequalities $\sum_{i=2}^{\infty} |x_i|^p \leq \sum_{i=1}^{\infty} |x_i|^p$ and $|x_1|^p \leq \sum_{i=1}^{\infty} |x_i|^p$. \hfill $\square$

**Proposition 4.4.** The space $l_p$ ($1 \leq p < \infty$) can be renormed with the new equivalent norm $\| \cdot \|'_p$ defined by (11) so that the Lorentz cone $C_p$ given by (9) is a BP cone $C(\ell)$ in the renormed space $l_p$ with $\ell = e_1$ and $\| \ell \|'_p = 1$.

$$C_p = \{ x \in l_p : \| x \|'_p = \ell(x) \} = \{ x \in l_p : \max\{\| (x_2, x_3, \ldots) \|_p, |x_1|\} = x_1 \}. \hfill (13)$$

Moreover, this BP cone has a nonempty interior in the topology induced by the new norm.

Here, as before, $\| \ell \|'_p$ stands for the norm of the functional $\ell$ on $(l_p, \| \cdot \|'_p)$.

**Proof.** Recall that by Proposition 4.2, the cone $C_p$ is a BP cone $C(\bar{\ell})$ with $\bar{\ell} = \sqrt{2} e_1$ and hence, it has a nonempty interior. Now, let $\ell = e_1$. Then for any $x \in l_p$, we have $\ell(x) = x_1$. The functional $\ell$ is strictly positive on $C_p$, and the set

$$B := \{ x \in C_p : \ell(x) = 1 \} = \{ x \in l_p : \| (x_2, x_3, \ldots) \|_p \leq 1, x_1 = 1 \}$$

is a bounded base of $C_p$. Following Theorem 4.1, we can renorm the space $l_p$ by using the Minkowski functional associated to the set

$$F := \text{cl conv } (-B \cup B) = \{ x \in l_p : \| (x_2, x_3, \ldots) \|_p \leq 1, |x_1| \leq 1 \}, \hfill (14)$$

which is the closed unit ball of the equivalent norm $\| \cdot \|'_p$ as in Theorem 4.1. Using (14), it is evident that the new norm has the form given in (11). By Lemma 4.1, the functional $\ell = e_1$ has the norm 1 in the dual space of the renormed space $l_p$. Theorem 4.1 then implies that $C_p$ is the BP cone $C(\ell)$ and the equality (13) holds.
Finally, since the norm given by (11) is equivalent to the original one by Lemma 4.2, the interior of $C_p$ in the new topology coincides with its interior in the original one and is nonempty by Proposition 4.2. □

The diagram in Fig. 3 illustrates the approach of Proposition 4.4 in the following sense: The space $l_p$ ($1 \leq p < \infty$) is renormed so that the Lorentz cone $C_p$ is a BP cone given by an equation. Recall that $\|\ell\|$ stands for the norm of $\ell$ in the dual space of $l_p$, which equals the sequence space $l_q$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $\|\ell\|'$ stands for the norm of $\ell$ in the dual space of $l_p$ equipped with the new norm (see Fig. 3).

Example 4.1. We illustrate Propositions 4.2 and 4.4 in the case $X = \mathbb{R}^3$ equipped with the $l_1$-norm or the $l_2$-norm. Here, for $x \in \mathbb{R}^3$ we mean the vector $(x_1, x_2, x_3)$.

(a) Let $X = \mathbb{R}^3$ be equipped with the $l_1$-norm. The Lorentz cone $C_1$ has the form

$$C_1 = \{x \in \mathbb{R}^3 : \|(x_2, x_3)\|_1 \leq x_1\}$$

(see Fig. 4).

Let $\ell = (1, 0, 0) \in \mathbb{R}^3$ with

$$\ell(x) = x_1 \text{ for all } x \in \mathbb{R}^3$$

be given. By the same arguments used in the proof of Proposition 4.4, in $\mathbb{R}^3$ equipped with the new norm

$$\|x\|'_1 := \max\{\|(x_2, x_3)\|_1, |x_1|\} = \max\{|x_2| + |x_3|, |x_1|\},$$

the Lorentz cone $C_1$ is the BP cone $C(\ell)$ and has the form

$$C_1 = \{x \in \mathbb{R}^3 : \max\{|x_2| + |x_3|, |x_1|\} = x_1\}.$$

Note that the closed unit ball in the new norm is the cuboid given by

$$F = \{x \in \mathbb{R}^3 : \max\{|x_2| + |x_3|, |x_1|\} \leq 1\} = \{x \in \mathbb{R}^3 : |x_2| + |x_3| \leq 1, |x_1| \leq 1\}.$$

see Fig. 5.
Fig. 4. Illustration of the Lorentz cone $C_1$ in Example 4.1,(a) (one looks into this cone). The set $B$ denotes the base used in the proof of Proposition 4.4.

Fig. 5. Illustration of the unit ball $F$ of the equivalent norm $\| \cdot \|'$ in $\mathbb{R}^3$ equipped with the $l_1$ norm in Example 4.1,(a). $s := (0.65, -0.5, 0.5)$ illustrates a point on the sphere of this ball.

(b) Let $X = \mathbb{R}^3$ be equipped with the euclidean $l_2$-norm. This case has already been considered in [12, Example 4.6 (a)]. Note that in this case the new norm has the form

$$\| x \|'_2 := \max \{ \| (x_2, x_3) \|_2, |x_1| \} = \max \{ \sqrt{x_2^2 + x_3^2}, |x_1| \},$$
and the closed unit ball in this norm is the cylinder
\[ F = \{ x \in \mathbb{R}^3 : \max\{ \sqrt{x_2^2 + x_3^2}, |x_1| \} \leq 1 \} \]
The reader is referred to [12, Fig. 1] for the closed unit ball and the Lorentz cone in this case.

Since all the components \( x_i \) of \( x = (x_i)_{i\in\mathbb{N}} \) play an equal role, we can define a Lorentz cone in the considered Banach spaces of sequences in a more general form. Let \( x_{(i)} \) be the sequence obtained from \( (x_i)_{i\in\mathbb{N}} \) by deleting the \( i \)-th-component of \( x \).

**Definition 4.2.** Let \( X \) be one of the Banach spaces of sequences \( l_p \) \((1 \leq p \leq \infty), c_0 \) or \( c \) with the norm \( \| \cdot \|_X \). The **generalized Lorentz cone** in \( X \), denoted by \( C_{X}^{i,j} \) for an arbitrary pair \((i, j)\), where \( i \in \mathbb{N} \) and \( j \in \{1, 2\} \), is the set
\[ C_{X}^{i,j} := \{ x \in X : \| x_{(i)} \|_X \leq (-1)^j x_i \}. \] (15)

The Lorentz cone defined by (8) is a special case of the one defined by (15) with \( i = 1 \) and \( j = 2 \). One can obtain analogous versions of Propositions 4.2 - 4.4 for the cones \( C_{X}^{i,j} \).

Note that instead of the norm (11), we will consider the following equivalent norm \( \| \cdot \|'_p \) in \( l_p \) \((1 \leq p < \infty)\)
\[ \| x \|'_p = \max\{ \| x_{(i)} \|_p, |x_i| \}. \] (16)

Note that the new norm defined by (16) is equivalent to the original norm and satisfies the inequalities stated in (12).

**Proposition 4.5.** The **generalized Lorentz cone** in \( l_p \) \((1 \leq p < \infty)\), here denoted by \( C_{p}^{i,j} \), is a BP cone \( C(\ell) \) generated by the functional \( \ell = (-1)^j 2^{1/p} e_i \) (with \( \| \ell \| = \sqrt{2} \))
\[ C_{p}^{i,j} = \{ x \in l_p : \| x \|_p \leq (-1)^j 2^{1/p} x_i \} \]
and it has the nonempty interior given by
\[ \text{int} C_{p}^{i,j} = \{ x \in l_p : \| x \|_p < (-1)^j 2^{1/p} x_i \} = \{ x \in l_p : \| x_{(i)} \|_p < (-1)^j x_i \}. \]

In the space \( l_p \) renormed with the equivalent norm \( \| \cdot \|'_p \) defined by (16), this cone is a BP cone \( C(\ell) \) with \( \ell = (-1)^j e_i \), \( \| \ell \|' = 1 \) (and a nonempty interior)
\[ C_p = \{ x \in l_p : \| x \|'_p = (-1)^j x_i \}. \]

**Proposition 4.6.** Let \( X \) be one of the Banach spaces of sequences \( l_\infty \), \( c_0 \) or \( c \) with the norm \( \| \cdot \|_X \). The **generalized Lorentz cone** \( C_{X}^{i,j} \) in \( X \) is a BP cone \( C(\ell) \) generated by the functional \( \ell = (-1)^j e_i \) (with \( \| \ell \| = 1 \))
\[ C_{X}^{i,j} = \{ x \in X : \| x \|_X = (-1)^j x_i \} = \{ x \in X : |x_k| \leq (-1)^j x_i \text{ for all } k \in \mathbb{N} \} \] (17)
and it has the nonempty interior given by
\[ \text{int } C^i_X = \{ x \in X : \|x_i\|_X < (-1)^j x_1 \}. \]

Further relations between Lorentz cones and BP cones in the Banach spaces \( c_0, c \) and \( l_\infty \) will be established in the next section.

5. **Representations of BP cones given by an equation in some Banach spaces**

In this section, we provide representations of all BP cones given by an equation in \( l_1, c_0, c, L^1_{[0,1]} \), and of some ones in \( l_\infty, L^\infty_{[0,1]}, C_{[0,1]} \). Along the way, we study the emptiness of the interior of these cones. As a byproduct, we show that the dual space of \( l_\infty \) and \( L^\infty_{[0,1]} \) are broader than \( l_1 \) and \( L^1_{[0,1]} \), respectively.

5.1. **The Banach space \( l_1 \)**. Recall that the dual space of the Banach space \( l_1 \) is the Banach space \( l_\infty \). Given \( x = (x_i)_{i \in \mathbb{N}} \in l_1 \) and \( \ell = (\ell_i)_{i \in \mathbb{N}} \in l_\infty \), the dual pair is presented in the form
\[
\ell(x) = \langle \ell, x \rangle = \sum_{i=1}^{\infty} \ell_i x_i.
\]
The Banach spaces \( l_1 \) and \( l_\infty \) are neither reflexive nor strictly convex.

Let an arbitrary \( \ell = (\ell_i)_{i \in \mathbb{N}} \in l_\infty \) with \( \|\ell\| = 1 \) be given. Then we have \( \sup_{i \in \mathbb{N}} |\ell_i| = 1 \). We associate to \( \ell \) the following sets of indexes
\[
I_+(\ell) = \{ i \in \mathbb{N} : \ell_i = 1 \}, \\
I_-(\ell) = \{ i \in \mathbb{N} : \ell_i = -1 \}, \\
I_0(\ell) = \{ i \in \mathbb{N} : |\ell_i| < 1 \}. 
\]
We present the form of a BP cone \( C(\ell) \) with \( \|\ell\| = 1 \) in \( l_1 \).

**Proposition 5.1.** Let \( \ell = (\ell_i)_{i \in \mathbb{N}} \in l_\infty \) with \( \|\ell\| = 1 \) be given.

(i) The BP cone \( C(\ell) \) coincides with the set
\[
U(\ell) := \{ x = (x_i)_{i \in \mathbb{N}} \in l_1 : x_i \geq 0 \text{ if } i \in I_+(\ell), \ x_i \leq 0 \text{ if } i \in I_-(\ell) \text{ and } x_i = 0 \text{ if } i \in I_0(\ell) \}.
\]

(ii) The BP cone \( C(\ell) \) is nontrivial if and only if
\[
I_+(\ell) \cup I_-(\ell) \neq \emptyset
\]
(i.e., there exists at least one index \( j \) such that \( |\ell_j| = 1 \)).

**Proof.** (i) Let us show the equality \( C(\ell) = U(\ell) \). It is clear that \( 0 \in U(\ell) \), i.e. \( U(\ell) \) is nonempty. To prove that \( U(\ell) \subset C(\ell) \), take an arbitrary \( x = (x_i)_{i \in \mathbb{N}} \in U(\ell) \). Then one has \( \ell_i x_i = |x_i| \) for all \( i \in \mathbb{N} \) and \( \|x\| = \sum_{i=1}^{\infty} |x_i| = \sum_{i=1}^{\infty} \ell_i x_i = \ell(x) \). This means that \( x \in C(\ell) \). Hence, \( U(\ell) \subset C(\ell) \).
It remains to show that $C(\ell) \subset U(\ell)$. Let $x = (x_i)_{i \in \mathbb{N}} \in C(\ell)$ be arbitrarily chosen. By the definition of the BP cone, we have $\ell(x) = \|x\|$. Assume to the contrary that $x \notin U(\ell)$. Then for at least one index $j$ one has either $x_j < 0$, $\ell_j = 1$ or $x_j > 0$, $\ell_j = -1$ or $x_j \neq 0$, $|\ell_j| < 1$. Correspondingly, we have $\ell_j x_j = x_j < 0 < |x_j|$ in the first case, $\ell_j x_j = -x_j < 0 < |x_j|$ in the second case and $|\ell_j x_j| < |x_j|$ in the last case. Since $|\ell_i||x_i| \leq |x_i|$ for all $i \in \mathbb{N}$, we obtain

$$\|x\| = \sum_{i=1}^{\infty} |x_i| > \sum_{i=1}^{\infty} \ell_i x_i = \ell(x),$$

a contradiction.

(ii) The “only if” part. If the condition (20) does not hold, the set $U(\ell)$ contains only the vector zero and the assertion (i) implies that $C(\ell)$ is trivial.

The “if” part. Let the condition (20) be fulfilled with $j \in I_+(\ell) \cup I_-(\ell)$, i.e. we have $|\ell_j| = 1$. Next, we define $x := (0, \ldots, 0, \text{sgn}(\ell_j), 0, \ldots) \in l_1$ where the $j$th component equals the signum $\text{sgn}(\ell_j)$. It is clear that $x \neq 0$ and $x \in U(\ell)$. The assertion (i) then implies that $x \in C(\ell) \setminus \{0\}$, and hence $C(\ell)$ is nontrivial $\square$

Remark 5.1. (i) When $\ell = (1, 1, \ldots)$, $C(\ell)$ is the nonnegative orthant of $l_1$.

(ii) One can check that the cone $U(\ell)$ defined by (19) is a set of additivity of the norm in $l_1$. Since the index sets given by (18) may be different, there are various sets of additivity of the norm in $l_1$.

Next, we use the representation (19) of a BP cone $C(\ell)$ with $\|\ell\| = 1$ established in Proposition 5.1,(i) to show that this cone has an empty interior.

Proposition 5.2. Every BP cone $C(\ell)$ in $l_1$ with $\|\ell\| = 1$ has an empty interior.

Proof. Let $C(\ell)$ with $\|\ell\| = 1$ be an arbitrary BP cone. Then $C(\ell)$ coincides with the set $U(\ell)$ given by (19). Suppose to the contrary that some $k = (k_i)_{i \in \mathbb{N}} \in C(\ell)$ is an interior point of $C(\ell)$. Then there exists a scalar $\rho > 0$ such that $\mathbb{B}(k, \rho) \subset C(\ell)$. Since $k \in l_1$, it is evident that $\lim_{i \to \infty} |k_i| = 0$. Let $j$ be an index such that $|k_j| \leq \rho/2$. If $j \in I_+(\ell)$, then the $j$th component of the vector $k - \rho e_j$ is $k_j - \rho < 0$ and the representation (19) implies that $k - \rho e_j \notin C(\ell)$. If $j \in I_-(\ell)$ or $j \in I_0(\ell)$, then the $j$th component of the vector $k + \rho e_j$ is $k_j + \rho > 0$ or is $\rho > 0$ and the representation (19) implies that $k + \rho e_j \notin C(\ell)$. In any case, we have $\mathbb{B}(k, \rho) \not\subset C(\ell)$, a contradiction $\square$

Remark 5.2. (i) Although BP cones given by an equation in $l_1$ have an empty interior by Proposition 5.2, there are other cones with a nonempty interior. Let $D$ be the cone

$$D := \{x \in l_1 : \text{all partial sums of } x \text{ are nonnegative}\}.$$

It is known that $(1, 0, 0, \ldots) \in \text{int } D$. By Proposition 5.2, $D$ cannot be a BP cone $C(\ell)$ with $\|\ell\| = 1$. 
(ii) The results of this subsection are extensions of some ones obtained in [12] for $\mathbb{R}^n$ equipped with the $l_1$ norm.

5.2. **The Banach space $c_0$ or $c$.** Recall that the dual space of both Banach spaces $c_0$ and $c$ is the Banach space $l_1$.

Let $\ell = (\ell_i)_{i \in \mathbb{N}} \in l_1$ with $\|\ell\| = 1$ be given. We associate to $\ell$ the following sets of indexes

$$I_+(\ell) = \{i \in \mathbb{N} : \ell_i = 1\},$$
$$I_-(\ell) = \{i \in \mathbb{N} : \ell_i = -1\},$$
$$I_0(\ell) = \{i \in \mathbb{N} : |\ell_i| < 1\}.$$

Since $\|\ell\| = \sum_{i \in \mathbb{N}} |\ell_i| = 1$, there are only two possible cases: $I_+(\ell) \cup I_-(\ell) = \emptyset$ or $I_+(\ell) \cup I_-(\ell)$ is a singleton.

**Proposition 5.3.** Let $X$ be either the Banach space $c_0$ or the Banach space $c$ and let $\ell = (\ell_i)_{i \in \mathbb{N}} \in l_1$ with $\|\ell\| = 1$ be given.

(i) If $I_+(\ell) \cup I_-(\ell) = \emptyset$, then the BP cone $C(\ell)$ is trivial.

(ii) If $I_+(\ell) \cup I_-(\ell) = \{i\}$ for some $i \in \mathbb{N}$, then the BP cone $C(\ell)$ is a Lorentz cone $C^i_{\mathbb{X}}$ defined by (17) for some $j = 1$ or $j = 2$ depending on $i \in I_-(\ell)$ or $i \in I_+(\ell)$.

**Proof.** (i) It is clear that $I_0(\ell) = \mathbb{N}$ or $|\ell_i| < 1$ for all $i \in \mathbb{N}$. We show that $C(\ell) = \{0\}$. Suppose to the contrary that there exists $x = (x_i)_{i \in \mathbb{N}} \in C(\ell)$ such that for some $j \in \mathbb{N}$ one has $|x_j| > 0$. Since $|\ell_i| < 1$ for all $i \in \mathbb{N}$, we get $|\ell_i||x_i| \leq |x_i|$ for all $i \in \mathbb{N}$ and $|\ell_j||x_j| < |x_j|$. Therefore, we obtain

$$\ell(x) = \sum_{i \in \mathbb{N}} \ell_i x_i \leq \sum_{i \in \mathbb{N}} |\ell_i||x_i| < \sum_{i \in \mathbb{N}} |x_i| = \|x\|,$$

which is a contradiction to $x \in C(\ell)$.

(ii) Let $\ell = (-1)^j \ell_i$ for $j = 1$ or $j = 2$ depending on $i \in I_+(\ell)$ or $i \in I_-(\ell)$. By Proposition 4.6, $C(\ell)$ is the Lorentz cone $C^i_{\mathbb{X}}$ as in (17) and it is nontrivial because $\ell$ itself belongs to $C(\ell)$. \qed

**Remark 5.3.** By Propositions 5.3 and 4.6, all nontrivial BP cones given by an equation in the Banach spaces $c_0$ and $c$ have a nonempty interior.

5.3. **The Banach space $l_\infty$.** It is known that the dual space of $l_\infty$ contains $l_1$ but does not coincide with the latter because there are bounded functionals $\phi$ on $l_\infty$, which are not of the form

$$\phi(x) = \sum_{i=1}^{\infty} \ell_i x_i \text{ for } x = (x_i)_{i \in \mathbb{N}} \in l_\infty$$

for some $(\ell_1, \ell_2, \ldots) \in l_1$. Such functionals exist (one can prove it by using the Hahn-Banach extension theorem) but they cannot be given explicitly. Motivated by this fact, we restrict
ourselves to a presentation for the BP cone $C(\ell)$ with $\|\ell\| = 1$ in $l_\infty$ in the case $\ell \in l_1$. As a byproduct, our result also implies the existence of a bounded linear functional $\ell : l_\infty \to \mathbb{R}$, which does not belong to $l_1$.

We need the following well-known special version of the Hahn-Banach theorem.

**Proposition 5.4.** Let $X$ be a Banach space. For every vector $x \in X$, there exists some $x^* \in X^*$ with $\|x^*\| = 1$ such that $x^*(x) = \|x\|$.

**Proposition 5.5.** For every $\ell \in l_1$ with $\|\ell\| = 1$, the corresponding BP cone $C(\ell)$ in $l_\infty$ is nontrivial.

**Proof.** Applying Proposition 5.4 to $X = l_1$ and $x = \ell$, we find $u \in l_\infty$ with $\|u\| = 1$ such that $\langle u, x \rangle = \|x\| = \|\ell\| = 1$. Then $u \in C(\ell)$ because $\ell(u) = \langle u, x \rangle = \langle u, x \rangle = 1 = \|u\|$. □

We associate to any $\ell = (\ell_i)_{i \in \mathbb{N}} \in l_1$ with $\|\ell\| = 1$ the following sets of indexes

$I_+ (\ell) = \{ i \in \mathbb{N} : \ell_i > 0 \}$,
$I_- (\ell) = \{ i \in \mathbb{N} : \ell_i < 0 \}$,
$I_0 (\ell) = \{ i \in \mathbb{N} : \ell_i = 0 \}$.

Observe that since $\|\ell\| = 1$, we always have

$I_+(\ell) \cup I_-(\ell) \neq \emptyset$. \hspace{1cm} (21)

To illustrate Proposition 5.5, let $x = (x_i)_{i \in \mathbb{N}} \in l_\infty$ be the vector satisfying $x_i = 1$ for all $i \in I_+(\ell)$, $x_i = -1$ for all $i \in I_-(\ell)$ and $x_i = 0$ for all $i \in I_0(\ell)$. Clearly, the vector $x$ is nonzero and $\|x\| = 1$. Moreover, $x \in C(\ell)$ because

$\ell(x) = \sum_{i=1}^{\infty} \ell_i x_i = \sum_{i=1}^{\infty} |\ell_i| = \|\ell\| = 1 = \|x\|$.

Next, we establish a representation for a BP cone in $l_\infty$ with $\ell \in l_1$ and $\|\ell\| = 1$.

**Proposition 5.6.** For every $\ell \in l_1$ with $\|\ell\| = 1$ we have

$C(\ell) = \{ x = (x_i)_{i \in \mathbb{N}} \in l_\infty : x_i = \|x\| \text{ for } i \in I_+(\ell), \ x_i = -\|x\| \text{ for } i \in I_-(\ell) \}$.

**Proof.** As it has been shown before, the BP cone $C(\ell)$ is nontrivial. Denote by $U(\ell)$ the set in the right-hand side of the desired equality. Note that the definition of $U(\ell)$ is meaningful because the relation (21) holds. For an arbitrary $x = (x_i)_{i \in \mathbb{N}} \in U(\ell)$ we obtain

$\ell(x) = \sum_{i=1}^{\infty} \ell_i x_i = \sum_{i \in I_+(\ell) \cup I_-(\ell) \cup I_0(\ell)} \ell_i x_i = \sum_{i=1}^{\infty} |\ell_i| \|x\| = \|x\| \sum_{i=1}^{\infty} |\ell_i| = \|x\|.$

Hence, $x \in C(\ell)$. Thus, $U(\ell) \subset C(\ell)$ is shown.
Further, let \( x = (x_i)_{i \in \mathbb{N}} \in C(\ell) \) be arbitrarily chosen. Assume to the contrary that \( x \notin U(\ell) \). It follows from Propositions 5.6 and 4.6 that the BP cone \( C(\ell) \) is one of the generalized Lorentz cones \( C_{\infty}^i \), there exists a functional arbitrary \( k \in C(\ell) \). Let \( \ell(x) = \sum_{i=1}^{\infty} \ell_i x_i \leq \sum_{i=1}^{\infty} |\ell_i x_i| < \sum_{i=1}^{\infty} |\ell_i| \|x\| = \|x\| \sum_{i=1}^{\infty} |\ell_i| = \|x\|, \) a contradiction. Thus, \( C(\ell) \subset U(\ell) \). \( \square \)

**Remark 5.4.** (i) It follows from Propositions 5.6 and 4.6 that the BP cone \( C(\ell) \) is one of the generalized Lorentz cones \( C_{\infty}^i \) if and only if
\[
I_+(\ell) \cup I_-(\ell) = \{i\},
\]
i.e. the set \( I_+(\ell) \cup I_-(\ell) \) is a singleton. In this case, \( C(\ell) \) is one of the generalized Lorentz cones \( C_{\infty}^i \), where \( j = 1 \) if \( i \in I_+(\ell) \) and \( j = 2 \) if \( i \in I_-(\ell) \).

(ii) Let \( K \) be the set of additivity of the norm presented in Example 3.1, (iii). By Theorem 3.1, there exists a functional \( \ell \in (l_\infty)^* \) with \( \|\ell\| = 1 \) such that \( K \subset C(\ell) \). Note that for an arbitrary \( k = (k_i)_{i \in \mathbb{N}} \in K \setminus \{0\} \) one has \( k_i < \|k\| \) for all \( i \) and Proposition 5.6 implies that \( K \) cannot be included in a BP cone \( C(\ell) \) with \( \ell \in l_1 \). As a byproduct, this fact implies that the dual of \( l_\infty \) is broader than \( l_1 \).

**Proposition 5.7.** Let \( C(\ell) \) be a nontrivial BP cone in \( l_\infty \) with \( \ell \in l_1 \) and \( \|\ell\| = 1 \). Then the interior of \( C(\ell) \) is empty unless \( C(\ell) \) is one of the generalized Lorentz cones \( C_{\infty}^i \) defined by (15).

**Proof.** If \( C(\ell) \) is one of the generalized Lorentz cones \( C_{\infty}^i \), then it has a nonempty interior, due to Proposition 4.6. Now assume that \( C(\ell) \) is not a generalized Lorentz cone. Then, as noted in Remark 5.4, (i), the set \( I_+(\ell) \cup I_-(\ell) \) is not a singleton and it must contain at least two different indexes \( j_1 \) and \( j_2 \). Suppose to the contrary that there exists some \( k = (k_i)_{i \in \mathbb{N}} \in \text{int} C(\ell) \). Then \( B(k, \rho) \subset C(\ell) \) for some scalar \( \rho > 0 \). Proposition 5.6 implies that \( |k_{j_1}| = |k_{j_2}| = \|k\| \). Let \( u = (u_i)_{i \in \mathbb{N}} \in l_\infty \) be the vector \( u := k + \rho e_{j_1} \) in the case \( j_1 \in I_+(\ell) \) and \( u = k - \rho e_{j_1} \) in the case \( j_1 \in I_-(\ell) \). Clearly, \( u \in B(k, \rho) \). In the first case, we have \( k_{j_1} = \|k\|, u_{j_1} = k_{j_1} + \rho = \|k\| + \rho \) and \( u_i = k_i \) for all \( i \in \mathbb{N} \setminus \{j_1\} \). In particular, we have \( u_{j_2} = k_{j_2} \) and \( |u_{j_2}| = |k_{j_2}| = \|k\| \). On the other hand, it is easy to see that \( \|u\| = \sup_{i \in \mathbb{N}} |u_i| = \|k\| + \rho \) and since \( j_2 \in I_+(\ell) \cup I_-(\ell) \) and \( u \in C(\ell) \), Proposition 5.6 implies that \( |u_{j_2}| = \|u\| = \|k\| + \rho \). This contradiction implies that \( u \notin C(\ell) \) and, therefore, \( k \notin \text{int} C(\ell) \). The case \( j_1 \in I_-(\ell) \) can be proved similarly. \( \square \)

**Example 5.1.** We provide two examples to illustrate Propositions 5.6 and 5.7.

(i) Let \( \ell = e_1 \). Then \( I_+(\ell) \cup I_-(\ell) = \{1\} \), and \( \text{int} C(\ell) \neq \emptyset \) as one can see in Proposition 4.3.
(ii) Let $\ell := (1/2, 1/4, 1/8, \ldots)$. Note that $\ell \in l_1$, $\|\ell\| = 1$ and $I_+(\ell) \cup I_-(\ell) = \mathbb{N}$. The corresponding BP cone $C(\ell)$ has the form $C(\ell) = \text{cone}\{e\}$ with $e := (1, 1, 1, \ldots)$. It is clear that $\text{int}C(\ell) = \emptyset$.

5.4. The Banach space $L^1_{[0,1]}$. Let us begin with some notation. To any function $f : [0,1] \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$, denote

\[
[f > a] := \{A \subseteq [0,1]: f(t) > a \text{ a.e. on } A \text{ and } f(t) \leq a \text{ a.e. on } [0,1] \setminus A\}.
\]

One can prove that $\text{mes } A_1 = \text{mes } A_2$ for all $A_1, A_2 \in [f > a]$ ($\text{mes}(\cdot)$ denotes the measure of a set). Similarly, we can define the families of sets $[f \geq a]$, $[f < a]$, $[f = a]$ or $[f \neq a]$.

Recall that the norm of $f \in L^1_{[0,1]}$ is given by

\[
\|f\| = \int_0^1 |f(t)| \, dt
\]

and the norm of $f \in L^\infty_{[0,1]}$ is given by

\[
\|f\| = \text{ess sup}_{t \in [0,1]} |f(t)| = \inf\{a \in \mathbb{R}: \text{mes } A = 0 \text{ for } A \in \{[f > a]\}\},
\]

where by $|f|$ we mean the function defined by $|f|(t) = |f(t)|$ for all $t \in [0,1]$. For $\ell \in L^\infty_{[0,1]}$ and $x \in L^1_{[0,1]}$, the dual pair is given by

\[
\ell(x) = \langle \ell, x \rangle = \int_0^1 \ell(t)x(t) \, dt.
\]

The space $L^1_{[0,1]}$ and its dual are neither reflexive nor strictly convex. Then for an arbitrary $\ell \in L^\infty_{[0,1]}$ with $\|\ell\| = 1$ the BP cone

\[
C(\ell) = \{x \in L^1_{[0,1]}: \ell(x) = \|x\|\}
\]

may be trivial, i.e., it contains only the zero element. Below we will establish necessary and sufficient conditions for this cone to be nontrivial.

For an arbitrary $\ell \in L^\infty_{[0,1]}$ with $\|\ell\| = 1$ we define the family $\mathcal{F}(\ell)$ of tuples $(I_+(\ell), I_-(\ell), I_0(\ell))$ of sets with the following properties

(i) $I_+(\ell), I_-(\ell), I_0(\ell) \subset [0,1]$
(ii) $\text{mes}(I_+(\ell) \cup I_-(\ell) \cup I_0(\ell)) = 1$
(iii) $\ell(t) = 1$ a.e. on $I_+(\ell)$,
\[
\ell(t) = -1 \text{ a.e. on } I_-(\ell),
\]
\[
|\ell(t)| < 1 \text{ a.e. on } I_0(\ell).
\]

The family $\mathcal{F}(\ell)$ is nonempty as we can take the sets $I_+(\ell) \in [\ell = 1]$, $I_-(\ell) \in [\ell = -1]$, $I_0(\ell) \in [\|\ell\| < 1]$, which satisfy the properties (i)-(iii) due to the definition of the sets $[\ell = 1]$, $[\ell = -1]$, $[\|\ell\| < 1]$ and the fact that for $\ell \in L^\infty_{[0,1]}$ with $\|\ell\| = 1$ one has $\text{mes } A = 1$ for any $A \in [\|\ell\| \leq 1]$. 
The following result provides a representation of a BP cone \(C(\ell)\) with \(\|\ell\| = 1\) in \(L_{[0,1]}^1\).

**Proposition 5.8.** Let \(\ell \in L_{[0,1]}^\infty\) with \(\|\ell\| = 1\) be given, and let \((I_+(\ell), I_-(\ell), I_0(\ell)) \in \mathcal{F}(\ell)\) be arbitrarily chosen.

(i) The BP cone \(C(\ell)\) coincides with the set
\[
U(\ell) := \{ x(t) \in L_{[0,1]}^1 : x(t) \geq 0 \text{ a.e. on } I_+(\ell), \ x(t) \leq 0 \text{ a.e. on } I_-(\ell), \ x(t) = 0 \text{ a.e. on } I_0(\ell) \}. \tag{22}
\]

(ii) The BP cone \(C(\ell)\) is nontrivial if and only if the set \(I_+(\ell) \cup I_-(\ell)\) has a positive measure.

Note that in the Lebesgue spaces \(L_{[0,1]}^1\) and \(L_{[0,1]}^\infty\), any function \(g\), which equals \(f\) a.e. on \([0,1]\) (i.e. \(f(t) = g(t)\) a.e. on \([0,1]\)), is identified as \(f\). Hence, the set \(U(\ell)\) does not depend on the choice of the tuple \((I_+(\ell), I_-(\ell), I_0(\ell))\) in \(\mathcal{F}(\ell)\).

**Proof.** (i) First, we show that \(U(\ell) \subseteq C(\ell)\). It is clear that the zero function belongs to \(U(\ell)\). Let \(\bar{x} \in U(\ell)\) be given. The definition of the set \(U(\ell)\) and the properties of the sets \(I_+(\ell), I_-(\ell)\) and \(I_0(\ell)\) imply that
\[
\ell(t)\bar{x}(t) = |\ell(t)\bar{x}(t)| = |\bar{x}(t)| \quad \text{a.e. on } I_+(\ell) \cup I_-(\ell)
\]
and \(\ell(t)\bar{x}(t) = 0 = |\bar{x}(t)|\) a.e. on \(I_0(\ell)\). Therefore, we have
\[
\|\bar{x}\| = \int_0^1 |\bar{x}(t)| \, dt = \int_{I_+(\ell) \cup I_-(\ell) \cup I_0(\ell)} |\bar{x}(t)| \, dt \\
= \int_{I_+(\ell) \cup I_-(\ell) \cup I_0(\ell)} \ell(t)\bar{x}(t) \, dt \\
= \int_0^1 \ell(t)\bar{x}(t) \, dt = \ell(\bar{x}),
\]
which implies that \(\bar{x} \in C(\ell)\).

Next, we show by contraposition that \(C(\ell) \subseteq U(\ell)\). Let \(\bar{x} \notin U(\ell)\). Then there exist sets \(I_+ \subseteq I_+(\ell), I_- \subseteq I_-(\ell)\) and \(I_0 \subseteq I_0(\ell)\) such that \(\text{mes}(I_+ \cup I_- \cup I_0) > 0\) and
\[
\ell(t)\bar{x}(t) = \bar{x}(t) < 0 < |\bar{x}(t)| \quad \text{a.e. on } I_+, \\
\ell(t)\bar{x}(t) = -\bar{x}(t) < 0 < |\bar{x}(t)| \quad \text{a.e. on } I_-,
\]
\[
0 = \ell(t)\bar{x}(t) < |\bar{x}(t)| \quad \text{a.e. on } I_0.
\]

Denote \(V := I_+ \cup I_- \cup I_0\). It follows that
\[
\ell(t)\bar{x}(t) = |\bar{x}(t)| \quad \text{a.e. on } (I_+(\ell) \cup I_-\ell) \cup I_0(\ell)) \setminus V,
\]
and
\[
\ell(t)\bar{x}(t) < |\bar{x}(t)| \quad \text{a.e. on } V.
\]
Then \(\bar{x} \notin C(\ell)\) because
\[
\|\bar{x}\| = \int_0^1 |\bar{x}(t)| \, dt = \int_{I_+ (\ell) \cup I_- (\ell) \cup I_0 (\ell)} |\bar{x}(t)| \, dt
\]
\[
= \int_{(I_+ (\ell) \cup I_- (\ell) \cup I_0 (\ell)) \setminus V} |\bar{x}(t)| \, dt + \int_V |\bar{x}(t)| \, dt
\]
\[
> \int_{(I_+ (\ell) \cup I_- (\ell) \cup I_0 (\ell)) \setminus V} \ell(t)|\bar{x}(t)| \, dt + \int_V \ell(t)|\bar{x}(t)| \, dt
\]
\[
= \int_{I_+ (\ell) \cup I_- (\ell) \cup I_0 (\ell)} \ell(t)|\bar{x}(t)| \, dt = \ell(\bar{x}).
\]

(ii) The “only if” part follows from the representation of the set \(U(\ell)\) and the equality \(C(\ell) = U(\ell)\). The “if” part follows from the fact that \(x(\cdot) \in C(\ell)\), where \(x\) is the nonzero function defined by \(x(t) = 1\) for \(t \in I_+ (\ell)\) and \(x(t) = -1\) for \(t \in I_- (\ell)\) and \(x(t) = 0\) for \(t \in [0, 1] \setminus (I_+ (\ell) \cup I_- (\ell))\).

\[\Box\]

**Remark 5.5.** When \(\ell \in L_1^{\infty}([0,1])\) is given by \(\ell(t) = 1\) a.e. on \([0,1]\), the representation (22) implies that \(C(\ell)\) is the nonnegative orthant of \(L_1^{0}([0,1])\).

**Proposition 5.9.** For every \(\ell \in L_1^{\infty}([0,1])\) with \(\|\ell\| = 1\) the BP cone \(C(\ell)\) has an empty interior.

**Proof.** Suppose to the contrary that there exists some \(k \in \text{int} C(\ell)\). Since \(\text{int} C(\ell) \cup \{0\}\) is a cone, we may assume, without loss of generality, that \(\|k\| = 1\). Let \(\rho \in [0,1]\) be a scalar such that \(B(k, \rho) \subset C(\ell)\). Let \((I_+ (\ell), I_- (\ell), I_0 (\ell)) \in \mathcal{F}(\ell)\) be arbitrarily chosen. We will consider separately the following two possible cases: (a) \(\text{mes}(I_+ (\ell) \cup I_- (\ell)) < 1\); (b) \(\text{mes}(I_+ (\ell) \cup I_- (\ell)) = 1\).

*Case (a).* In this case, we have \(\gamma := \text{mes} I_0 (\ell) > 0\). Recall that by Proposition 5.8, \(k(t) = 0\) a.e. on \(I_0 (\ell)\). Let \(u \in L_1^{1}([0,1])\) be the function defined by \(u(t) = \frac{1}{\gamma}\) a.e. on \(I_0 (\ell)\) and \(u(t) = 0\) a.e. on \([0, 1] \setminus I_0 (\ell)\). Since
\[
\int_0^1 |u(t)| \, dt = \int_{I_0 (\ell)} |u(t)| \, dt + \int_{[0,1] \setminus I_0 (\ell)} |u(t)| \, dt = \int_{I_0 (\ell)} \frac{1}{\gamma} \, dt = \frac{1}{\gamma} \gamma = 1,
\]
it follows that \(k - \rho u \in B(k, \rho)\). Since \((k - \rho u)(t) = -\frac{\rho}{\gamma} < 0\) a.e. on \(I_0 (\ell)\), Proposition 5.8 implies that \(k - \rho u \notin C(\ell)\), a contradiction to \(B(k, \rho) \subset C(\ell)\).

*Case (b).* Since \(1 = \|k\| = \int_0^1 |k(t)| \, dt\), there is some set \(I \subset [0,1]\) with positive measure such that
\[
|k(t)| \leq 2 \text{ a.e. on } I.
\]
Therefore, the equality \(\text{mes}(I_+ (\ell) \cup I_- (\ell)) = 1\) implies that at least one of the inequalities \(\text{mes}(I \cap I_+ (\ell)) > 0\) and \(\text{mes}(I \cap I_- (\ell)) > 0\) holds.

Assume first that \(\text{mes}(I \cap I_+ (\ell)) > 0\). Choose a measurable set \(J\) such that \(J \subset I \cap I_+ (\ell)\) and \(0 < \xi := \text{mes} J < \frac{\rho}{4}\). Let \(u \in L_1^{1}([0,1])\) be the function defined by \(u(t) = \frac{3}{\rho}\) a.e. on \(J\) and...
\[ u(t) = \frac{1}{1-\xi}(1 - \frac{3\xi}{\rho}) \text{ a.e. on } [0, 1] \setminus J. \] Note that \( 1 - \xi > 0 \) because \( \xi < \frac{\xi}{4} \leq \frac{1}{4} \) and \( 1 - \frac{3\xi}{\rho} > 0 \) because \( \frac{3\xi}{\rho} < \frac{3}{4} \). Further, since

\[
\|u\| = \int_0^1 |u(t)| \, dt = \int_J |u(t)| \, dt + \int_{[0,1] \setminus J} |u(t)| \, dt
\]

\[
= \frac{3}{\rho} \text{mes}(J) + \frac{1}{1-\xi} \left(1 - \frac{3\xi}{\rho}\right) \text{mes}([0,1] \setminus J)
\]

\[
= \frac{3\xi}{\rho} + \frac{1}{1-\xi} \left(1 - \frac{3\xi}{\rho}\right)(1 - \xi) = 1,
\]

it follows that \( k - \rho u \in B(k, \rho) \). Moreover, \( \rho u(t) = 3 \) a.e. on \( J \). Observe that for the function \( k - \rho u \), we have \( (k - \rho u)(t) = k(t) - \rho u(t) = k(t) - 3 \) a.e. on \( J \). Since \( J \subset I \), it follows that \( (k - \rho u)(t) < 0 \) a.e. on \( J \) and since \( J \subset I_+(\ell) \) Proposition 5.8 implies that \( k - \rho u \notin C(\ell) \), a contradiction to \( B(k, \rho) \subset C(\ell) \).

The case \( \text{mes}(I \cap I_-(\ell)) > 0 \) can be considered similarly. \( \square \)

5.5. The Banach space \( L^\infty_{[0,1]} \). It is known that the dual space of \( L^\infty_{[0,1]} \) contains \( L^1_{[0,1]} \) but does not coincide with the latter. Using the Hahn-Banach extension theorem, one can show that there are bounded functionals \( \phi \) on \( L^\infty_{[0,1]} \), which are not of the form

\[
\phi(x) = \int_0^1 \ell(t)x(t) \, dt \quad \text{for all } x \in L^\infty_{[0,1]}
\]

for some \( \ell \in L^1_{[0,1]} \) and these functionals cannot be given explicitly. Hence, we restrict ourselves to a BP cone \( C(\ell) \) with \( \|\ell\| = 1 \) in \( L^\infty_{[0,1]} \) only for \( \ell \in L^1_{[0,1]} \). Note that our result also implies the existence of a bounded linear functional \( \ell : L^\infty_{[0,1]} \to \mathbb{R} \), which does not belong to \( L^1_{[0,1]} \).

In analogy to Subsection 5.4 for an arbitrary \( \ell \in L^1_{[0,1]} \) with \( \|\ell\| = 1 \) we consider the set-families \( [\ell > 0] \), \( [\ell < 0] \) and \( [\ell = 0] \) and the family \( \mathcal{F}(\ell) \) of tuples \((I_+(\ell), I_-(\ell), I_0(\ell))\).

Observe that since \( \|\ell\| = 1 \), it follows that

\[
\text{mes}(I_+(\ell) \cup I_-(\ell)) > 0. \quad (23)
\]

**Proposition 5.10.** Let \( \ell \in L^1_{[0,1]} \) with \( \|\ell\| = 1 \) be given, and let \((I_+(\ell), I_-(\ell), I_0(\ell)) \in \mathcal{F}(\ell) \) be arbitrarily chosen. Then the BP cone \( C(\ell) \) is nontrivial and it coincides with the set

\[
U(\ell) := \{x \in L^\infty_{[0,1]} : x(t) = \|x\| \text{ a.e. on } I_+(\ell), x(t) = -\|x\| \text{ a.e. on } I_-(\ell)\}.
\]

**Proof.** Note that Proposition 5.4 implies that the cone \( C(\ell) \) is nontrivial.
First we show that $U(\ell) \subset C(\ell)$. Let $x \in U(\ell)$. Observe that $x(t)\ell(t) = \|x\|\ell(t)$ a.e. on $I_+^{1}(\ell) \cup I_-^{1}(\ell)$ and $x(t)\ell(t) = \|x\|\ell(t)$ a.e. on $I_0(\ell)$. Hence, we have
\[
\ell(x) = \int_0^1 x(t)\ell(t) \, dt = \int_{I_+^{1}(\ell) \cup I_-^{1}(\ell) \cup I_0(\ell)} x(t)\ell(t) \, dt
\]
\[= \int_{I_+^{1}(\ell) \cup I_-^{1}(\ell)} \|x\|\ell(t) \, dt + \int_{I_0(\ell)} \|x\|\ell(t) \, dt
\]
\[= \|x\| \int_0^1 \ell(t) \, dt = \|x\| \|\ell\| = \|x\|.
\]
This means that $x \in C(\ell)$ and hence, $U(\ell) \subset C(\ell)$.

Next, we show by contraposition that $C(\ell) \subset U(\ell)$. Take an arbitrary $x \notin U(\ell)$. Then one can find a measurable set $J \subset I_+^{1}(\ell) \cup I_-^{1}(\ell)$ such that $\text{mes} J > 0$ and $|x(t)| < \|x\|$ a.e. on $J$. Therefore, we have $|x(t)\ell(t)| < \|x\|\ell(t)$ a.e. on $J$. Then $x \notin C(\ell)$ because
\[
\ell(x) = \int_0^1 x(t)\ell(t) \, dt = \int_J x(t)\ell(t) \, dt + \int_{[0,1] \setminus J} x(t)\ell(t) \, dt
\]
\[\leq \int_J |x(t)\ell(t)| \, dt + \int_{[0,1] \setminus J} |x(t)\ell(t)| \, dt
\]
\[< \int_J \|x\|\ell(t) \, dt + \int_{[0,1] \setminus J} \|x\|\ell(t) \, dt
\]
\[= \|x\| \int_0^1 \ell(t) \, dt = \|x\| \|\ell\| = \|x\|.
\]
\[
\square
\]

Example 5.2. (i) Let $\ell \in L_{[0,1]}^{1}$ be the function defined by $\ell(t) = 0.5$ a.e. on $[0,0.5]$ and $\ell(t) = -1.5$ a.e. on $[0.5,1]$. Then $\|\ell\| = 1$. The inequality (23) holds and Proposition 5.10 implies that $C(\ell)$ is nontrivial. Namely, we have for some scalar $c \geq 0$

\[
C(\ell) = \{x \in L_{[0,1]}^{\infty} \mid x(t) = c \text{ a.e. on } [0,0.5], x(t) = -c \text{ a.e. on } [0.5,1]\}.
\]

(ii) Let $K$ be the set of additivity of the norm presented in Example 3.1, (iv). By Theorem 3.1, there exists a functional $\ell \in (L_{[0,1]}^{\infty})^*$ with $\|\ell\| = 1$ such that $K \subset C(\ell)$. Observe that for each function $u_i$ ($i \in \mathbb{N}$), which belongs to $K$, one has $u_i(t) < \|u_i\| = 1$ for all $t \in [0,1]$. Then Proposition 5.10 implies that $K$ cannot be included in a BP cone $C(\ell)$ with $\ell \in L_{[0,1]}^{1}$. As a byproduct, this fact implies that the dual of $L_{[0,1]}^{\infty}$ is broader than $L_{[0,1]}^{1}$.

**Proposition 5.11.** Suppose that $\ell \in L_{[0,1]}^{1}$ with $\|\ell\| = 1$. Then the BP cone $C(\ell)$ has an empty interior.

**Proof.** Suppose to the contrary that there exist $k \in \text{int } C(\ell)$ and $\rho \in [0,1]$ such that $B(k, \rho) \subset C(\ell)$. Without loss of generality, we may assume that $\|k\| = 1$.

By (23), at least one of the inequalities $\text{mes } I_+^{1}(\ell) > 0$ and $\text{mes } I_-^{1}(\ell) > 0$ holds. Consider first the case $\text{mes } I_+^{1}(\ell) > 0$. Let $J \subset I_+^{1}(\ell)$ be a measurable set such that $0 < \text{mes } J < \text{mes } I_+^{1}(\ell)$. Then we also have $\text{mes}(I_+^{1}(\ell) \setminus J) > 0$. Denote by $v \in L_{[0,1]}^{\infty}$ the function defined by: $v(t) = 1$ a.e. on $J$, $v(t) = -1$ a.e. on $I_+^{1}(\ell) \setminus J$ and $v(t) = 0$ a.e. in $[0,1] \setminus I_+^{1}(\ell)$. It is clear that $\|v\| = 1$. Let $u := k + \rho v$. Then $u \in B(k, \rho)$ and therefore, $u \in C(\ell)$. On the other hand, observe that $u(t) = k(t) + \rho v(t) = \|k\| + \rho$ a.e. on $J$, $u(t) = k(t) + \rho v(t) = \|k\| - \rho$ a.e. on
\(I_+(\ell) \setminus J\) because \(k \in C(\ell)\) and \(k(t) = \|k\|\) on \(I_+(\ell)\) by Proposition 5.10. This means that \(u\) takes two different values \(\|k\| + \rho\) and \(\|k\| - \rho\) on two measurable subsets of positive measure of \(I_+(\ell)\) and Proposition 5.10 implies that \(u \notin C(\ell)\). Thus, we obtain a contradiction. The case \(\text{mes } I_-(\ell) > 0\) can be considered similarly.

5.6. The Banach space \(C_{[0,1]}\). The Banach space \(C_{[0,1]}\) consists of continuous functions \(x : [0, 1] \to \mathbb{R}\) with the max norm: for any \(x \in C_{[0,1]}\)

\[
\|x\| = \max_{t \in [0,1]} |x(t)|.
\]

The dual space of \(C_{[0,1]}\) can be identified with the linear space of finite signed Radon measures on \([0, 1]\). In this case a continuous linear functional on \(C_{[0,1]}\) is associated with a unique Radon measure on \([0, 1]\). The space \(C_{[0,1]}\) and its dual are neither reflexive nor strictly convex. Here, we restrict ourselves to the case when \(\ell\) is the “Dirac mass” functional, because the corresponding BP cone is simpler to be represented.

**Proposition 5.12.** Let \(\ell\) be the “Dirac mass” functional \(\delta_{t_0} (t_0 \in [0, 1])\) defined by: for \(x \in C_{[0,1]}\)

\[
\delta_{t_0}(x) = x(t_0).
\]

Then \(\|\delta_{t_0}\| = 1\) and the corresponding BP cone \(C(\ell)\) has the form

\[
C(\ell) = \{x \in C_{[0,1]} : x(t_0) = \|x\|\}.
\]

Note that in the case \(t_0 = 1\), we get the BP cone \(C(\ell)\) mentioned in Example 2.2, (c). We omit the proof of Proposition 5.12.

**Remark 5.6.** The research carried out in Sections 4 and 5 allows us to conclude that BP cones given by an equation with a nonempty interior exist in the Banach spaces \(l_\infty, c_0\) and \(c\) (Lorentz cones in these spaces are examples of such cones) but they do not exist in the Banach spaces \(l_p (1 \leq p < \infty)\) and \(L^p_{[0,1]} (1 \leq p < \infty)\). We do not know whether there exist such cones in the Banach spaces \(L^\infty_{[0,1]}\) and \(C_{[0,1]}\) or not.

6. Applications

In this last section we present some short applications in optimal control and approximation in infinite dimensional spaces.

6.1. Application to optimal control problems. Let us consider an optimal control problem with a control satisfying a positive constraint; the objective and the system of differential equations are not explicitly specified. But we assume that every control \(u \in L^\infty_{[0,T]}\) for some final time \(T > 0\) is positive in the sense that for some \(\ell \in L^1_{[0,T]}\) with \(\|\ell\| = 1\) the condition \(u \in C(\ell)\) holds for the associated BP cone \(C(\ell)\).
Proposition 5.10 can be easily extended from functions defined on $[0, 1]$ to those on $[0, T]$. The constraint $u \in C(\ell)$ then means $u \in U(\ell)$ for every tuple $(I_+(\ell), I_-(\ell), I_0(\ell)) \in \mathcal{F}(\ell)$. Consequently, these positive controls have the property

$$u(t) = \begin{cases} \|u\| & \text{a.e. on } I_+(\ell) \\ -\|u\| & \text{a.e. on } I_-(\ell). \end{cases}$$

(24)

This is some type of a bang-bang principle known from optimal control (e.g. compare [21, 31]). Although a bang-bang principle in its original form is an optimality condition, the variant given in (24) is a result of the positivity constraint and an objective is not required. This result shows that positivity of an $L^\infty_{[0,T]}$ control in the aforementioned sense is already a strong requirement in optimal control.

6.2. **Application to approximation problems.** In a strictly convex Banach space $X$ let an arbitrary element $\hat{x} \in X \setminus \{0\}$ and an arbitrary continuous linear functional $\ell \in X^*$ with $\|\ell\| = 1$ be given. Let $C(\ell)$ denote the associated BP cone in $X$. Then we consider the approximation problem with positivity constraint

$$\inf_{x \in C(\ell)} \|x - \hat{x}\|.$$  

(25)

By Proposition 3.1 there is some $\bar{x} \in X$ with $\|\bar{x}\| = 1$ and $C(\ell) = \text{cone}\{\bar{x}\}$. Hence, every $x \in C(\ell)$ can be written in the form $x = \lambda \bar{x}$ for some $\lambda \geq 0$, and the approximation problem (25) is equivalent to the problem

$$\inf_{\lambda \geq 0} \|\lambda \bar{x} - \hat{x}\|.$$  

(26)

Problem (26) is an optimization problem with only one real variable and, therefore, it is simple to solve.

In addition, let $X$ be a Hilbert space. Then it is a well-known optimality condition that $\bar{x}$ and $\lambda \bar{x} - \hat{x}$ (for the optimal point $\lambda \bar{x}$) are perpendicular (as a result of [6, Prop. 2.1]). Consequently, we have with $\langle \bar{x}, \bar{x} \rangle = \|\bar{x}\|^2 = 1$

$$0 = \langle \bar{x}, \lambda \bar{x} - \hat{x} \rangle = \lambda \langle \bar{x}, \bar{x} \rangle - \langle \bar{x}, \hat{x} \rangle = \lambda - \langle \bar{x}, \hat{x} \rangle$$

So, we conclude $\lambda = \langle \bar{x}, \hat{x} \rangle$ and the element $x = \langle \bar{x}, \hat{x} \rangle \bar{x}$ solves the approximation problem (25) in a Hilbert space $X$.

**Conclusion**

This paper answers an open question concerning BP cones given by an equation in Banach spaces. Key statements refer to properties of these cones, a characterization of the strict convexity of the space, generalized Lorentz cones in sequence spaces and representations of these cones in concrete infinite dimensional Banach spaces. The investigations reveal the rich mathematical structure of these cones. The benefit of this theory is that the specific
structure of BP cones given by an equation is now known for many sequence and function spaces used in applications.

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