Fixed point and coincidence problems of set-valued mappings via regularity in metric spaces

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Abstract In this paper, by constructing a common iterative scheme, we unify two important results given recently by Ioffe [13], [14] and Mansour, Bahraoui and Bekkali [20] on the fixed point theorem and approximate fixed point theorem by using a regular property called “orbital regularity” of set-valued mappings suggested both in complete metric spaces and in non-complete metric spaces. Some applications to (approximate) coincidence/ double fixed point problems as well as the stability of (approximate) Milyutin regularity of set-valued mappings in metric spaces are given.

Keywords Orbital regularity · orbital pseudo-Lipschitz · (approximate) fixed point · coincidence/double fixed point · iterate scheme · approximate/ Milyutin regularity

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1 Introduction

Fixed point problem of mappings between metric spaces is one of important problems of mathematics: “Find a point $x$ belonging to a metric space $X$ such that

$$x = f(x),$$

where $f$ maps from $X$ to itselfs”. In other words, among elements of $X$, find a special point which is invariant under acting by $f$. The well-known result about this topic is Banach-Caccioppoli’s fixed point theorem ([2], [3]) saying that a mapping $f$ maps from a complete metric space $X$ to itselfs which is contractive, i.e., there is a constant $\kappa \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \kappa d(x, y), \quad \forall x, y \in X$$

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then it has a unique fixed point in $X$. This result is a crucial tool in metric spaces. It provides a constructive method to find those fixed points of mappings defined in metric spaces. It has many important applications in various areas of mathematics and in other sciences. A standard application is the proof of the Picard-Lindelöf theorem about the existence and uniqueness of solutions to certain ordinary differential equations [17]. The sought solution of the differential equation is expressed as a fixed point of a suitable integral operator which transforms continuous functions into continuous functions. Banach’s fixed-point theorem is then used to show that this integral operator has a unique fixed point. One consequence of the Banach fixed-point theorem is that small Lipschitz perturbations of the identity are bi-Lipschitz homeomorphisms. A direct consequence of this result yields the proof of the inverse function theorem. Besides, it can be also used to give sufficient conditions under which Newton’s method of successive approximations is guaranteed to work, and similarly for Chebyshev’s third order method. Proving existence and uniqueness of solutions to integral equations or giving a proof to the Nash embedding theorem [9] were also made use it. Additionally, it can be used to prove existence and uniqueness of solutions to value iteration, policy iteration, and policy evaluation of reinforcement learning [19] and so on. Day by day, this result was interested and studied in many different directions. The interest reader can refer many results about this topic, for instance, in [1], [18], [16], [8], and so on.

Because of many practical requirements, many mathematical problems transfer to find fixed points of set-valued mappings, i.e., find a point $x \in X$ such that

$$x \in F(x)$$

with $F : X \rightrightarrows X$ be set-valued mapping. The famous theorem on this problem is Nadler’s multi-valued theorem [21]: “Let $X$ be a complete metric space and suppose that $F$ maps $X$ into the set of closed subsets of $X$ and is Lipschitz continuous in the sense of Pompeiu-Hausdorff distance on $X$ with Lipschitz constant $\lambda \in (0, 1)$, i.e.,

$$H(F(x), F(y)) \leq \lambda d(x, y), \quad \forall x, y \in X.$$  

Then $F$ has a fixed point.” This theorem provides an important tool in solving the inclusions, generalized equations or differential inclusions. Since its important role, in last years, many directions of study on this topic have explored; mainly concentrating on the regularity/Lipschitzness of the considered mapping $F$ and/or the properties of the primal space $X$.

Recently, Ioffe [15], [13], [14], [11], [12] used Milyutin regularity of a set-valued mapping on a fixed set to give the existence of the fixed point of the set-valued mapping on the considered set and estimate the distance from a given point to the fixed point set via known data. Due to the orbital regularity is weaker than the Lipschitz continuity, so Ioffe’s result is an extension of Nadler’s result on an arbitrary set of $X$. Technique was used this work of Ioffe to be the well-known Ekeland variational principle [7]. Another one was separately given recently by Mansour, Bahraoui and Bekkali [20], by using the global metric regularity of set-valued mapping and an iterative technique, they obtained the existence of approximate fixed points of set-valued mappings defined on metric space without using its completeness.
Observed from these two results, we construct a common iterate which unifies these ones in the sense that when $X$ is a metric space (to be not complete) one gets the result by Mansour, Bahraboui and Bekkali; and when $X$ is a complete metric space, we reach Ioffe’s result. Using the obtained results, we obtain important results on existence (approximate) double fixed/ coincidence points of a pair of set-valued mappings; simultaneously, we also get the result on the stability of (approximate) Milyutin regularity given recently by Ioffe ([15], Theorem 2.78) and Tron, Han, Ngai [23] in metric spaces both complete and non-complete ones.

The organization of the paper is as follows. In Section 2, we give the definition of the orbital regularity as well as its equivalent versions, and then we establish the relationship between them. In the third section, we construct a general iterative scheme to ensure the existence of fixed points and to give an estimate of the distance from it through the original data under the assumption on the orbital regularity of considered set-valued mappings. Our results unify two recent ones given separately by Ioffe [13], [15] and Mansour, Bahraboui and Bekkali [20] and it made to weaken some important results existed in the literature, for instance, Nadler’s multi-valued fixed point theorem and its corollaries. In the last one, using the obtained results, we derive the results on (approximate) double fixed/ coincidence theorems of two set-valued mappings as well as the stability of (approximate) Milyutin regularity given recently by Ioffe ([15], Theorem 2.78) and Tron, Han, Ngai [23] in complete metric spaces. Our results also establish for the case of non-complete of the considered spaces.

2 Orbital regularity and equivalences

For a set-valued mapping, we recall some regular properties which is well-known in the literature: $\gamma$-regularity, $\gamma$-openness, and $\gamma$-pseudo-Lipschitzness.

**Definition 1** Let $F : X \rightarrow X$ be a set-valued mapping between metric spaces, $U$ be a subset of $X$ and $\gamma : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a gauge function which is positive on $U$. $F$ is said $\gamma$-regular on $U$ with modulus $\tau$ if

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x))$$

for all $x, y \in U$ with $0 < \tau d(y, F(x)) < \gamma(x)$. We denote $\operatorname{reg}_\gamma F(U)$ the lowest bound of $\tau$. If no such $\tau$ exists, we set $\operatorname{reg}_\gamma F(U) = +\infty$. We call $\operatorname{reg}_\gamma F(U)$ the Milyutin regular modulus of $F$ on $U$.

**Definition 2** $F$ is said $\gamma$-pseudo-Lipschitz on $U$ with modulus $\theta$ if

$$x \in U, y \in F(u), \theta d(x, u) < \gamma(x) \Rightarrow d(y, F(x)) \leq \theta d(u, x).$$

We denote $\operatorname{lip}_\gamma F(U)$ the lowest bound of $\tau$. If no such $\tau$ exists, we set $\operatorname{lip}_\gamma F(U) = +\infty$. We call $\operatorname{lip}_\gamma F(U)$ the $\gamma$-pseudo-Lipschitz modulus of $F$ on $U$.

**Definition 3** $F$ is said $\gamma$-open on $U$ with modulus $\tau$ if

$$x, y \in U, d(y, F(x)) < \tau t < \gamma(x) \Rightarrow y \in F(B(x, t)).$$

We denote $\operatorname{sur}_\gamma F(U)$ the upper bound of $\tau$. If no such $\tau$ exists, we set $\operatorname{sur}_\gamma F(U) = -\infty$. We call $\operatorname{sur}_\gamma F(U)$ the $\gamma$-surjectivity modulus of $F$ on $U$. 

These three properties have a close relationship: 
\( F \) is \( \gamma \)-regular on \( U \) with modulus \( \tau \) if and only if \( F \) is \( \gamma \)-open on \( U \) with modulus \( \gamma^{-1} \) if and only if \( F \) is \( \gamma \)-pseudo-Lipschitz on \( U \) with modulus \( \tau \).

The reader is interested in these properties as well as their relationships can refer to recent works of Ioffe, for instance, [11], [12], [15], [13], [14].

In three definitions above, by taking \( \gamma(x) = d(x, X \setminus U) \) one gets concepts of Milyutin regularities. Milyutin regularity is one of the most important nonlinear regular model in variational analysis.

**Definition 4** Let \( F : X \rightrightarrows X \) be a set-valued mapping between metric spaces. \( F \) is said Milyutin regular on \( U \) with modulus \( \tau \) if

\[
d(x, F^{-1}(y)) \leq \tau d(y, F(x))
\]

for all \( x, y \in U \) with \( 0 < \tau d(y, F(x)) < m(x) := d(x, X \setminus U) \). We denote \( \text{reg}_M F(U) \) the lowest bound of \( \tau \). If no such \( \tau \) exists, we set \( \text{reg}_M F(U) = +\infty \). We call \( \text{reg}_M F(U) \) the Milyutin regular modulus of \( F \) on \( U \).

**Definition 5** \( F \) is said Milyutin pseudo-Lipschitz on \( U \) with modulus \( \theta \) if

\[
x \in U, y \in F(u), \theta d(x, u) < m(x) \Rightarrow d(y, F(x)) \leq \theta d(u, x).
\]

We denote \( \text{lip}_M F(U) \) the lowest bound of \( \tau \). If no such \( \tau \) exists, we set \( \text{lip}_M F(U) = +\infty \). We call \( \text{lip}_M F(U) \) the Milyutin pseudo-Lipschitz modulus of \( F \) on \( U \).

**Definition 6** \( F \) is said Milyutin open on \( U \) with modulus \( \tau \) if

\[
x, y \in U, d(y, F(x)) < rt, t < m(x) \Rightarrow y \in F(B(x, t))
\]

We denote \( \text{sur}_M F(U) \) the upper bound of \( \tau \). If no such \( \tau \) exists, we set \( \text{sur}_M F(U) = -\infty \). We call \( \text{sur}_M F(U) \) the Milyutin surjectivity modulus of \( F \) on \( U \).

As observed by Ioffe in his works [11], [12], [15], [13], [14] that for the fixed point problem of set-valued mappings, we do not really need the whole metric regularity, we only use a weaker version of it called the orbital regularity now given below. It means that we consider the regularity on a fixed set in which one variable changes only.

**Definition 7** Let \( F : X \rightrightarrows X \) be a set-valued mapping between metric spaces. \( F \) is said orbital regular on \( U \) with modulus \( \tau \) if

\[
d(x, F^{-1}(x)) \leq \tau d(x, F(x))
\]

for all \( x \in U \) with \( 0 < \tau d(x, F(x)) < m(x) := d(x, X \setminus U) \). We denote \( \text{reg}_O F(U) \) the lowest bound of \( \tau \). If no such \( \tau \) exists, we set \( \text{reg}_O F(U) = +\infty \). We call \( \text{reg}_O F(U) \) the orbital regular modulus of \( F \) on \( U \).

**Definition 8** \( F \) is said orbital open on \( U \) with modulus \( \tau \) if

\[
x \in U, d(x, F(x)) < rt, t < m(x) \Rightarrow x \in F(B(x, t))
\]

Denote by \( \text{sur}_O F(U) \) the upper bound of \( \tau \). If no such \( \tau \) exists, we set \( \text{sur}_O F(U) = -\infty \). We call \( \text{sur}_O F(U) \) the orbital surjectivity/openness modulus of \( F \) on \( U \).
Definition 9 $F$ is said orbital pseudo-Lipschitz on $U$ with modulus $\theta$ if

$$x \in U, x \in F(u), \theta d(x, u) < m(x) \Rightarrow d(x, F(x)) \leq \theta d(u, x).$$

We denote $\operatorname{lip}_{O} F(U)$ the lowest bound of $\tau$. If no such $\tau$ exists, we set $\operatorname{lip}_{O} F(U) = +\infty$. We call $\operatorname{lip}_{O} F(U)$ the orbital pseudo-Lipschitz modulus of $F$ on $U$.

Three properties above are equivalent as given in the proposition below.

Proposition 1 (Equivalences) Let $F : X \rightrightarrows X$ be a set-valued mapping from metric space to itself and $U \subset X$. The following three statements are equivalent:

(a) $F$ is orbital regular on $U$ with modulus $\tau$.
(b) $F$ is orbital open on $U$ with modulus $\tau^{-1}$.
(c) $F^{-1}$ is orbital pseudo-Lipschitz on $U$ with modulus $\tau$.

Proof $\bullet$ (a) $\Rightarrow$ (b). Suppose that $F$ is orbital regular on $U$ with modulus $\tau$. We take now $x \in U$ such that $d(x, F(x)) < \tau^{-1} t, t < m(x)$. It implies $\tau d(x, F(x)) < m(x)$. By the orbital regularity of $F$, one gets the estimate $d(x, F^{-1}(x)) \leq \tau d(x, F(x)) < m(x)$. Hence, we can choose a point $u \in F^{-1}(x)$, i.e., $x \in F(u)$ satisfying $d(x, u) < t < m(x)$. This means $x \in F(B(x, t))$. In other words, $F$ is orbital open on $U$ with modulus $\tau^{-1}$.

$\bullet$ (b) $\Rightarrow$ (c). Take $x \in U, x \in F^{-1}(u)$ (i.e., $u \in F(x)$) such that $\tau^{-1} d(x, u) < m(x)$. It follows $\tau^{-1} d(x, F(x)) \leq \tau^{-1} d(x, u) < \tau m(x)$. From this, taking an arbitrary positive real $t$ such that $\tau^{-1} d(x, F(x)) \leq \tau^{-1} d(x, u) < t < m(x)$. Thus, by the orbital openness of $F$, one has $x \in F(B(x, \tau t))$. So, there exists $v \in B(x, t)$ with $x \in F(v)$. It derives that $d(x, F^{-1}(x)) \leq d(x, v) < t$. Since $t$ is arbitrarily close $\tau^{-1} d(x, u)$, we get that $d(x, F^{-1}(x)) \leq \tau^{-1} d(x, u)$. That is $F^{-1}$ is orbital pseudo-Lipschitz on $U$ with modulus $\tau$.

$\bullet$ (c) $\Rightarrow$ (a). Let $x \in U$ be such that $0 < \tau d(x, F(x)) < m(x)$. Thus, we can find a point $u \in F(x)$ such that $\tau d(x, F(x)) \leq \tau d(x, u) < m(x)$. It follows by the orbital pseudo-Lipschitzness of $F^{-1}$ that one has $d(x, F^{-1}(x)) \leq \tau d(u, x)$. It results that $d(x, F^{-1}(x)) \leq \tau d(x, F^{-1}(x))$. The proof is completed.

3 Orbital regularity and fixed point problem in metric space

Given $\varepsilon \geq 0$, the $\varepsilon$-neighborhood of a subset $A$ of metric space $X$ is defined by

$$\varepsilon - A = \{ x \in X | d(x, A) \leq \varepsilon \}.$$  

Let $F : X \rightrightarrows X$ be a set-valued mapping. A point $x \in X$ is called to be a $\varepsilon$-fixed point of $F$ if $d(x, F(x)) \leq \varepsilon$. When $\varepsilon = 0$ and $F$ has closed-valued (i.e., $F(x)$ is closed in $X$ for every $x \in X$), this notion reduces to the fixed point of $F$, i.e., $x \in F(x)$. The sets of all of such $\varepsilon$-fixed points and fixed points are denoted by $\varepsilon - \operatorname{Fix}(F)$ and $\operatorname{Fix}(F)$, respectively and defined by

$$\varepsilon - \operatorname{Fix}(F) := \{ x \in X : d(x, F(x)) \leq \varepsilon \}, \operatorname{Fix}(F) := \{ x \in X : x \in F(x) \}.$$  

Obviously, $0 - \operatorname{Fix}(F) = \operatorname{Fix}(F)$ if $F$ has closed-valued.
\textbf{Theorem 1} Let $X$ be a metric space, $F : X \rightrightarrows X$ be a set-valued mapping. Let further $U$ be an open subset of $X$. Suppose that there are $\bar{x} \in U$ and $\theta \in (0, 1)$ such that $F$ is orbital regular on $U$ with $\reg(F(U)) < \theta$. Under these conditions, if
\begin{equation}
  d(\bar{x}, F^{-1}(\bar{x})) < (1 - \theta) d(\bar{x}, X \setminus U),
\end{equation}
then
\begin{enumerate}[(a)]
  \item for every $\varepsilon > 0$, $\varepsilon - \Fix(F) \cap U \neq \emptyset$ and
  \begin{equation}
    d(\bar{x}, \varepsilon - \Fix(F)) \leq \frac{d(\bar{x}, F^{-1}(\bar{x}))}{1 - \theta};
  \end{equation}
  \item suppose further that $X$ is complete and $F : X \to X$ has closed graph, i.e., its graph is a closed set in the product space $X \times Y$. Then, $\Fix(F) \cap U \neq \emptyset$ and
  \begin{equation}
    d(\bar{x}, \Fix(F)) \leq \frac{d(\bar{x}, F^{-1}(\bar{x}))}{1 - \theta}.
  \end{equation}
\end{enumerate}

\textbf{Proof} Let $l > \frac{d(\bar{x}, F^{-1}(\bar{x}))}{1 - \theta}$. With the hypoteses of the theorem, by induction, we shall construct a sequence $\{x_n\}$ in $X$ with $x_0 := \bar{x}$ satisfying the following properties for $n \geq 1$:
\begin{align}
  x_n &\in F^{-1}(x_{n-1}), x_n \in U, \quad (2) \\
  d(x_n, x_{n-1}) &< \min\{\theta^{n-1}(1 - \theta) d(\bar{x}, X \setminus U), l \theta^{n-1}(1 - \theta)\}. \quad (3)
\end{align}

The fact $l > \frac{d(\bar{x}, F^{-1}(\bar{x}))}{1 - \theta}$, namely $d(\bar{x}, F^{-1}(\bar{x})) < l(1 - \theta)$ along with the assumption (1) give us
\begin{equation}
  d(\bar{x}, F^{-1}(\bar{x})) < \min\{(1 - \theta) d(\bar{x}, X \setminus U), l(1 - \theta)\}. \quad (4)
\end{equation}

Based on this estimation, we choose a point $x_1 \in F^{-1}(\bar{x})$ such that
\begin{equation}
  d(\bar{x}, x_1) < \min\{(1 - \theta) d(\bar{x}, X \setminus U), l(1 - \theta)\}. \quad (5)
\end{equation}

And then, $d(\bar{x}, x_1) < (1 - \theta) d(\bar{x}, X \setminus U) < d(\bar{x}, X \setminus U)$, and hence $x_1 \in U$. Furthermore, because of $\bar{x} \in F(x_1)$ and the relation (5) one gets the estimation
\begin{align*}
  d(x_1, F(x_1)) &\leq d(x_1, \bar{x}) \\
  &< \min\{(1 - \theta) d(\bar{x}, X \setminus U), l(1 - \theta)\} \\
  &< (1 - \theta) d(\bar{x}, X \setminus U).
\end{align*}

It implies that
\begin{align}
  \theta d(x_1, F(x_1)) &\leq \theta d(x_1, \bar{x}) \quad (7) \\
  &< \theta (1 - \theta) d(\bar{x}, X \setminus U) \quad (8) \\
  &\leq \theta (1 - \theta) (d(x_1, \bar{x}) + d(x_1, X \setminus U)). \quad (9)
\end{align}
Hence, \( \theta d(x_1, \bar{x}) \leq (1 - \theta)d(x_1, X \setminus U) \) and
\[
\theta d(x_1, F(x_1)) \leq (1 - \theta)d(x_1, X \setminus U) < d(x_1, X \setminus U).
\]
Based on this inequality and the orbital regularity of \( F \), one gets the estimate
\[
d(x_1, F^{-1}(x_1)) \leq \theta d(x_1, F(x_1)).
\]
(10)
Using this and (6), one derives that
\[
d(x_1, F^{-1}(x_1)) \leq \theta d(x_1, F(x_1)) \leq \theta d(x_1, \bar{x})
\leq \min \{ \theta(1 - \theta)d(\bar{x}, X \setminus U), l\theta(1 - \theta) \},
\]
and thus we can take \( x_2 \in F^{-1}(x_1) \) such that
\[
d(x_2, x_1) \leq \theta d(x_1, F(x_1)) \leq \theta d(x_1, \bar{x})
\leq \min \{ \theta(1 - \theta)d(\bar{x}, X \setminus U), l\theta(1 - \theta) \}.
\]
From the relation, it follows that
\[
d(x_2, \bar{x}) \leq d(x_2, x_1) + d(x_1, \bar{x})
\leq \theta(1 - \theta)d(\bar{x}, X \setminus U) + (1 - \theta)d(\bar{x}, X \setminus U) < d(\bar{x}, X \setminus U),
\]
which implies \( x_2 \in U \). In addition,
\[
d(x_2, F(x_2)) \leq d(x_2, x_1) < \min \{ \theta(1 - \theta)d(\bar{x}, X \setminus U), l\theta(1 - \theta) \}.
\]
Suppose that one has already determined the sequence \( (x_k) \) satifying (2)-(3) for \( k = 2, 3, ..., n \) for some \( n > 1 \). Accordingly,
\[
d(x_n, F(x_n)) \leq d(x_n, x_{n-1}) < \min \{ \theta^{n-1}(1 - \theta)d(\bar{x}, X \setminus U), l\theta^{n-1}(1 - \theta) \} < d(\bar{x}, X \setminus U)
\]
which implies by the orbital regularity of \( F \)
\[
d(x_n, F^{-1}(x_n)) \leq \theta d(x_n, F(x_n)).
\]
From here one chooses \( x_{n+1} \in F^{-1}(x_n) \) such that
\[
d(x_n, x_{n+1}) \leq \theta d(x_n, F(x_n)) < \min \{ \theta^n(1 - \theta)d(\bar{x}, X \setminus U), l\theta^n(1 - \theta) \} < d(\bar{x}, X \setminus U).
\]
This also gives us that \( x_{n+1} \in U \). In conclusion, we constructed the sequence \( (x_n) \) satifying (2)-(3) for all \( n \in \mathbb{N} \).

With the properties of sequence \( (x_n) \), it results for all \( m, n \in \mathbb{N} \) with \( m > n \) that
\[
d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1})
\leq \sum_{i=n}^{m-1} \left( \min \{ \theta^i(1 - \theta)d(\bar{x}, X \setminus U), l\theta^i(1 - \theta) \} \right)
= \left( \min \{ \theta^n(1 - \theta^{m-n+1})d(\bar{x}, X \setminus U), l\theta^n(1 - \theta^{m-n+1}) \} \right).
\]
(11)
Let $\varepsilon > 0$. Taking $n = 0$ in the estimate, one sees that
\[
d(x_m, \bar{x}) < \min \{ (1 - \theta^{m+1})d(\bar{x}, X \setminus U), l(1 - \theta^{m+1}) \} \tag{12}
\]
\[
< \min \{ d(\bar{x}, X \setminus U), l \}. \tag{13}
\]
Since for $m$ large enough, $x_m \in \varepsilon - \text{Fix}(F)$, $x_m \in U$ and $d(\bar{x}, \varepsilon - \text{Fix}(F)) \leq d(x_m, \bar{x}) < l$. Let $l$ tend to $\frac{d(\bar{x}, F^{-1}(\bar{x}))}{1 - \theta}$, one concludes that
\[
d(\bar{x}, \varepsilon - \text{Fix}(F)) \leq \frac{d(\bar{x}, F^{-1}(\bar{x}))}{1 - \theta}.
\]
This proves (a).

(b) If $X$ is complete then by the estimate (11), the sequence $(x_n)$ is a Cauchy one, thus there exists an element $u \in X$ such that $x_n$ converging to $u$. By passing the limit in (2), (12) and (13) as $n, m$ tending to infinity, one obtains that $u \in F(u)$ and $d(u, \bar{x}) < \min \{ d(\bar{x}, X \setminus U), l \}$. The last relation implies $u \in U$ and $d(u, \bar{x}) < l$. Thus, $\text{Fix}(F) \cap U \neq \emptyset$ and $d(\bar{x}, \text{Fix}(F)) \leq d(u, \bar{x}) < l$. Since $l$ is a positive real which is arbitrarily close to $\frac{d(\bar{x}, F^{-1}(\bar{x}))}{1 - \theta}$, it leads to the estimate
\[
d(\bar{x}, \text{Fix}(F)) \leq \frac{d(\bar{x}, F^{-1}(\bar{x}))}{1 - \theta}.
\]
This establishes (b) and the theorem is proved.

**Theorem 2** Let $X$ be a metric space, $G : X \rightrightarrows X$ be a set-valued mapping. Let further $U$ be an open subset of $X$. Assume that there are $\bar{x} \in U$ and $\theta \in (0, 1)$ such that $G$ is orbital pseudo-Lipschitz on $U$ with $\text{lip}_{G} U \setminus U < \theta$ and
\[
d(\bar{x}, G(\bar{x})) < (1 - \theta)d(\bar{x}, X \setminus U). \tag{14}
\]
Then,

(a) for every $\varepsilon > 0$, $\varepsilon - \text{Fix}(G) \cap U \neq \emptyset$ and
\[
d(\bar{x}, \varepsilon - \text{Fix}(G)) \leq \frac{d(\bar{x}, G(\bar{x}))}{1 - \theta}.
\]

(b) If $X$ is complete and $G : X \rightarrow X$ has closed-graph, then $\text{Fix}(G) \cap U \neq \emptyset$ and
\[
d(\bar{x}, \text{Fix}(G)) \leq \frac{d(\bar{x}, G(\bar{x}))}{1 - \theta}.
\]

**Proof** Let now $l$ be an arbitrary positive number such that $l > \frac{d(\bar{x}, G(\bar{x}))}{1 - \theta}$, with the hypotheses of the theorem, with a similar argument as in Theorem 1, we also construct a sequence $(x_n)$ with $x_0 := \bar{x}$ satisfying the following properties for $n \geq 1$:
\[
x_n \in G(x_{n-1}), x_n \in U, \tag{15}
\]
\[
d(x_n, x_{n-1}) < \min \{ \theta^{n-1}(1 - \theta)d(\bar{x}, X \setminus U), l\theta^{n-1}(1 - \theta) \}. \tag{16}
\]
Accordingly, for all \( m,n \in \mathbb{N} \) with \( m > n \) that

\[
d(x_n,x_m) \leq \sum_{i=n}^{m-1} d(x_i,x_{i+1})
\]

\[
< \sum_{i=n}^{m-1} \left( \min\left\{ \theta^i(1-\theta)d(\bar{x},X \setminus U),l\theta^i(1-\theta) \right\} \right)
\]

\[
= \left( \min\left\{ \theta^n(1-\theta^{m-n+1})d(\bar{x},X \setminus U),l\theta^n(1-\theta^{m-n+1}) \right\} \right). \quad (17)
\]

(a) Taking \( n=0 \) in the estimate, one receives that

\[
d(x_m,\bar{x}) < \min\{1-\theta^m\}d(\bar{x},X \setminus U),l(1-\theta^m) \}
\]

\[
< \min\{d(\bar{x},X \setminus U),l\}. \quad (18)
\]

Since for \( m \) large enough, \( x_m \in e-\text{Fix}(G) \), one results that \( e-\text{Fix}(G) \cap U \neq \emptyset \) and

\[
d(\bar{x},e-\text{Fix}(G)) \leq d(x_m,\bar{x}) < l.
\]

Due to the arbitrary closeness of \( l \) to \( d(\bar{x},G^{-1}(\bar{x})) \), it turns out

\[
d(\bar{x},e-\text{Fix}(G)) \leq \frac{d(\bar{x},G^{-1}(\bar{x}))}{1-\theta}. \quad (19)
\]

(b) If \( X \) is complete then by the estimate (17), the sequence \( (x_n) \) is a Cauchy one, thus there is an element \( u \in X \) such that \( x_n \) converging to \( u \). Let \( n \) tend to \( \infty \) in (15), one gets that \( u \in G(u) \) and let \( m \) tend to \( \infty \) in (18) and (19), one obtains

\[
d(u,\bar{x}) < \min\{d(\bar{x},X \setminus U),l\}. \quad (18)
\]

The latter implies \( u \in U \) and \( d(u,\bar{x}) < l \). Thus, \( \text{Fix}(G) \cap U \neq \emptyset \) and \( d(\bar{x},\text{Fix}(G)) \leq d(u,\bar{x}) < l \). Since \( l \) is arbitrarily close to \( d(\bar{x},G^{-1}(\bar{x})) \), the desired inequality is established

\[
d(\bar{x},\text{Fix}(G)) \leq \frac{d(\bar{x},G^{-1}(\bar{x}))}{1-\theta}.
\]

The proof is complete.

From two theorems above, one establishes a general result as follows when the set-valued mapping satisfies one of three properties: orbital regularity, orbital openness, and orbital pseudo-Lipschitzness.

**Theorem 3** If \( F \) is orbital regularity or orbital openness and fulfills the assumption (1); or, if \( F \) is orbital pseudo-Lipschitz and satisfies the assumption (14) then we have the same conclusions (a) and (b) as in Theorem 1 and 2.

In the special case of \( U = B(\bar{x},b) \) and \( m(x) = b-d(x,\bar{x}) \) of Theorem 2, one obtains the following result which is a weaker version of Theorem 5E.2 (6], see also in [5]; in turn, it is an extension of Nadler’s multi-valued contraction theorem [21] which is well known in the literature.
Corollary 1 Let $X$ be a metric space, let $F : X \rightrightarrows X$ be a set-valued mapping, $\bar{x} \in X$, a positive number $b$ be given. Assume that

(i) $F$ is orbital pseudo-Lipschitz on $U$ with $\text{lip}_O(G)(U|U) < \kappa < 1$;
(ii) $d(\bar{x}, F(\bar{x})) < b(1 - \kappa)$.

Then,

(a) for every $\varepsilon > 0$, $\varepsilon - \text{Fix}(F) \cap U \neq \emptyset$ and

$$d(\bar{x}, \varepsilon - \text{Fix}(F)) \leq \frac{1}{1 - \kappa} d(\bar{x}, F(\bar{x})).$$

(b) Moreover, if $X$ is complete and $F$ has closed-graph then $\text{Fix}(F) \cap U \neq \emptyset$ and

$$d(\bar{x}, \text{Fix}(F)) \leq \frac{1}{1 - \kappa} d(\bar{x}, F(\bar{x})).$$

From this corollary by taking $U \equiv X$, we can obtain the following interesting result which is well-known in the literature.

Corollary 2 Let $X$ be a metric space, let $F : X \rightrightarrows X$ be a set-valued mapping and $\bar{x} \in X$. Assume that $F$ is contractive on $X$ with constant $\kappa$. Then,

(a) for every $\varepsilon > 0$, $\varepsilon - \text{Fix}(F) \neq \emptyset$ and

$$d(\bar{x}, \varepsilon - \text{Fix}(F)) \leq \frac{1}{1 - \kappa} d(\bar{x}, F(\bar{x}));$$

(b) moreover, if $X$ is complete and $F$ has closed-graph then $\text{Fix}(F) \neq \emptyset$ and

$$d(\bar{x}, \text{Fix}(F)) \leq \frac{1}{1 - \kappa} d(\bar{x}, F(\bar{x})).$$

Note that the conclusion (a) in Corollary 2 is the main result in the recent work of Mansour, Bahraoui and Bekkali ([20], Theorem 4), and the conclusion (b) is Nadler’s well-known theorem [21].

Below we give a result on the solution of fixed point problem of the product mapping

$$F = (F_1, F_2, \ldots, F_n),$$

where $F_i : X_i \rightrightarrows Y_i$ are set-valued mappings between metric spaces $X_i, Y_i$ for $i = 1, 2, \ldots, n$.

Proposition 2 Let $F_i : X_i \rightrightarrows Y_i$ be set-valued mappings between metric spaces $X_i, Y_i$ and $U_i$ be open subsets of $X_i$ for $i = 1, 2, \ldots, n$. If $F_i$ is orbital Milyutin regular on $U_i$ with modulus $\tau_i$ for $i = 1, 2, \ldots, n$ then the product $F = (F_1, F_2, \ldots, F_n)$ is also orbital Milyutin regular on $\prod_{i=1}^n U_i$ with modulus $\tau = \max \{ \tau_i : i = 1, 2, \ldots, n \}$ for the distance $d$ in the product space $\prod_{i=1}^n X_i$ defined by

$$d((x_i), (y_i)) = \max_{i=1,\ldots,n} d_i(x_i, y_i)$$

for $(x_i), (y_i) \in \prod_{i=1}^n X_i$ with Milyutin function $\rho(x_i) = \min_{i=1,\ldots,n} d_i(x_i, X_i \setminus U_i)$.
Proof Let \( x = (x_i) \in \prod_{i=1}^n U_i \) be such that \( \tau d(x, F(x)) < \rho(x_i) \). The latter implies
\[
\tau \max_{i=1,\ldots,n} d_i(x_i, F_i(x_i)) < \min_{i=1,\ldots,n} d_i(x_i, X_i \setminus U_i).
\]
It results that
\[
\tau d_i(x_i, F_i(x_i)) \leq \tau d_i(x_i, F_i(x_i)) < d_i(x_i, X_i \setminus U_i), \quad \text{for } i = 1, 2, \ldots, n.
\]
By the orbibal Milyutin regularities of \( F_i \), one gets the estimates
\[
d_i(x, F_i^{-1}(x_i)) \leq \tau d_i(x_i, F_i(x_i)) \quad \text{for } i = 1, 2, \ldots, n.
\]
Consequently,
\[
\max_{i=1,\ldots,n} d_i(x, F_i^{-1}(x_i)) \leq \tau \max_{i=1,\ldots,n} d_i(x_i, F_i(x_i)).
\]
Namely,
\[
d(x, F^{-1}(x)) \leq \tau d(x, F(x)).
\]
This derives the conclusion of the theorem. The proof is complete.

Theorem 4 With the assumptions as in Proposition 2, if there is a \( \tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \in \prod_{i=1}^n X_i \) such that
\[
d(\tilde{a}, F^{-1}(\tilde{a})) < (1 - \max_{i=1,\ldots,n} \tau_i) \rho(\tilde{a}),
\]
where the distance \( d \) defined as (20). Then,
(a) for every \( \varepsilon > 0 \), \( \varepsilon - \text{Fix}(F) \neq \emptyset \) and
\[
d(\tilde{a}, \varepsilon - \text{Fix}(F)) \leq \frac{1}{(1 - \max_{i=1,\ldots,n} \tau_i)} d(\tilde{a}, F^{-1}(\tilde{a})),
\]
(b) moreover, if \( X_i \) is complete and \( F_i \) has closed-graph for \( i = 1, 2, \ldots, n \) then \( \text{Fix}(F) \neq \emptyset \), i.e., \( \text{Fix}(F_i) \neq \emptyset \), for \( i = 1, 2, \ldots, n \) and
\[
d(\tilde{a}, \text{Fix}(F)) \leq \frac{1}{(1 - \max_{i=1,\ldots,n} \tau_i)} d(\tilde{a}, F^{-1}(\tilde{a})).
\]

4 Regularity and coincidence/double fixed point problems

In this part, we shall study coincidence and double/fixed point problems of set valued mappings via their regularities mentioned above. Let \( G : X \rightrightarrows Y, H : Y \rightrightarrows X \) be set-valued mappings. Our first aim is to find a pair \((\hat{x}, \hat{y})\) \( \in X \times Y \) such that \( \hat{x} \in H(\hat{y}) \) and \( \hat{y} \in G(\hat{x}) \), i.e., to look for a double fixed point of pair \((G, H)\). The set of all such points is said \((G, H)\) and is defined by
\[
\text{DFix}(G, H) := \{ (x, y) \in X \times Y : y \in G(x), x \in H(y) \}.
\]
The second one is to look for some point \( \hat{x} \) belonging to the coincidence set of two set-valued mappings \( T : X \rightrightarrows Y, S : X \rightrightarrows Y \), that is the set
\[
\text{Coin}(T, S) := \{ x \in X : T(x) \cap S(x) \neq \emptyset \}.
\]
Besides, we also give some estimate from a given point to these sets via known data. Concretely, to solve these problems, one explores the regularities of the Descartes product mapping of $G$ and $H$ denoted by $G \times H$ and defined by

$$(G \times H)(x_1, x_2) := F(x_1) \times G(x_2) = \{(y_1, y_2) \in Y_1 \times Y_2 : y_1 \in G(x_1), y_2 \in H(x_2)\}.$$ 

Afterwards applying the obtained results in Section 3 to establish the desired purposes.

The propositions below establish metric regularity and pseudo Lipschitzness of product mapping through the ones of component mappings.

**Proposition 3** Let $X, Y$ be metric spaces, $F_1 : X \rightrightarrows Y, F_2 : Y \rightrightarrows X$ be set-valued mappings and $U, U'$ be open sets in $X, V, V'$ be open sets in $Y$. For any $x, v$, set $\gamma(x) = d_X(x, X \setminus U), \delta(v) = d_Y(v, Y \setminus V')$. Consider the product mapping $T : (x, v) \rightrightarrows (F_1(x), F_2(v))$ and assume that there exist $r > 0, s > 0$ such that

(i) $F_1$ is $(\gamma, r)$-metrically regular on $U \times V$,

(ii) $F_2$ is $(\delta, s)$-metrically regular on $V' \times U'$.

Then, $T$ is $(\rho_*, (\sqrt{rs}))$-metrically regular on $(U \times V') \times (V \times U')$ for the distance $d_*$ defined by

$$d_*((x, v), (u, y)) = \max \left\{1/\sqrt{\gamma}(x, u), 1/\sqrt{s}\delta(v, y)\right\}$$

for any $(x, v), (u, y) \in X \times Y$ with

$$\rho_*(x, v) = \min \left\{1/\sqrt{\gamma}(x), 1/\sqrt{s}\delta(v)\right\}.$$ 

**Proof** Take $(x, v) \in U \times V', (u, y) \in U' \times V$ such that

$$d_*((y, u), T(x, v)) < 1/\sqrt{rs} \rho_*(x, v).$$

It means that

$$\max \left\{\frac{1}{\sqrt{\gamma}} d_X(u, F_2(v)), \frac{1}{\sqrt{s}} d_Y(y, F_1(x))\right\} < \frac{1}{\sqrt{rs}} \min \left\{\frac{1}{\sqrt{\gamma}} \gamma(x), \frac{1}{\sqrt{s}} \delta(v)\right\},$$

which follows that

$$d_X(u, F_2(v)) < s^{-1} \delta(v), \quad \text{and} \quad d_Y(y, F_1(x)) < r^{-1} \gamma(x).$$

By the metric regularity of $F_1, F_2$, we have

$$d_X(x, F^{-1}_1(y)) \leq r d_Y(y, F_1(x)), \quad \text{and} \quad d_Y(v, F^{-1}_2(u)) \leq s d_X(u, F_2(v)),$$

which imply that

$$d_*((x, v), T^{-1}(u, y)) = \max \left\{1/\sqrt{\gamma} d_X(x, F^{-1}_1(y)), 1/\sqrt{s} d_Y(v, F^{-1}_2(u))\right\} \leq \max \left\{1/\sqrt{r} d_Y(y, F_1(x)), 1/\sqrt{s} d_X(u, F_2(v))\right\} = \sqrt{rs} \max \left\{1/\sqrt{r} d_Y(y, F_1(x)), 1/\sqrt{s} d_X(u, F_2(v))\right\} = \frac{1}{\sqrt{rs}} = d_*((u, y), T(x, v)).$$

This is the desired conclusion. The proposition is proved.
Remark 1 The conclusion of the proposition also holds if the distance $d_s$ is replaced by the $d_*$ defined by

$$d_*(x, y) = \frac{1}{\sqrt{r}}d_X(x, u) + \frac{1}{\sqrt{s}}d_Y(v, y)$$

(22)

with the same gauge function $\rho_*$. 

Proposition 4 With the assumptions as in Proposition 3, if

(i) $F_1$ is $(\delta, r)$-pseudo Lipschitz on $U \times V$,
(ii) $F_2$ is $(\gamma, s)$-pseudo Lipschitz on $V' \times U'$,

then, $T$ is $(\rho^*, (\sqrt{rs}))$-pseudo Lipschitz on $(U \times V') \times (V \times U')$ for the distance $d^*$ defined by

$$d^*((x, v), (u, y)) = \max \{\sqrt{r}d_X(x, u), \sqrt{s}d_Y(v, y)\}$$

(23)

for any $(x, v), (u, y) \in X \times Y$ with

$$\rho^*(x, v) = \min \{\sqrt{r}\gamma(x), \sqrt{s}\delta(v)\}.$$ 

Proof Take $(x, v), (x', v') \in U \times V', (y, u) \in T(x', v') \cap (V \times U')$ such that

$$\sqrt{rs}d^*((x', v'), (x, v)) < \rho^*(y, u).$$

That is

$$\max \{\sqrt{r}d_X(x', x), \sqrt{s}d_Y(v', v)\} < (\sqrt{rs})^{-1} \min \{\sqrt{r}\gamma(x), \sqrt{s}\delta(y)\},$$

which results

$$rd_X(x', x) < \delta(y), \quad \text{and} \quad sd_Y(v', v) < \gamma(u).$$

By the pseudo Lipschitzness of $F_1, F_2$,

$$d_Y(y, F_1(x)) \leq rd_X(x', x), \quad \text{and} \quad d_X(u, F_2(v)) \leq sd_Y(v', v),$$

one receives

$$d^*((y, u), T(x, v)) = \max \{\sqrt{r}d_X(u, F_2(v)), \sqrt{s}d_Y(y, F_1(x))\}$$

$$\leq \max \{\sqrt{r}sd_Y(v', v), \sqrt{s}rd_X(x', x)\}$$

$$= \sqrt{rs} \max \{\sqrt{s}d_Y(v', v), \sqrt{r}d_X(x', x)\}$$

$$= \sqrt{rs}d^*((x', v'), (x, v)).$$

This is the conclusion and hence the proof ends.

Remark 2 The conclusion of the proposition does not change if the distance $d^*$ is replaced by the $d^{**}$ defined by

$$d^{**}((x, v), (u, y)) = \sqrt{r}d_X(x, u) + \sqrt{s}d_Y(v, y)$$

(24)

with the same gauge function $\rho^*$. 

Theorem 5  Let $X, Y$ be metric spaces, $F_1 : X \rightrightarrows Y, F_2 : Y \rightrightarrows X$ be set-valued mappings and $U, U'$ be open sets in $X, V, V'$ be open sets in $Y$. For any $x, v$, set $\gamma(x) = d_Y(x, X \setminus U), \delta(v) = d_Y(v, Y \setminus V')$. Assume that there exist $r > 0, s > 0$ with $rs < 1$ such that

(a) $F_1$ is $(γ, r)$-metrically regular on $U \times V$,
(b) $F_2$ is $(δ, s)$-metrically regular on $V' \times U'$,
(c) there is $(\hat{u}, \hat{y}) \in U \times V'$ such that

\[
\max \left\{ \frac{1}{\sqrt{r}}d(\hat{u}, F_1^{-1}(\hat{y})), \frac{1}{\sqrt{s}}d(\hat{y}, F_2^{-1}(\hat{u})) \right\} < (1 - \sqrt{rs}) \min \left\{ \frac{1}{\sqrt{r}}γ(\hat{u}), \frac{1}{\sqrt{s}}δ(\hat{y}) \right\}.
\]

then,

(i) for every $ε > 0$, $ε - \text{Fix}(F_1, F_2) \cap [U \times V'] \neq \emptyset$ and

\[
d_*((\hat{u}, \hat{y}), ε - \text{Fix}(F_1, F_2)) \leq \frac{d_*(((\hat{u}, \hat{y}), (F_1^{-1}, F_2^{-1}))(\hat{u}, \hat{y}))}{1 - \sqrt{rs}}.
\]

(ii) furthermore, if $X, Y$ are complete and $F_1, F_2$ have closed graphs then $\text{Fix}(F_1 \times F_2) \cap [U \times V'] \neq \emptyset$ and

\[
d_*((\hat{u}, \hat{y}), \text{Fix}(F_1, F_2)) \leq \frac{d_*(((\hat{u}, \hat{y}), (F_1^{-1}, F_2^{-1}))(\hat{u}, \hat{y}))}{1 - \sqrt{rs}}.
\]

Consequently, there is a pair $(\hat{u}, \hat{y}) \in U \times V'$ satisfying $\hat{u} \in F_2(\hat{y}), \hat{y} \in F_1(\hat{u})$ and

\[
d_*((\hat{u}, \hat{y}), (\hat{u}, \hat{y})) \leq \frac{d_*(((\hat{u}, \hat{y}), (F_1^{-1}, F_2^{-1}))(\hat{u}, \hat{y}))}{1 - \sqrt{rs}}.
\]

Proof  Let $T := (F_1, F_2)$. Then, by Proposition 3, $T$ is $(ρ_*, (√rs))$-metrically regular on $[U \times V'] \times (V \times U')$. Moreover, the condition (c) means that

\[
d_*((\hat{u}, \hat{y}), (T^{-1}(\hat{u}, \hat{y}))) < (1 - √rs) ρ_*(\hat{u}, \hat{y}).
\]

We are now ready to apply Theorem 1 for the map $T$, one gets the conclusion of the theorem. The proof ends.

Theorem 6  Let $X, Y$ be metric spaces, $F_1 : X \rightrightarrows Y, F_2 : Y \rightrightarrows X$ be set-valued mappings and $U, U'$ be open sets in $X, V, V'$ be open sets in $Y$. For any $x, v$, set $\gamma(x) = d_Y(x, X \setminus U), \delta(v) = d_Y(v, Y \setminus V')$. Assume that there exist $r > 0, s > 0$ with $rs < 1$ such that

(a) $F_1$ is $(δ, r)$-pseudo Lipschitz on $U \times V$,
(b) $F_2$ is $(γ, s)$-pseudo Lipschitz on $V' \times U'$,
(c) there is $(\hat{u}, \hat{y}) \in U \times V'$ such that

\[
\max \left\{ \sqrt{r}d(\hat{u}, F_2(\hat{y})), \sqrt{s}d(\hat{y}, F_1(\hat{u})) \right\} < (1 - \sqrt{rs}) \min \left\{ \sqrt{r}γ(\hat{u}), \sqrt{s}δ(\hat{y}) \right\}.
\]

Then,

(i) for every $ε > 0$, $ε - \text{Fix}(F_1, F_2) \cap [U \times V'] \neq \emptyset$ and

\[
d^*(((\hat{u}, \hat{y}), ε - \text{Fix}(F_1, F_2)) \leq \frac{d^*((((\hat{u}, \hat{y}), (F_1, F_2))(\hat{u}, \hat{y}))}{1 - \sqrt{rs}};
\]

...
(ii) if $X, Y$ are complete and $F_1, F_2$ have closed graphs then $\text{Fix}(F_1, F_2) \cap [U \times V'] \neq \emptyset$ and

$$d^*(\overline{[\hat{u}, \hat{y}]}, \text{Fix}(F_1, F_2)) \leq \frac{d^*(\overline{[\hat{u}, \hat{y}]}, (F_1, F_2)(\hat{u}, \hat{y}))}{1 - \sqrt{r_s}}.$$  

Consequently, there is a pair $(\hat{u}, \hat{y}) \in U \times V'$ satisfying $\hat{u} \in F_2(\hat{y}), \hat{y} \in F_1(\hat{u})$ and

$$d^*((\hat{u}, \hat{y}), (\hat{u}, \hat{y})) \leq \frac{d^*((\hat{u}, \hat{y}), (F_1, F_2)(\hat{u}, \hat{y}))}{1 - \sqrt{r_s}}.$$  

Proof Let $T := F_1 \times F_2$. Then, by Proposition 4, $T$ is $(\gamma^*, (\sqrt{r_s}))$-pseudo Lipschitz on $(U \times V') \times (V \times U')$ for the distance $d^*$. Moreover, the condition (c) means that

$$d^*((\hat{u}, \hat{y}), T(\hat{u}, \hat{y})) < (1 - \sqrt{r_s}) \rho^*(\hat{u}, \hat{y}).$$  

We now apply Theorem 2 for the map $T$, one gets the conclusion of the theorem. The proof is completed.

Results below establish the existence of approximate/coincidence points of set-valued mappings.

**Theorem 7** Let $X, Y$ be metric spaces, $F_1 : X \rightrightarrows Y, F_2 : X \rightrightarrows Y$ be set-valued mappings and $U, U'$ be open sets in $X, V, V'$ be open sets in $Y$. For any $x, y$, set $\gamma(x) = d_\delta(x, X \setminus U), \delta(v) = d_\gamma(y, Y \setminus V')$. Consider the mapping $G : X \times Y \rightrightarrows X \times Y$ defined by $G(x, y) = (F_1, F_2^{-1})(x, y) = F_1(x) \times F_2^{-1}(y)$. Assume that there exist $r > 0, s > 0$ and $rs < 1$ such that

(a) $F_1$ is $(\gamma, r)$-metrically regular on $U \times V$,
(b) $F_2$ is $(\delta, s)$-pseudo Lipschitz on $U' \times V'$,
(c) there is $(\hat{u}, \hat{y}) \in U \times V'$ such that

$$d_\gamma((\hat{u}, \hat{y}), G^{-1}(\hat{y}, \hat{u})) < (1 - \sqrt{r_s}) \rho_\delta(\hat{u}, \hat{y}).$$  

Then,

(i) for every $\varepsilon > 0$, $\varepsilon - \text{Fix}(G) \cap U \neq \emptyset$ and

$$d_\gamma((\hat{u}, \hat{y}), \varepsilon - \text{Fix}(G)) \leq \frac{d_\gamma((\hat{u}, \hat{y}), G^{-1}(\hat{y}, \hat{u}))}{1 - \sqrt{r_s}}.$$  

(ii) if $X, Y$ are complete and $F_1, F_2$ have closed graphs then $\text{Coin}(F_1, F_2) \cap U \neq \emptyset, \text{Coin}(F_1^{-1}, F_2^{-1}) \cap V' \neq \emptyset$ and

$$d_\gamma((\hat{u}, \hat{y}), \text{Coin}(F_1, F_2) \times \text{Coin}(F_1^{-1}, F_2^{-1})) \leq \frac{d_\gamma((\hat{u}, \hat{y}), G^{-1}(\hat{u}, \hat{y}))}{1 - \sqrt{r_s}}.$$  

As a result,

$$d_X(\hat{u}, \text{Coin}(F_1^{-1}, F_2^{-1})) \leq \frac{\sqrt{r}}{1 - \sqrt{r_s}}d_\gamma((\hat{u}, \hat{y}), G^{-1}(\hat{u}, \hat{y})), $$

and,

$$d_Y(\hat{y}, \text{Coin}(F_1, F_2)) \leq \frac{\sqrt{s}}{1 - \sqrt{r_s}}d_\gamma((\hat{u}, \hat{y}), G^{-1}(\hat{u}, \hat{y})).$$
Proof Invoking Theorem 5 for the map \( G \), one gets the conclusion. The proof is finished.

**Theorem 8** Let \( X, Y \) be metric spaces, \( F_1 : X \rightrightarrows Y, F_2 : X \rightrightarrows Y \) be set-valued mappings and \( U, U' \) be open sets in \( X, V, V' \) be open sets in \( Y \). For any \( x, v \), set \( \gamma(x) = d_X(x, X \setminus U), \delta(v) = d_Y(v, Y \setminus V') \). Consider the mapping \( H : X \times Y \rightrightarrows X \times Y \) defined by \( H(x, y) = (F_1^{-1}, F_2)(x, y) = F_1^{-1}(x) \times F_2(y) \). Assume that there exist \( r > 0, s > 0 \) and \( rs < 1 \) such that

(a) \( F_1 \) is \((\gamma, r)\)-metrically regular on \( U \times V \),
(b) \( F_2 \) is \((\delta, s)\)-pseudo Lipschitz on \( U' \times V' \),
(c) there is \((\bar{u}, \bar{y}) \in U \times V' \) such that

\[
d^*(((\bar{u}, \bar{y}), H(\bar{u}, \bar{y}))) < (1 - \sqrt{rs}) p^*(\bar{u}, \bar{y}),
\]

then,

(i) for every \( \varepsilon > 0 \), \( \varepsilon - \text{Fix}(H) \cap U \neq \emptyset \)

\[
d^*((\bar{u}, \bar{y}), \varepsilon - \text{Fix}(H)) \leq \frac{d^*((\bar{u}, \bar{y}), H(\bar{u}, \bar{y}))}{1 - \sqrt{rs}}.
\]

(ii) if \( X, Y \) are complete and \( F, G \) have closed graphs then \( \text{Coin}(F_1, F_2) \cap U \neq \emptyset, \text{Coin}(F_1^{-1}, F_2^{-1}) \cap V' \neq \emptyset \)

\[
d^*((\bar{u}, \bar{y}), \text{Coin}(F_1, F_2) \times \text{Coin}(F_1^{-1}, F_2^{-1})) \leq \frac{d^*((\bar{u}, \bar{y}), H(\bar{u}, \bar{y}))}{1 - \sqrt{rs}}.
\]

Accordingly,

\[
dx(\bar{u}, \text{Coin}(F_1^{-1}, F_2^{-1})) \leq \frac{1}{\sqrt{r}(1 - \sqrt{rs})} dx(\bar{u}, F_2(\bar{y})),
\]

and,

\[
dy(\bar{y}, \text{Coin}(F_1, F_2)) \leq \frac{1}{\sqrt{s}(1 - \sqrt{rs})} dy(\bar{y}, F_1(\bar{u})).
\]

Proof Applying Theorem 6 for the mapping \( H : X \times Y \rightrightarrows X \times Y \) we obtain the conclusion. The proof finishes.

**Remark 3** The conclusion of Theorems 5, 8 does not change if the distance \( d_* \) is replaced by \( d_+ \) as defined in (22) and similarly, the one Theorems 6, 7 also holds if the distance \( d^* \) is replaced by \( d^*_r \) as defined in (24).

Below we give an interesting corollary of Theorem 6 on the existence of double fixed points of set-valued mappings when we take open subsets in the hypotheses of this theorem being open balls. Corollaries of Theorem 5, 7 and 8 were also followed in the same way.

**Corollary 3** Let \( X, Y \) be metric spaces, \( F_1 : X \rightrightarrows Y, F_2 : Y \rightrightarrows X \) be set-valued mappings and let \( x, u \in X, \bar{x}, \bar{y} \in Y \), positive reals \( \alpha_1, \alpha_2, \beta_1, \beta_2 \). Set \( U = B(\bar{x}, \alpha_1), U' = B(\bar{u}, \alpha_2), V = B(\bar{y}, \beta_1), \text{ and } V' = B(\bar{v}, \beta_2) \). For any \( x, v \), set \( \gamma(x) = (\alpha_1 - d(x, \bar{x}))^+, \delta(v) = (\beta_1 - d(v, \bar{v}))^+ \). Assume that there exist \( r > 0, s > 0 \) with \( rs < 1 \) such that
(a) $F_1$ is $(\delta,r)$-pseudo Lipschitz on $U \times V$.
(b) $F_2$ is $(\gamma,s)$-pseudo Lipschitz on $V' \times U'$.
(c) $\sqrt{r_{d}(\bar{x},F_2(\bar{v}))} + \sqrt{s_{d}(\bar{v},F_1(\bar{x}))} < (1 - \sqrt{r_{s}^2})\min\{\sqrt{r_{a_1}},\sqrt{s_{B_2}}\}$.

Then,

(i) for every $\varepsilon > 0$, $\varepsilon = \text{Fix}(F_1, F_2) \cap [U \times V'] \neq \emptyset$ and

$$d_+(((\bar{u},\bar{y}),\varepsilon - \text{Fix}(F_1, F_2))) \leq \frac{\sqrt{r_{d}(\bar{x},F_2(\bar{v}))} + \sqrt{s_{d}(\bar{v},F_1(\bar{x}))}}{1 - \sqrt{r_{s}^2}}.$$

(ii) if $X,Y$ are complete and $F_1, F_2$ have closed graphs then $\text{Fix}(F_1, F_2) \cap [U \times V'] \neq \emptyset$ and

$$d_+((\bar{x},\bar{y}), \text{Fix}(F_1, F_2)) \leq \frac{\sqrt{r_{d}(\bar{x},F_2(\bar{v}))} + \sqrt{s_{d}(\bar{v},F_1(\bar{x}))}}{1 - \sqrt{r_{s}^2}}.$$

Consequently, there is a pair $(\bar{u}, \bar{y}) \in U \times V'$ satisfying $\bar{u} \in F_2(\bar{y}), \bar{y} \in F_1(\bar{u})$ and

$$d_+(((\bar{u},\bar{y}), (\bar{u},\bar{y}))) \leq \frac{\sqrt{r_{d}(\bar{x},F_2(\bar{v}))} + \sqrt{s_{d}(\bar{v},F_1(\bar{x}))}}{1 - \sqrt{r_{s}^2}}.$$

5 Application to Milyutin’s theorem on a fixed set in metric spaces

This section is devoted to give an important application of fixed point theorems established as above in Milyutin’s theorems. Besides results given recently in works [23], [13] and [15], we obtain also approximate results in noncomplete metric spaces.

In this section, we need an important proposition as given by Mansour and al. [20] below.

**Proposition 5** Let $X$ be a metric space, $x \in X$ and $\Phi : X \rightrightarrows X$ be a contraction mapping with a constant $\kappa$. Then, for every $\varepsilon > 0$, there exists a sequence $(u_n) \in X$ such that for all $n$, one has

$$u_{n+1} \in \Phi(u_n), \lim d(u_n,u_{n+1}) = 0, d(x,u_{n+1}) \leq \frac{1}{1 - \kappa}(d(x,\Phi(x)) + \varepsilon).$$

The main result of this section is as follows.

**Theorem 9** Let $X$ be a metric space, $Y$ be a metric space with the invariant-shift distance, $U,V$ be open subsets of $X$ and $Y$, respectively and let $F : X \rightrightarrows Y$ be a set-valued mapping, $g : X \to Y$ be a single-valued mapping. Assume that $F$ is Milyutin regular on $U \times V$ with modulus $\tau > 0$ and $g$ is Lipschitz continuous on $U$ with constant $\lambda$ such that $\tau \lambda < 1$ and there exists an open set $W \subset Y$ such that $W - g(U) \subset V$.

Then, for $(x,y) \in U \times W$ with $0 < (\tau^{-1} - \lambda)^{-1}d(y,F(x) + g(x)) < m_U(x)$, and set $T_y = F^{-1} \circ (-g + y)$, one has the following statements.
(a) for every $\varepsilon > 0$, $\varepsilon - \text{Fix}(T_y) \cap U \neq \emptyset$ and

$$d(x, \varepsilon - \text{Fix}(T_y)) \leq \frac{\tau}{1 - \tau \lambda} d(y, F(x) + g(x)).$$

Additionally, there exists a sequence $(y_\alpha)$ converging to $y$ such that

$$d(x, (F + g)^{-1}(y_\alpha)) \leq \frac{\tau}{1 - \tau \lambda} (d(y, F(x) + g(x)) + \varepsilon).$$

(b) If $X$ is complete then one concludes that $\text{Fix}(T_y) \cap U \neq \emptyset$ and

$$d(x, \text{Fix}(T_y)) \leq \frac{\tau}{1 - \tau \lambda} d(y, F(x) + g(x)).$$

Namely, $F + g$ is Milyutin regular on $(U, W)$ with modulus $\frac{\tau}{1 - \tau \lambda}$.

**Proof** We firstly claim that $T_y$ is orbital pseudo-Lipschitz on $U$ with modulus $\tau \lambda$. Indeed, take $x' \in U, x' \in T_y(u')$, $\tau \lambda d(x', u') < m_U(x')$, one has

$$d(x', T_y(x')) = d(x', (F^{-1} \circ (-g(x') + y)))$$

$$\leq \tau d(y - g(x'), F(x'))$$

$$\leq \tau d(y - g(x'), y - g(u'))$$

$$= \tau d(g(x'), g(u'))$$

$$\leq \tau \lambda d(x', u'),$$

(note that the estimate (25) is resulted from the following events: $x' \in U, y - g(x') \in W - g(U) \subset V$ and $\tau d(y - g(x'), F(x')) \leq d(y - g(x'), y - g(u')) \leq \tau \lambda d(x', u') < d(x', u') < m_U(x')$).

In addition, one has

$$d(x, T_y(x)) = d(x, F^{-1}(-g(x) + y))$$

$$\leq \tau d(y - g(x), F(x))$$

$$= \tau d(y, F(x) + g(x))$$

$$< \tau (\tau^{-1} - \lambda) m_U(x)$$

$$< (1 - \tau \lambda) m_U(x).$$

Hence, by Theorem 2, one obtains following two assertions:

(a) for every $\varepsilon > 0$, $\varepsilon - \text{Fix}(T_y) \cap U \neq \emptyset$ and

$$d(x, \varepsilon - \text{Fix}(T_y)) \leq \frac{d(x, T_y(x))}{1 - \tau \lambda} \leq \frac{\tau}{1 - \tau \lambda} d(y, F(x) + g(x)).$$

Besides, invoking Proposition 5, one chooses a sequence $(u_\alpha) \in X$ such that for all $n$,

$$u_{n+1} \in T_y(u_n), \lim d(u_n, u_{n+1}) = 0,$$
and
\[ d(x, u_{n+1}) \leq \frac{1}{1 - \tau \lambda} (d(x, T_y(x)) + \tau \epsilon). \] (26)

The condition \( u_{n+1} \in T_y(u_n) \) implies \( u_{n+1} \in F^{-1}(y - g(u_n)) \) which follows \( y - g(u_n) \in F(u_{n+1}) \). Thus, \( y \in g(u_n) + F(u_{n+1}) \). From the Lipschitzness of \( g \), one gets
\[
y \in g(u_{n+1}) + \lambda d(u_n, u_{n+1}) B_X + F(u_{n+1})
= (F + g)(u_{n+1}) + \lambda d(u_n, u_{n+1}) B_X.
\]

It follows that there exists a sequence \( (e_n) \in B_X \) such that
\[
y - \lambda d(u_n, u_{n+1}) e_n \in (F + g)(u_{n+1}).
\]

Setting \( y_n := y - \lambda d(u_n, u_{n+1}) e_n \), then certainly, \( y_n \) converges to \( y \) and one further has \( y_n \in (F + g)(u_{n+1}) \). This implies \( u_{n+1} \in (F + g)^{-1}(y_n) \). Eventually, by (26),
\[
d(x, (F + g)^{-1}(y_n)) \leq d(x, u_{n+1})
\leq \frac{1}{1 - \tau \lambda} (d(x, T_y(x)) + \tau \epsilon)
\leq \frac{1}{1 - \tau \lambda} (\tau d(y, F(x) + g(x)) + \tau \epsilon)
= \frac{\tau}{1 - \tau \lambda} (d(y, F(x) + g(x)) + \epsilon).
\]

(b) If \( X \) is complete then according to the second conclusion of Theorem 2, one reaches \( \text{Fix}(T_y) \cap U \neq \emptyset \) and
\[
d(x, \text{Fix}(T_y)) \leq \frac{d(x, T_y(x))}{1 - \tau \lambda} \leq \frac{\tau}{1 - \tau \lambda} d(y, F(x) + g(x)).
\]

Note that \( u \in \text{Fix}(T_y) \Leftrightarrow u \in T_y(u) \). That is \( u \in F^{-1}(y - g(u)) \), or, \( y - g(u) \in F(u) \), that is to say, \( y \in F(u) + g(u) \), and thus \( u \in (F + g)^{-1}(y) \). So, we get
\[
d(x, (F + g)^{-1}(y)) \leq \frac{\tau}{1 - \tau \lambda} d(y, F(x) + g(x)).
\]

That is \( F + g \) is Milyutin regular on \( U \) with modulus \( \frac{\tau}{1 - \tau \lambda} \).

The proof ends.

In a similar way as in Theorem 9, one establishes a result given by Tron, Ngai, Han [23] in both complete and non-complete metric spaces as follows.

Given now function \( g : X \to Y, U \subset X, V \subset Y \) and \( \epsilon > 0 \), we set
\[
W^{\epsilon, g} := \{(x, y) \in X \times Y : x \in U, B(y - g(x), \epsilon m_U(x)) \subset V \},
\]
where the function \( m_U \) is defined by \( m_U(x) := d(x, X \setminus U) \) to be called the Milyutin function. Given now a subset \( W \) of \( X \times Y \). For every \( y \in Y \), we associate it to set \( W_y = \{x \in X : (x, y) \in W\} \), and for every \( x \in X \), we associate it to set \( W_x = \{y \in Y : (x, y) \in W\} \). Then, we denote \( P_x W := \bigcup_{y \in Y} W_y \) and \( P_y W := \bigcup_{x \in X} W_x \). It is easy to see that when \( W = U \times V \), the sets \( W_y \) (with \( y \in V \)), \( P_y W \) coincide with \( U \) and the sets \( W_x \) (with \( x \in U \)), \( P_x W \) coincide with \( V \).
Theorem 10 Let $X$ be a metric space, $Y$ be a metric space with the invariant-shift distance, $U, V$ be open subsets of $X$ and $Y$, respectively and let $F : X \rightrightarrows Y$ be a set-valued mapping, $g : X \to Y$ be a single-valued mapping. Suppose that $F$ is Milyutin regular on $U \times V$ with modulus $\tau > 0$ and $g$ is Lipschitz continuous on $U$ with constant $\lambda$ such that $\tau \lambda < 1$.

Then, for $(x, y) \in W^{g, \lambda}$ satisfying $0 < (\tau^{-1} - \lambda)^{-1} [d(y, F(x) + g(x))] < m_{P_{X}, W^{g, \lambda}}(x)$.

Set

$$A := \{(x, y) \in W^{g, \lambda} : 0 < (\tau^{-1} - \lambda)^{-1} [d(y, F(x) + g(x))] < m_{P_{X}, W^{g, \lambda}}(x)\},$$

and fix such $y$, setting $T_y = F^{-1} \circ (-g + y)$, one has

(a) for every $\epsilon > 0$, $\varepsilon - \text{Fix}(T_y) \cap A_y \neq \emptyset$ and

$$d(x, \varepsilon - \text{Fix}(T_y)) \leq \frac{\tau}{1 - \tau \lambda} d(y, F(x) + g(x)).$$

Moreover, there exists a sequence $(y_n)$ converging to $y$ such that

$$d(x, (F + g)^{-1}(y_n)) \leq \frac{\tau}{1 - \tau \lambda} (d(y, F(x) + g(x)) + \epsilon).$$

(b) If $X$ is complete then one concludes that $\text{Fix}(T_y) \cap A_y \neq \emptyset$ and

$$d(x, \text{Fix}(T_y)) \leq \frac{\tau}{1 - \tau \lambda} d(y, F(x) + g(x)).$$

As a result, $F + g$ is Milyutin regular on $A$ with modulus $\frac{\tau}{1 - \tau \lambda}$.

Remark 4 (a) The second conclusions of (a) in Theorem 10, and Theorem 9 could be considered as a type of “approximate Milyutin regularity” of the map $F + g$.

With this name, we can restate the conclusion that $F + g$ is approximate Milyutin regularity on $A$ with modulus $\frac{\tau}{1 - \tau \lambda}$.

(b) By approaching of the fixed point theorem, one can obtain results in both complete metric and non-complete space. In the complete case of considered space, one reobtain the recent result of Milyutin of the sum given by Tron, Han, Ngai [23] and the one given by Ioffe [13] and [15].

6 Conclusion

With these obtained results, we hope we can unify many results existing in metric spaces by a common iterative scheme and opens the prospects to work with problems in metric settings without using the completeness of the considered spaces such as optimization problems, equilibrium problems, the convergence of approximate iterative algorithms, and so on.
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