A New Insight on Augmented Lagrangian Method and Its Extensions

Jianchao Bai

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Abstract. Motivated by the recent work [He-Yuan, Balanced Augmented Lagrangian Method for Convex Programming, arXiv: 2108.08554v1, (2021)], a novel Augmented Lagrangian Method (ALM) has been proposed for solving a family of convex optimization problem subject to equality or inequality constraint. This new method is then extended to solve the multi-block separable convex optimization problem, and two related primal-dual hybrid gradient algorithms are also discussed. Preliminary and some new convergence results are established with the aid of variational analysis for both the saddle point of the problem and the first-order optimality conditions of involved subproblems.

Key words: convex optimization, augmented Lagrangian method, primal-dual hybrid gradient algorithm, global convergence, complexity

Mathematics Subject Classification(2010): 65K10; 65Y20; 90C25

1 Introduction

The recent interesting work, that is, the Balanced Augmented Lagrangian Method (abbreviated by B-ALM), proposed by He-Yuan [11] is to solve the following convex optimization problem subject to linear equality or inequality constraints:

$$
\min \{ \theta(x) \mid Ax = b \ (or \geq b), \ x \in X \},
$$

where \( \theta(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) is a closed proper convex function (not necessarily strongly convex or smooth); \( X \subseteq \mathbb{R}^n \) is a closed convex set; \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) are given. Here and hereafter, the symbols \( \mathbb{R}, \mathbb{R}^n, \mathbb{R}^+ \), \( \mathbb{R}^{m \times n} \) denote the set of real numbers, the set of \( n \) dimensional nonnegative column vectors, and the set of \( m \times n \) dimensional real matrices respectively. The bold \( \mathbf{I} \) and \( \mathbf{0} \) stand for a specific identity matrix and zero matrix with proper dimensions, respectively, and \( Q \succ 0 \) means the matrix \( Q \) is symmetric positive definite. Throughout this paper, we assume that

- The solution set of the problem (1) is nonempty;

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†School of Mathematics and Statistics and the MIIT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi’an 710129, China. (jianchaobai@nwpu.edu.cn).
The proximal operator of $\theta$ defined as $\text{prox}_{\varrho, r}(x) = \arg \min_{x \in \mathcal{X}} \{ \theta(x) + \frac{r}{2} \| x - c \|^2 \}$ is available for given data $c \in \mathcal{R}^n$.

A benchmark method for solving the problem (1) is the so-called Augmented Lagrangian Method (ALM, [7, 13]) which reads

$$
\begin{align*}
\textbf{x}_{k+1} &= \arg \min_{x \in \mathcal{X}} \left\{ L(x, \lambda_k) + \frac{r}{2} \| Ax - b \|^2 \right\}, \\
\lambda_{k+1} &= \lambda_k - r (Ax_{k+1} - b),
\end{align*}
$$

where $r > 0$ denotes the penalty parameter and

$$
L(x, \lambda) = \theta(x) - \langle \lambda, Ax - b \rangle
$$

denotes the Lagrangian function associated with (1). Ignoring some constants, it is easy to check that the above subproblem amounts to

$$
\textbf{x}_{k+1} = \arg \min_{x \in \mathcal{X}} \left\{ \theta(x) + \frac{r}{2} \| Ax - b \|^2 \right\}
$$

but is still difficult to be solved if without utilizing some linearization techniques or inner solvers. As described in [11], the framework of B-ALM reads the following iterates

$$
\begin{align*}
\textbf{x}_{k+1} &= \arg \min_{x \in \mathcal{X}} \left\{ L(x, \lambda_k) + \frac{r}{2} \| A(x - x_k) \|^2 + \frac{1}{2} \| x - x_k \|^2_{Q} \right\}, \\
\lambda_{k+1} &= \lambda_k - r \left[ A(2x_{k+1} - x_k) - b \right],
\end{align*}
$$

and its subproblem may have closed-form solution since its solution can be expressed as a proximal mapping. Convergence of B-ALM depends on the positive definiteness of the matrix

$$
\begin{bmatrix}
  rI & \frac{1}{r}AA^T \\
  A & 1_A A^T + \delta I
\end{bmatrix}, \forall r, \delta > 0.
$$

One may use a general form $\frac{1}{r}AA^T + Q$ for any $Q \succ 0$ to replace the above lower-upper block to guarantee the convergence of B-ALM. A major merit of B-ALM is that it greatly weakens the convergence conditions of some ALM and related first-order splitting algorithms [2, 5, 6, 8, 10, 12, 15]. Namely, the parameter $r$ does not depend on $\| A^TA \|$, where $\| A^TA \|$ represents the spectral norm of $A^TA$ and $^T$ denotes the transpose operator. However, it will take much time to update $\lambda_{k+1}$, and in practice it will take an inner solver to tackle the dual subproblem or use the well-known Cholesky factorization to deal with an equivalent linear equation of the dual subproblem.

In this note, we will propose and analyze a new Penalty ALM (abbreviated by P-ALM) for solving (1). For conciseness, we first present this P-ALM as the following:

**Initialize** $(x^0, \lambda^0)$ and choose $r > 0$, $Q \succ 0$.

**While** stopping criteria is not satisfied **do**

$$
\begin{align*}
\textbf{x}_{k+1} &= \arg \min_{x \in \mathcal{X}} \left\{ \theta(x) - \langle \lambda_k, Ax - b \rangle + \frac{r}{2} \| A(x - x_k) \|^2 + \frac{1}{2} \| x - x_k \|^2_{Q} \right\}, \\
\lambda_{k+1} &= \lambda_k - r \left[ A(2x_{k+1} - x_k) - b \right].
\end{align*}
$$

**End while**

For the problem with inequality constraints $Ax \geq b$, the dual update reads

$$
\lambda^{k+1} = \mathcal{P}_{\mathcal{R}^+} \left( \lambda_k - r \left[ A(2x_{k+1} - x_k) - b \right] \right).
$$

Some main features of P-ALM are summarized as three aspects:
Different from the construction of ALM, the $x$-subproblem of P-ALM uses two different quadratic penalty terms $\frac{r^2}{2} \|A(x - x^k)\|^2$ and $\frac{1}{2} \|x - x^k\|_Q^2$, where the later can be regarded as a metric proximal term, while the first could be treated as a new penalty term unlike $\frac{1}{2} \|Ax - b\|^2$. Equivalently,

$$x^{k+1} = \arg\min_{x \in \mathcal{X}} \left\{ \theta(x) - \langle \lambda^k, Ax - b \rangle + \frac{1}{2} \|x - x^k\|_r^2 A^T A + Q \right\}.$$  

In particular, taking $Q = \tau I - rA^T A$ with $\tau > r\|A^T A\|$ could convert the $x$-update to the following proximal mapping

$$\text{prox}_{\theta, \tau}(x) = \arg\min_{x \in \mathcal{X}} \left\{ \theta(x) + \frac{\tau}{2} \|x - x^k - \frac{1}{\tau} A^T \lambda^k\|^2 \right\}.$$  

which, by the second assumption, will have a closed-form solution since the objective function is strongly convex. If $\text{prox}_{\theta, \tau}(x)$ is not available but $\theta$ is smooth, then user can select an inner solver such as conjugate gradient method to solve $x$-subproblem inexactly.

- The dual update is the same as that in [5] and is comparatively much easier than that of B-ALM. In fact, the dual update combines the information of both the current iterate $x^{k+1}$ and an extrapolation iterate $x^{k+1} - x^k$.

- As said before, the global convergence of this P-ALM, compared with some existing splitting algorithms, will no longer depend on the value $\|A^T A\|$. We show that the primal residual and the objective gap converge in a sublinear convergence rate, see Theorem 2.2 and Corollary 2.1. Motivated by the structure of $H$ in (6), we also discuss a generalization of P-ALM and two new-types of Primal-Dual Hybrid Gradient algorithm (PDHG) for solving the multiple block separable convex optimization and the saddle-point problem, respectively.

The rest parts of this article are organized as follows. In Section 2, we analyze the global convergence and sublinear convergence rate of P-ALM from the prospective of variational inequality. Section 3 generalizes the proposed P-ALM to solve the multi-block separable convex programming problem. Based on the construction of P-ALM, in Section 4 we discuss two novel PDHG algorithms for solving a family of convex-concave saddle-point problems. Finally, we conclude the paper in Section 5.

## 2 Convergence analysis of P-ALM

From the perspective of optimization, a point

$$w^* = (x^*; \lambda^*) \in M := \mathcal{X} \times \Lambda,$$

where $\Lambda := \left\{ \begin{array}{ll} \mathbb{R}^m, & \text{if } Ax = b, \\ \mathbb{R}^m_+, & \text{if } Ax \geq b, \end{array} \right.$

is called the saddle-point of the Lagrangian function $L(x, \lambda)$ if

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*), \quad \forall w \in M,$$

hold.

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which can be separately rewritten as
\[
\begin{cases}
  x^* \in \mathcal{X} \quad \theta(x) - \theta(x^*) + \langle x - x^*, -A^T\lambda^* \rangle \geq 0, \quad \forall x \in \mathcal{X} \\
  \lambda^* \in \Lambda 
\end{cases}
\]

These inequalities can be further expressed as the following unified variational inequality
\[
\text{VI}(\theta, \mathcal{J}, \mathcal{M}) : \quad \theta(x) - \theta(x^*) + \langle w - w^*, \mathcal{J}(w^*) \rangle \geq 0, \quad \forall w \in \mathcal{M},
\]

where
\[
w = \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad \mathcal{J}(w) = \begin{pmatrix} -A^T\lambda \\ Ax - b \end{pmatrix}.
\]

An equivalent form of (2) is
\[
\theta(x) - \theta(x^*) + \langle w - w^*, \mathcal{J}(w) \rangle \geq 0, \quad \forall w \in \mathcal{M},
\]

since the affine mapping \( \mathcal{J}(w) \) is skew symmetric and satisfies
\[
\langle w - \bar{w}, \mathcal{J}(w) - \mathcal{J}(\bar{w}) \rangle = 0, \quad \forall w, \bar{w} \in \mathcal{M}.
\]

Obviously, the solution set of VI(\( \theta, \mathcal{J}, \mathcal{M} \)), denoted by \( \mathcal{M}^* \), is nonempty by the assumption on solution set of the problem (1). A straightforward conjecture is that convergence of P-ALM can be showed if its generated sequence is characterized by a similar inequality to (3) with an extra term converging to zero. Based on such observation, we next investigate some properties of the sequence generated by P-ALM with the aid of a \( H \)-weighted norm defined as \( \|w\|^2_H = \langle w, Hw \rangle \) for any \( H \succ 0 \).

**Lemma 2.1** The sequence \( \{w^k\} \) generated by P-ALM satisfies
\[
w^{k+1} \in \mathcal{M}, \quad \theta(x) - \theta(x^{k+1}) + \langle w - w^{k+1}, \mathcal{J}(w) \rangle \geq \langle w - w^{k+1}, H(w^k - w^{k+1}) \rangle
\]

for any \( w \in \mathcal{M} \), where
\[
H = \begin{bmatrix} rA^TA + Q & A^T \\ A & \frac{r}{2}I \end{bmatrix}
\]

is symmetric positive definite for any \( r > 0 \) and \( Q > 0 \). Moreover, we have
\[
\theta(x) - \theta(x^{k+1}) + \langle w - w^{k+1}, \mathcal{J}(w) \rangle \\
\geq \frac{1}{2} \left( \|w - w^{k+1}\|^2_H - \|w - w^{k}\|^2_H \right) + \frac{1}{2} \|w^k - w^{k+1}\|^2_H, \quad \forall w \in \mathcal{M}.
\]

Proof. First of all, the first-order optimality condition of the \( x \)-subproblem in P-ALM gives
\[
x^{k+1} \in \mathcal{X}, \quad \theta(x) - \theta(x^{k+1}) + \langle x - x^{k+1}, -A^T\lambda^k + (rA^TA + Q)(x^{k+1} - x^k) \rangle \geq 0, \quad \forall x \in \mathcal{X},
\]

that is,
\[
\theta(x) - \theta(x^{k+1}) + \langle x - x^{k+1}, -A^T\lambda^{k+1} \rangle \\
\geq \langle x - x^{k+1}, (rA^TA + Q)(x^k - x^{k+1}) + A^T(\lambda^k - \lambda^{k+1}) \rangle, \quad \forall x \in \mathcal{X}.
\]
Besides, it follows from the update of $\lambda^{k+1}$ that $\lambda^{k+1} \in \Lambda$ and
\[
\left\langle \lambda - \lambda^{k+1}, Ax^{k+1} - b \right\rangle = \left\langle \lambda - \lambda^{k+1}, A(x^k - x^{k+1}) + \frac{1}{r}(\lambda^k - \lambda^{k+1}) \right\rangle, \quad \forall \lambda \in \Lambda. \tag{9}
\]
Finally, by combining the above inequalities (8)-(9) together with the structure of $H$ in (6) and the property in (4), the conclusion (5) is proved.

Observing that the symmetric matrix $H$ has the following decomposition:
\[
H = \begin{bmatrix}
 rA^T & A \\
 A & \frac{1}{r}I
\end{bmatrix} + \begin{bmatrix}
 Q & 0 \\
 0 & 0
\end{bmatrix} = \left( \sqrt{r}A^T, \frac{1}{\sqrt{r}}I \right) + \begin{bmatrix}
 Q & 0 \\
 0 & 0
\end{bmatrix}.
\]
For any $w = (x; \lambda) \neq 0$, we have
\[
w^T H w = \left\| \sqrt{r}Ax + \frac{1}{\sqrt{r}}\lambda \right\|^2 + \|x\|_Q^2 > 0
\]
and hence the matrix $H$ is symmetric positive definite.

Now, by substituting $p := w$, $q = v := w^{k+1}$, and $u := w^k$ into the following identity
\[
\left\langle p - q, H(u - v) \right\rangle = \frac{1}{2} \left\{ \left\| p - v \right\|_H^2 - \left\| p - u \right\|_H^2 \right\} + \frac{1}{2} \left\{ \left\| q - u \right\|_H^2 - \left\| q - v \right\|_H^2 \right\},
\]
we have by (5) that the inequality (7) holds clearly.

The following theorem shows that the sequence $\{w^k\}$ generated by P-ALM is contractive under the $H$-weighted norm and thus converges globally to the solution point of VI($\theta, J, M$).

**Theorem 2.1** For any $r > 0$ and $Q > 0$, the sequence $\{w^k\}$ generated by P-ALM satisfies
\[
\left\| w^* - w^{k+1} \right\|_H^2 \leq \left\| w^* - w^k \right\|_H^2 - \left\| w^k - w^{k+1} \right\|_H^2, \quad \forall w^* \in \mathcal{M}^*.
\tag{10}
\]
Moreover, there exists a $w^\infty \in \mathcal{M}^*$ such that $\lim_{k \to \infty} w^k = w^\infty$.

Proof. Setting $w = w^*$ in (7) is to achieve
\[
\frac{1}{2} \left( \left\| w^* - w^k \right\|_H^2 - \left\| w^* - w^{k+1} \right\|_H^2 \right) - \frac{1}{2} \left\| w^k - w^{k+1} \right\|_H^2 \\
\geq \theta(x^{k+1}) - \theta(x^*) + \left\langle w^{k+1} - w^*, J(w^*) \right\rangle \geq 0,
\]
that is, the inequality (10) holds.

Note that the inequality (10) implies that the sequence $\{w^k\}$ is bounded and
\[
\lim_{k \to \infty} \left\| w^k - w^{k+1} \right\|_H^2 = 0 \iff \lim_{k \to \infty} (w^k - w^{k+1}) = 0. \tag{11}
\]
Let \( w^\infty \) be any accumulation point of \( \{ w^k \} \). Then, (taking a subsequence of \( \{ w^k \} \) if necessary) it follows from (11) that
\[
\theta(x) - \theta(x^\infty) + \langle w - w^\infty, J(w^\infty) \rangle = \theta(x) - \theta(x^\infty) + \langle w - w^\infty, J(w) \rangle \geq 0, \quad \forall w \in \mathcal{M}.
\]
This indicates \( w^\infty \in \mathcal{M}^* \) compared to (2). So, by (10) again, we have
\[
\| w^k - w^\infty \|^2_H \leq \| w^j - w^\infty \|^2_H, \quad \text{for all } k \geq j.
\]
Then, it follows from \( w^\infty \) being an accumulation point that \( \lim_{k \to \infty} w^k = w^\infty \). ■

Before establishing the sublinear convergence rate of P-ALM for the following general average iterates (seemingly, it was used firstly in [3] to accelerate convergence of a stochastic method numerically)
\[
w_T := \frac{1}{T+1} \sum_{k=T}^{T+\kappa} w^{k+1} \quad \text{and} \quad x_T := \frac{1}{T+1} \sum_{k=T}^{T+\kappa} x^{k+1},
\]
we analyze the convergence complexity of the pointwise iteration (see also [4, Theorem 6]) and the primal residual, where the notation \( \partial \theta(x) \) represents its sub-differential at \( x \), and \( N_\mathcal{X}(x) \) denotes the normal cone of \( \mathcal{X} \) at \( x \).

**Theorem 2.2** For any \( k > 0 \), there exists an integer \( t \leq k \) such that
\[
\| x^{t-1} - x^t \|^2 \leq \frac{\varrho}{k} \quad \text{and} \quad \| s^t \|^2 \leq \frac{\varrho}{k},
\]
where \( s^t \in \mathcal{R}^n \) satisfies \( A^T \lambda^t - s^t \in \partial \theta(x) + N_\mathcal{X}(x) \), and \( \varrho > 0 \) is a constant only depending on the problem data and the parameters of P-ALM.

**Proof.** Let \( k > 0 \) be a fixed constant and \( t \in [1, k] \) be an integer such that
\[
\| w^{t-1} - w^t \| = \min \{ \| w^{l-1} - w^l \| : l = 1, \ldots, k \}.
\]
Then, summing up (10) over \( k = 0, 1, \ldots, \infty \) gives
\[
\sum_{k=0}^{\infty} \| w^k - w^{k+1} \|^2_H \leq \| w^* - w^0 \|^2_H < \infty,
\]
which together with the positive definiteness of \( H \) shows
\[
\| w^{t-1} - w^t \|^2 \leq \frac{\varrho}{k} \iff \| x^{t-1} - x^t \|^2 \leq \frac{\varrho}{k} \quad \text{and} \quad \| \lambda^{t-1} - \lambda^t \|^2 \leq \frac{\varrho}{k},
\]
where and in the following of this proof, \( \varrho > 0 \) is a generic constant only depending on the problem data and the parameters of P-ALM.

Now, it follows from (8) that by defining
\[
s^t = (rA^TA + Q)(x^t - x^{t-1}) + A^T(\lambda^t - \lambda^{t-1}) \in \mathcal{R}^n,
\]
we have
\[
\theta(x) - \theta(x^t) - (x^t - x)^T(A^T \lambda^t - s^t) \geq 0, \quad \forall x \in \mathcal{X},
\]
which implies \( A^T \lambda^t - s^t \in \partial \theta(x^t) + N_\mathcal{X}(x^t) \). By (15) again, we have
\[
\| s^t \|^2 \leq \| x^{t-1} - x^t \|^2_{rA^TA+Q} + \| A^T(\lambda^t - \lambda^{t-1}) \|^2,
\]
which together with (14) ensures the right inequality in (13). ■
**Theorem 2.3** For any $r > 0, Q > 0$ and for any $T > 0, \kappa \geq 0$, the sequence $\{w^k\}$ generated by P-ALM satisfies

$$\theta(x_T) - \theta(x) + \langle w_T - w, J(w) \rangle \leq \frac{1}{2(T + 1)} \|w^\kappa - w\|^2_H, \forall w \in M. \quad (16)$$

**Proof.** The inequality (7) indicates

$$\theta(x) - \theta(x^{k+1}) + \langle w - w^{k+1}, J(w) \rangle + \frac{1}{2} \|w - w^k\|^2_H \geq \frac{1}{2} \|w - w^{k+1}\|^2_H.$$

Summing the above inequality over $k = \kappa, \kappa + 1, \ldots, \kappa + T$, we have

$$(T + 1)\theta(x) - \sum_{k=\kappa}^{\kappa+T} \theta(x^{k+1}) + \left\langle (T + 1)w - \sum_{k=\kappa}^{\kappa+T} w^{k+1}, J(w) \right\rangle + \frac{1}{2} \|w - w^\kappa\|^2_H \geq 0,$$

which, by the definition of $w_T$ and $x_T$, shows that

$$\frac{1}{T + 1} \sum_{k=\kappa}^{\kappa+T} \theta(x^{k+1}) - \theta(x) + \langle w_T - w, J(w) \rangle \leq \frac{1}{2(T + 1)} \|w - w^\kappa\|^2_H. \quad (17)$$

Because $\theta$ is convex function having the property

$$\theta(x_T) \leq \frac{1}{T + 1} \sum_{k=\kappa}^{\kappa+T} \theta(x^{k+1}),$$

then the inequality (16) is obtained by plugging the above inequality into (17). \hfill \square

The above Theorem 2.3 shows that P-ALM converges in a sublinear convergence rate in the ergodic sense. Furthermore, for any $\eta > 0$, let $\Gamma_\eta = \{\lambda | \|\lambda\| \leq \eta\}$ and

$$\gamma_\eta = \inf_{x^* \in X} \sup_{\lambda \in \Lambda} \|w - w^\kappa\|^2_H ; \quad (18)$$

we can get the following tight result whose proof is similar to that of [1, 14] and thus is omitted here for the sake of conciseness.

**Corollary 2.1** Let $\gamma_\eta$ be defined in (18) for any $\eta > 0$ and $x_T$ be defined in (12). Then, the sequence $\{w^k\}$ generated by P-ALM satisfies

$$\theta(x_T) - \theta(x^*) + \eta \|Ax_T - b\| \leq \frac{\gamma_\eta}{2(T + 1)}, \forall x^* \in X. \quad (19)$$

### 3 Spitting version of P-ALM for multi-block convex optimization

In this section, we consider a multi-block extension of the problem (1):

$$\min \left\{ \theta(x) := \sum_{i=1}^p \theta_i(x_i) \mid \sum_{i=1}^p A_i x_i = b \ (or \geq b), \ x_i \in X_i \right\}, \quad (20)$$
where $\theta_i(x_i) : \mathbb{R}^{n_i} \to \mathbb{R}, i = 1, 2, \ldots, p$ are closed proper convex functions (not necessarily strongly convex or smooth); $X_i \subseteq \mathbb{R}^{n_i}, i = 1, 2, \ldots, p$ are closed convex sets; $A_i \in \mathbb{R}^{m \times n_i}$ and $b \in \mathbb{R}^m$ are given. For this problem, we denote

$$M := \prod_{i=1}^p X_i \times \Lambda, \quad \text{where } \Lambda := \begin{cases} \mathbb{R}^m, & \text{if } \sum_{i=1}^p A_i x_i = b, \\ \mathbb{R}^m_+, & \text{if } \sum_{i=1}^p A_i x_i \geq b. \end{cases}$$

A natural extension of P-ALM (denoted by EP-ALM) is the following:

**Initialize** $(x_0^1, \ldots, x_0^p, \lambda^0)$ and choose $r_i > 0, Q_i > 0$ for $i = 1, 2, \ldots, p$;

**While** stopping criteria is not satisfied **do**

For $i = 1, 2, \ldots, p$,

$$x_i^{k+1} = \arg \min_{x_i \in X_i} \left\{ \theta_i(x_i) - \langle \lambda^k, A_i x_i - b \rangle + \frac{r_i}{2} \| A_i (x_i - x_i^k) \|^2 + \frac{1}{2} \| x_i - x_i^k \|^2_{Q_i} \right\}. $$

End for

$$\lambda^{k+1} = \lambda^k \sum_{j=1}^p \frac{r_j}{r_i} \left[ \sum_{i=1}^p A_i (2x_i^{k+1} - x_i^k) - b \right].$$

End while

Analogous to the analysis in Section 2, the saddle-point $w^* = (x_1^*; \ldots; x_p^*; \lambda^*) \in M$ of the Lagrangian function of (20) will satisfy

$$\theta(x) - \theta(x^*) + \langle w - w^*, J(w^*) \rangle \geq 0, \forall w \in M, \quad (21)$$

where

$$w = \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}, \quad J(w) = \begin{pmatrix} -A_1^T \lambda \\ \vdots \\ -A_p^T \lambda \\ \sum_{i=1}^p A_i x_i - b \end{pmatrix}. $$

We next give a brief analyze for the convergence of EP-ALM.

**Lemma 3.1** The sequence $\{w^k\}$ generated by EP-ALM satisfies

$$w^{k+1} \in M, \quad \theta(x) - \theta(x^{k+1}) + \langle w - w^{k+1}, J(w) \rangle \geq \langle w - w^{k+1}, H(w^k - w^{k+1}) \rangle \quad (22)$$

for any $w \in M$, where

$$H = \begin{bmatrix} r_1 A_1^T A_1 + Q_1 & \cdots & 0 & A_1^T \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & r_p A_p^T A_p + Q_p & A_p^T \\ A_1 & \cdots & A_p & \sum_{i=1}^p \frac{1}{r_i} I \end{bmatrix} \quad (23)$$
is symmetric positive definite for any \( r_i > 0 \) and \( Q_i \succ 0 \). Moreover, we have

\[
\theta(x) - \theta(x^{k+1}) + \langle w - w^{k+1}, J(w) \rangle \\
\geq \frac{1}{2} \left( \|w - w^{k+1}\|_H^2 - \|w - w^k\|_H^2 \right) + \frac{1}{2} \|w^k - w^{k+1}\|_H^2, \quad \forall w \in \mathcal{M}.
\]

(24)

Proof. First of all, for each \( i = 1, 2, \ldots, p \), the first-order optimality condition of the \( x_i \)-subproblem in EP-ALM gives that \( x_i^{k+1} \in \mathcal{X}_i \) and

\[
\theta_i(x_i) - \theta_i(x_i^{k+1}) + \langle x_i - x_i^{k+1}, -A_i^T \lambda^k + \left( r_i A_i^T A_i + Q_i \right) \left( x_i^{k+1} - x_i^k \right) \rangle \geq 0, \quad \forall x_i \in \mathcal{X}_i,
\]

that is,

\[
\theta_i(x_i) - \theta_i(x_i^{k+1}) + \langle x_i - x_i^{k+1}, -A_i^T \lambda^k + \left( r_i A_i^T A_i + Q_i \right) \left( x_i^{k+1} - x_i^k \right) \rangle \geq 0, \quad \forall x_i \in \mathcal{X}_i,
\]

that is,

\[
\theta_i(x_i) - \theta_i(x_i^{k+1}) + \langle x_i - x_i^{k+1}, -A_i^T \lambda^k \rangle \\
\geq \langle x_i - x_i^{k+1}, \left( r_i A_i^T A_i + Q_i \right) \left( x_i^{k+1} - x_i^k \right) + A_i^T (\lambda^k - \lambda^{k+1}) \rangle.
\]

(25)

Besides, it follows from the update of \( \lambda^{k+1} \) that \( \lambda^{k+1} \in \Lambda \) and

\[
\left\langle \lambda - \lambda^{k+1}, \sum_{i=1}^p A_i x_i^{k+1} - b \right\rangle = \left\langle \lambda - \lambda^{k+1}, \sum_{i=1}^p A_i (x_i^k - x_i^{k+1}) + \sum_{j=1}^p \frac{1}{r_j} (\lambda^k - \lambda^{k+1}) \right\rangle, \quad \forall \lambda \in \Lambda.
\]

(26)

Finally, by combining the above inequalities (25)-(26) together with the structure of \( H \) as well as the skew-symmetric property of \( J(w) \), the conclusion (22) is proved.

Observing that \( H \) has the following decomposition

\[
H = \hat{H} + \begin{bmatrix}
Q_1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & Q_p & 0 \\
0 & \cdots & \cdots & 0
\end{bmatrix}
\]

and

\[
\hat{H} = \begin{bmatrix}
r_1 A_1^T A_1 & \cdots & 0 & A_1^T \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & r_p A_p^T A_p & A_p^T \\
A_1 & \cdots & A_p & \sum_{i=1}^p \frac{1}{r_i} I
\end{bmatrix}
\]

\[
= \begin{bmatrix}
r_1 A_1^T A_1 & \cdots & 0 & A_1^T \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
A_1 & \cdots & \cdots & \frac{1}{r_1} I
\end{bmatrix} + \cdots + \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & r_p A_p^T A_p & A_p^T \\
0 & \cdots & A_p & \frac{1}{r_p} I
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\sqrt{r_1} A_1^T \\
\vdots \\
0 \\
\frac{1}{\sqrt{r_1}} I
\end{bmatrix} + \begin{bmatrix}
\sqrt{r_1} A_1, 0, \ldots, 0, \frac{1}{\sqrt{r_1}} I
\end{bmatrix} + \cdots + \begin{bmatrix}
0 \\
\vdots \\
0 \\
\frac{1}{\sqrt{r_p}} I
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 \\
\vdots \\
0 \\
\frac{1}{\sqrt{r_p}} I
\end{bmatrix}
\]

\[
\begin{bmatrix}
0, \ldots, 0, \sqrt{r_p} A_p, \frac{1}{\sqrt{r_p}} I
\end{bmatrix}
\]

\[
\begin{bmatrix}
0, \ldots, 0, \sqrt{r_p} A_p, \frac{1}{\sqrt{r_p}} I
\end{bmatrix}
\]

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For any $w = (x; \lambda) \neq 0$, we have

$$w^T H w = \sum_{i=1}^{p} \left( \sqrt{r_i} A_i x_i + \frac{1}{\sqrt{r_i}} \lambda \right)^2 + \sum_{i=1}^{p} \|x_i\|_Q^2 > 0$$

and hence the matrix $H$ is symmetric positive definite.

Now, by substituting $p := w$, $q := v := w^{k+1}$, and $u := w^k$ into the following identity

$$\langle p - q, H(u - v) \rangle = \frac{1}{2} \left( \|p - v\|_H^2 - \|p - u\|_H^2 \right) + \frac{1}{2} \left( \|q - u\|_H^2 - \|q - v\|_H^2 \right),$$

we have by (22) that the inequality (24) holds clearly.

Finally, we can establish the global convergence and the sublinear convergence rate of EP-ALM by Lemma 3.1 as that in the rest parts of Section 2.

### 4 Further discussions on two new PDHG

In this section, we discuss two new types of PDHG algorithm for the following convex-concave saddle-point problem

$$\min_{x \in X} \max_{y \in Y} \Phi(x, y) := \theta_1(x) - y^T A x - \theta_2(y), \quad (27)$$

where $A \in \mathbb{R}^{m \times n}$, $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ are closed convex sets, and both $\theta_1 : \mathbb{R}^n \to \mathbb{R}$ and $\theta_2 : \mathbb{R}^m \to \mathbb{R}$ are convex but possibly nonsmooth functions. The solution set of this problem is assumed to be nonempty throughout the forthcoming discussions.

The original PDHG proposed in [16] is to solve some TV image restorations models. Extending it to the problem (27), we get the following scheme:

$$\begin{cases} 
  x^{k+1} = \arg \min_{x \in X} \Phi(x, y^k) + \frac{r}{2} ||x - x^k||^2, \\
  y^{k+1} = \arg \max_{y \in Y} \Phi(2x^{k+1} - x^k, y) - \frac{s}{2} ||y - y^k||^2,
\end{cases}$$

where $r, s$ are positive scalars. He, et al. [10] pointed out that convergence of the above PDHG can be shown if $\theta_1$ is strongly convex and $rs > \rho(A^T A)$ (these conditions are very strict). Here $\rho(A^T A)$ denotes the spectral radius of $A^T A$. To weaken these convergence conditions, e.g., the function $\theta_1$ is only convex and the parameters $r, s$ do not depend on $\rho(A^T A)$, we would develop the following novel PDHG (abbreviated by N-PDHG1) for solving the problem (27):

<table>
<thead>
<tr>
<th>Initialize $(x^0, \lambda^0)$ and choose $r &gt; 0, Q \succ 0$:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>While</strong> stopping criteria is not satisfied do</td>
</tr>
<tr>
<td>$x^{k+1} = \arg \min_{x \in X} \Phi(x, y^k) + \frac{1}{r}</td>
</tr>
<tr>
<td>$y^{k+1} = \arg \max_{y \in Y} \Phi(2x^{k+1} - x^k, y) - \frac{1}{2r}</td>
</tr>
<tr>
<td><strong>End while</strong></td>
</tr>
</tbody>
</table>
Another algorithm (denoted by N-PDHG2) is just to modify the final subproblem of PDHG, whose framework is described as follows:

\[
\text{Initialize } (x^0, \lambda^0) \text{ and choose } r > 0, Q \succ 0; \\
\text{While stopping criteria is not satisfied do} \\
x^{k+1} = \arg \min_{x \in X} \Phi(x, y^k) + \frac{r}{2} \|x - x^k\|^2. \\
y^{k+1} = \arg \max_{y \in Y} \Phi(2x^{k+1} - x^k, y) - \frac{1}{2} \|y - y^k\|^2_{rA^T + Q}. \\
\text{End while}
\]

Because the above two algorithms are very similar, in the following parts we just analyze the convergence properties of N-PDHG1 and then briefly discuss the convergence of the second algorithm. For convenience, we denote \( M := X \times Y \) and

\[
u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad u^k = \begin{pmatrix} x^k \\ y^k \end{pmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}.
\]

**Lemma 4.1** The sequence \( \{u^k\} \) generated by N-PDHG1 satisfies

\[
u^{k+1} \in M, \quad \Phi(u) - \Phi(u^{k+1}) + \left\langle u - u^{k+1}, M u \right\rangle \geq \left\langle u - u^{k+1}, H(u^k - u^{k+1}) \right\rangle \quad (28)
\]

for any \( u \in M \), where \( H \) is given by (6). Moreover, we have

\[
\Phi(u) - \Phi(u^{k+1}) + \left\langle u - u^{k+1}, M u \right\rangle \geq \frac{1}{2} \left( \|u - u^{k+1}\|^2_H - \|u - u^k\|^2_H \right) + \frac{1}{2} \|u^k - u^{k+1}\|^2_H, \quad \forall u \in M. \quad (29)
\]

**Proof.** According to the first-order optimality condition of the \( x \)-subproblem in N-PDHG1, we have \( x^{k+1} \in X \) and

\[
\theta_1(x) - \theta_1(x^{k+1}) + \left\langle x - x^{k+1}, -A^T y^k + \left( rA^T A + Q \right)(x^{k+1} - x^k) \right\rangle \geq 0, \quad \forall x \in X,
\]

that is,

\[
\theta_1(x) - \theta_1(x^{k+1}) + \left\langle x - x^{k+1}, -A^T y^{k+1} \right\rangle \\
\geq \left\langle x - x^{k+1}, \left( rA^T A + Q \right)(x^k - x^{k+1}) + A^T(y^k - y^{k+1}) \right\rangle. \quad (30)
\]

Similarly, we have by the first-order optimality condition of \( y \)-subproblem that \( y^{k+1} \in Y \) and

\[
\theta_2(y) - \theta_2(y^{k+1}) + \left\langle y - y^{k+1}, A(2x^{k+1} - x^k) + \frac{1}{r}(y^{k+1} - y^k) \right\rangle \geq 0, \quad \forall y \in Y,
\]

that is,

\[
\theta_2(y) - \theta_2(y^{k+1}) + \left\langle y - y^{k+1}, A^T x^{k+1} \right\rangle \geq \left\langle y - y^{k+1}, A(x^k - x^{k+1}) + \frac{1}{r}(y^k - y^{k+1}) \right\rangle. \quad (31)
\]

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Combining the inequalities (30)-(31) together with the structure of $H$ given by (6), we have

$$
\Phi(u) - \Phi(u^{k+1}) + \left< u - u^{k+1}, Mu^{k+1} \right> \geq \left< u - u^{k+1}, H(u^k - u^{k+1}) \right>,
$$

which together with the property $\left< u - u^{k+1}, M(u - u^{k+1}) \right> = 0$ ensures that the inequality (28) holds. The inequality (29) can be obtained similar to the proof of Lemma 2.1 and thus is omitted here.

Now, we briefly discuss the convergence of N-PDHG1. Let $u^* = (x^*; y^*) \in M$ be a solution point of the problem (27). Then, it holds

$$
\Phi_{y \in Y}(x^*, y) \leq \Phi(x^*, y^*) \leq \Phi_{x \in X}(x, y^*),
$$

namely,

$$
\begin{cases}
  x^* \in X, & \theta_1(x) - \theta_1(x^*) + \left< x - x^*, -A^T y^* \right> \geq 0, \quad \forall x \in X, \\
  y^* \in Y, & \theta_2(y) - \theta_2(y^*) + \left< y - y^*, Ax^* \right> \geq 0, \quad \forall y \in Y.
\end{cases}
$$

So, finding a solution point of (27) amounts to finding $u^* \in M$ such that

$$
u^* \in M, \quad \Phi(u) - \Phi(u^*) + \left< u - u^*, Mu^* \right> \geq 0, \quad \forall u \in M.
$$

Setting $u := u^*$ in (29) together with (32) shows

$$
\left\| u^* - u^{k+1} \right\|_H^2 \leq \left\| u^* - u^k \right\|_H^2 - \left\| u^k - u^{k+1} \right\|_H^2,
$$

that is, the sequence generated by N-PDHG1 is contractive that thus N-PDHG1 converges globally. The sublinear convergence rate of N-PDHG1 is similar to the proof of P-ALM. Note that convergence of N-PDHG1 does not need the strongly convexity of $\theta_1$ and allows more flexibility on choosing the proximal parameter $r$.

Finally, it is not difficulty from the first-order optimality conditions of the involved subproblems in N-PDHG2 that

$$
u^{k+1} \in M, \quad \Phi(u) - \Phi(u^{k+1}) + \left< u - u^{k+1}, Mu \right> \geq \left< u - u^{k+1}, \widetilde{H}(u^k - u^{k+1}) \right>
$$

for any $u \in M$, where

$$
\widetilde{H} = \begin{bmatrix}
  rI & A^T \\
  A & \frac{1}{\tau}AA^T + Q
\end{bmatrix}
$$

and this matrix $\widetilde{H}$ is positive definite for any $r > 0$ and $Q > 0$. So, N-PDHG2 also converges globally with a sublinear convergence rate. This matrix $H$ is what we discussed in Section 1 and could reduce to that in [11] with $Q = \delta I$ for any $\delta > 0$.

5 Concluding remarks

In this paper, we have proposed a novel augmented Lagrangian method for solving a family of convex optimization problem subject to linear equality or inequality constraints. An extension of P-ALM and two related PDHG algorithms are also studied for solving a multi-block separable convex optimization and a saddle-point problem, respectively. The global convergence and the sublinear convergence rate of the proposed algorithm have been established in terms
of objective gap plus constraint residual and the primal residual. Our proposed algorithm is motivated by the resent work [11] from a different point of view for modifying the involved x-subproblem, while the work [11] aims to modify the dual update to weaken the convergence conditions. Our proposed algorithm and the method B-PLM will make their respective advantages complementary to each other, and users could select each of them for a specified problem. If we denote the output of P-ALM (or other three algorithms) as \( \tilde{w}^k \), then the convergence of a direct extension of the proposed P-ALM with the following relaxation step

\[
\tilde{w}^{k+1} = \tilde{w}^k + \gamma (\tilde{w}^k - w^k), \quad \gamma \in (0, 2)
\]

can be proved similarly. And finally, it is easy to get the worst-case \( O\left(\frac{1}{2\gamma(1+\gamma)}\right) \) convergence rate in the ergodic sense. For the general correction step as mentioned in [9]

\[
w^{k+1} = w^k + \gamma M (\tilde{w}^k - w^k),
\]

where \( M \) is a nonsingular matrix and \( \gamma > 0 \), we believe it also converges.

References


