New IP-based lower bounds for small Ramsey numbers using circulant graphs

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Abstract

In this article we address the problem of finding lower bounds for small Ramsey numbers $R(m,n)$ using circulant graphs. Our constructive approach is based on finding feasible colorings of circulant graphs using Integer Programming (IP) techniques. First we show how to model the problem as a Stackelberg game and, using the tools of bilevel optimization, we transform it into a single-level IP problem with an exponential number of constraints. Using related results from graph theory, we provide strengthening valid inequalities for whose separation we develop a tailored branch-and-bound algorithm. With our new method, we improve 17 best-known lower bounds for $R(3,n)$ where n ranges between 26 and 46. The graphs representing feasible circulant (m,n)-colorings used to prove these new lower bounds are made publicly available. To the best of our knowledge, this is a first IP-based approach to tackle this very challenging combinatorial optimization problem.

Key words: Combinatorial optimization, Ramsey numbers, Bilevel optimization, Graph theory, Maximum clique problem.

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1. Introduction

In 1928, the British mathematician Frank Plumton Ramsey proved a theorem showing that "any structure will necessarily contain an orderly substructure" [10]. This theorem is at the core of Ramsey theory and can be interpreted in various ways. In this article we focus on the Ramsey's theorem stated in the context of graph theory and we develop a new branch-and-cut algorithm, based on Integer Programming (IP) techniques, to compute lower bounds for Ramsey numbers using circulant graphs.

Let K_q denote a complete graph with q vertices (i.e., of order q). A 2-coloring of a graph is a partition of its set of edges in two subsets. Accordingly, the Ramsey's theorem for 2-colorings of graphs can be stated as follows:

Theorem 1 (Ramsey's theorem [22], 2-coloring version)**.** *Given $m \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that any 2-colored complete graph K_q contains a monochromatic subgraph K_m.*

The essence of Ramsey theory is in finding patterns amongst apparent chaos, or, according to Theodore Motzkin, "Ramsey theory implies that a complete disorder is an impossibility" [10]. Indeed, the above result states that for any $m \in \mathbb{N}$, there exists $q \in \mathbb{N}$, such that, no matter how we color the edges of K_q, it will always contain a monochromatic m-clique (i.e., a K_m). The smallest such number q is called the diagonal Ramsey number $R(m, m)$. A more general definition of Ramsey numbers is given below.

Definition 1 (Ramsey number $R(m, n)$, 2-color version)**.** *Given $m, n \in \mathbb{N}$, a Ramsey number, written as $R(m, n)$, is the smallest $q \in \mathbb{N}$ such that the 2-colored graph K_q, whose edges are colored in either blue or red, implies a blue subgraph K_m, or a red subgraph K_n.*

Applications of Ramsey theory can be found in fields of communications, information retrieval in computer science, or in decision making ([23]). In simple terms, the Ramsey number $R(m, n)$ gives the solution to the "party problem", which asks for the minimum number of guests $R(m, n)$ that must be invited so that at least m will know each other or at least n will not know each other. In the language of graph theory, the Ramsey number is the minimum number of vertices $R(m, n)$ such that all undirected simple graphs of order $R(m, n)$ contain a clique of order m or an independent set of order n. We recall that an independent set is a subset of vertices inducing a graph with no edges and it corresponds to the monochromatic red subgraph K_n (since independent sets are cliques in complement graphs). On the other hand, the clique corresponds to the monochromatic blue subgraph K_m. Ramsey's theorem states that $R(m, n)$ exists for all m and n. Moreover, the Ramsey number $R(m, 2)$ is equal to m and, by symmetry, it is true that

$$R(m, n) = R(n, m).$$

The values of Ramsey numbers are known only for very small values of m and n, cf. Table 1. For the remaining combinations of m and n, the values of $R(m, n)$ remain unknown and only lower and upper bounds for $R(m, n)$ are available. Up-to-date bounds are provided in a dynamic survey by Radziszowski [21].

As observed by Paul Erdös, the problem of finding Ramsey numbers is not only difficult from

m	n	$R(m,n)$
3	4	9
3	5	14
3	6	18
3	7	23
3	8	28
3	9	36
4	4	18
4	5	25

Table 1: The known values for $R(m,n)$.

from a theoretical perspective but also extremely difficult from computational point of view. In what follows we report a famous citation of Paul Erdös which gives an idea of how difficult the problem is [29]:

> "Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number $R(5,5)$. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number $R(6,6)$, however, we would have no choice but to launch a preemptive attack."

One of the principal reasons for this computational difficulty is that the number of possible 2-colorings of a graph increases exponentially with just a small increase in either m or n and with the size q of K_q to be colored (see the computational Section 5).

1.1. Basic notation, definitions and properties

Before we state our contribution, we introduce some notation. Given a simple undirected graph G, let $V(G)$ denote its set of vertices and $E(G)$ its set of edges. Two vertices u and v of V are called *neighbours* or *adjacent vertices* if there is an edge $\{u,v\} \in E(G)$. Let $N(v) = \{u \in V | \{u,v\} \in E\}$ denote the *neighborhood* of a vertex $v \in V(G)$, i.e., the set of vertices adjacent to a vertex v. In addition, let $G(v)$ denote the graph induced by the neighborhood of the vertex $v \in V(G)$, i.e., with vertex set $V(G(v)) = N(v)$ and edge set $E(G(v))$ comprising those edges in $E(G)$ with both endpoints in $N(v)$. The *Maximum Clique Problem* (MCP) asks for determining a largest clique in a graph. The size of this clique, i.e., the solution value of the MCP, is known as the *clique number* $\omega(G)$ of the graph .

Definition 2 ((m,n)-coloring of K_q)**.** *Given $m,n \in \mathbb{N}$, and $q \in \mathbb{N}$, an (m,n)-coloring is the coloring of edges of K_q into red and blue such that the maximum clique in the blue colored subgraph is smaller than m, and the maximum clique in the red colored subgraph is smaller than n.*

The following proposition allows to state the problem of finding the Ramsey Number $R(m,n)$ as a combinatorial optimization problem:

Proposition 1. *Given $m,n \in \mathbb{N}$, if $q \in \mathbb{N}$ is the order of the largest complete graph K_q that admits a feasible (m,n)-coloring, then $R(m,n) = q+1$.*

In the literature, *circulant graphs* are often successfully used to construct feasible (m, n)-colorings (see, e.g., [21]).

Definition 3 (Circulant graph)**.** *A simple undirected graph G is called circulant with respect to a list $L \subseteq \{1, \ldots, \lfloor \frac{|V(G)|}{2} \rfloor\}$, if i-th vertex is adjacent to the vertex $(i + j - 1)$ and vertex $(i - j - 1)$, for each $j \in L$.*

In the above definition, we assume that indices are always calculated with respect to mod $|V(G)| + 1$.

Definition 4 (Circulant (m, n)-coloring of K_q)**.** *A (m, n)-coloring of K_q is called circulant if the underlying blue (red) graph is circulant. The largest $q \in \mathbb{N}$ such that K_q admits a circulant (m, n)-coloring is denoted by $C(m, n)$.*

The following observation allows to establish a relation between Ramsey numbers $R(m, n)$ and the values $C(m, n)$:

Observation 1. *Given $q \in \mathbb{N}$, if there exists a feasible (m, n)-coloring of K_q, then $R(m, n) > q$. Moreover, $R(m, n) \geq C(m, n) + 1$.*

An example of a circulant coloring of K_{17} is given in Figure 1. For the blue subgraph, $L = \{1, 2, 4, 8\}$, and for the red one, $L = \{3, 5, 6, 7\}$. One of the largest cliques for each coloring is represented by solid lines, i.e., $\{4, 8, 17\}$ and $\{6, 11, 17\}$, in blue and red, respectively. The remaining edges are depicted with dashed lines. In Figure 2 we show the adjacency matrix (with dark blue (red) cells associated to the solid lines of the blue (red) clique in K_{17}). Hence, according to Observation 1, $R(4, 4) \geq 18$. As stated in [14], this coloring is unique for K_{17}. Furthermore, in [11] it has been established that $R(4, 4) \leq 18$ (due to the recursion $R(m, n) \leq R(m - 1, n) + R(m, n - 1)$). In the Appendix, see Section 8, we report for $m = 5$ and $n = 5$ the adjacency matrix of K_{41} corresponding to a feasible circulant $(5, 5)$-coloring ($C(5, 5) = 41$ and $43 \leq R(5, 5) \leq 48$).

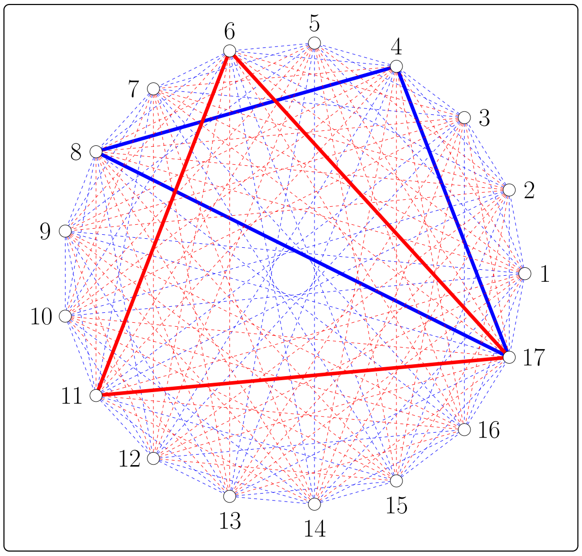

Figure 1: Circulant $(4, 4)$-coloring of K_{17}.

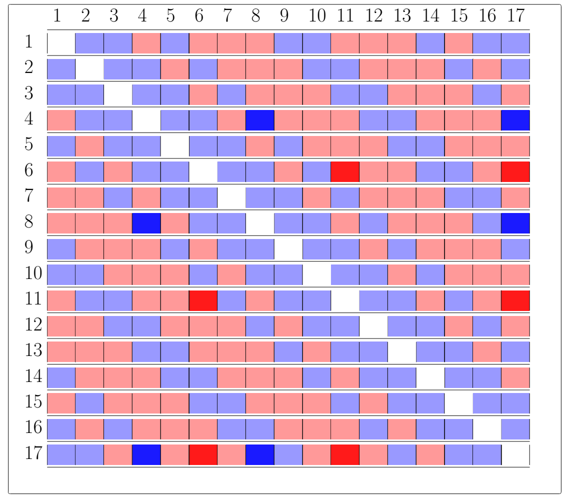

Figure 2: Circular adjacency matrix for the circulant (m, n)-coloring of Figure 1.

1.2. Related works

There exists a large body of literature on finding bounds for Ramsey numbers and an up-to-date list of best-known lower and upper bounds is regularly updated in a dynamic survey provided by Radziszowski [21]. In terms of methodology for finding lower bounds, we distinguish between nonconstructive and constructive methods. The former ones use counting and probabilistic arguments (see, e.g., [4]). The latter ones explicitly construct a feasible (m, n)-coloring of K_q (for a given $q \in \mathbb{N}$), thus implying that $R(m, n) \geq q + 1$.

The first results concerning circulant colorings date back to 1966, when in the Ph.D. dissertation by J.G. Kalbfleisch [14], some first circulant colorings have been established. Many additional results have been made available in [12]. Concerning the most recent constructive approaches for circulant graphs (which is the focus of our paper), a method based on edge coloring of circulant graphs is provided in [31]. In [13] the authors proposed an approach for generating circulant triangle-free graphs of prime order, with which state-of-the-art lower bounds for $R(3, n)$ and $34 \leq n \leq 49$ are obtained. Starting from (block) circulant graphs and either by applying heuristics or by modifying the construction of circulant graphs, Exoo and Tatarevic [5] provided some state-of-the-art lower bounds for $R(4, n)$ with $n \geq 8$. Finally, a relaxation of a circulant coloring, known as a distance coloring, has been used in [17] to establish some additional lower bounds for $R(m, n)$ with $m \geq 5$ and $n \geq 9$.

1.3. Contribution of the paper

We propose a branch-and-cut method to calculate lower bounds for small Ramsey numbers $R(m, n)$ based on circulant coloring. First we show how to model the problem as a Stackelberg game with a single leader and two independent followers, using the tools of bilevel optimization. We then reformulate the problem as a single-level optimization problem and provide an Integer Programming (IP) formulation with an exponential number of constraints for which we develop a branch-and-cut approach. Using the related results of graph theory, we derive strengthening valid inequalities. To separate them, we develop a tailored branch-and-bound procedure. Specifically, we focus on circulant graphs, whose coloring provides valid lower bounds for Ramsey numbers. With our branch-and-cut method for circulant graphs, we manage to improve 17 best-known lower bounds for $R(3, n)$ where n ranges between 26 and 46. The graphs representing feasible (m, n)-colorings used to prove these new bounds are available online at `https://github.com/psanse/Ramsey-Numbers`. To the best of our knowledge, this is a first IP-based approach to solve this difficult combinatorial optimization problem.

As mentioned in the previous paragraph, thanks to the newly developed branch-and-cut method, we managed to construct feasible circulant colorings which are used to prove the major result of our work. Precisely, the list of the 17 new lower bounds for small Ramsey numbers $R(m, n)$ is summarized in the following theorem:

Theorem 2.

$$\begin{array}{llll} R(3,26) \geq 161, & R(3,28) \geq 179, & R(3,30) \geq 197, & R(3,31) \geq 208 \\ R(3,33) \geq 226, & R(3,34) \geq 233, & R(3,35) \geq 244, & R(3,36) \geq 255 \\ R(3,37) \geq 265, & R(3,38) \geq 275, & R(3,39) \geq 286, & R(3,40) \geq 297 \\ R(3,41) \geq 311, & R(3,42) \geq 320, & R(3,43) \geq 333, & R(3,44) \geq 339 \\ R(3,46) \geq 362. & & & \end{array}$$

Proof. The constructive proof of the theorem is provided in the Appendix, see Section 8. □

Moreover, improving the lower bounds for $R(3, n)$ recursively and automatically improves other lower bounds for Ramsey numbers for larger values of m and n. This can be done using the following formulas:

$$R(3, 4n + 1) \geq 6R(3, n + 1) - 5 \quad [2] \tag{1}$$

$$R(5, n) \geq 4R(3, n - 1) - 3 \quad [33] \tag{2}$$

$$R(3, n, l + 1) \geq 4R(n, l) - 3 \quad [33] \tag{3}$$

Next to each of these three formulas we report the article in which the result has been proposed and proven. The formula (3) refers to generalized Ramsey numbers related to 3-coloring of the graphs.

The article is organized as follows. In Section 2, we state the problem of finding Ramsey numbers as a bilevel optimization problem, and show the related single-level IP formulation. In Section 3 we focus on finding an (m, n)-coloring in circulant graphs. Section 4 explains a tailored branch-and-bound procedure used to separate inequalities of our IP model. In Section 5 we report the obtained computational results, and we draw final conclusions in Section 6.

2. Bilevel Formulation for Improving the Lower Bound

Following Proposition 1, we can state the problem of finding the Ramsey number $R(m, n)$ as a Stackelberg game. In a Stackelberg game, there are two players, a leader and a follower, the leader moves first and, after observing the action of the leader, the follower maximizes her payoff. The goal of the leader is to maximize her payoff, while anticipating the best response of the follower. Stackelberg games can be modeled using the tools of bilevel optimization, and a state-of-the-art computational method for bilevel mixed integer programs can be found in [6, 7]. See also the recent survey by Kleinert et al. [16] for further references.

Let $\underline{R}(m,n)$ refer to the best known lower bound, and $\bar{R}(m,n)$ to the best known upper bound for $R(m,n)$. Furthermore, let $\underline{q} = \underline{R}(m,n) - 1$ and $\bar{q} = \bar{R}(m,n) - 1$. Finding the value q in Proposition 1 can be stated as a leader-follower game. The leader chooses a clique K_q of $K_{\bar{q}}$ of maximum size, and colors the edges of K_q into blue and red. There are two followers: The first follower takes the blue edges and searches for a largest clique in the blue subgraph. The second follower searches for a largest clique in the red subgraph. The leader maximizes q, and searches for a 2-coloring of the edges, under the constraints that the optimal response of the first follower (i.e., the size of the maximum clique in the blue graph) is at most $m-1$, and the optimal response of the second follower (i.e., the size of the maximum clique in the red graph) is at most $n-1$.

This bilevel optimization problem can be modeled using the following binary variables:

$$x_i = \begin{cases} 1, & \text{if vertex } i \text{ is part of } K_q \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, \dots, \bar{q}$$

$$b_e = \begin{cases} 1, & \text{if edge } e \text{ is colored blue in } K_q \\ 0, & \text{otherwise} \end{cases} \qquad e \in E(K_{\bar{q}})$$

$$r_e = \begin{cases} 1, & \text{if edge } e \text{ is colored red in } K_q \\ 0, & \text{otherwise} \end{cases} \qquad e \in E(K_{\bar{q}})$$

The problem of the leader is given as:

$$\begin{aligned}
q := \max \sum_{i=1}^{\bar{q}} x_i & & \text{(4a)}\\
\text{s.t.} \quad r_e + b_e \geq x_i + x_j - 1 & \quad e = \{i,j\} \in E(K_{\bar{q}}) & \text{(4b)}\\
r_e + b_e \leq 1 & \quad e \in E(K_{\bar{q}}) & \text{(4c)}\\
r_e \leq \min\{x_i, x_j\} & \quad e = \{i,j\} \in E(K_{\bar{q}}) & \text{(4d)}\\
b_e \leq \min\{x_i, x_j\} & \quad e = \{i,j\} \in E(K_{\bar{q}}) & \text{(4e)}\\
\Phi_b(x,r,b) \leq m - 1 & & \text{(4f)}\\
\Phi_r(x,r,b) \leq n - 1 & & \text{(4g)}\\
(x,r,b) \in \{0,1\}^{\bar{q}+2\binom{\bar{q}}{2}} & & \text{(4h)}
\end{aligned}$$

Additional valid inequalities are added to break symmetries, i.e., we fix the first $\underline{q}$ vertices to one, and ensure that the vertices with indices $\underline{q}+1$ to q are part of the optimal solution:

$$x_i = 1 \qquad i \in \{1, \dots, \underline{q}\} \tag{5}$$

$$x_i \leq x_{i-1} \qquad i \in \{\underline{q}+1, \dots, \bar{q}\} \tag{6}$$

Inequalities (4b) together with (4c) ensure that for the subgraph K_q, each edge is colored either red or blue. For the edges of $E(K_{\bar{q}}) \setminus E(K_q)$ (for which at least one of the end points is not in K_q), we use constraints (4d)-(4e) to fix the variables r and b to zero. Finally, $\Phi_b(x, r, b)$ (respectively $\Phi_r(x, r, b)$) represents the value of the maximum clique on the blue, respectively, red subgraph. Constraints (4f) and (4g) correspond to the value-function reformulation of the lower-level subproblems, representing the two followers.

The lower-level subproblems correspond to the $\mathcal{NP}$-hard problem of finding a maximum clique, and they are modeled as integer programs. Let y_i, $1 \leq i \leq \bar{q}$, be a binary variable indicating whether the vertex i belongs to a maximum clique found in the blue subgraph. Then, for a given decision $(\bar{x}, \bar{r}, \bar{b})$ of the leader, the lower-level subproblem of the first follower corresponds to the following IP:

$$\Phi_b(\bar{x}, \bar{r}, \bar{b}) = \max \sum_{i=1}^{\bar{q}} y_i \tag{7a}$$

$$\text{s.t. } y_i \leq \bar{x}_i \qquad 1 \leq i \leq \bar{q} \tag{7b}$$

$$y_i + y_j \leq 1 + \bar{b}_e \qquad e = \{i, j\} \in E(K_{\bar{q}}) \tag{7c}$$

$$y \in \{0, 1\}^{\bar{q}} \tag{7d}$$

Constraints (7b) ensure that vertex i can belong to a maximum clique only if it belongs to the subgraph K_q chosen by the leader. Constraints (7c) guarantee that two vertices i and j can belong to a maximum clique only if the edge connecting them is colored in blue.

Similarly, using z_i binary variables to indicate whether the vertex i, $1 \leq i \leq \bar{q}$, is part of the maximum clique in the red subgraph, we model the second lower-level subproblem as:

$$\Phi_r(\bar{x}, \bar{r}, \bar{b}) = \max \sum_{i=1}^{\bar{q}} z_i \tag{8a}$$

$$\text{s.t. } z_i \leq \bar{x}_i \qquad 1 \leq i \leq \bar{q} \tag{8b}$$

$$z_i + z_j \leq 1 + \bar{r}_e \qquad e = \{i, j\} \in E(K_{\bar{q}}) \tag{8c}$$

$$z \in \{0, 1\}^{\bar{q}} \tag{8d}$$

In the following subsection, we show how this bilevel integer programming problem can be reformulated as a single-level IP with an exponential number of constraints.

The problem can also be interpreted as an interdiction problem: coloring an edge in blue, means "interdicting" this edge for the red follower, and vice versa. Contrary to standard network interdiction problems known from the literature (see, e.g., the recent surveys in [16, 28]), the leader does not have a fixed budget in terms of the number of edges that can be interdicted. Instead, the budget is imposed per every single edge, i.e., the leader is forced to interdict either the blue or the red follower, but it is not possible to interdict both of them at the same time. For related studies on clique-interdiction problems, see [8, 9] and further references therein.

2.1. Single-level reformulation

In order to convexify constraints (4f) and (4g) of model (4), we show the following result:

Proposition 2. *Given a decision of the leader determined by the vector* $(\bar{x}, \bar{r}, \bar{b}) \in \{0,1\}^{\bar{q}+2\binom{\bar{q}}{2}}$, *there always exists an optimal solution of the problem* (7) *which can be also obtained by solving*

$$\max \left\{ \sum_{i=1}^{\bar{q}} \bar{x}_i y_i - \sum_{ij \in E(K_{\bar{q}})} \bar{r}_{ij} y_i y_j : y \in \{0,1\}^{\bar{q}} \right\}. \tag{9}$$

Proof. Let $S_x = \{i \in V(K_{\bar{q}}) : \bar{x}_i = 1\} \subseteq V(K_{\bar{q}})$ be the subset of vertices chosen by the leader. Then, the problem (9) can also be restated as

$$\max_{S \subseteq S_x} \left\{ |S| - \sum_{i,j \in S} \bar{r}_{ij} \right\}.$$

It is not difficult to see that the value of this problem is precisely the clique number of the graph induced by the edges colored in blue by the leader (see, e.g., Proposition 1 in [9]). □

Hence, we can convexify the constraints $\Phi_b(x, r, b) \leq m - 1$ by using the following exponential family of clique-cuts:

$$\sum_{i \in S} x_i - \sum_{i \in S} \sum_{j \in S, j > i} r_{ij} \leq m - 1 \quad S \subseteq V(K_{\bar{q}}), |S| \geq m. \tag{10}$$

These cuts are stating that the size of any clique in the blue subgraph of K_q cannot be larger than $m - 1$. Similarly, to replace $\Phi_r(x, r, b) \leq n - 1$ in the above model, we will add the following cuts:

$$\sum_{i \in S} x_i - \sum_{i \in S} \sum_{j \in S, j > i} b_{ij} \leq n - 1 \quad S \subseteq V(K_{\bar{q}}), |S| \geq n. \tag{11}$$

That way, by replacing constraints (4f) and (4g) with clique-cuts (10) and (11), respectively, we obtain a single-level problem reformulation with an exponential number of constraints.

2.2. Kalbfleisch Inequalities

Kalbfleisch [15] provides the following result which is valid for feasible 2-colorings of a clique.

Proposition 3 ([11, 15])**.** *In a feasible* (m, n)*-coloring, the number of blue edges incident to any vertex is at most* $R(m-1, n) - 1$.

Proof. We repeat the proof by Kalbfleisch here: assume there is a vertex v with $R(m-1, n)$ or more incident blue edges. Let N_b be the blue neighbors of this vertex. Since $|N_b| \geq R(m-1, n)$, then the subgraph induced by N_b contains either a blue clique of size $m-1$, or a red clique of size n. Hence, in both cases, by connecting the vertex v with blue edges to all vertices from N_b, we are guaranteed to obtain a blue clique of size m or a red clique of size n, hence violating the (m,n)-coloring property. □

This translates into following valid inequalities:

$$\sum_{e \in \delta(i)} b_e \leq x_i \cdot [\bar{R}(m-1,n)-1] \qquad i = 1,\dots,\bar{q} \tag{12}$$

In particular, when $m = 3$, this constraint states that the maximum blue-degree of every vertex is at most $n-1$, since the blue neighborhood set of any vertex must be an independent set. We use the best known upper bounds for $R(m,n)$ in these constraints, as for most of the relevant cases, the exact value remains unknown.

The same arguments from the proof of Proposition 3 can be repeated for the red color, which means that we can also derive the corresponding inequalities:

$$\sum_{e \in \delta(i)} r_e \leq x_i \cdot [\bar{R}(m,n-1)-1] \qquad i = 1,\dots,\bar{q} \tag{13}$$

2.3. Turán inequalities

A graph G is said to be m-clique-free if $\omega(G) \leq m-1$. We now attempt to answer the following question: what is the maximum number of edges of a graph with q vertices which is m-clique-free? The answer to this question was given by Turán in 1941:

Proposition 4 ([30]). *Among all m-clique-free graphs on q vertices, Tourán graph $T_{q,m-1}$ has the most edges.*

Turán graph $T_{q,m-1} = (V,E)$ is constructed as follows: we create $m-1$ disjoint sets of vertices, $V = V_1 \cup \cdots \cup V_{m-1}$, and insert all edges between vertices in different sets. Obviously, such constructed graph is m-clique-free, since any set of m vertices has two vertices in the same set V_i. The number of edges in such a graph is maximized, when the sets V_i are as evenly sized as possible, i.e., when $||V_i| - |V_j|| \leq 1$ for all $1 \leq i,j \leq m-1$. We call such a graph on q vertices Turán's graph $T_{q,m-1}$.

The number of edges in Turán's graph $T_{q,m-1}$ can easily be calculated (see, e.g., [1]):

Proposition 5. *The maximum number of edges in a graph with q vertices which is m-clique-free is*

$$\left\lfloor \left(1 - \frac{1}{m-1}\right) \frac{q^2}{2} \right\rfloor.$$

Hence, the following valid inequalities provide an upper bound on the number of blue/red edges in any feasible solution:

$$\sum_{e \in E(K_{\bar{q}})} b_e \leq \left\lfloor \frac{(m-2)\bar{q}^2}{2(m-1)} \right\rfloor \tag{14}$$

$$\sum_{e \in E(K_{\bar{q}})} r_e \leq \left\lfloor \frac{(n-2)\bar{q}^2}{2(n-1)} \right\rfloor \tag{15}$$

Rather than having on the right-hand-side the upper bound on the size of the input graph, $\bar{q}$, once could obtain much tighter bounds by using the real size of the graph, which is given as $\sum_i x_i$. Unfortunately, replacing $\bar{q}$ with $\sum_{i=1}^{\bar{q}} x_i$ in the above formulas would lead to non-linear constraints. Instead, we can look at the problem from an alternative perspective and ask: what is the minimum number of edges that has to be removed from K_q, to make sure that the remaining graph is m-clique-free?

Corollary 1. *The minimum number of edges that has to be removed from K_q ($q \geq m$) to ensure that the remaining graph is m-clique free is given as*

$$\tau(m, K_q) = \binom{q}{2} - \left\lfloor \frac{(m-2)q^2}{2(m-1)} \right\rfloor.$$

Using the latter result, we can impose the following constraints for any subset of vertices $S \subseteq V(K_{\bar{q}})$ whose cardinality is larger than $m-1$:

$$\sum_{i,j \in S} r_{ij} \geq \tau(m, K_{|S|}) \left(\sum_{i \in S} x_i - |S| + 1 \right) \qquad S \subseteq V(K_{\bar{q}}), |S| \geq m. \tag{16}$$

If we interpret the coloring of edges in red as "interdiction" of the blue subgraph, then these inequalities impose the minimum number of edges that need to be interdicted in the blue subgraph, to ensure that it is m-clique-free. The inequalities are not binding if the vertices of S are not fixed to one, i.e., if $\sum_{i \in S} x_i < |S|$. Similarly, for "interdicting" the red subgraph, we impose the following constraints:

$$\sum_{i,j \in S} b_{ij} \geq \tau(n, K_{|S|}) \left(\sum_{i \in S} x_i - |S| + 1 \right) \qquad S \subseteq V(K_{\bar{q}}), |S| \geq n. \tag{17}$$

3. Computing lower bounds for Ramsey numbers on circulant graphs

The mathematical model given in Section 2 works in a search space of size $O(2^{\binom{\bar{q}}{2}})$, and can be applied to find Ramsey numbers for very small values of m and n. In what follows, we provide an adaptation of the latter model with which we improved some of the state-of-the art lower bounds for Ramsey numbers $R(3, n)$. The major key elements of this adaptation are: removal of x variables thanks to the multi-branching approach, and reduction of the search space by focusing on circulant colorings only.

3.1. Multi-branching strategy

Rather than working in the space of (x, r, b) variables, as given in Model (4), we apply a multi-branching approach in which we branch on the size of the graph colored by the leader. That way, we branch on the auxiliary variable corresponding to $q := \sum_{i=1}^{\bar{q}} x_i$. Once the value of q is fixed, the variables x can be eliminated (due to constraints (5)-(6)), and we obtain the following simplified set of constraints describing a feasible (m, n)-coloring of K_q:

$$r_e + b_e = 1 \qquad e \in E(K_q) \tag{18a}$$

$$\sum_{i \in S} \sum_{j \in S, j > i} r_{ij} \geq \tau(m, K_{|S|}) \qquad S \subseteq V(K_q), |S| \geq m \tag{18b}$$

$$\sum_{i \in S} \sum_{j \in S, j > i} b_{ij} \geq \tau(n, K_{|S|}) \qquad S \subseteq V(K_q), |S| \geq n \tag{18c}$$

$$\sum_{j \in N(i)} b_{ij} \leq \bar{R}(m-1, n) - 1 \qquad i = 1, \ldots, q \tag{18d}$$

$$\sum_{j \in N(i)} r_{ij} \leq \bar{R}(n-1, m) - 1 \qquad i = 1, \ldots, q \tag{18e}$$

$$(r, b) \in \{0, 1\}^{2\binom{q}{2}} \tag{18f}$$

That way, our problem turns into a problem of finding a feasible solution satisfying the constraints of Model (18). We notice that in this model we are not including clique-cuts, as they are dominated by Tourán's inequalities, which is shown in Proposition 6.

Proposition 6. *For a subset $S \subseteq V(K_q)$ such that $|S| \geq m$, inequalities* (18b) *dominate the clique-cuts*

$$\sum_{ij \in S} r_{ij} \geq |S| - m + 1.$$

Proof. Indeed, as the left-hand sides of both constraints are the same, this follows directly from the fact that $\tau(m, K_q) \geq q - m + 1$, for any $q \geq m$. □

Separation of Turán's inequalities (18b) *and* (18c). Let $(\bar{r}, \bar{b})$ be a binary vector representing the current solution of the leader. We build the blue graph using the edges $e \in E(K_q)$ such that $\bar{b}_e = 1$, and search for a maximum clique, say $\bar{K}$ in this subgraph. If $|V(\bar{K})| \geq m$, we calculate $\tau(m, \bar{K})$ (this can be done in constant time) and insert the cut (18b) associated to $S = V(\bar{K})$ into the relaxed master problem. A similar procedure can be applied to separate constraints (18c). Hence, we notice that the separation of the Turán's inequalities (18b) and (18b) boils down to solving the maximum clique problems in two circulant monochromatic graphs. To this end we employ the tailored branch-and-bound procedure for finding maximum cliques described in Section 4.

3.2. Circulant Colorings

Some of the best-known lower bounds for $R(m, n)$ are obtained from a circulant (m, n)-coloring (for example, $R(4, 4) = C(4, 4) + 1$ or $\underline{R}(6, 6) = 102$, see [15, 21] for further details). By focusing on circulant matrices, we can reduce the size of the search space from $2^{\binom{q}{2}}$ (for the general case) to $2^{\lfloor \frac{q}{2} \rfloor}$. This is one of the reasons why most of the successful approaches for improving lower bounds for Ramsey numbers heavily exploit the circulant matrix property (see [5, 13, 17] for some recent examples).

The adjacency matrix of a circulant graph is called circulant matrix. Besides being symmetric, we observe that the entries of this matrix are completely determined by the first half of its first row. We therefore extend the Model (18) with the inequalities (19) and (20), to which we refer as *circulant constraints*:

$$b_{ij} = b_{i-1,j-1} \quad 2 \le i \le q-1, i+1 \le j \le q-1. \tag{19}$$

As the coloring matrix has to be symmetric, and we are working on the upper diagonal matrix only, we also have to ensure that the first raw is a palindrome, i.e., we also need to add constraints:

$$b_{1i} = b_{1,q-i+2} \quad 2 \le i \le \left\lfloor \frac{q}{2} \right\rfloor. \tag{20}$$

Circulant constraints are added in the initialization phase of the branch-and-cut approach. The advantage of including circulant constraints and focusing on circulant matrices only is that the respective solutions can be calculated very fast. For example, for K_{17} and $(m, n) = (4, 4)$, several hours of CPU time are necessary to find a feasible coloring, whereas after adding the circulant constraints, the CPU time is reduced to few seconds.

3.3. Computing $C(m, n)$ values

With the multi-branching strategy, we are also able to establish the values of $C(m, n)$, provided that our algorithm terminates within the given time limit. More precisely, we start the multi-branching approach by choosing q from the interval $[\underline{C}(m, n), \bar{R}(m, n) - 1]$ (in increasing order, where $\underline{C}(m, n)$ refers to the known lower bound for $C(m, n)$). Our branch-and-cut method on circulant graphs can then determine infeasibility for a given choice of q, in which case we proceed with $q + 1$. It is well-known (see, e.g., [15]) that circulant lower bounds are not consecutive, and hence, after multiple iterations of detecting infeasibility, it might be possible to find another larger value of q for which a circulant (m, n)-coloring exists.

4. Separation routines

The separation of Turán's inequalities (18b) and (18c) requires finding a maximum clique in the circulant monochromatic graphs obtained from the given coloring provided by the leader variables (as explained in Section 3.1). To this end, we developed an efficient branch-and-bound procedure (enhanced by a heuristic preprocessing phase) whose major ingredients are described in this section. The procedure is inspired by the state-of-the-art MCP algorithms, see, e.g., [18], [25], [26] and [27], and adapted to the specific nature of the circulant graphs.

The nodes of the branching tree required to solve the IP Model (18) (in which all the blue b and red r variables take integer values) are associated to monochromatic blue graphs (G_b) and monochromatic red graphs (G_r). Precisely, G_b has the vertex set $V(G_b) = V(K_q)$ and edge set $E(G_b) = \{\{i, j\} \in E(K_q) : b_{ij} = 1\}$; G_r has the vertex set $V(G_r) = V(K_q)$ and the edge set $E(G_r) = \{\{i, j\} \in E(K_q) : r_{ij} = 1\}$. Note that these graphs are circulant since constraints (19) and (20) are imposed. An example of a circulant monochromatic graph G_b (corresponding to the circulant coloring of K_{17} shown in Figure 1) is provided in part (a) of Figure 3.

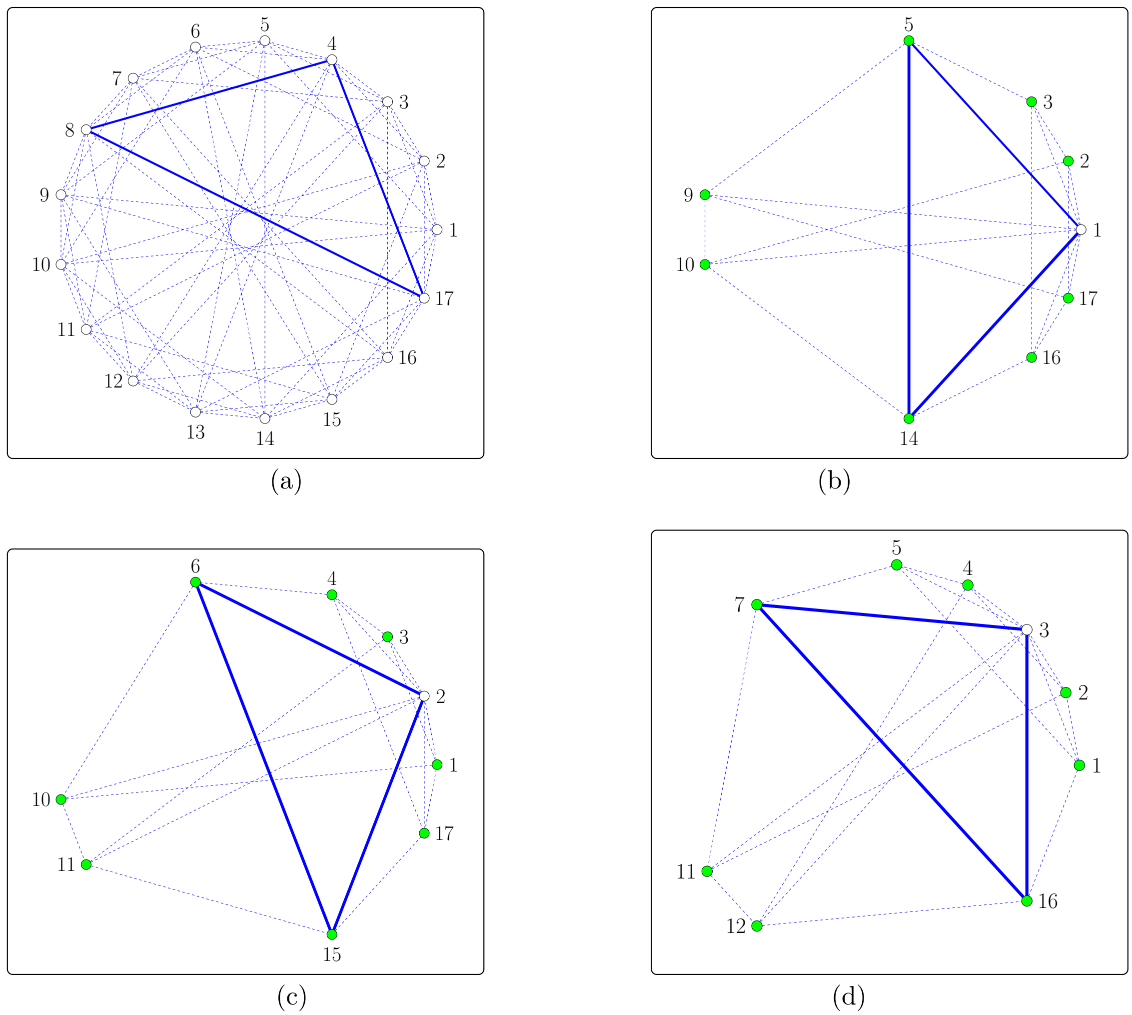

Figure 3: An example of a circulant graph with respect to the list $L = \{1, 2, 4, 8\}$ featuring a maximum clique of size 3, i.e., the set $\{4, 8, 17\}$ (part (a)). In parts (b), (c) and (d) we show the isomorphic graphs induced by the vertices 1, 2 and 3, and their respective neighbourhoods.

The separation problem calls for solving the maximum clique problem in G_b and in G_r, i.e., to determine $\omega(G_b)$ and $\omega(G_r)$. Computing the clique number of a circulant graph is known to be $\mathcal{NP}$-hard as shown in Theorem 1 of [3]; therefore it is very unlikely that a polynomial-time separation algorithm exists. However, we exploit the structure of the circulant graphs G_b and G_r to design an efficient branch-and-bound separation algorithm as described in the remainder of the section.

A relevant property of circulant graphs is that the graphs induced by the neighborhood of each individual vertex are isomorphic. We visually demonstrate this property with the help of Figure 3. Part (b) of the figure depicts the vertex 1 together with its neighborhood, i.e., the set of vertices $\{2, 3, 5, 9, 10, 14, 16, 17\}$ and also the graph induced by this set of vertices. The vertices of the neighborhood appear in green. In a similar fashion, parts (c) and (d) depict the vertices 2 and 3 respectively, together with their neighborhoods and the graphs they induce. By construction, the 3 induced graphs represented are isomorphic and appear simply "rotated". A maximum clique in the graph G_b of the example (part (a)) is drawn using thicker edges, i.e., the set of vertices $\{4, 8, 17\}$; consequently $\omega(G_b) = 3$. Parts (b), (c) and (d) further show 3 other maximum cliques of G_b, this time containing the vertices 1, 2 and 3 respectively. The largest cliques in these graphs are: in (b) the set $\{1, 5, 14\}$, in (c) the set $\{2, 6, 15\}$ and (d) the set $\{3, 7, 16\}$. The example illustrates that, in circulant graphs, a maximum clique can be derived from the graph induced by the neighborhood of any one of its vertices. This fact is summarized in the following observation:

Observation 2. *Given a circulant graph G, the clique number $\omega(G) = \omega(G(v)) + 1$ for any $v \in V(G)$.*

Recalling that $G(v)$ is the graph induced by the neighborhood of a vertex $v \in V(G)$, Observation 2 states that in order to find $\omega(G)$ of a circulant graph G it is sufficient to compute the clique number in any graph induced by the neighborhood of one of its vertices. Once this maximum clique is computed, a maximum clique in G can be obtained by simply adding the vertex v. It is also worth noting that, for a given circulant graph G, the graphs $G(v)$, $v \in V(G)$, are not circulant in the general case, as shown, e.g., in Figure 3. So after selecting one vertex $v \in V(G)$, a general purpose MCP algorithm must be executed to compute a maximum clique in $G(v)$.

In what follows, we use the simplified notation G to refer either to the graphs G_b and G_r, since the separation procedure is designed for a generic circulant graph. The algorithm has an initial preprocessing heuristic phase, where the clique heuristic algorithm AMTS, described in [32], attempts to find a large clique in G. The exact branch-and-bound phase is only executed if AMTS fails to find a violated cut. If this occurs, our separation algorithm exploits Observation 2 by selecting the first vertex $v_1 \in V(G)$ and computing the induced graph $G(v_1)$, which, as mentioned previously, is not circulant in the general case. $G(v_1)$ is the input to the branch-and-bound procedure that is briefly described next. We note that the choice of vertex v_1 could also have been done randomly. Indeed, because of the properties of circulant graphs, $G(v_1)$ does not contain any of the other isomorphic subgraphs induced by the neighborhoods of the other vertices in $V(G)$.

It is worth noting at this point that, for the G_b graphs associated to small values of m, i.e., $m = 3$ and $m = 4$, it is more efficient to enumerate all possible cliques of size 3 and 4 respectively than to call the maximum clique procedure, all the more so since we are operating with $G(v_1)$ and not G. Specifically, in the case of $m = 3$ the algorithm checks if $G(v_1)$ is edge-free, and triangle-free in the case of $m = 4$. For these graphs the exact branch-and-bound procedure is not required.

Our exact procedure is of the constructive type, and it iteratively builds a clique C, starting from the empty set, while branching in a depth-first manner. Each branching node is associated to a subproblem (subgraph) $\hat{G} \subseteq G$. The candidate vertices for branching at each node are those vertices adjacent to every vertex in C; precisely, the branching operation on a vertex $v \in V(\hat{G})$ results in adding v to the clique C and associating the subproblem graph $\hat{G}(v), v \in V(G)$ to the new open node. The procedure backtracks when all the candidate vertices have been explored. During the search, the procedure keeps track of the largest (incumbent) clique found so far, and those branching nodes such that the value $|C| + \omega(\hat{G}(v))$ is not greater than the size of the incumbent clique can be pruned. Since computing $\omega(\hat{G}(v))$ can be computationally challenging, finding tight upper bounds for the clique number $\omega(\hat{G}(v))$ that can be efficiently computed is critical for the performance of the procedure. Specifically, the main sources of efficiency of the algorithm are: *i*) the use of the sequential greedy independent set coloring heuristic as a bound for $\omega(\hat{G}(v))$, see, [24]; *ii*) the use of an additional SAT-based bound in case the previous bound fails to prune the node. The latter bound exploits the encoding of the maximum clique problem to a partial maximum satisfiability problem, see, e.g., [18]; *iii*) the use of bitstrings to encode the adjacency matrix in memory, combined with efficient bitmasking operations to compute both the branching and the computation of both bounds (cf. [24] for a detailed description of this encoding).

5. Computational results

In this work, all the experiments have been performed on a 20-core Intel(R) Xeon(R) CPU E5-2690 v2@3.00GHz, using 128 GB of main memory and running a 64 bit Linux operating system. The source code was compiled with gcc 5.4.0 and the -o3 optimization flag. In all the tests, a time limit of 96 hours was set for each Ramsey instance. Cplex 12.8 was used as a general-purpose mixed integer programming solver. All Cplex parameters were set to their default values, except the strategy used to explore the branching tree which was set to *depth-first*. The rationale behind this choice was to limit the number of open nodes in the queue and accordingly memory usage. In addition, this diving strategy allows to quickly explore many leaves of the branching tree with the goal of finding feasible solutions.

We decided on the following test campaign for our IP-based algorithm. For the case in which a (non-trivial) lower bound was available in the literature, i.e., a feasible 2-coloring for K_q was known for $R(m,n)$, we tested the following Ramsey instances: *i*) $m = 3$ and every value of n in the interval $[20, 48]$; *ii*) the diagonal Ramsey instances $R(6,6)$, $R(7,7)$, $R(8,8)$, $R(9,9)$ and $R(10,10)$. The reason for this selection was driven by the order of the graphs, since in extensive preliminary experiments it was established that our efficient separation procedure was more effective over (relatively) large graphs, and, accordingly, there was more potential to find new Ramsey lower bounds in such scenarios. Specifically, we considered a minimum threshold of 100 for the order of the graphs, and attempted to improve the known Ramsey lower bounds by at most 3 units.

In addition, we also tested Ramsey number instances for which a (non-trivial) lower bound was unavailable in the literature. Specifically, we ran our IP-based algorithm on the instances $m = 4$, $n = 23$ and $n = 24$, with q values in the interval $[314, 339]$, and $[315, 340]$ respectively. The initial value for q was determined from the available lower bound for the $R(m, n-1)$ instance, i.e., $R(4,22) \geq 314$.

The result of this campaign are the 17 new Ramsey bounds listed in the contribution Section 1.3. These correspond to instances with a value $m = 3$. For the instances with $m = 4$ as well as the diagonal Ramsey numbers, the algorithm was not able to find new lower bounds within the imposed time limit of 96 hours.

We report in Table 2 additional information concerning the 17 new lower bounds on the Ramsey numbers $R(m,n)$ found thanks to our new IP-based approach. The first two columns of the table show the values m and n; the next column reports the best lower bounds

		$R(m,n)$ lower bounds		
m	n	best-known lower bound [21]	new lower bound	improvement
3	26	159 [31]	**161**	2
3	28	177 [19]	**179**	2
3	30	195 [19]	**197**	2
3	31	206 [19]	**208**	2
3	33	224 [19]	**226**	2
3	34	230 [13]	**233**	3
3	35	242 [13]	**244**	2
3	36	252 [13]	**255**	3
3	37	264 [13]	**265**	1
3	38	272 [13]	**275**	3
3	39	284 [13]	**286**	2
3	40	294 [13]	**297**	3
3	41	308 [13]	**311**	3
3	42	318 [13]	**320**	2
3	43	332 [13]	**333**	1
3	44	338 [13]	**339**	1
3	46	360 [13]	**362**	2

Table 2: The list of the 17 new bounds for $R(3,n)$ with n in the range $[26,46]$ and a comparison with the previously best-known lower bounds from the literature. The corresponding article where the previous lower bound was found is cited next to it.

(LBs) known in the literature, see [21]; the last two columns show the values of the new lower bounds and the difference between them (column "improvement"). The last column demonstrates that the improvements obtained range from 1 to 3 units. Specifically, all the improvements refer to Ramsey numbers $R(m,n)$ where $m=3$ and n ranges from 26 to 46. In this interval, all the lower bounds have been improved with the exception of $n=27,29,32$ and 45. As mentioned in Section 3, the improvements have been obtained by computing feasible circulant 2-colorings for graphs of order equal to the value of the new lower bound minus one. These feasible circular 2-colorings are publicly available and can be downloaded at `https://github.com/psanse/Ramsey-Numbers`. It is worth mentioning, that these improvements have been obtained for complete graphs which are considerably large, i.e., the smallest one is of order 160 and the largest one is of order 361. Determining maximum cliques in such large 2-colored graphs can be computationally demanding; indeed this operation is one of the main drivers of the performance of the new algorithm. In what follows we report the details of the key features related to its performance.

We report in Table 3 detailed information regarding the performance of the proposed IP-based algorithm. Besides the values of m and n, the table shows the total computing time in seconds (column "total"), the time necessary to separate the Tourán inequalities (18b) and (18c), using the separation procedure described in Section 4 (column "for separation"). In addition,

m	n	CPU time in seconds total	CPU time in seconds for separation	number of explored nodes	number of cuts blue $\to m$	number of cuts red $\to n$
3	26	51,963.1	33,629.6	1,468,734	1,014	77,543
3	28	23,995.9	18,844.6	739,688	1,196	13,704
3	30	22,502.6	22,083.2	171,565	888	6,359
3	31	22,084.1	21,991.0	44,420	746	2,287
3	33	59,880.1	59,790.1	40,828	694	1,869
3	34	29,311.5	29,302.5	6,463	460	722
3	35	360.0	214.0	13,494	375	2,403
3	36	17,396.0	458.3	163,058	594	27,054
3	37	4,103.6	493.2	52,255	486	10,781
3	38	21,281.1	1,964.6	114,957	655	24,881
3	39	39,400.4	1,707.6	128,084	664	28,773
3	40	7,347.0	4,888.8	63,955	547	9,722
3	41	84,733.8	27,487.0	294,425	763	46,480
3	42	79,756.7	14,660.0	152,005	522	33,861
3	43	75,948.2	55,279.6	202,878	633	24,620
3	44	308,992.6	220,784.6	207,484	814	39,642
3	46	151,056.0	143,522.0	80,559	535	13,614

Table 3: Detailed information concerning the performance of the new algorithm to compute the 17 new lower bounds for the Ramsey numbers $R(m, n)$ of Theorem 2 which are also reported in Table 2.

we report the total number of nodes explored during the branching procedure and, finally, the number of separated cuts found during the execution of the algorithm (columns "number of cuts"). Specifically, the column "blue $\to m$" refers to the number of constraints (18b) separated, while the column "red $\to n$" refers to the number of constraints (18c) separated.

Before commenting the results shown in Table 3, it is worth mentioning that the IP-based algorithm employs the heuristic separation algorithm denoted AMTS (described in Section 4) for the instances with $n \geq 35$. This choice is empirically motivated by the fact that when the order of the graphs is bigger than 240, the time spent by the separation procedure becomes predominant. For this reason, and in order to speed up the algorithm, we first run the AMTS heuristic to attempt to separate the violated cuts, and run the exact separation procedure only in case the heuristic one fails. The exact separation is also called to prove that no violated cuts exist. Extensive preliminary results showed that by setting a time limit of 0.05 seconds to the heuristic separation procedure, the proposed algorithm achieved the best performance.

We recall that the time limit was set to 96 hours for each run, but only the instances $R(3, 44)$ and $R(3, 46)$ took more than one day of computation, i.e., $R(3, 46)$ took 42h and $R(3, 44)$ took 85h; the rest of the instances were solved in less than 24 hours. Moreover, some of the instances were solved much faster, i.e., $R(3, 35)$ was solved in 6 minutes and $R(3, 37)$ as well as $R(3, 40)$ took less than 3 hours. For the instances with $n \leq 34$, i.e., where the heuristic separation is not employed, most of the time is taken by the exact separation procedure.

The average time for separating one cut when $n = 26$ is 0.42 seconds (graphs of order 160). The average time grows consistently as the values of n increases, reaching an average time of 24.8 seconds when $n = 35$ (graphs of order 245). For the remaining instances, i.e., with $n \geq 35$ except $n = 46$, the use of the heuristic separation allows to decrease the average separation time to values below 1 second for n up to 42 (graphs of order up to 320) and below 3 seconds for $n = 43$ (graphs of order 332). The largest instance $R(3, 46)$ has the highest average separation time of 10.1 seconds. In this particular case, AMTS often fails in computing violated cuts, and consequently the algorithm has to resort in many branching nodes to the exact separation procedure. Given that the separation problem is $\mathcal{NP}$-hard, our tests show that for graphs of order higher than 330, the execution of the separation procedure can, in some specific branching nodes, take many hours of computing time. This explains why the algorithm is unable to find new lower bounds for Ramsey numbers with values of n greater than 46.

Even though the dimension of the search space for circulant 2-colorings for a graph of order q is $2^{\lfloor \frac{q}{2} \rfloor}$, Table 3 demonstrates that the number of branching nodes explored by our method is exceedingly small compared to $2^{\lfloor \frac{q}{2} \rfloor}$. For example, for the problem instance $R(3, 26)$, the IP-algorithm examined $1,468,734$ nodes out of a possible 2^{80}. This shows that the added inequalities are effective in excluding infeasible configurations from the branching tree.

As mentioned in Section 3.3, the proposed IP-based algorithm can also be used to determine the size of feasible circulant 2-colorings $C(m, n)$. This can be done by determining a feasible circulant 2-coloring (alternatively, its non-existence) for every graph of order up to $R(m, n) - 1$ (alternatively, any upper bound on $R(m, n)$). The value $C(m, n)$ is the size of the largest feasible circulant 2-coloring obtained in this way.

The Figures 4, 5 and 6 show the number of explored branching nodes and the computing times taken to solve the largest graphs (in increasing order) required to prove the values $C(4, 7) = 46$, $C(4, 8) = 51$ and $C(5, 5) = 41$, respectively. The best known upper bounds on the Ramsey numbers for these three cases are: $R(4, 7) \leq 61$, $R(4, 8) \leq 84$ and $R(5, 5) \leq 48$, see [21].

Determining feasible circulant 2-colorings is relatively easy from a computational point of view in all these three cases. More precisely, the algorithm spends slightly over 5 seconds with $m = 4$ and $n = 7$ to find a feasible coloring for K_{46}, 3.5 seconds with $m = 4$, $n = 8$ to find a feasible coloring for K_{51}, and 8.8 seconds with $m = 5$, $n = 5$ to find a feasible coloring for K_{41}. However, and as can be seen in the three figures, determining infeasibiliy for circulant 2-colorings may become computationally challenging when increasing the order of the graphs. Specifically, Figures 4 and 5 clearly show an exponential growth of the number of explored nodes (and, accordingly, the computational time) as the order of the graphs rises. For example, in the case of $m = 4$ and $n = 8$ the number of nodes is $200,000$ for K_{72}, but goes up to 1.8×10^6 for K_{83}. Accordingly, the computational time rises from 400 seconds to $8,500$ seconds. These figures again illustrate that with our method we explore a significantly smaller portion of the search space, which helps us to quickly find the largest 2-colorings while proving that for certain values of $q \in \mathbb{N}$, circulant 2-colorings of K_q are infeasible.

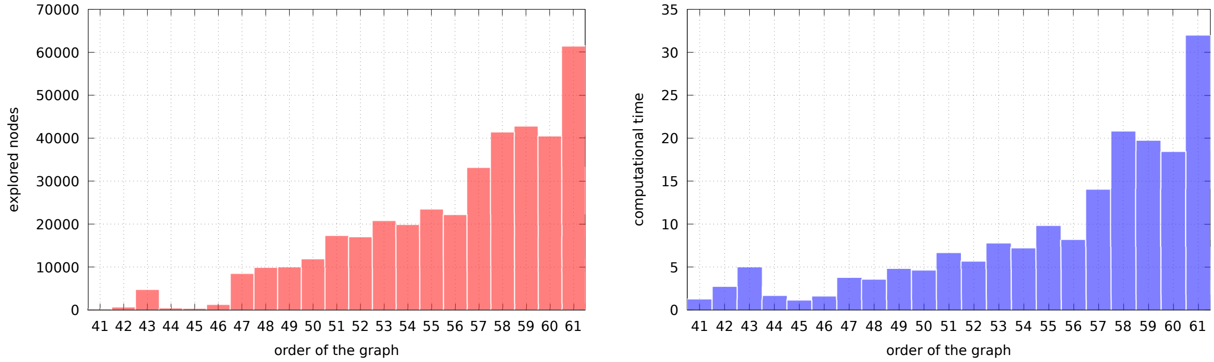

Figure 4: Number of explored nodes and computational times for the calculation of $C(4,7)$.

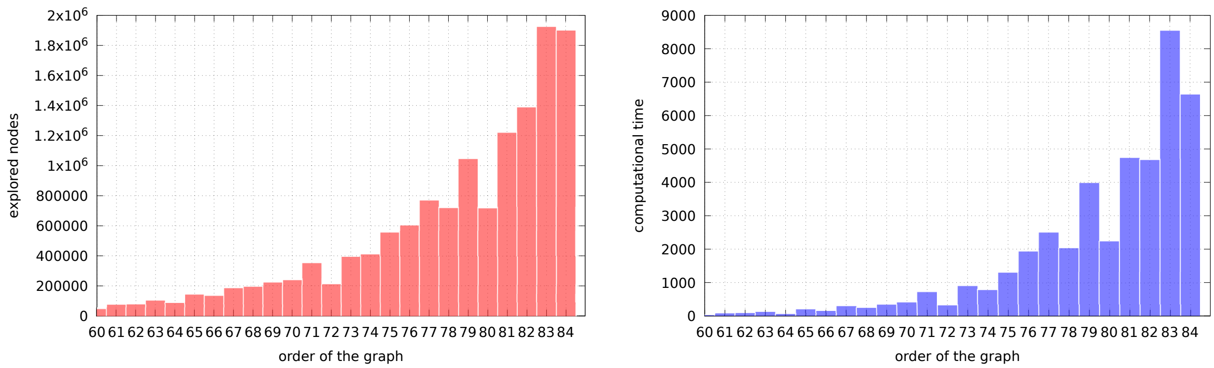

Figure 5: Number of explored nodes and computational times for the calculation of $C(4,8)$.

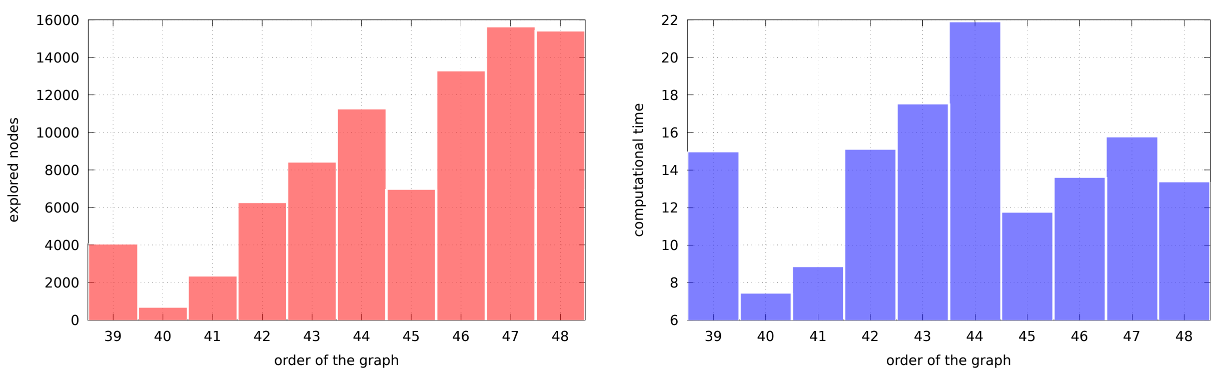

Figure 6: Number of explored nodes and computational times for the calculation of $C(5,5)$.

6. Conclusions

In this article we proposed an IP-based approach for calculating small Ramsey numbers with which we were able to improve 17 state-of-the-art bounds for $R(3,n)$ with n ranging between 26 and 46. Our method is effective in exploring the search space when we focus on circulant graphs. Kalbfleisch [15] stated the hypothesis that critical 2-colorings (i.e., those that prove Ramsey numbers) are "close" to circulant ones, but the precise definition of this "closeness" does not exist. For example, for $R(5,5)$ it is known that $C(5,5) = 41$ and there are 656 different $(5,5)$ colorings of K_{42} (proving the currently best-known lower bound of 43) none of which is circulant [21]. In [20], the authors show the adjacency matrix of a 2-coloring of K_{42} which has an almost-circulant structure. Hence, exploring the almost-circulant property using the IP methodology seems like a promising direction for future research.

One possible way to search for almost-circulant colorings is to start with a circulant coloring of a smaller order, and iteratively insert vertices, while limiting the Hamming distance between the two graphs. Similar local search strategies based on vertex insertion starting from (block) circulant colorings have been proposed in [5]. To the best of our knowledge, IP-based methods for finding Ramsey numbers have not been used in the past. Given the promising results obtained on circulant graphs, our IP model could be a starting point to efficiently explore the neighborhoods of circulant colorings in combination with other matheuristics like feasibility pump or local branching.

7. Acknowledgments

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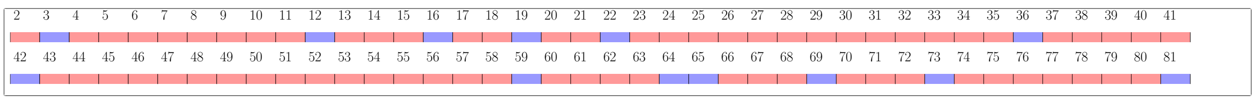
8. Appendix.

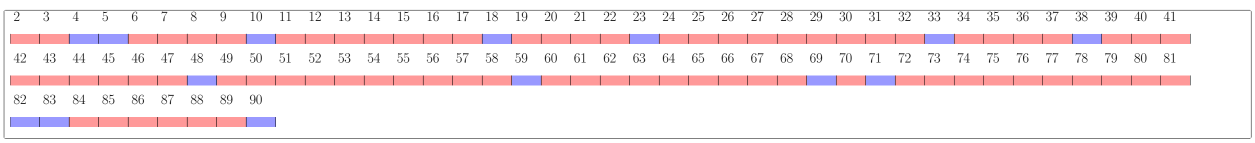
8.1. Proof of Theorem 2

Proof. To prove the theorem, we provide in what follows the 16 feasible circulant 2-colorings of the theorem. For a given pair (m, n), the feasible circulant 2-coloring of the graph K_q is given by showing the coloring of the first half of the edges incident to the first vertex of K_q, i.e., the set of edges $\{1, i\}$, $i = 2, \ldots, \lfloor \frac{q}{2} \rfloor$. The second half of the edges can be obtained by setting the color of the edge $\{1, q-i+2\}$ equal to the color of the edge $\{1, i\}$ with $2 \leq i \leq \lfloor \frac{q}{2} \rfloor$. The color of the remaining edges $\{i, j\}$ in the upper triangular part of the circulant adjacency matrix, i.e., with $i = 2, \ldots, q-1$ and $j = i+1, \ldots, q$, are obtained by setting the color of the edge $\{i, j\}$ equal to color of the edge $\{i-1, j-1\}$.

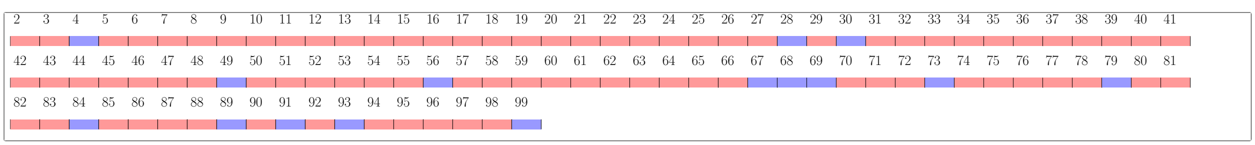
Feasible circulant 2-coloring for K_{160}, $m = 3$ and $n = 26$.



Feasible circulant 2-coloring for K_{178}, $m = 3$ and $n = 28$.



Feasible circulant 2-coloring for K_{196}, $m = 3$ and $n = 30$.



Feasible circulant 2-coloring for K_{207}, $m = 3$ and $n = 31$.

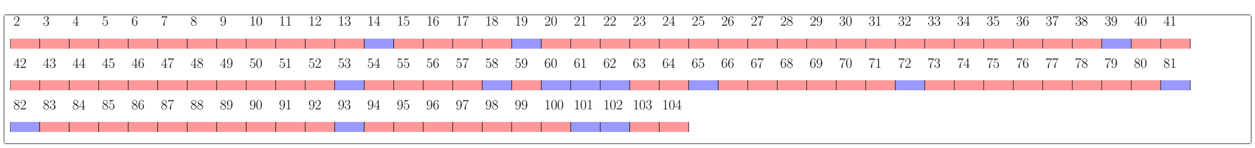
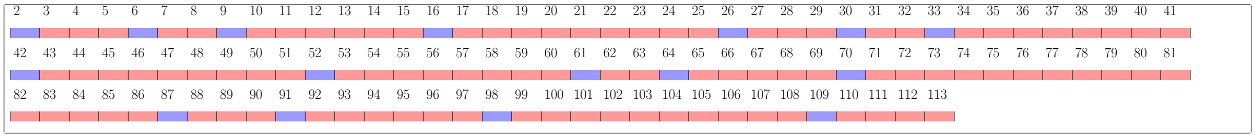
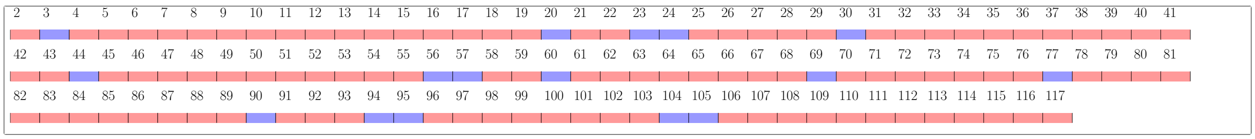


Figure 7: Feasible circulant 2-colorings for $m = 3$ and $n \in \{26, 28, 30, 31\}$.

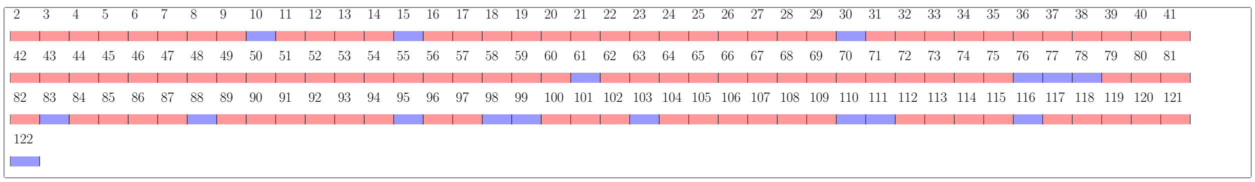
Feasible circulant 2-coloring for K_{225}, $m = 3$ and $n = 33$.



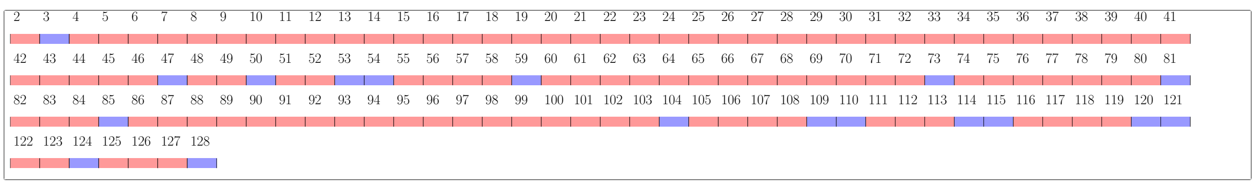
Feasible circulant 2-coloring for K_{232}, $m = 3$ and $n = 34$.



Feasible circulant 2-coloring for K_{243}, $m = 3$ and $n = 35$.



Feasible circulant 2-coloring for K_{254}, $m = 3$ and $n = 36$.



Feasible circulant 2-coloring for K_{264}, $m = 3$ and $n = 37$.

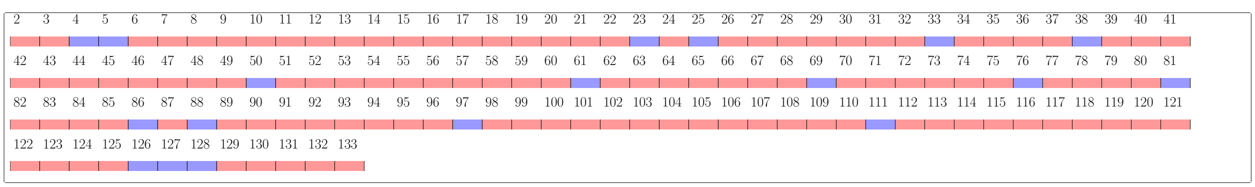
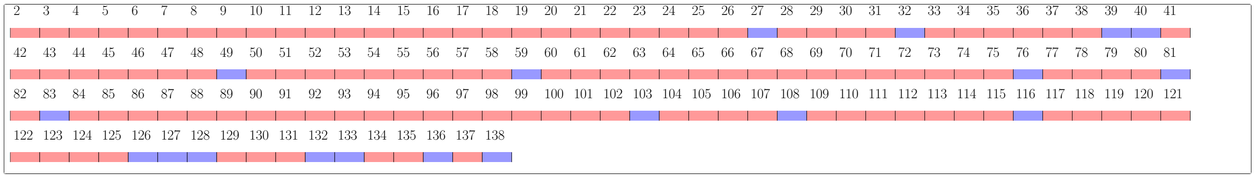
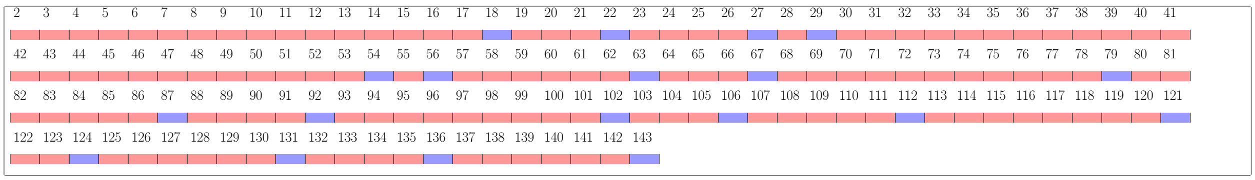


Figure 8: Feasible circulant 2-colorings for $m = 3$ and $n \in \{33, 34, 35, 36, 37\}$.

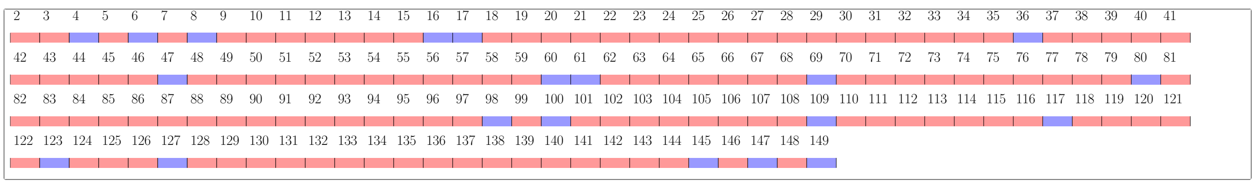
Feasible circulant 2-coloring for K_{274}, $m = 3$ and $n = 38$.



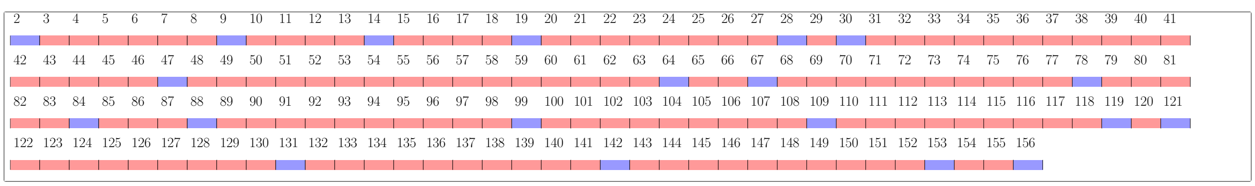
Feasible circulant 2-coloring for K_{285}, $m = 3$ and $n = 39$.



Feasible circulant 2-coloring for K_{296}, $m = 3$ and $n = 40$.



Feasible circulant 2-coloring for K_{310}, $m = 3$ and $n = 41$.



Feasible circulant 2-coloring for K_{319}, $m = 3$ and $n = 42$.

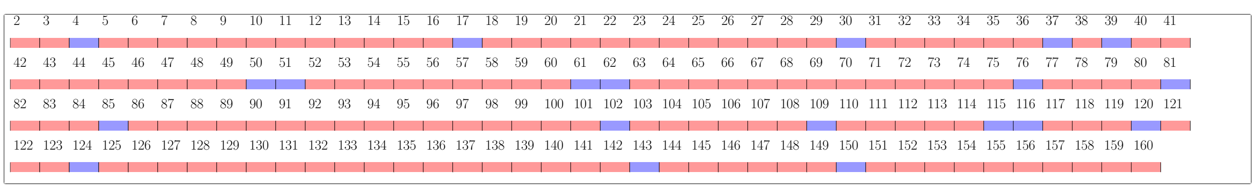
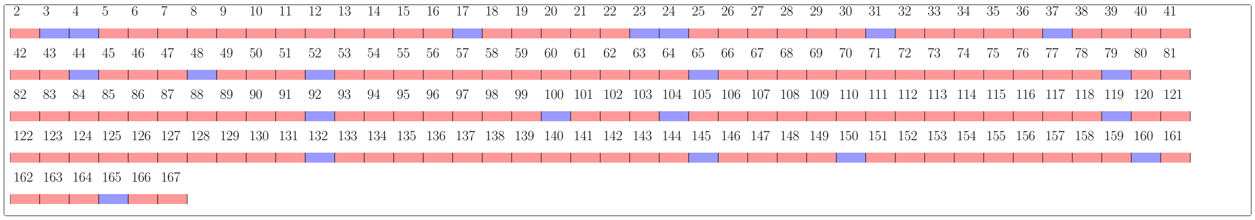
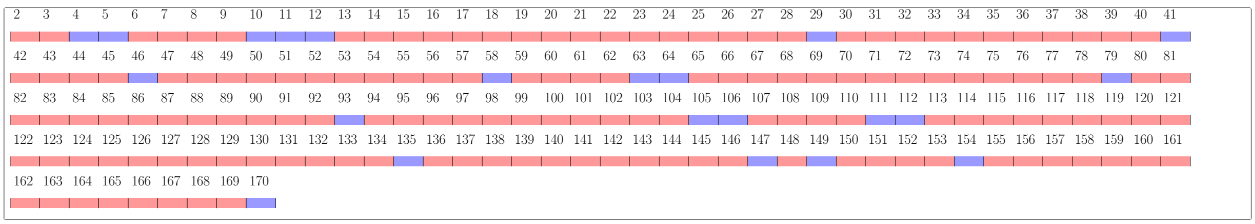


Figure 9: Feasible circulant 2-colorings for $m = 3$ and $n \in \{38, 39, 40, 41, 42\}$.

Feasible circulant 2-coloring for K_{332}, $m = 3$ and $n = 43$.



Feasible circulant 2-coloring for K_{338}, $m = 3$ and $n = 44$.



Feasible circulant 2-coloring for K_{361}, $m = 3$ and $n = 46$.

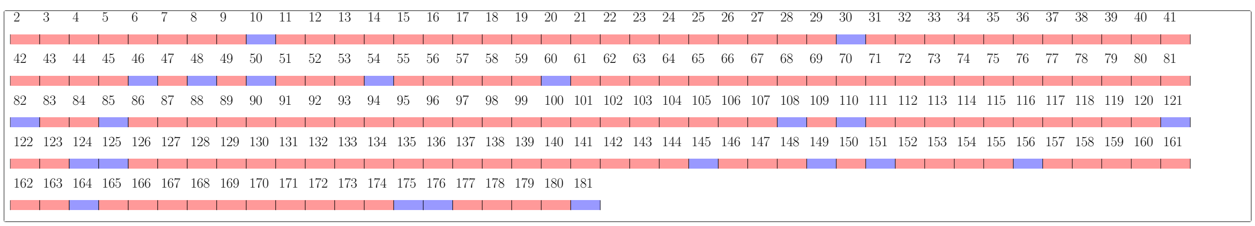


Figure 10: Feasible circulant 2-colorings for $m = 3$ and $n \in \{43, 44, 46\}$.

$\square$

8.2. *Circulant* $(5,5)$*-coloring of* K_{41}

In Figure 11 we show the adjacency matrix of K_{41} for $m = n = 5$ corresponding to a feasible circulant coloring. No feasible circulant 2-coloring exists for larger graphs and $C(5,5) = 41$, see [21]. The darker cells represent the edges of the maximum red and blue cliques of size 4.

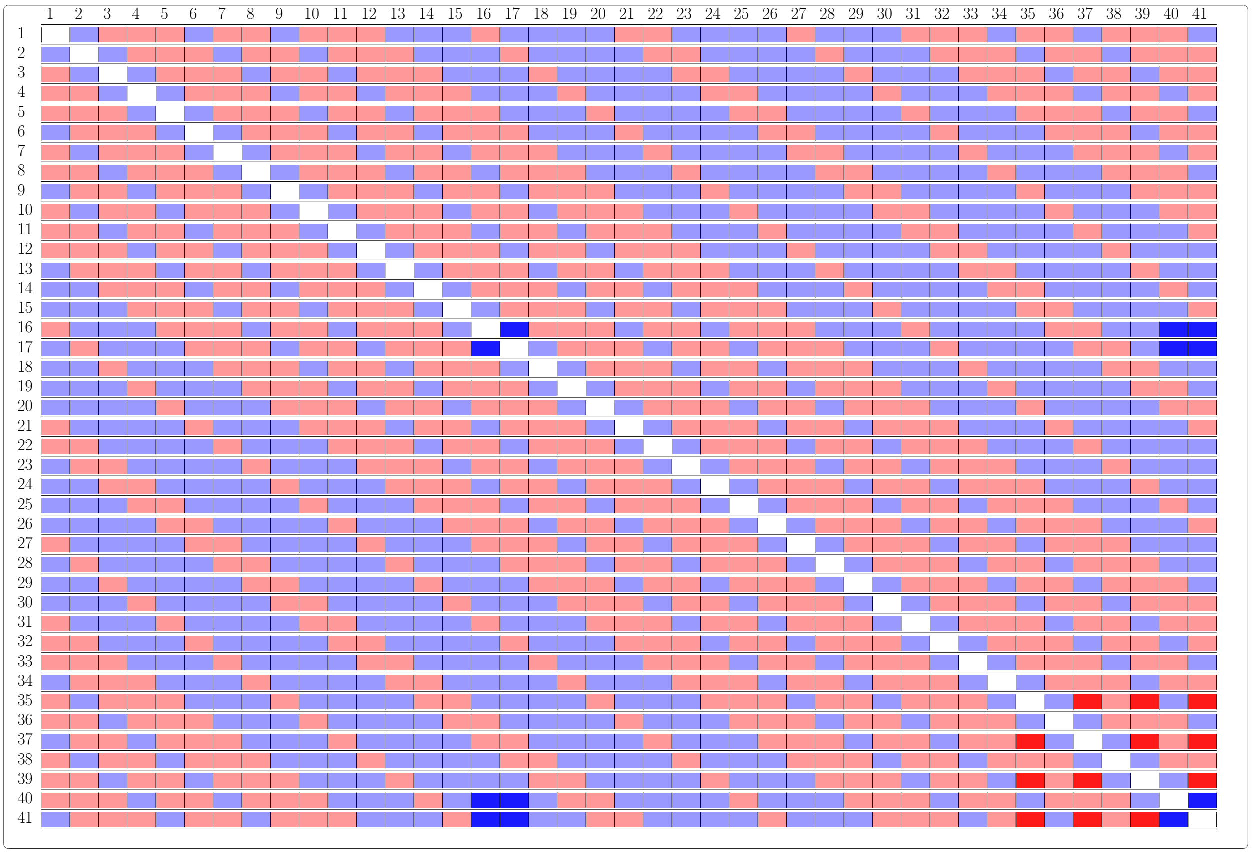


Figure 11: Circulant adjacency matrix of K_{41}.