A very simple analysis of higher order liftings for binary problems

Preprint, Oct. 7, 2021 (with changes of Nov. 6, 2021)
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Abstract

Based on the observation that the max-cut-polytope is the projection of a higher-dimensional regular simplex, and the fact that this simplex coincides with the $n$-th semidefinite lifting, a simple proof is given that the $n$-th lifting for the max-cut polytope is exact. An approach to reduce the dimension of higher order liftings concludes this short note.

Key words: Binary problems, higher order liftings.

1 Introduction

Lovasz and Schrijver, [4] and Lasserre, [3], proposed convex exact higher order liftings of hard combinatorial problems. This paper presents a closely related and very simple analysis of higher order liftings for binary problems.

1.1 Notation

Given $n \in \mathbb{N}$, let $N := \{1, \ldots, n\}$. For $I, J \subseteq N$ let $I \triangle J := (I \cup J) \setminus (I \cap J)$ be the symmetric difference of $I$ and $J$. For $x \in \{\pm 1\}^n$ let $\bar{x} \in \mathbb{R}^{2^n}$ denote the augmented vector with components $\bar{x}_I := \prod_{i \in I} x_i$ for $I \subseteq N$: (By convention, when $I$ is the empty set, $\bar{x}_\emptyset := 1$.) The space of real symmetric $k \times k$ matrices is denoted by $S^k$ and the cone of real symmetric positive semidefinite $k \times k$ matrices is denoted by $S^k_+$. The trace inner product of $X, Y \in S^k$ inducing the Frobenius norm $\| \cdot \|_F$ is denoted by $\langle X, Y \rangle$. The all-one-vector is denoted by $e$, its dimension being evident from the context. The max-cut-polytope is denoted by $MC := \text{conv}(\{xx^T \mid x \in \{\pm 1\}^n\})$, see e.g. [1, 2, 5].
2 A High-Dimensional Simplex

Consider the convex hull of all augmented \( \{ \pm 1 \} \)-vectors, i.e. the polytope

\[
S := \text{conv}( \{ \bar{x}\bar{x}^T \mid x \in \{ \pm 1 \}^n \} ) \subset S^{2n}.
\]

As there are \( 2^n \) such vectors, \( S \) is contained in a \( (2^n-1) \)-dimensional affine subspace. (Note that for \( I, J \subseteq N \), the product \( \bar{x}_I \bar{x}_J \) is given by \( \bar{x}_I \bar{x}_J = \bar{x}_{I \setminus J} \).)

**Proposition 2.1** For \( x, y \in \{ \pm 1 \}^n \) with \( x \neq y \) it always follows

\[
\| \bar{x} - \bar{y} \|_2 = 2^{(n+1)/2}. \quad \text{And for any two vertices } \bar{x}\bar{x}^T \text{ and } \bar{y}\bar{y}^T \text{ of } S \text{ it follows that } \| \bar{x}\bar{x}^T - \bar{y}\bar{y}^T \|_F = 2^{(2n+1)/2}.
\]

**Proof.** Consider the case that \( y \) differs from \( x \) exactly in the components \( 1, \ldots, k \), i.e.

\[
x_1y_1 = \ldots = xky_k = -1.
\]

The number of subsets of \( \{1, \ldots, k\} \) with an even number of elements is \( 2^{k-1} \) and the number of subsets of \( \{1, \ldots, k\} \) with an odd number of elements also is \( 2^{k-1} \). (See the remark below.) If an even number of elements from \( \{1, \ldots, k\} \) is contained in \( I \subseteq N \), then \( \bar{x}_I = \bar{y}_I \) else \( \bar{x}_I = -\bar{y}_I \). Similarly when some subset \( \{i_1, \ldots, i_k\} \) of \( N \) is considered in place of \( \{1, \ldots, k\} \). Hence exactly half the entries of \( \bar{x} \) and \( \bar{y} \) differ, and the absolute value of their difference is always 2 leading to

\[
\| \bar{x} - \bar{y} \|_2 = \| \bar{x} + \bar{y} \|_2 = 2\sqrt{2^{n-1}} = 2^{(n+1)/2}.
\]

Hence, for each of the \( 2^n \) columns of \( \bar{x}\bar{x}^T \) and \( \bar{y}\bar{y}^T \) there are \( 2^{n-1} \) entries that differ, so that

\[
\| \bar{x}\bar{x}^T - \bar{y}\bar{y}^T \|_F = 2\sqrt{2^n \cdot 2^{n-1}} = 2^{(2n+1)/2}.
\]

**Remark:** It is a common exercise to show by induction that the number of subsets of \( N = \{1, \ldots, n\} \) with an even number of elements is \( 2^{n-1} \). Indeed for \( n = 1 \) the two subsets of \( N \) are \( \emptyset \) and \( N \).

Now let \( \hat{n} := n-1 \geq 1 \) and \( \hat{N} := \{1, \ldots, \hat{n}\} \). The subsets of \( N \) are given by \( I \) and \( I \cup \{n\} \) where \( I \subseteq \hat{N} \). By induction hypothesis, \( 2^{n-1} \) of the sets \( I \) and also \( 2^{n-1} \) of the sets \( I \cup \{n\} \) have even cardinality. Thus, the claim follows from \( 2^{\hat{n}-1} + 2^{n-1} = 2^{n-1} \).

Thus, the polytope \( S \) has \( 2^n \) vertices lying in a \( (2^n-1) \)-dimensional affine space. Within this affine space the set \( S \) is a regular simplex. In particular, the vertices are affinely independent. (Again a straightforward exercise.) The first central observation of this note is:

- The projection of \( S \) onto the rows and columns associated with \( \bar{x}_{\{1\}}, \ldots, \bar{x}_{\{n\}} \) is the max-cut-polytope \( \text{MC} = \text{conv}(\{xx^T \mid x \in \{\pm 1\}^n\}) \) in \( S^n \).

The max-cut-polytope has \( 2^{n-1} \) vertices as \( x \) and \( -x \) generate the same vertex, i.e. \( xx^T = (-x)(-x)^T \) while \( \bar{x} \) and \( \bar{z} \) for \( z := -x \) do not, \( \bar{x}\bar{x}^T \neq \bar{z}\bar{z}^T \).

The second key observation of this short note (detailed below) is that the semidefinite relaxation \( \hat{S} \) of \( S \) coinciding with the \( n \)-th lifting also coincides with \( S \). This observation implies the known fact that the semidefinite liftings of sufficiently high order do represent the exact convex hull, and thus also the max-cut-polytope. Observe that the \( n \)-th lifting in [4] is a high dimensional linear extension of \( \text{MC} \) as introduced in Theorem 3 of [6]; the above representation of \( S \) via its \( 2^n + 1 \) facets is a simpler form of such linear extension.
3 Semidefinite Representation

Note that $S \subseteq \tilde{S}$ where the semidefinite relaxation $\tilde{S}$ is given by

$$\tilde{S} := \{ X \in S_{+}^{2n} \mid X_{\emptyset, \emptyset} = 1, \ X_{I,J} = X_{K,L} \ for \ any \ I, J, K, L \subseteq N \ with \ I \triangle J = K \triangle L \}.$$ 

The equations relating $X_{I,J}$ and $X_{K,L}$ represent simple equalities such as $\vec{x}_{\{i,j\}} = \vec{x}_{\{i,k\}} = \vec{y}_{\{j,k\}}$. Thus, since $X_{\emptyset, \emptyset} = 1$, it follows in particular that $X_{I,I} = 1$ for all $I \subseteq N$ so that there are $2^n - 1$ “free” matrix entries $X_{I,J}$ of $X$ in $\tilde{S}$. Same as $S$, also $\tilde{S}$ is contained in an $(2^n - 1)$-dimensional affine space. In fact, as shown next, both sets coincide.

Lemma 3.1 The sets $S$ and $\tilde{S}$ coincide.

Proof. Both, $S$ and $\tilde{S}$ are convex subsets of the $(2^n - 1)$ dimensional affine subspace

$$\{ X \in S_{+}^{2n} \mid X_{\emptyset, \emptyset} = 1, \ X_{I,J} = X_{K,L} \ for \ any \ I, J, K, L \subseteq N \ with \ I \triangle J = K \triangle L \}$$

and the identity matrix in $S_{+}^{2n}$ is a point in the relative interior of $\tilde{S}$. It suffices to show that all relative boundary points $X \in \partial S$ have rank at most $2^n - 1$, and thus are also at the relative boundary of $\tilde{S}$.

Let $X$ be a boundary point of the simplex $S$, i.e. $X$ is a convex combination of all vertices of $S$ except one vertex $\vec{y}_{\emptyset}$ of $S$. Let $\vec{x} \vec{z}^T$ be some other vertex. By Proposition 2.1,

$$2^{2n+1} = \| \vec{x} \vec{z}^T - \vec{y}_{\emptyset} \vec{y}_{\emptyset}^T \|_{F}^2 = \| \vec{x} \vec{z}^T \|_{F}^2 + \| \vec{y}_{\emptyset} \vec{y}_{\emptyset}^T \|_{F}^2 - 2(\vec{x}^T \vec{y}_{\emptyset})^2 = 2^{2n+1} - 2(\vec{x}^T \vec{y})^2$$

so that $(\vec{x}^T \vec{y})^2 = \vec{y}^T(\vec{x} \vec{z}^T) \vec{y} = 0$. As this is true for all other vertices $\vec{x} \vec{z}^T$ it follows that $\vec{y}^T X \vec{y} = 0$, i.e. $X$ has rank at most $2^n - 1$.

Corollary 3.1 The usual definitions of higher order liftings contain some redundancies such as identical rows and columns. The set $\tilde{S}$ is the $n$-th lifting after eliminating identical rows and columns. For $1 \leq k < n$, liftings of order $k$ can be defined in a similar way by considering augmented vectors $\vec{x}$ with components $\vec{x}_I$ where $I \subseteq N$ has cardinality at most $k$. The corresponding semidefinite approximation of the max-cut-polytope is defined in an analogous way as the projection of the semidefinite relaxation for $\vec{x} \vec{x}^T$ onto rows and columns associated with $\vec{x}_{\{1\}}, \ldots, \vec{x}_{\{n\}}$. The previous lemma implies for any subset $M \subseteq N$ of cardinality at most $k$ that the restriction of the $k$-th lifting to the matrix with entries $X_{I,J}$ for $I, J \subseteq M$ is exact, indicating that the accuracy of the lifting is improving when increasing $k$.

4 Reduced Representations

In this section the size of the representation is reduced by eliminating half of the subsets of $\vec{x}$: A reduced representation of the max-cut-polytope is obtained, for example, when
considering vectors \( \vec{y} \in \{\pm 1\}^{2n-1} \) with components \( \vec{y}_i := \prod_{i \in J} x_i \) where \( x \in \{\pm 1\}^n \) is as before and \( I \subseteq N \) has odd cardinality only. For sets \( I, J \subseteq N \) with odd cardinality it follows that

\[
|I \triangle J| = |I| + |J| - 2|I \cap J|
\]

is even so that the rank-1-matrix \( Y = \vec{y}\vec{y}^T \in S^{2n-1} \) only has entries \( Y_{I,J} = \vec{x}_{I \triangle J} \) with subsets \( I \triangle J \) of even cardinality \( |I \triangle J| \). The approximation of the max-cut-polytope is evident.

Similarly, by considering only even-cardinality subsets \( I \) (including the empty set) for the definition of a second vector \( \vec{z} \) with entries \( \vec{z}_i := \prod_{i \in I} x_i \). The resulting matrix \( Z = \vec{z}\vec{z}^T \) has entries \( Z_{I,J} = \vec{x}_{I \triangle J} \) with subsets \( I \triangle J \) that are also of even cardinality \( |I \triangle J| \). It turns out that for \( k = n \) also the liftings above are exact:

To see this let \( S_1 := \text{conv}( \{ \vec{y}\vec{y}^T \mid x \in \{\pm 1\}^n \} \) and \( S_2 := \text{conv}( \{ \vec{z}\vec{z}^T \mid x \in \{\pm 1\}^n \} \) where \( \vec{y} \) and \( \vec{z} \) are defined as above.

**Lemma 4.1** When \( n \) is odd \( S_1 \) coincides with its semidefinite relaxation \( \tilde{S}_1 \), and when \( n \) is even \( S_2 \) coincides with its semidefinite relaxation \( \tilde{S}_2 \).

**Proof.** For \( i \in \{1,2\} \) let \( x^{(i)} \in \{\pm 1\}^n \) and \( y^{(i)} \) with components \( y_j^{(i)} := \prod_{j \in J} x_j^{(i)} \) for odd-cardinality \( J \) be given and assume that \( (x^{(1)}_1, \ldots, x^{(1)}_{|I|}) \) and \( (x^{(2)}_1, \ldots, x^{(2)}_{|J|}) \) differ in the components \( 1 \leq k \leq n - 1 \). (The case \( k = n \) generates the same matrix \( \vec{y}^{(1)}(\vec{y}^{(1)})^T = \vec{y}^{(2)}(\vec{y}^{(2)})^T \).) As in the proof of Proposition 2.1, if an even number of elements from \( \{1, \ldots, k\} \) is contained in \( I \subseteq N \), then \( y_i^{(1)} = y_i^{(2)} \) else \( y_i^{(1)} = -y_i^{(2)} \), and thus, again as in the proof of Proposition 2.1,

\[
\|\vec{y}^{(1)} - \vec{y}^{(2)}\|_2 = \|\vec{y}^{(1)} + \vec{y}^{(2)}\|_2 = 2\sqrt{2^{n-2}} = 2^{n/2}.
\]

Since \( \vec{y} \) and \( \vec{z} \) are based on a disjoint union of all indices \( I \subseteq N \) it follows from the above and from Proposition 2.1 that also \( \|\vec{z}^{(1)} - \vec{z}^{(2)}\|_2 = 2^{n/2} \) for different \( \vec{z}^{(1)}, \vec{z}^{(2)} \in \{\pm 1\}^{2n-1} \).

Thus, in both cases Proposition 2.1 is valid just with a different constant distance. Since there are only \( 2^{n-1} \) vertices in \( S_1 \) or \( S_2 \), the proof of Lemma 3.1 is applicable as well. \( \square \)

Observe that both relaxations can be combined by relating equivalent entries of both reduced representations \( Y = \vec{y}\vec{y}^T \) and \( Z = \vec{z}\vec{z}^T \) with equality constraints. A larger matrix inequality is thus replaced with two smaller matrix inequalities (namely \( Y \) and \( Z \) being positive semidefinite). This will be referred to as mixed reduced lifting below. It turns out that for even numbers \( n \) that the above mixed reduced lifting of order \( n/2 \) is exact:

First, for even numbers \( n \) let \( \vec{z} \) have components \( \vec{z}_i := \prod_{i \in I} x_i \) where \( x \in \{\pm 1\}^n \) and \( I \subseteq N \) with \( |I| \leq n/2 \) only. Then observe that the proof of Lemma 3.1 also applies in the lower-dimensional setting \( \hat{X} = \vec{x}\vec{x}^T \in S^{2n-1} \) with \( 2^{n-1} \) equidistant extreme points. This implies the known strengthening of Lemma 3.1, namely for even \( n \) the lifting of order \( n/2 \) is exact.

Now let \( \vec{y} \) have components \( y_i := \prod_{i \in I} x_i \) where \( x \in \{\pm 1\}^n \) as before and \( I \subseteq N \) has odd cardinality only and \( |I| \leq n/2 \). Likewise assume that \( \vec{z} \) has components \( \vec{z}_i := \prod_{i \in I} x_i \) where \( x \in \{\pm 1\}^n \) and \( I \subseteq N \) has even cardinality only and \( |I| \leq n/2 \). Thus,

\[
\vec{z} = \Pi \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix}
\]
for some permutation matrix $\Pi$.

Consider the block-structured simplex

$$\mathcal{BS} := \text{conv}\left( \left\{ \begin{pmatrix} \hat{y} \hat{y}^T & 0 \\ 0 & \hat{z} \hat{z}^T \end{pmatrix} \mid x \in \{\pm 1\}^n \right\} \right)$$

where the dimensions of the matrix blocks follow from the context. As $x$ and $-x$ generate the same matrix, $\mathcal{BS}$ is the convex hull of $2^n - 1$ points contained in an $(2^n - 1)$-dimensional affine subspace of $\mathcal{S}^{2^n}$. Up to a permutation, the mixed reduced lifting of $\mathcal{BS}$ coincides with the semidefinite approximation of $\hat{x} \hat{x}^T$ projected to a block diagonal format. This projection is consistent with respect to the semidefinite ordering and with respect to the equality constraints $"\hat{X}_{I,J} = \hat{X}_{K,L}"$ for $I \triangle J = K \triangle L$, so that the arguments in the proof of Lemma 3.1 are applicable again.

Acknowledgment: The author wishes to thank Melinda Hagedorn and Manuel Bomze for many helpful comments on this short note.

References


