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Robust CARA Optimization

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We propose robust optimization models and their tractable approximations that cater for ambiguity-averse decision makers whose underlying risk preferences are consistent with *constant absolute risk aversion* (CARA). Specifically, we focus on maximizing the worst-case expected exponential utility where the underlying uncertainty is generated from a set of stochastically independent factors with ambiguous marginals. To obtain computationally tractable formulations, we propose a hierarchy of approximations, starting from formulating the objective function as tractable concave functions in affinely perturbed cases, developing approximations in concave piecewise affinely perturbed cases, and proposing new multi-deflected linear decision rules for adaptive optimization models. We also extend the framework to address a multi-period consumption model. The resultant models would take the form of an exponential conic optimization problem (ECOP), which can be practicably solved using current off-the-shelf solvers. We present numerical examples including project management and multi-period inventory management with financing to illustrate how our approach can be applied to obtain high-quality solutions that could outperform current stochastic optimization approaches, especially in situations with high risk aversion levels.

Key words: robust optimization, constant absolute risk aversion, exponential cone programming
History: October 6, 2021.

1. Introduction

Optimization under uncertainty is a fundamental problem in operations research that can have significant practical impact. Unlike a deterministic optimization problem, the *objective* function in a decision model facing uncertainty is *subjective* and contingent on the preference of the decision maker. Indeed, decision preference concerning uncertainty can be associated with *risk* and *ambiguity*. In this paper, we refer to risk in situations where the true probability distribution of the underlying random variable is known, while in ambiguity or *Knightian uncertainty* (Knight 1921), the true distribution would be unknown. For a random payoff with known distribution, risk aversion relates to the preference of a more predictable, but possibly lower payoff, over a highly unpredictable, but possibly higher payoff. However, if the true distribution is unknown, the actual payoff risk would also be unknown. Ambiguity aversion relates to the cautious behavior of evaluating the payoff risk using the worst distribution among an acceptable set of uncertain priors. Both risk and ambiguity aversion are sensible and relevant in decision making, not only for individuals or firms, but also for some government agencies where their policies could impact public health or safety.

In articulating the risk preference of the decision maker, we focus on the paradigm of expected utility introduced by Von Neumann and Morgenstern (1947), which has been the *de facto* decision criterion for rational agents in evaluating risky payoffs. Risk aversion is associated with an increasing and concave utility function. In particular, we will focus on the exponential utility function, which is uniquely associated with decision makers whose preferences are consistent with *constant absolute risk aversion* (CARA) (Arrow 1965, Pratt 1964). In such preferences, the risk tolerance level, which is defined by the magnitude of the ratio of the first to the second derivative of the utility function, is a constant and does not depend on the payoff amount. The risk tolerance level can also be interpreted as the payoff amount for which the decision maker would be roughly indifferent between accepting or rejecting a gamble involving a 50-50 chance of winning that amount and losing half that amount (see, *e.g.*, Delquié 2008). Because decision models with an exponential utility function are often analytically tractable, it is used as an adequate approximation of general utility functions (see Kirkwood 2004, for more details). For instance, the exponential utility preference preserves the structural properties of Markov decision process (Howard and Matheson 1972). In addition, if the distribution of the random payoffs is Gaussian, then the CARA preferences would be consistent with the mean-variance preferences (Markowitz 1952); this property has been used to reveal the high risk aversion of investors in the well-known *equity premium puzzle* of Mehra and Prescott (1985). As such, CARA preferences are commonly assumed in the literature of economics, finance and operations research (see, *e.g.*, Abbas and Howard 2015, Holmstrom and Milgrom 1991, Bouakiz and Sobel 1992, Veronesi 1999, Feng and Xiao 2008, Hall et al. 2015). In the literature review on the application of risk aversion, Corner and Corner (1995) note that the CARA utility function is about five times more commonly adopted than other types of utility functions combined.

Dantzig (1955) proposes the stochastic linear optimization model as a computational framework for optimization under uncertainty. A stochastic linear optimization model is typically a linear optimization problem; it has a risk-neutral objective function and the random variable is limited to discrete probability distribution with a modest number of scenarios. However, if the number of scenarios is infinite or exponential, solving the stochastic optimization model would be computationally intractable (Dyer and Stougie 2006, Hanasusanto et al. 2016). Nevertheless, we can still obtain approximate solutions via sample average approximation (SAA) approach, which is a popular randomized approximation technique that improves with the number of samples (see, *e.g.*

Shapiro et al. 2014). For large sample size, many large-scale linear optimization techniques have also been developed for solving the stochastic optimization problem more efficiently (see, *e.g.,* Kall et al. 1994, Birge and Louveaux 2011, Prékopa 2013). Although we can extend this framework to an objective function based on the expected exponential utility, it would result in a nonlinear convex exponential conic optimization problem (ECOP), whose formulation size depends on the number of scenarios (see, *e.g.,* Dowson et al. 2020). Unlike the linear optimization models, such problems may not scale as well computationally with the number of samples; moreover, the quality of the approximation may depend on the risk tolerance level. Consequently, because of the computational limit on the number of samples, solutions obtained vian SAA may suffer from the *optimizer's curse* and result in poor out-of-sample performance (Smith and Winkler 2006). Hence, apart from SAA approximation, there is a need to also consider other approximation approaches that would scale well computationally.

There is also a need to address ambiguity because in real world optimization problems under uncertainty, the true probability distribution of the underlying random variable is often unavailable. Gilboa and Schmeidler (1989) axiomatize the preference of an ambiguity-averse decision maker and propose the decision criterion that evaluates the worst-case expected utility over an ambiguity set of prior distributions, though the earliest application of such decision preference may be traced to Scarf (1957). Similar decision criteria have also been proposed in the *coherent risk measures* of Artzner et al. (1999) within the mathematical finance community, and *distributionally robust optimization* within the operations research community (see, *e.g.*, Goh and Sim 2010, Delage and Ye 2010, Esfahani and Kuhn 2018, Chen et al. 2019).

In this paper, we focus on developing tractable robust optimization frameworks that cater for CARA preferences; while there are tractable distributionally robust optimization frameworks for piecewise linear utility functions (see, *e.g.*, Bertsimas et al. 2010, Wiesemann et al. 2014), they have yet been extended to exponential utility. Specifically, we propose tractable frameworks to address distributionally robust adaptive optimization problems with CARA preferences, which can be applied to solving a variety of dynamic decision problems such as, network lot-sizing, project management and inventory management, among others. The underlying model of uncertainty is represented by a set of independent factors for which their marginal distributions are partially characterized via an ambiguity set. To obtain computationally tractable formulations, we propose a hierarchy of approximations, starting from formulating the objective function as tractable concave functions in affinely perturbed cases, developing approximations in concave piecewise affinely perturbed cases, and extending the approximations to address recourse using linear decision rule techniques (see, *e.g.*, Ben-Tal et al. 2004, Chen et al. 2008, Goh and Sim 2010, Georghiou et al. 2015, Kuhn et al. 2011, Bertsimas et al. 2019). We also extend the framework to address a multi-period consumption model. The resultant models would take the form of exponential conic optimization problems, which can be practicably solved using current off-the-shelf solvers. In our computational studies, we present numerical examples including project management and multi-period inventory management with financing to illustrate how our approach can be applied to obtain high-quality solutions that could outperform current stochastic optimization approaches, especially in situations with high risk aversion levels.

Notations We use boldface lowercase letters to represent vectors such as $\boldsymbol{a}$, and calligraphic font to denote a set such as $\mathcal{Z}$. We denote by $\mathbb{R}$, $\mathbb{R}_+$ the set of all and non-negative real numbers, respectively. We denote by $[N] \triangleq \{1, 2, ..., N\}$ the set of positive running indices up to N. We denote by $|\mathcal{Z}|$ the cardinality of the set $\mathcal{Z}$. We use $\|\cdot\|$ to denote Euclidean norm of a vector. We denote by $\mathcal{P}_0(\mathcal{Z})$ the set of all probability distributions on support set $\mathcal{Z}$. We use tilde $(\tilde{\cdot})$ to denote uncertain parameters and use $\mathbb{P}$ to denote probability measure on sample space $\mathcal{Z}$. We denote by $\mathbb{E}_{\mathbb{P}}[\tilde{\boldsymbol{z}}]$ the expectation of $\tilde{\boldsymbol{z}}$ under probability distribution $\tilde{\boldsymbol{z}} \sim \mathbb{P}$. The inequality between two uncertain parameters $\tilde{t} \geq \tilde{v}$ describes state-wise dominance, *i.e.*, $\tilde{t}(\omega) \geq \tilde{v}(\omega)$ for all $\omega \in \Omega$. We denote $\mathcal{R}^{k,n}$ as the space of all measurable functions from $\mathbb{R}^k$ to $\mathbb{R}^n$ that are bounded on compact sets. We define plus function $(x)^+ \triangleq \max\{x, 0\}$.

2. Robust optimization with CARA preferences

We first focus on a decision model where we denote $\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^{I_x}$ as a vector of *here-and-now* decision variables and $f(\boldsymbol{x}, \boldsymbol{z})$ being the payoff function in which the second argument, $\boldsymbol{z} \in \mathcal{Z} \subseteq \mathbb{R}^{I_z}$ represents the model's parameters that are subject to uncertainty. After the decision $\boldsymbol{x}$ has been made, the payoff function would be randomly perturbed by the uncertain parameters. We next present the following assumption on the model of uncertainty.

Assumption 1 **(Independent factors with ambiguous marginals)**. *We assume that the model's uncertainty is generated from a set of stochastically independent factors with ambiguous marginals, denoted by $\tilde{\boldsymbol{z}} = (\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_{I_z})$ where $\tilde{z}_j$, $j \in [I_z]$ has an unknown distribution $\mathbb{P}_j$ partially characterized by an ambiguity set $\mathcal{F}_j$, i.e., $\tilde{z}_j \sim \mathbb{P}_j \in \mathcal{F}_j \subseteq \mathcal{P}_0([\underline{z}_j, \bar{z}_j])$, $\underline{z}_j < \bar{z}_j$. The distribution of $\tilde{\boldsymbol{z}}$ is the product distribution $\mathbb{P} \in \mathcal{F}$, where $\mathcal{F} \triangleq \times_{j \in [I_z]} \mathcal{F}_j$. We denote $\mathcal{Z} = [\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]$ as the support set of all distributions in $\mathcal{F}$, i.e., $\mathcal{F} \subseteq \mathcal{P}_0(\mathcal{Z})$. We also partition the index set $[I_z] = \mathcal{J}^+ \cup \mathcal{J}^- \cup \mathcal{J}$ so that $j \in \mathcal{J}^+$ if and only if $\underline{z}_j \geq 0$, $j \in \mathcal{J}^-$ if and only if $\bar{z}_j \leq 0$.*

CARA certainty equivalent

Observe that under Assumption 1, the payoff function, $f(\boldsymbol{x}, \tilde{\boldsymbol{z}})$ is a random variable with ambiguous probability distributions. In considering both risk and ambiguity, we adopt the preference relation in Gilboa and Schmeidler (1989), such that for a given increasing utility function, $u : \mathbb{R} \to \mathbb{R}$ and ambiguity set, $\mathcal{F} \subseteq \mathcal{P}_0(\mathcal{Z})$, the random payoff $f(\boldsymbol{x}_1, \tilde{\boldsymbol{z}})$ is preferred over $f(\boldsymbol{x}_2, \tilde{\boldsymbol{z}})$ if and only if the worst-case expected utility function $\inf_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}[u(f(\boldsymbol{x}_1, \tilde{\boldsymbol{z}}))] \geq \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}[u(f(\boldsymbol{x}_2, \tilde{\boldsymbol{z}}))]$. Therefore, the goal of our robust decision model is to maximize the worst-case expected utility of the payoff as follows:

$$\max_{\boldsymbol{x}\in\mathcal{X}} \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}\left[u(f(\boldsymbol{x}, \tilde{\boldsymbol{z}}))\right],$$

or equivalently,

$$\max_{\boldsymbol{x}\in\mathcal{X}} \inf_{\mathbb{P}\in\mathcal{F}} u^{-1}\left(\mathbb{E}_{\mathbb{P}}\left[u(f(\boldsymbol{x}, \tilde{\boldsymbol{z}}))\right]\right), \tag{1}$$

for which the objective function $u^{-1}\left(\mathbb{E}_{\mathbb{P}}\left[u(f(\boldsymbol{x}, \tilde{\boldsymbol{z}}))\right]\right)$ is referred to the *certainty equivalent* of the random payoffs under the utility function u and distribution $\mathbb{P} \in \mathcal{F}$. In articulating risk, certainty equivalent has the benefit of being more interpretable than expected utility. As a consequence of Jensen's inequality, for concave utility representing risk aversion, the certainty equivalent would not exceed the risk-neutral expected payoffs (see, *e.g.*, Mas-Colell et al. 1995). Specific to the CARA preferences, we have the exponential utility $u(v) = 1 - e^{-v/\kappa}$ with $\kappa > 0$ being the risk tolerance level. Hence, we focus on the following robust CARA optimization model:

$$\max_{\boldsymbol{x}\in\mathcal{X}} \inf_{\mathbb{P}\in\mathcal{F}} -\kappa \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{f(\boldsymbol{x}, \tilde{\boldsymbol{z}})}{\kappa}\right)\right]. \tag{2}$$

Definition 1. For a given random variable $\tilde{v}$, $\tilde{v} \sim \mathbb{P}$, the *CARA certainty equivalent* is defined as

$$\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right] \triangleq \begin{cases} \operatorname{ess\,inf}_{\mathbb{P}}\left[\tilde{v}\right] & \text{if } \kappa = 0 \\ \mathbb{E}_{\mathbb{P}}\left[\tilde{v}\right] & \text{if } \kappa = \infty \\ -\kappa \log \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\dfrac{\tilde{v}}{\kappa}\right)\right] & \text{if } \kappa \in (0, \infty). \end{cases}$$

Given an ambiguity set of probability distributions $\mathcal{F}$, the *ambiguity-averse CARA certainty equivalent* is defined as

$$\mathbb{C}^{\kappa}_{\mathcal{F}}\left[\tilde{v}\right] \triangleq \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}\right].$$

Hence, Problem (2) can be written as

$$\max_{\boldsymbol{x}\in\mathcal{X}} \mathbb{C}^{\kappa}_{\mathcal{F}}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right]. \tag{3}$$

Note that we extend the definition of CARA certainty equivalent to the cases of zero and infinite risk tolerance level, which would correspond to the worst-scenario and risk-neutral preferences, respectively. Specifically, our robust CARA optimization model recovers the distributionally robust optimization model of the form

$$\max_{\boldsymbol{x}\in\mathcal{X}} \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right] \tag{4}$$

when $\kappa = \infty$ where the decision maker is risk-neutral and ambiguity-averse. When $\kappa \in \mathbb{R}_+$, the robust CARA optimization model captures the preference of risk and ambiguity aversion, and the degree of risk aversion increases as the risk tolerance level κ decreases. In the extreme case of $\kappa = 0$, the model would coincide with the classical stochastic-free robust optimization model as follows:

$$\max_{\boldsymbol{x} \in \mathcal{X}} \inf_{\boldsymbol{z} \in \mathcal{Z}} f(\boldsymbol{x}, \boldsymbol{z}). \tag{5}$$

We note that as it is more common in the optimization literature to consider minimizing a cost function, we can also consider the following minmax certainty equivalent problem associated with exponential disutility:

$$\min_{\boldsymbol{x} \in \mathcal{X}} \sup_{\mathbb{P} \in \mathcal{F}} \kappa \log \mathbb{E}_{\mathbb{P}} \left[\exp\left(\frac{f(\boldsymbol{x}, \tilde{\boldsymbol{z}})}{\kappa} \right) \right], \tag{6}$$

where the exponential disutility has the form $u(v) = e^{v/\kappa} - 1$. Hence, analogous to the definitions of CARA certainty equivalent and ambiguity-averse CARA certainty equivalent, we also define

$$\overline{\mathbb{C}}^{\kappa}_{\mathbb{P}}[\tilde{v}] \triangleq -\mathbb{C}^{\kappa}_{\mathbb{P}}[-\tilde{v}], \qquad \overline{\mathbb{C}}^{\kappa}_{\mathcal{F}}[\tilde{v}] \triangleq \sup_{\mathbb{P} \in \mathcal{F}} \overline{\mathbb{C}}^{\kappa}_{\mathbb{P}}[\tilde{v}] = -\mathbb{C}^{\kappa}_{\mathcal{F}}[-\tilde{v}].$$

In the remaining of the paper, we will adopt the payoff maximization decision model of Problem (2), although all the following development can easily be transformed to the cost minimization decision model of Problem (6).

We review some useful properties of the CARA certainty equivalent as follows:

Proposition 1. *Consider the random variable $\tilde{v}, \tilde{v} \sim \mathbb{P}$. The CARA certainty equivalent has the following properties:*

1. $\lim_{\kappa \to \infty} \mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}] = \mathbb{E}_{\mathbb{P}}[\tilde{v}]$.
2. $\lim_{\kappa \downarrow 0} \mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}] = \operatorname{ess\,inf}_{\mathbb{P}}[\tilde{v}]$.
3. $\mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}]$ *is non-decreasing in* $\kappa > 0$.
4. $\mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}]$ *is jointly concave in* $\tilde{v}$ *and* $\kappa > 0$.
5. *For all* $\nu \in \mathbb{R}$, $\mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v} + \nu] = \mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}] + \nu$.

These properties hold for $\mathbb{C}^{\kappa}_{\mathcal{F}}[\cdot]$ *as well.*

Remark 1. The first two properties justify the definition of the CARA certainty equivalent at its limits. The third property relates to the monotonicity of the CARA certainty equivalent with regards to the degree of risk aversion. As we will show, we can exploit the concavity property for computational tractability. For a fix risk tolerance, the concavity of the CARA certainty equivalent implies a preference for diversification, which is also associated with risk aversion behavior. The last property implies *translation invariance*, *i.e.*, adding a constant amount to the random payoff would increase the CARA certainty equivalent by exactly the same amount. Note that the last two properties can be associated with convex risk measures (see, *e.g.*, Artzner et al. 1999, Föllmer and Schied 2002), which is unique for CARA among all certainty equivalents.

We also present the following properties, which are useful for deriving exact and tractable approximations of robust CARA optimization models.

PROPOSITION 2. *Consider the random variables $\tilde{v}, \tilde{\nu}$, $(\tilde{v}, \tilde{\nu}) \sim \mathbb{P}$.*

1. $\mathbb{C}_{\mathbb{P}}^{\kappa}[\tilde{v}]$ *is super-additive in $(\kappa, \tilde{v})$, i.e.,*

$$\mathbb{C}_{\mathbb{P}}^{\kappa_1+\kappa_2}[\tilde{v}+\tilde{\nu}] \geq \mathbb{C}_{\mathbb{P}}^{\kappa_1}[\tilde{v}] + \mathbb{C}_{\mathbb{P}}^{\kappa_2}[\tilde{\nu}]$$

for any $\kappa_1, \kappa_2 \in \mathbb{R}_+$.

2. *If $\tilde{v}$, $\tilde{\nu}$ are also independently distributed, then*

$$\mathbb{C}_{\mathbb{P}}^{\kappa}[\tilde{v}+\tilde{\nu}] = \mathbb{C}_{\mathbb{P}}^{\kappa}[\tilde{v}] + \mathbb{C}_{\mathbb{P}}^{\kappa}[\tilde{\nu}].$$

These properties hold for $\mathbb{C}_{\mathcal{F}}^{\kappa}[\cdot]$ as well.

Payoff functions with affine perturbations

We now focus on robust CARA optimization models with affinely perturbed payoff functions, which allow us to reformulate such models as tractable exponential conic optimization problems. Specifically, we consider the payoff function of the following form,

$$f(\boldsymbol{x}, \boldsymbol{z}) = a^0(\boldsymbol{x}) + \sum_{j\in[I_z]} a^j(\boldsymbol{x}) z_j. \tag{7}$$

We assume the function $a^j(\boldsymbol{x}) : \mathbb{R}^{I_x} \to \mathbb{R}$ is concave for all $j \in \{0\} \cup \mathcal{J}^+$, convex for all $j \in \mathcal{J}^-$, and affine for all $j \in \mathcal{J}$ so that $f(\boldsymbol{x}, \boldsymbol{z})$ is concave in $\boldsymbol{x}$ for any $\boldsymbol{z} \in \mathcal{Z}$.

PROPOSITION 3. *The ambiguity-averse CARA certainty equivalent of the payoff function (7), $\mathbb{C}_{\mathcal{F}}^{\kappa}[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})]$ is equal to the optimal value of the following optimization problem:*

$$\begin{array}{lll} \max\limits_{\boldsymbol{\lambda}\in\mathbb{R}^{1+I_z}} & \lambda^0 + \sum\limits_{j\in[I_z]} \phi_j(\kappa, \lambda^j) & \\ \text{s.t.} & a^j(\boldsymbol{x}) \geq \lambda^j & \forall j \in \{0\} \cup \mathcal{J}^+ \\ & a^j(\boldsymbol{x}) \leq \lambda^j & \forall j \in \mathcal{J}^- \\ & a^j(\boldsymbol{x}) = \lambda^j & \forall j \in \mathcal{J}, \end{array} \tag{8}$$

where we define the function $\phi_j : [0, \infty] \times \mathbb{R} \to \mathbb{R}$,

$$\phi_j(\kappa, \lambda) \triangleq \mathbb{C}_{\mathcal{F}_j}^{\kappa}[\lambda \tilde{z}_j]. \tag{9}$$

As we will see from Proposition 3, having stochastic independent factors would help us decompose the multi-dimensional integration problems associated with evaluating the CARA certainty equivalent to more analytically tractable single-dimensional integration problems. To appreciate the simplification, we consider the following example.

Example 1. Let $\tilde{z}_j$ be independent uniform random variables over the unit interval $[0,1]$, $a^j(\boldsymbol{x}) \equiv a_j < 0$ for any $j \in [I_z]$ and $a^0(\boldsymbol{x}) \equiv a_0 > 0$, then

$$\mathbb{C}^{\kappa}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right] = a_0 - \sum_{j\in[I_z]} \kappa \log \int_0^1 \exp\left(-\frac{a_j z_j}{\kappa}\right) \mathrm{d}z_j = a_0 - \sum_{j\in[I_z]} \kappa \log \left(\frac{\kappa - \kappa e^{-a_j/\kappa}}{a_j}\right).$$

Hence the computation of $\mathbb{C}^{\kappa}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]$ can be obtained in closed form. In contrast, due to multi-dimensional integration, evaluating an expected concave piecewise linear utility such as

$$\mathbb{E}_{\mathbb{P}}\left[\min\left\{a_0 + \sum_{j\in[I_z]} a_j \tilde{z}_j, 0\right\}\right]$$

is known to be a #P-hard problem (Hanasusanto et al. 2016).

Finally, we point out the unique computational challenge of robust CARA optimization compared with distributionally robust optimization through the following example.

Example 2. We consider a simple mean-support ambiguity set as follows:

$$\mathcal{G} = \left\{\mathbb{P}\in\mathcal{P}_0(\mathbb{R}^{I_z}) \;\middle|\; \begin{array}{l} \tilde{\boldsymbol{z}} \sim \mathbb{P} \\ \mathbb{E}_{\mathbb{P}}\left[\tilde{\boldsymbol{z}}\right] = \boldsymbol{\mu} \\ \mathbb{P}\left[\tilde{\boldsymbol{z}} \in [\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]\right] = 1 \end{array}\right\}.$$

Note that to evaluate the ambiguity-averse CARA certainty equivalent $\mathbb{C}^{\kappa}_{\mathcal{G}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]$ of an affine function $f(\boldsymbol{x},\boldsymbol{z}) = \boldsymbol{a}^\top\boldsymbol{z}$, we need to solve the optimization problem

$$\sup_{\mathbb{P}\in\mathcal{G}} \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{f(\boldsymbol{x},\tilde{\boldsymbol{z}})}{\kappa}\right)\right] = \begin{array}{ll} \inf\limits_{\alpha,\boldsymbol{\beta}} & \alpha + \boldsymbol{\beta}^\top\boldsymbol{\mu} \\ \text{s.t.} & \alpha + \boldsymbol{\beta}^\top\boldsymbol{z} \geq \exp\left(\frac{-f(\boldsymbol{x},\boldsymbol{z})}{\kappa}\right) \;\forall \boldsymbol{z} \in [\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}] \end{array}$$

However, the constraint involves a convex maximization problem $\sup_{\boldsymbol{z}\in[\underline{\boldsymbol{z}},\bar{\boldsymbol{z}}]} \exp\left(\frac{-\boldsymbol{a}^\top\boldsymbol{z}}{\kappa}\right) - \boldsymbol{\beta}^\top\boldsymbol{z}$, which, in general, may not be amendable to a tractable reformulation. In contrast, evaluating the worst-case expectation $\inf_{\mathbb{P}\in\mathcal{G}} \mathbb{E}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]$ is known to be a tractable linear optimization problem. This shows the difficulty of applying standard techniques from the distributionally robust optimization to address the computation of robust CARA optimization problems. Moreover, from the comparison of Example 2 with Proposition 3, we can see stochastic independence in Assumption 1 is essential for tractability of the robust CARA optimization problems.

Practicably solvable reformulations

Our main interest is to obtain solutions to the robust CARA optimization problems reliably and within reasonable time so that the solutions can be implemented in practice.

Definition 2 (Practicably solvable problem). We refer to an optimization problem being *practicably solvable* as one that can be formulated and solved to optimality within reasonable time for their purpose using current available solvers.

In particular, a convex optimization problem that can be formulated using a modest number of decision variables, and a modest number of linear, convex quadratic, second-order conic constraints would be a practicably solvable problem. In this list of practicably solvable constraint types, we also include exponential conic constraints, which are essential in modeling exponential and logarithms arising from the robust CARA optimization model. Exponential conic constraints are now supported in Mosek, and they can also be approximated fairly accurately via second-order conic constraints, which are broadly supported in solvers such as Gurobi, CPLEX and SDPT3.

Definition 3. A convex set $\mathcal{W} \subseteq \mathbb{R}^I$ is *exponential cone representable ($\mathcal{K}_{\exp}$-representable)* if it is conic representable with exponential cones, *i.e.*,

$$\boldsymbol{x} \in \mathcal{W} \Longleftrightarrow \exists\ \boldsymbol{u} \in \mathbb{R}^J : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{b} \in \mathcal{K}_{\exp}^K$$

with $\boldsymbol{A} \in \mathbb{R}^{3K\times I}$, $\boldsymbol{B} \in \mathbb{R}^{3K\times J}$, $\boldsymbol{b} \in \mathbb{R}^{3K}$ and $\mathcal{K}_{\exp}^K$ is the Cartesian product of K *exponential cones*

$$\mathcal{K}_{\exp} \triangleq \{(x_1, x_2, x_3) | x_1 \geq x_2 \exp(x_3/x_2), x_2 > 0\} \cup \{(x_1, 0, x_3) | x_1 \geq 0, x_3 \leq 0\}. \tag{10}$$

We also say that a convex (concave) function is $\mathcal{K}_{\exp}$-representable if its epigraph (hypograph) is a $\mathcal{K}_{\exp}$-representable set.

Theorem 1. *Let $g(\boldsymbol{x}, \kappa) = -\kappa \log \sum_{i\in[I]} p_i e^{-x_i/\kappa}$ with $\kappa > 0$ and $p_i > 0$ for all $i \in [I]$, then the closure of its hypograph $\{(\boldsymbol{x}, \kappa, y) : y \leq g(\boldsymbol{x}, \kappa), \kappa > 0\}$ can be represented by*

$$\left\{ (\boldsymbol{x}, \kappa, y) \;\middle|\; \exists\ \boldsymbol{q} \in \mathbb{R}^I\ :\ \sum_{i\in[I]} p_i q_i \leq \kappa,\ (q_i, \kappa, y - x_i) \in \mathcal{K}_{\exp}\quad \forall i \in [I] \right\}.$$

For Problem (8) to be practicably solvable, we note that the functions $\phi_j(\kappa, \lambda^j)$, $j \in [I_z]$ are $\mathcal{K}_{\exp}$-representable for some choice of ambiguity sets, as we will show in the following example.

Example 3. Consider an ambiguity set with mean, mean absolute deviation, and support information

$$\mathcal{G} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}) \;\middle|\; \begin{array}{l} \tilde{z} \sim \mathbb{P} \\ \mathbb{E}_{\mathbb{P}}[\tilde{z}] = \mu \\ \mathbb{E}_{\mathbb{P}}[|\tilde{z} - \mu|] \leq \delta \\ \mathbb{P}[\tilde{z} \in [-1, 1]] = 1 \end{array} \right\}$$

where the support is normalized without loss of generality, otherwise we can consider transformation $z \mapsto \frac{2z - (\underline{z} + \overline{z})}{\overline{z} - \underline{z}}$ if $\mathbb{P}[\tilde{z} \in [\underline{z}, \overline{z}]] = 1$. As in Postek et al. (2018) where the inequality in $\mathcal{G}$ is replaced by equality, we show

$$\phi(\kappa, \lambda) = -\kappa \log \left(\frac{\delta}{2(\mu+1)} e^{\lambda/\kappa} + \frac{\delta}{2(1-\mu)} e^{-\lambda/\kappa} + \left(1 - \frac{\delta}{2(\mu+1)} - \frac{\delta}{2(1-\mu)}\right) e^{-\mu\lambda/\kappa} \right),$$

where

$$\phi(\kappa, \lambda) \triangleq \mathbb{C}_{\mathcal{G}}^{\kappa}[\lambda \tilde{z}]$$

is hence $\mathcal{K}_{\exp}$-representable based on the Theorem 1. The proof is relegated to Appendix A.

We refer interested readers to Nemirovski and Shapiro (2007) for more examples, which we also summarize in Table 1.

Table 1 Equivalent representations of $\phi(\kappa,\lambda)$

Ambiguity set	$\phi(\kappa,\lambda)$
$\{\mathbb{P}[\tilde{z}\in[-1,1]]=1\}$	$-\lvert\lambda\rvert$
$\left\{\begin{array}{l}\mathbb{P} \text{ is symmetric}\\ \mathbb{P}[\tilde{z}\in[-1,1]]=1\end{array}\right\}$	$-\kappa\log\left(\frac{e^{\lambda/\kappa}+e^{-\lambda/\kappa}}{2}\right)$
$\left\{\begin{array}{l}\mathbb{P} \text{ is unimodal w.r.t. } 0\\ \mathbb{P}[\tilde{z}\in[-1,1]]=1\end{array}\right\}$	$-\kappa\log\int_0^1 e^{s\lvert\lambda\rvert/\kappa}\mathrm{d}s$
$\left\{\begin{array}{l}\mathbb{P} \text{ is symmetric,}\\ \text{unimodal w.r.t. } 0\\ \mathbb{P}[\tilde{z}\in[-1,1]]=1\end{array}\right\}$	$-\lambda-\kappa\log\int_0^1 e^{-2\lambda s/\kappa}\mathrm{d}s$
$\left\{\begin{array}{l}\mathbb{E}_{\mathbb{P}}[\tilde{z}]\in[\underline{\mu},\overline{\mu}]\\ \mathbb{P}[\tilde{z}\in[-1,1]]=1\end{array}\right\}$	$\min\left\{\begin{array}{l}-\kappa\log\left(\frac{(1+\underline{\mu})e^{-\lambda/\kappa}+(1-\underline{\mu})e^{\lambda/\kappa}}{2}\right),\\ -\kappa\log\left(\frac{(1+\overline{\mu})e^{-\lambda/\kappa}+(1-\overline{\mu})e^{\lambda/\kappa}}{2}\right)\end{array}\right\}$
$\left\{\begin{array}{l}\mathbb{E}_{\mathbb{P}}[\tilde{z}]=\mu\\ \mathbb{E}_{\mathbb{P}}[\lvert\tilde{z}-\mu\rvert]\le\delta\\ \mathbb{P}[\tilde{z}\in[-1,1]]=1\end{array}\right\}$	$-\kappa\log\left(\frac{\delta}{2(\mu+1)}e^{\lambda/\kappa}+\frac{\delta}{2(1-\mu)}e^{-\lambda/\kappa}+\left(1-\frac{\delta}{2(\mu+1)}-\frac{\delta}{2(1-\mu)}\right)e^{-\mu\lambda/\kappa}\right)$
$\left\{\begin{array}{l}\mathbb{E}_{\mathbb{P}}[\tilde{z}]=\mu\\ \mathbb{E}_{\mathbb{P}}[\lvert\tilde{z}\rvert^2]\le\sigma^2\\ \mathbb{P}[\tilde{z}\in[-1,1]]=1\end{array}\right\}$	$\min\left\{\begin{array}{l}-\kappa\log\left(\frac{(1-\mu)^2\exp\left(\frac{-(\mu-\sigma^2)\lambda}{(1-\mu)\kappa}\right)+(\sigma^2-\mu^2)\exp(-\lambda/\kappa)}{1-2\mu+\sigma^2}\right),\\ -\kappa\log\left(\frac{(1+\mu)^2\exp\left(\frac{-(\mu+\sigma^2)\lambda}{(1+\mu)\kappa}\right)+(\sigma^2-\mu^2)\exp(\lambda/\kappa)}{1+2\mu+\sigma^2}\right)\end{array}\right\}$
$\left\{\begin{array}{l}\mathbb{P} \text{ is symmetric,}\\ \mathbb{E}_{\mathbb{P}}[\lvert\tilde{z}\rvert^2]\le\sigma^2\\ \mathbb{P}[\tilde{z}\in[-1,1]]=1\end{array}\right\}$	$-\kappa\log\left(\frac{\sigma^2(e^{\lambda/\kappa}+e^{-\lambda/\kappa})}{2}+1-\sigma^2\right)$

Gaussian quadrature approximation

However, some of the reformations in Table 1 may not admit obvious conic representations. For example, when $\mathcal{G}$ is the set containing all unimodal distributions with bounded support, we have

$$\phi(\kappa,\lambda)=-\kappa\log\int_0^1 e^{s|\lambda|/\kappa}\mathrm{d}s=-\kappa\log\left(\frac{\kappa e^{|\lambda|/\kappa}-\kappa}{|\lambda|}\right).$$

Ben-Tal et al. (2009) derive its quadratic lower bound $-|\lambda|/2 - \lambda^2/(24\kappa)$ based on Taylor's expansion. Nevertheless, we can use the Gaussian quadrature approximation (see, *e.g.*, Trefethen 2019)

$$\phi(\kappa,\lambda) = -\kappa \log \int_0^1 e^{s|\lambda|/\kappa} \mathrm{d}s \approx -\kappa \log \sum_{\ell \in [n]} \omega_\ell \exp\left(\frac{s_\ell |\lambda|}{\kappa}\right)$$

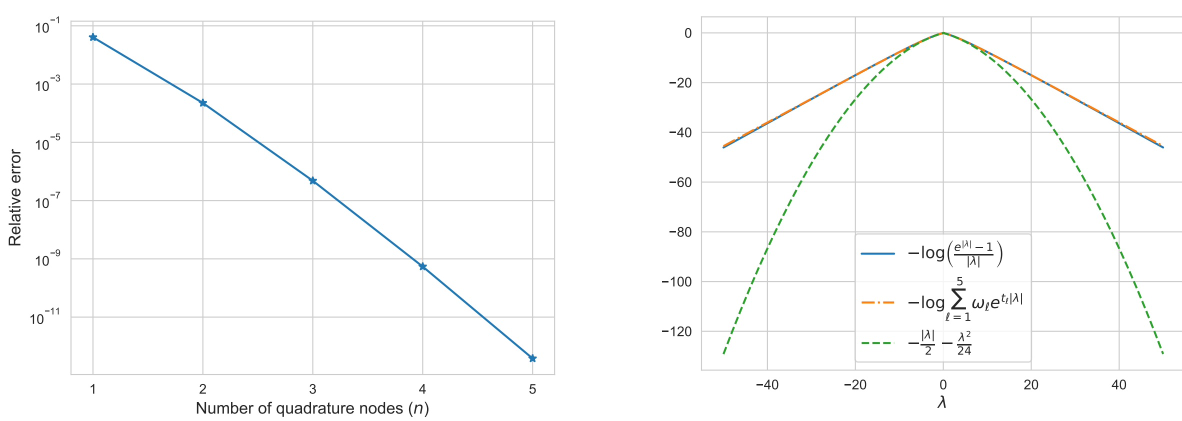
where ω_ℓ, s_ℓ, $\ell \in [n]$ are the quadrature weights and nodes on interval $[0,1]$. Clearly by Theorem 1 the quadrature approximation is $\mathcal{K}_{\text{exp}}$-representable.

In practice, we can choose $n = 5$ with parameters

$$\boldsymbol{\omega} = [0.1185, 0.2393, 0.2844, 0.2393, 0.1185], \quad \boldsymbol{s} = [0.0469, 0.2308, 0.5000, 0.7692, 0.9531],$$

which already provide very accurate estimation, see Figure 1a where we plot the relative approximation error of integral $\int_0^1 e^x \mathrm{d}x$ by quadrature approximation with different number of nodes, and Figure 1b for a comparison of quadrature approximation and the quadratic lower bound of $-\log\left(\frac{e^{|\lambda|}-1}{|\lambda|}\right)$.

Figure 1 **Illustration of Gaussian quadrature approximation**



(a) Approximation of $\int_0^1 e^x \mathrm{d}x$

(b) Approximation of $-\log\left(\frac{e^{|\lambda|}-1}{|\lambda|}\right)$

Payoff functions with concave piecewise affine perturbations

We have shown the robust CARA optimization is practicably tractable for concave payoff functions with affine perturbations. However, in practice the payoff functions can be nonlinear in the uncertain factors, in which case the robust CARA optimization model would be intractable. Example 1 shows that even evaluating the CARA certainty equivalent of a simple concave piecewise affine function under known distribution can be #P-hard. Therefore, there is a need for deriving tractable approximations of robust CARA optimization models with payoff functions that are commonly used in practice.

We now consider robust CARA optimization for payoff functions with concave piecewise affine perturbations, *i.e.*,

$$f(\boldsymbol{x},\boldsymbol{z}) = \min_{i\in\mathcal{I}} \left\{ a_i^0(\boldsymbol{x}) + \sum_{j\in[I_z]} a_i^j(\boldsymbol{x}) z_j \right\}. \tag{11}$$

For each $i\in\mathcal{I}$, we assume the function $a_i^j(\boldsymbol{x}): \mathbb{R}^{I_x} \to \mathbb{R}$ is concave for all $j \in \{0\}\cup\mathcal{J}^+$, convex for all $j\in\mathcal{J}^-$, and affine for all $j\in\mathcal{J}$. Moreover, we assume the optimization problem $\max_{\boldsymbol{x}\in\mathcal{X}} f(\boldsymbol{x},\boldsymbol{z})$ is practicably solvable for any $\boldsymbol{z}\in\mathcal{Z}$.

We next show the ambiguity-averse CARA certainty equivalent of the payoff function (11) has a practicably solvable lower bound.

Theorem 2. *The ambiguity-averse CARA certainty equivalent of the payoff function* (11), $\mathbb{C}_{\mathcal{F}}^{\kappa}[f(\boldsymbol{x},\tilde{\boldsymbol{z}})]$ *has a practicably solvable lower bound:*

$$\begin{aligned} \Lambda(\kappa,\boldsymbol{x}) \triangleq \max_{\boldsymbol{\alpha},\boldsymbol{\beta}} \ & \Phi(\kappa,\boldsymbol{\alpha},\boldsymbol{\beta}) \\ \text{s.t.} \ & a_i^0(\boldsymbol{x}) \geq \alpha_i && \forall i\in\mathcal{I} \\ & a_i^j(\boldsymbol{x}) \geq \beta_i^j && \forall i\in\mathcal{I}, j\in\mathcal{J}^+ \\ & a_i^j(\boldsymbol{x}) \leq \beta_i^j && \forall i\in\mathcal{I}, j\in\mathcal{J}^- \\ & a_i^j(\boldsymbol{x}) = \beta_i^j && \forall i\in\mathcal{I}, j\in\mathcal{J} \\ & \boldsymbol{\alpha}\in\mathbb{R}^{|\mathcal{I}|}, \boldsymbol{\beta}\in\mathbb{R}^{|\mathcal{I}|\times I_z}, \end{aligned} \tag{12}$$

where the objective function is

$$\begin{aligned} \Phi(\kappa,\boldsymbol{\alpha},\boldsymbol{\beta}) \triangleq \max_{\rho,\boldsymbol{\gamma},\boldsymbol{r},\boldsymbol{q},\boldsymbol{\kappa}} \ & r_0 + \rho \\ \text{s.t.} \ & \kappa_0 + \kappa_1 = \kappa \\ & \sum_{i\in\mathcal{I}} q_i \leq \kappa_1 \\ & (q_i, \kappa_1, \rho - r_i) \in \mathcal{K}_{\exp} && \forall i\in\mathcal{I} \\ & \sum_{j\in[I_z]} \phi_j(\kappa_0, \gamma^j) \geq r_0 \\ & \alpha_i + \sum_{j\in[I_z]} \phi_j(\kappa_1, \beta_i^j - \gamma^j) \geq r_i && \forall i\in\mathcal{I} \\ & \boldsymbol{\gamma}\in\mathbb{R}^{I_z}, \boldsymbol{r}\in\mathbb{R}^{1+|\mathcal{I}|}, \boldsymbol{\kappa}\in\mathbb{R}^2_+, \rho\in\mathbb{R}, \boldsymbol{q}\in\mathbb{R}^{|\mathcal{I}|}. \end{aligned} \tag{13}$$

Observe that despite not being able to determine the precise value of $\mathbb{C}_{\mathcal{F}}^{\kappa}[f(\boldsymbol{x},\tilde{\boldsymbol{z}})]$, the following result provides conditions when the approximation can be exact.

Theorem 3. *For any $\boldsymbol{x}\in\mathcal{X}$, the function $\Lambda(\kappa,\boldsymbol{x})$ is non-decreasing in $\kappa\in[0,\infty]$ and satisfies $\Lambda(\kappa,\boldsymbol{x})\geq\inf_{\boldsymbol{z}\in\mathcal{Z}} f(\boldsymbol{x},\boldsymbol{z})$. Moreover, $\mathbb{C}^{\kappa}_{\mathcal{F}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]=\Lambda(\kappa,\boldsymbol{x})$ if there exists some $i^*\in\mathcal{I}$ such that*

$$a^0_{i^*}(\boldsymbol{x})+\sum_{j\in[I_z]} a^j_{i^*}(\boldsymbol{x})z_j\leq a^0_i(\boldsymbol{x})+\sum_{j\in[I_z]} a^j_i(\boldsymbol{x})z_j \qquad \forall\boldsymbol{z}\in\mathcal{Z}, i\in\mathcal{I}.$$

Remark 2. In the extreme risk aversion where $\kappa=0$, Theorem 3 implies that

$$\inf_{\boldsymbol{z}\in\mathcal{Z}} f(\boldsymbol{x},\boldsymbol{z})=\mathbb{C}^0_{\mathcal{F}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]=\Lambda(0,\boldsymbol{x})\geq\inf_{\boldsymbol{z}\in\mathcal{Z}} f(\boldsymbol{x},\boldsymbol{z}),$$

alluding to the improving accuracy of the approximation $\Lambda(\kappa,\boldsymbol{x})$ as the risk tolerance, κ decreases. The exact result could also occur when there exists a dominant payoff component, $i^*\in\mathcal{I}$ as defined in Theorem 3, which could arise in situations with low coefficient of variations among the payoff components, or when there are identical variations such that $a^j_{i_1}(\boldsymbol{x})=a^j_{i_2}(\boldsymbol{x})$ for all $i_1,i_2\in\mathcal{I}, j\in[I_z]$.

We next illustrate the approximation in Theorem 2 through two examples. First we note that Theorem 2 provides a new tractable lower bound for evaluating $\mathbb{E}_{\mathbb{P}}\left[\min\left\{\lambda^0+\sum_{j\in[I_z]}\lambda^j\tilde{z}_j,0\right\}\right]$, which is shown to be #P-hard in Example 1 under independent identically distributed (i.i.d.) uniform random factors. One well-known tractable lower bound (Nemirovski and Shapiro 2007, Chen et al. 2008) is based on the observation of $-\min\{-z,0\}=(z)^+\leq\kappa\exp\left(\frac{z}{\kappa}-1\right)$ for any $\kappa>0$. We show in the following simple example that our approximation scheme may improve that in certain cases.

Example 4. Let $\tilde{z}$ be a standard normal random variable. Consider approximations of $\mathbb{E}_{\mathbb{P}}\left[(\tilde{z}+\mu)^+\right]=\mu F(\mu)+\frac{e^{-\mu^2/2}}{\sqrt{2\pi}}$ where $F(\cdot)$ is the cumulative distribution function of $\tilde{z}$. The popular upper bound in existing literature is

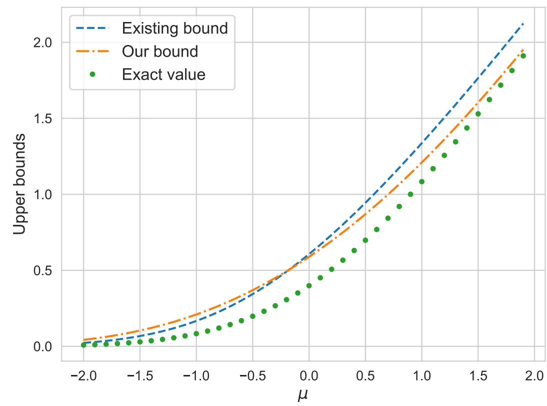
$$\mathbb{E}_{\mathbb{P}}\left[(\tilde{z}+\mu)^+\right]\leq\inf_{\kappa>0}\mathbb{E}_{\mathbb{P}}\left[\kappa\exp\left(\frac{\tilde{z}+\mu}{\kappa}-1\right)\right]=\inf_{\kappa>0}\frac{\kappa}{e}\exp\left(\frac{\mu}{\kappa}+\frac{1}{2\kappa^2}\right), \tag{14}$$

while based on Theorem 2 we obtain the upper bound

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(\tilde{z}+\mu)^+\right]&=\overline{\mathbb{C}}^{\infty}_{\mathbb{P}}\left[(\tilde{z}+\mu)^+\right]=-\mathbb{C}^{\infty}_{\mathbb{P}}\left[\min\{-\tilde{z}-\mu,0\}\right]\\&\leq\inf_{\kappa_1>0,\gamma}\mathbb{E}_{\mathbb{P}}\left[\gamma\tilde{z}\right]+\kappa_1\log\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{(1-\gamma)\tilde{z}+\mu}{\kappa_1}\right)+\exp\left(\frac{-\gamma\tilde{z}}{\kappa_1}\right)\right]\\&=\inf_{\kappa_1>0,\gamma}\kappa_1\log\left(\exp\left(\frac{\mu}{\kappa_1}+\frac{(1-\gamma)^2}{2\kappa_1^2}\right)+\exp\left(\frac{\gamma^2}{2\kappa_1^2}\right)\right).\end{aligned} \tag{15}$$

See Figure 2 for a comparison of the existing bound (14) and our bound (15) where $\mu\in[-2,2]$. We can see our bound is better than bound (14) when μ is mostly positive, though the bound does not perform as well when μ is mostly negative. We can improve the bound of $\mathbb{E}_{\mathbb{P}}\left[(\tilde{z}+\mu)^+\right]$ using infimal convolution of the two bounds (see, *e.g.,* Chen and Sim 2009). Unfortunately, for $\kappa<\infty$, we are not able to extend the bound (14) for evaluating the certainty equivalent, $\overline{\mathbb{C}}^{\kappa}_{\mathbb{P}}\left[(\tilde{z}+\mu)^+\right]$.

Figure 2 **Upper bounds of** $\mathbb{E}_{\mathbb{P}}\left[(\tilde{z}+\mu)^{+}\right]$



On Monte-Carlo approximation

We next compare our bound with Monte-Carlo approximation, which is a typical way of evaluating the CARA certainty equivalent $\mathbb{C}^{\kappa}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]$ under distribution $\mathbb{P}$. Basically, we generate S i.i.d. samples from the distribution $\mathbb{P}$ and construct the random approximation of $\mathbb{C}^{\kappa}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right]$ as follows:

$$-\kappa\log\frac{1}{S}\sum_{s\in[S]}\exp\left(-\frac{f(\boldsymbol{x},\hat{\boldsymbol{z}}^s)}{\kappa}\right)$$

where $\hat{\boldsymbol{z}}^s$, $s\in[S]$ are realized samples independently drawn from $\tilde{\boldsymbol{z}}\sim\mathbb{P}$. We show the Monte-Carlo approximation is upward biased as follows,

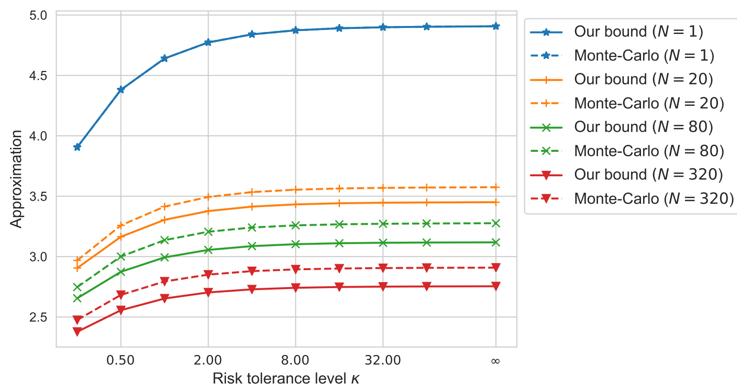
Proposition 4. *Consider the random variable* $(\tilde{\boldsymbol{z}}^1,\ldots,\tilde{\boldsymbol{z}}^S)\sim\mathbb{P}^S$*, then*

$$\mathbb{E}_{\mathbb{P}^S}\left[-\kappa\log\frac{1}{S}\sum_{s\in[S]}\exp\left(-\frac{f(\boldsymbol{x},\tilde{\boldsymbol{z}}^s)}{\kappa}\right)\right]\geq\mathbb{C}^{\kappa}_{\mathbb{P}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right].$$

Although the Monte-Carlo approximation is upward biased, based on the law of large numbers, one can expect to obtain reasonably good approximation if the sample size S is large enough. This intuition holds true for high risk tolerance level. However, when risk tolerance level is low, we will show in the following example that the upward bias can be pronounced even with large sample size.

Example 5. We consider approximations of CARA certainty equivalent of the minimum of N weighted sum of $I_z=20$ independently distributed random variables, for $N\in\{1,20,80,320\}$. Specifically, we evaluate $\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\min_{i\in[N]}\left\{\boldsymbol{a}_i^{\top}\tilde{\boldsymbol{z}}\right\}\right]$, where $\tilde{z}_j$, $j\in[I_z]$ are i.i.d. uniformly distributed random variables on $[0,1]$, and the weight vector $\boldsymbol{a}_i$ is randomly generated from the uniformly distributed unit hypercube $[0,1]^{I_z}$. We vary κ in $[0.25,64]$ and include $\kappa=\infty$. The results are

Figure 3 Comparison of our bound and Monte-Carlo approximation (10^6 samples) for $\kappa \geq 0.25$



obtained by averaging over 50 random instances and presented in Figure 3. We see the Monte-Carlo approximation coincides with our lower bound when $N = 1$, suggesting both are accurate. For $N > 1$, the gap between the two approximations becomes larger as κ increases and stabilizes when $\kappa \to \infty$.

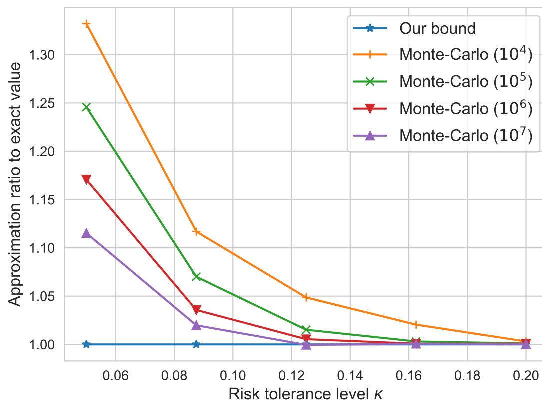
We observe that when the risk tolerance κ is low, the Monte-Carlo method may not provide accurate estimate; we show in Figure 4 where we plot the ratio of the Monte-Carlo approximations with $S \in \{10^4, 10^5, 10^6, 10^7\}$ samples to our bound of $\mathbb{C}^{\kappa}_{\mathbb{P}}[\boldsymbol{a}_1^\top \tilde{\boldsymbol{z}}]$ with $\kappa \in [0.05, 0.2]$. Note that our bound is exact at $N = 1$. We see the upward bias of Monte-Carlo approximation is more pronounced as S decreases, especially when $\kappa \leq 0.1$. What is surprising is the bias remains noticeable even with 10^7 samples. Hence, when the risk tolerance is low, the Monte-Carlo approximation would significantly overestimate the risk adjusted payoffs, while, as noted in Theorem 3, our deterministic approximation would provide a lower bound that is close to the actual CARA certainty equivalent. It is important to note that solving a stochastic optimization using SAA is a form of Monte-Carlo approximation, which yields random solutions with indicative objective values that are not achievable by the solutions. In contrast, our deterministic approximation would provide a pessimistic solution with an achievable indicative objective value.

3. Adaptive optimization and tractable approximations

We now propose a more general framework for adaptive robust CARA optimization that has provisions for recourse. We focus on a two-stage adaptive optimization problem with payoff function defined by the optimal value of a linear optimization problem as follows,

$$
\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{z}) = \max_{\boldsymbol{y}} \ & \boldsymbol{c}^\top \boldsymbol{y} \\
\text{s.t.} \ & \boldsymbol{b}_i^\top \boldsymbol{y} \leq a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \ \forall i \in \mathcal{I}, \\
& \boldsymbol{y} \in \mathbb{R}^{I_y}
\end{aligned}
\tag{16}
$$

Figure 4 **Ratio of Monte-Carlo approximation to our bound for $\kappa \leq 0.2$ at $N = 1$ where our bound is exact.**



where $\mathcal{I}$ is the index set of constraints and $\boldsymbol{a}_i(\boldsymbol{x})$ is the vector of $a_i^j(\boldsymbol{x})$ of $j \in [I_z]$ for each $i \in \mathcal{I}$. In this problem, $\boldsymbol{x}$ is the *here-and-now* decision and $\boldsymbol{y}$ is the *wait-and-see* or *recourse* decision adapted to uncertain parameter $\tilde{\boldsymbol{z}}$. As before, for each $i \in \mathcal{I}$, we assume the function $a_i^j(\boldsymbol{x}) : \mathbb{R}^{I_x} \to \mathbb{R}$ is concave for all $j \in \{0\} \cup \mathcal{J}^+$, convex for all $j \in \mathcal{J}^-$, and affine for all $j \in \mathcal{J}$.

Observe that with $I_y = 1$, $\boldsymbol{c} = 1$, and $\boldsymbol{b}_i = 1$, $i \in \mathcal{I}$, the concave piecewise affine payoff function (11) is a special case of (16). Moreover, we can assume without any loss of generality that the objective function in Problem (16) contains only the recourse decision. Otherwise, for the objective function $\boldsymbol{c}^\top \boldsymbol{y} + a_0^0(\boldsymbol{x}) + \boldsymbol{a}_0^\top(\boldsymbol{x})\boldsymbol{z}$, we can introduce another auxiliary variable y_{I_y+1} to replace it, and add the constraint $y_{I_y+1} - \boldsymbol{c}^\top \boldsymbol{y} \leq a_0^0(\boldsymbol{x}) + \boldsymbol{a}_0^\top(\boldsymbol{x})\boldsymbol{z}$. We assume without loss of generality that $\boldsymbol{b}_i \neq \boldsymbol{0}$, $i \in \mathcal{I}$; otherwise such a constraint can always be incorporated in $\mathcal{X}$, which describes the feasible set of the here-and-now decision.

Drawing from the insights of Zhen et al. (2018), we can always improve the formulation of an adaptive optimization problem via *Fourier-Motzkin elimination* of the recourse variables whenever it is computationally viable to do so. In particular, we highlight that Problem (16) does not have any equality constraint because for each equality constraint, we can eliminate a recourse variable without increasing the size of the formulation. The two-stage optimization problem is general enough to cover many practical optimization problems such as appointment scheduling, network lot-sizing, projection management, and so forth.

Definition 4. We say Problem (16) has *complete recourse* if and only if for any $\boldsymbol{d} \in \mathbb{R}^{|\mathcal{I}|}$, there exists some $\boldsymbol{y} \in \mathbb{R}^{I_y}$ such that $\boldsymbol{b}_i^\top \boldsymbol{y} \leq d_i$ for all $i \in \mathcal{I}$ (see, *e.g.,* Birge and Louveaux 2011).

Assumption 2. *We assume that the optimization problem* $\max_{\boldsymbol{x} \in \mathcal{X}} f(\boldsymbol{x}, \boldsymbol{z})$ *is practicably solvable and is bounded from above for any* $\boldsymbol{z} \in \mathcal{Z}$.

Note that we can express the ambiguity-averse CARA certainty equivalent as the following optimization problem,

$$\begin{aligned}\mathbb{C}^{\kappa}_{\mathcal{F}}\left[f(\boldsymbol{x},\tilde{\boldsymbol{z}})\right] = \max_{\boldsymbol{y}} \ & \mathbb{C}^{\kappa}_{\mathcal{F}}\left[\boldsymbol{c}^\top \boldsymbol{y}(\tilde{\boldsymbol{z}})\right] \\ \text{s.t.} \ & \boldsymbol{b}_i^\top \boldsymbol{y}(\boldsymbol{z}) \leq a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \quad \forall \boldsymbol{z} \in \mathcal{Z}, \quad \forall i \in \mathcal{I} \\ & \boldsymbol{y} \in \mathcal{R}^{I_z, I_y}.\end{aligned} \tag{17}$$

Since $\boldsymbol{y}$ is a function map instead of a finite vector of decision variables, the above problem is generally intractable.

A common approach to solve the adaptive optimization problem approximately is to use *linear decision rule (LDR)* to restrict the recourse function map to be affinely dependent on the uncertain parameters, *i.e.*, $\boldsymbol{y} \in \mathcal{L}^{I_z, I_y}$ where

$$\mathcal{L}^{I_z, I_y} \triangleq \left\{ \boldsymbol{y} \in \mathcal{R}^{I_z, I_y} \ \middle| \ \exists\, \boldsymbol{y}^0, \boldsymbol{y}^1, ..., \boldsymbol{y}^{I_z} : \boldsymbol{y}(\boldsymbol{z}) = \boldsymbol{y}^0 + \sum_{j \in [I_z]} z_j \boldsymbol{y}^j \right\}. \tag{18}$$

However, it has been well known that such approximation can be rather conservative and in some situations, we may sacrifice too much for tractability (Garstka and Wets 1974). To further improve the approximation, Chen et al. (2008), Goh and Sim (2010) propose the *deflected linear decision rule (DLDR)*, which we can adopt to provide a tractable approximation of Problem (17). For this purpose, we solve for each $i \in \mathcal{I}$:

$$\begin{aligned}\max_{\boldsymbol{y} \in \mathbb{R}^{I_y}} \ & \boldsymbol{c}^\top \boldsymbol{y} \\ & \boldsymbol{b}_k^\top \boldsymbol{y} \leq 0 \quad \forall k \in \mathcal{I} \backslash \{i\} \\ & \boldsymbol{b}_i^\top \boldsymbol{y} = -1\end{aligned} \tag{19}$$

and define $\mathcal{I}^o \subseteq \mathcal{I}$ as the index set of i such that the above optimization problem is feasible and $\boldsymbol{y}^i_\diamond$ as the corresponding optimal solution. In the case of complete recourse, Chen et al. (2008) note that Problem (19) would always be feasible. The DLDR has the following form:

$$\boldsymbol{y}^\dagger(\boldsymbol{z}) \triangleq \bar{\boldsymbol{y}}(\boldsymbol{z}) + \sum_{i \in \mathcal{I}^o} \boldsymbol{y}^i_\diamond \left(h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})\right)^+ \tag{20}$$

where

$$\begin{aligned}\bar{\boldsymbol{y}}(\boldsymbol{z}) &\triangleq \boldsymbol{y}^0 + \boldsymbol{Y}\boldsymbol{z} \\ h_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &\triangleq \boldsymbol{b}_i^\top \boldsymbol{y} - a_i^0(\boldsymbol{x}) - \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \ \forall i \in \mathcal{I},\end{aligned}$$

with $\boldsymbol{y}^0 \in \mathbb{R}^{I_y}$ and $\boldsymbol{Y} \triangleq [\boldsymbol{y}^1, ..., \boldsymbol{y}^{I_z}] \in \mathbb{R}^{I_y \times I_z}$.

Multi-deflected linear decision rule

We now propose the *multi-deflected linear decision rule (MLDR)* that improves upon DLDR. Specifically, we first solve for each $i \in \mathcal{I}$:

$$\begin{aligned} \max_{\boldsymbol{y} \in \mathbb{R}^{I_y}} \; & \boldsymbol{c}^\top \boldsymbol{y} \\ & \boldsymbol{b}_k^\top \boldsymbol{y} \leq 0 \qquad \forall k \in \mathcal{I} \backslash \{i\} \\ & \boldsymbol{b}_i^\top \boldsymbol{y} = -\|\boldsymbol{b}_i\| \end{aligned} \tag{21}$$

and denote $\boldsymbol{y}_*^i$ as its optimal solution for each $i \in \mathcal{I}^o$. Observe that $\boldsymbol{y}_*^i = \|\boldsymbol{b}_i\| \boldsymbol{y}_\diamond^i$. Then we partition the index set $\mathcal{I}^o$ as

$$\mathcal{I}^o = \bigcup_{\ell \in [m]} \mathcal{I}_\ell^o$$

such that $\boldsymbol{y}_*^{i_1} = \boldsymbol{y}_*^{i_2}$ if and only if i_1 and i_2 are in the same $\mathcal{I}_\ell^o$. We denote $\boldsymbol{y}_*^\ell$ as any $\boldsymbol{y}_*^i$ with $i \in \mathcal{I}_\ell^o$ and define MLDR as follows:

$$\hat{\boldsymbol{y}}(\boldsymbol{z}) \triangleq \bar{\boldsymbol{y}}(\boldsymbol{z}) + \sum_{\ell \in [m]} \boldsymbol{y}_*^\ell \left(\max_{i \in \mathcal{I}_\ell^o} \left\{ \frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|} \right\} \right)^+ . \tag{22}$$

In the following, we show how the MLDR can improve over the DLDR.

THEOREM 4. *Under Assumption 2, for any distribution* $\mathbb{P} \in \mathcal{P}_0(\mathcal{Z})$ *and* $\kappa \in \mathbb{R}_+$, *we have* $\boldsymbol{c}^\top \boldsymbol{y}_*^\ell \leq 0$ *for all* $\ell \in [m]$ *and*

$$\mathbb{C}_{\mathbb{P}}^\kappa \left[\boldsymbol{c}^\top \hat{\boldsymbol{y}}(\tilde{\boldsymbol{z}}) \right] \geq \mathbb{C}_{\mathbb{P}}^\kappa \left[\boldsymbol{c}^\top \boldsymbol{y}^\dagger(\tilde{\boldsymbol{z}}) \right].$$

We also show that the MLDR can replicate the optimal recourse function for the simplest class of adaptive optimization problems with complete recourse.

THEOREM 5. *Suppose Problem* (16) *has complete recourse and* $I_y = 1$, *then there exists an MLDR that is optimal in Problem* (16) *for all* $\boldsymbol{z} \in \mathcal{Z}$.

REMARK 3. We remark that for the same class of adaptive optimization problems with complete recourse, Bertsimas et al. (2019) has also proposed the lifted *affine recourse adaptation* (ARA), which can achieve the optimal worst-case risk-neutral objective value under a moment-based ambiguity set. However, unlike MLDR, the lifted ARA may not necessarily replicate the optimal recourse function that we have for MLDR in Theorem 5. Hence, Bertsimas et al. (2019) cautioned the use of lifted ARA as a form of decision rule or policy for multi-period decision making.

In line with Theorem 5, we illustrate the advantage of MLDR over DLDR in the following example.

EXAMPLE 6. Consider the payoff function as follows:

$$\begin{aligned} f(x,z) = \max_{y} \ & y \\ \text{s.t.} \ & y \le 2 \\ & y \le z \\ & y \le 2z-1 \\ & y \le 3z \end{aligned}$$

where the optimal decision rule is $y^{OPT}(z) = \min\{2, z, 2z-1, 3z\}$. Note that $y_*^i = -1$ for all $i \in \mathcal{I}^o = \mathcal{I}$. It is easy to see $y^{OPT}(z)$ can be expressed as an MLDR, such as

$$\hat{y}(z) = 2 - (\max\{0, 2-z, 3-2z, 2-3z\})^+.$$

However, it cannot be represented by any DLDR, which has the form

$$y^\dagger(z) = \bar{y}(z) - (\bar{y}(z)-2)^+ - (\bar{y}(z)-z)^+ - (\bar{y}(z)-2z+1)^+ - (\bar{y}(z)-3z)^+$$

for any $\bar{y}(z) = y^0 + y^1 z$. To see this, note that

$$y^\dagger(z) \le \min\{y^{OPT}(z), 5z-1-\bar{y}(z), 3z+1-2\bar{y}(z)\}.$$

To guarantee $y^\dagger(z) = y^{OPT}(z)$ for all $z \in \mathbb{R}$, the slope of $5z-1-\bar{y}(z)$ and $3z+1-2\bar{y}(z)$ must lie in the interval $[0,3]$, which implies $5-y^1 \le 3$ and $3-2y^1 \ge 0$, a contradiction. Hence there always exists $z \in \mathbb{R}$ such that $y^\dagger(z) < y^{OPT}(z)$ under any choice of $y^0, y^1 \in \mathbb{R}$.

In the following proposition, we establish the conditions of feasibility of the MLDR in Problem (17).

PROPOSITION 5. *Suppose* $\bar{\boldsymbol{y}} \in \mathcal{L}^{I_z, I_y}$ *satisfies*

$$\boldsymbol{b}_i^\top \bar{\boldsymbol{y}}(\boldsymbol{z}) \le a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \quad \forall \boldsymbol{z} \in \mathcal{Z}, \quad \forall i \in \mathcal{I} \backslash \mathcal{I}^o,$$

Then the MLDR, $\hat{\boldsymbol{y}}$ *satisfies*

$$\boldsymbol{b}_i^\top \hat{\boldsymbol{y}}(\boldsymbol{z}) \le a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \quad \forall \boldsymbol{z} \in \mathcal{Z}, \quad \forall i \in \mathcal{I}.$$

Therefore, by applying the MLDR to Problem (17), we obtain a lower bound of (17) as follows:

$$\begin{aligned} \mathbb{C}_{\mathcal{F}}^{\kappa}[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})] \ge \max_{\bar{\boldsymbol{y}}} \ & \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{c}^\top \left(\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}}) + \sum_{\ell \in [m]} \boldsymbol{y}_*^\ell \left(\max_{i \in \mathcal{I}_\ell^o}\left\{\frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}}), \tilde{\boldsymbol{z}})}{\|\boldsymbol{b}_i\|}\right\}\right)^+\right)\right] \\ \text{s.t.} \ & h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z}) \le 0 \qquad \forall \boldsymbol{z} \in \mathcal{Z}, \quad \forall i \in \mathcal{I} \backslash \mathcal{I}^o \\ & \bar{\boldsymbol{y}} \in \mathcal{L}^{I_z, I_y}. \end{aligned} \tag{23}$$

However, Problem (23) involves evaluation of ambiguity-averse CARA certainty equivalent of a sum of concave piecewise affine (*i.e.*, sum-of-min) functions. Theoretically, we can write the sum-of-min functions as concave piecewise affine functions so that the problem can be approximated using techniques in Theorem 2. However, this may not be practical since the piecewise affine reformulation might involve exponentially many pieces. Nevertheless, we provide a tractable lower bound as follows.

THEOREM 6. *Under Assumption 2, the ambiguity-averse CARA certainty equivalent* (17) *has a practicably solvable lower bound:*

$$
\begin{array}{lll}
\max\limits_{\substack{\boldsymbol{\kappa},\boldsymbol{r},\boldsymbol{\alpha},\boldsymbol{\beta}\\ \boldsymbol{y}^0,\boldsymbol{Y},\bar{\boldsymbol{\lambda}},\underline{\boldsymbol{\lambda}}}} & r_0 - \sum\limits_{\ell\in[m]} (\boldsymbol{c}^\top \boldsymbol{y}_*^\ell) r_\ell & \\
\text{s.t.} & \kappa_0 - \sum\limits_{\ell\in[m]} (\boldsymbol{c}^\top \boldsymbol{y}_*^\ell)\kappa_\ell = \kappa & \\
& \boldsymbol{c}^\top \boldsymbol{y}^0 + \sum\limits_{j\in[I_z]} \phi_j(\kappa_0, \boldsymbol{c}^\top \boldsymbol{y}^j) \geq r_0 & \\
& \Phi(\kappa_\ell,\ \bar{\boldsymbol{\alpha}}_{\mathcal{I}_\ell^o} - \bar{\boldsymbol{b}}_{\mathcal{I}_\ell^o}^\top \boldsymbol{y}^0,\ \bar{\boldsymbol{\beta}}_{\mathcal{I}_\ell^o} - \bar{\boldsymbol{b}}_{\mathcal{I}_\ell^o}^\top \boldsymbol{Y}) \geq r_\ell & \forall \ell \in [m] \\
& \bar{\boldsymbol{\lambda}}_i^\top \bar{\boldsymbol{z}} - \underline{\boldsymbol{\lambda}}_i^\top \underline{\boldsymbol{z}} \leq \alpha_i - \boldsymbol{b}_i^\top \boldsymbol{y}^0 & \forall i \in \mathcal{I}\backslash\mathcal{I}^o \\
& \boldsymbol{Y}^\top \boldsymbol{b}_i - \boldsymbol{\beta}_i = \bar{\boldsymbol{\lambda}}_i - \underline{\boldsymbol{\lambda}}_i & \forall i \in \mathcal{I}\backslash\mathcal{I}^o \\
& a_i^0(\boldsymbol{x}) \geq \alpha_i & \forall i \in \mathcal{I} \\
& a_i^j(\boldsymbol{x}) \geq \beta_i^j & \forall i \in \mathcal{I}, j \in \mathcal{J}^+ \\
& a_i^j(\boldsymbol{x}) \leq \beta_i^j & \forall i \in \mathcal{I}, j \in \mathcal{J}^- \\
& a_i^j(\boldsymbol{x}) = \beta_i^j & \forall i \in \mathcal{I}, j \in \mathcal{J} \\
& \boldsymbol{\alpha} \in \mathbb{R}^{|\mathcal{I}|}, \boldsymbol{\beta} \in \mathbb{R}^{|\mathcal{I}|\times I_z}, \boldsymbol{\kappa} \in \mathbb{R}_+^{m+1}, \boldsymbol{r} \in \mathbb{R}^{m+1} & \\
& \boldsymbol{y}^0 \in \mathbb{R}^{I_y}, \boldsymbol{Y} \in \mathbb{R}^{I_y\times I_z}, \bar{\boldsymbol{\lambda}}, \underline{\boldsymbol{\lambda}} \in \mathbb{R}_+^{|\mathcal{I}\backslash\mathcal{I}^o|\times I_z} &
\end{array}
\tag{24}
$$

where for any index set $\mathcal{M}$, *we denote* $\bar{\boldsymbol{\alpha}}_{\mathcal{M}} \triangleq \begin{bmatrix} 0 \\ \boldsymbol{\alpha}_{\mathcal{M}} \end{bmatrix} \in \mathbb{R}^{|\mathcal{M}|+1}$, $\bar{\boldsymbol{\beta}}_{\mathcal{M}} \triangleq \begin{bmatrix} \boldsymbol{0}^\top \\ \boldsymbol{\beta}_{\mathcal{M}} \end{bmatrix} \in \mathbb{R}^{(|\mathcal{M}|+1)\times I_z}$, $\bar{\boldsymbol{b}}_{\mathcal{M}} \triangleq \begin{bmatrix} \boldsymbol{0} & \boldsymbol{b}_{\mathcal{M}} \end{bmatrix} \in \mathbb{R}^{I_y\times(|\mathcal{M}|+1)}$ *and* $\boldsymbol{\alpha}_{\mathcal{M}}$, $\boldsymbol{\beta}_{\mathcal{M}}$, $\boldsymbol{b}_{\mathcal{M}}$ *are the stacked vectors or matrices of* $\alpha_i/\|\boldsymbol{b}_i\|$, $\boldsymbol{\beta}_i^\top/\|\boldsymbol{b}_i\|$, $\boldsymbol{b}_i/\|\boldsymbol{b}_i\|$ *for* $i \in \mathcal{M}$, *respectively.*

4. Multi-period consumption model

We now extend to a T-period problem where information is revealed at every period and decisions are made dynamically to maximize the total utilities of consumption across all periods. At each period, $t \in [T]$, up to I_{ξ_t} of the independently distributed random factors, *i.e.*, $\tilde{z}_1, ..., \tilde{z}_{I_{\xi_t}}$ would be realized, with I_{ξ_t} increasing in t and $I_{\xi_T} = I_z$. For convenience, we define the random variable $\tilde{\boldsymbol{\xi}}_t \triangleq (\tilde{z}_1, ..., \tilde{z}_{I_{\xi_t}})$ and the vector $\boldsymbol{\xi}_t \triangleq (z_1, ..., z_{I_{\xi_t}})$ to denote a realization of $\tilde{\boldsymbol{\xi}}_t$. At each period $t \in [T]$, $\boldsymbol{\xi}_t$ is realized and a non-anticipative consumption of $v_t(\boldsymbol{\xi}_t)$ is made as a function of $\boldsymbol{\xi}_t$ and it is not influenced by the future uncertain outcomes of $\tilde{z}_{I_{\xi_t}+1},\dots,\tilde{z}_{I_z}$. In evaluating the utility of the consumption profile, we adopt the time-additive exponential utility preference (Varian 1992, Chapter 19) as follows,

$$u\left(v_1(\boldsymbol{\xi}_1),\dots,v_T(\boldsymbol{\xi}_T)\right) = \sum_{t\in[T]} \theta_t \left(1-\exp\left(-\frac{v_t(\boldsymbol{\xi}_t)}{\kappa}\right)\right),$$

where we can specify the temporal discounting via the weights θ_t, $\theta_t \geq 0$. Since we are maximizing the utility, without any loss of generality, we will normalize the weights so that $\sum_{t\in[T]}\theta_t = 1$.

We next generalize the notion of CARA certainty equivalent to the multi-period setting, which we associate with a constant consumption of $v = \mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}]$ at every period so that

$$u(v,\dots,v) = \mathbb{E}_{\mathbb{P}}\left[u\left(v_1(\tilde{\boldsymbol{\xi}}_1),\dots,v_T(\tilde{\boldsymbol{\xi}}_T)\right)\right].$$

Definition 5. For a given random variable, $\tilde{\boldsymbol{z}} \sim \mathbb{P}$, let $\tilde{\boldsymbol{v}} \triangleq \left(v_1(\tilde{\boldsymbol{\xi}}_1),\dots,v_T(\tilde{\boldsymbol{\xi}}_T)\right)$ denote the random non-anticipative consumption profile over time. We define the following *multi-period CARA certainty equivalent*

$$\mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] \triangleq \begin{cases} \displaystyle\min_{t\in[T]:\theta_t>0}\left\{\operatorname{ess\,inf}_{\mathbb{P}}[\tilde{v}_t]\right\} & \text{if } \kappa = 0 \\ \displaystyle\sum_{t\in[T]}\theta_t\mathbb{E}_{\mathbb{P}}[\tilde{v}_t] & \text{if } \kappa = \infty \\ \displaystyle -\kappa\log\left(\sum_{t\in[T]}\theta_t\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\tilde{v}_t}{\kappa}\right)\right]\right) & \text{if } \kappa\in(0,\infty). \end{cases}$$

Proposition 6. *The multi-period CARA certainty equivalent has the following properties:*

1. $\mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}]$ *is non-decreasing in* $\kappa\in[0,\infty]$.
2.
$$\min_{t\in[T]:\theta_t>0}\left\{\operatorname{ess\,inf}_{\mathbb{P}}[\tilde{v}_t]\right\} \leq \mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] \leq \sum_{t\in[T]}\theta_t\mathbb{E}_{\mathbb{P}}[\tilde{v}_t],$$
and the bounds are achievable.
3. $\mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}]$ *is jointly concave in* $\tilde{\boldsymbol{v}}$ *and* $\kappa > 0$.
4.
$$\begin{aligned} \mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] = \max\ & -\kappa\log\left(\sum_{t\in[T]}\theta_t\exp\left(-\frac{\nu_t}{\kappa}\right)\right) \\ \text{s.t.}\ & \mathbb{C}_{\mathbb{P}}^{\kappa}[\tilde{v}_t] \geq \nu_t \qquad \forall t\in[T] \\ & \boldsymbol{\nu}\in\mathbb{R}^T. \end{aligned} \tag{25}$$
5. *For all* $\nu\in\mathbb{R}$,
$$\mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}+\nu\mathbf{1}] = \mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}]+\nu.$$

Remark 4. The first two properties show the preservation of monotonicity with regards to the risk tolerance level and justify the definition of the multi-period CARA certainty equivalent at its limits. The joint concavity property is also preserved, together with the fourth property showing the connection to CARA certainty equivalent in each period, are essential for tractability of multi-period CARA optimization problems. The last property is the extension of the translation invariance to multi-period so that if each period is increased by the same certain amount, then the multi-period certainty equivalent should also increase by the same amount. This property is sensible and unique to the choice of exponential utility.

In considering ambiguity aversion, it may seem natural to evaluate the worst-case expected cumulative utility for the entire horizon as follows

$$\inf_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}\left[\sum_{t\in[T]} \theta_t \left(1-\exp\left(-\frac{\tilde{v}_t(\tilde{\boldsymbol{\xi}}_t)}{\kappa}\right)\right)\right].$$

However, we do not know how to tractably evaluate this criterion even if the consumption functions are affinely dependent on the random factors. Instead, we propose to evaluate the Gilboa and Schmeidler (1989) worst-case expected utility for every period in the following criterion,

$$\sum_{t\in[T]} \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}\left[\theta_t \left(1-\exp\left(-\frac{\tilde{v}_t(\tilde{\boldsymbol{\xi}}_t)}{\kappa}\right)\right)\right].$$

Apart from the computational benefits, we can also justify this approach as being more prudent in mitigating the ambiguous risk of under consumption or starvation that may occur in any period. Consequently, we propose the following multi-period ambiguity-averse CARA certainty equivalent.

Definition 6. Given an ambiguity set of probability distributions $\mathcal{F}$, we define the *multi-period ambiguity-averse CARA certainty equivalent* as follows

$$\mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] \triangleq \begin{cases} \min\limits_{t\in[T]:\theta_t>0}\left\{\inf\limits_{\mathbb{P}\in\mathcal{F}} \operatorname{ess\,inf}_{\mathbb{P}}[\tilde{v}_t]\right\} & \text{if } \kappa=0 \\ \sum\limits_{t\in[T]} \theta_t \inf\limits_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}[\tilde{v}_t] & \text{if } \kappa=\infty \\ -\kappa\log\left(\sum\limits_{t\in[T]} \theta_t \sup\limits_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\tilde{v}_t}{\kappa}\right)\right]\right) & \text{if } \kappa\in(0,\infty). \end{cases}$$

Proposition 7. *The multi-period ambiguity-averse CARA certainty equivalent has the following properties:*

1. $\mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}]$ *is non-decreasing in* $\kappa\in[0,\infty]$.
2.
$$\min_{t\in[T]:\theta_t>0}\left\{\inf_{\mathbb{P}\in\mathcal{F}} \operatorname{ess\,inf}_{\mathbb{P}}[\tilde{v}_t]\right\} \le \mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] \le \sum_{t\in[T]} \theta_t \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}[\tilde{v}_t],$$

   and the limits are achievable.

3. $\mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}]$ *is jointly concave in* $\tilde{\boldsymbol{v}}$ *and* $\kappa > 0$.
4.
$$\begin{aligned} \mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] = \max \ & -\kappa \log\left(\sum_{t\in[T]} \theta_t \exp(-\nu_t/\kappa)\right) \\ \text{s.t.} \ & \mathbb{C}_{\mathcal{F}}^{\kappa}[\tilde{v}_t] \geq \nu_t \qquad \forall t \in [T] \\ & \boldsymbol{\nu} \in \mathbb{R}^T. \end{aligned} \tag{26}$$
5. *For all* $\nu \in \mathbb{R}$,
$$\mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}} + \nu \mathbf{1}] = \mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] + \nu.$$
6.
$$\mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}] \leq \inf_{\mathbb{P}\in\mathcal{F}} \mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}[\tilde{\boldsymbol{v}}].$$

Remark 5. We see the multi-period ambiguity-averse CARA certainty equivalent preserves the salient properties of multi-period CARA certainty equivalent in Proposition 6. The last property shows that the multi-period ambiguity-averse CARA certainty equivalent is a more conservative (or robust) evaluation of the worst-case achievable multi-period CARA certainty equivalent evaluated at the beginning of the time horizon.

We are now ready to propose our robust CARA multi-period consumption model as follows,

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}_1,\dots,\boldsymbol{y}_T} \ & \mathbb{C}_{\mathcal{F}}^{\kappa,\boldsymbol{\theta}}\left[\boldsymbol{c}_1^\top \boldsymbol{y}_1(\tilde{\boldsymbol{\xi}}_1), \dots, \boldsymbol{c}_T^\top \boldsymbol{y}_T(\tilde{\boldsymbol{\xi}}_T)\right] \\ \text{s.t.} \ & \sum_{\tau\in[t]} \boldsymbol{b}_{t,i,\tau}^\top \boldsymbol{y}_\tau(\boldsymbol{\xi}_\tau) \leq a_{t,i}^0(\boldsymbol{x}) + \boldsymbol{a}_{t,i}^\top(\boldsymbol{x})\boldsymbol{\xi}_t \qquad \forall t \in [T], \ \forall i \in \mathcal{I}_t, \ \forall \boldsymbol{z} \in \mathcal{Z} \\ & \boldsymbol{y}_t \in \mathcal{R}^{I_{\xi_t}, I_{y_t}} \qquad \forall t \in [T]. \end{aligned} \tag{27}$$

where we denote $\boldsymbol{x}$ as the here-and-now decision and $\boldsymbol{y}_t$ as the t-th period recourse decision adapted to $\tilde{\boldsymbol{\xi}}_t$. Here, $\mathcal{I}_t$ is the set of constraint indices in period t. Note that the set of uncertain factors $\tilde{\boldsymbol{z}}$ is exactly $\tilde{\boldsymbol{\xi}}_T$. For each $t \in [T], i \in \mathcal{I}_t$, we assume $\boldsymbol{b}_{t,i,t} \neq \mathbf{0}$, and the function $a_{t,i}^j(\boldsymbol{x}) : \mathbb{R}^{I_x} \to \mathbb{R}$ are concave for all $j \in \{0\} \cup \mathcal{J}_t^+$, convex for all $j \in \mathcal{J}_t^-$, and affine for all $j \in \mathcal{J}_t$ where we denote $\mathcal{J}_t^+ \triangleq \mathcal{J}^+ \cap [I_{\xi_t}]$, $\mathcal{J}_t^- \triangleq \mathcal{J}^- \cap [I_{\xi_t}]$, $\mathcal{J}_t \triangleq \mathcal{J} \cap [I_{\xi_t}]$.

Observe that Problem (27) generalizes the robust CARA optimization with two-stage payoff function (16). We can also consider maximizing the expected CARA utility of the total payoffs in the following multi-period optimization model,

$$\begin{aligned} \max_{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}_1,\dots,\boldsymbol{y}_T} \ & \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\sum_{t\in[T]} \boldsymbol{c}_t^\top \boldsymbol{y}_t(\tilde{\boldsymbol{\xi}}_t)\right] \\ \text{s.t.} \ & \sum_{\tau\in[t]} \boldsymbol{b}_{t,i,\tau}^\top \boldsymbol{y}_\tau(\boldsymbol{\xi}_\tau) \leq a_{t,i}^0(\boldsymbol{x}) + \boldsymbol{a}_{t,i}^\top(\boldsymbol{x})\boldsymbol{\xi}_t \ \forall t \in [T], \ \forall i \in \mathcal{I}_t, \ \forall \boldsymbol{z} \in \mathcal{Z} \\ & \boldsymbol{y}_t \in \mathcal{R}^{I_{\xi_t}, I_{y_t}} \qquad \forall t \in [T]. \end{aligned} \tag{28}$$

Incidentally, this is also a special case of the T-period consumption model of Problem (27) with $\theta_t = 0$ for all $t \in [T-1]$, $\theta_T = 1$, an auxiliary recourse decision $y_{T,I_{y_T}+1}$ in period T as the consumption, and one more constraint $y_{T,I_{y_T}+1}(\boldsymbol{\xi}_T) - \sum_{t\in[T]} \boldsymbol{c}_t^\top \boldsymbol{y}_t(\boldsymbol{\xi}_t) \leq 0$ for all $\boldsymbol{z} \in \mathcal{Z}$.

We next extend the MLDR to Problem (27) with non-anticipativity consideration. For this purpose, we first solve for each $t \in [T]$, $i \in \mathcal{I}_t$, the problem

$$\begin{aligned}
\max_{\boldsymbol{y}_t,\dots,\boldsymbol{y}_T} \quad & \sum_{\tau=t}^{T} \theta_\tau \left(1 - \exp\left(-\boldsymbol{c}_\tau^\top \boldsymbol{y}_\tau / \kappa\right)\right) \\
\text{s.t.} \quad & \sum_{\tau=t}^{s} \boldsymbol{b}_{s,k,\tau}^\top \boldsymbol{y}_\tau \leq 0 && \forall s \in \{t,\dots,T\}, k \in \mathcal{I}_s \\
& \boldsymbol{b}_{t,i,t}^\top \boldsymbol{y}_t = -\|\boldsymbol{b}_{t,i,t}\| \\
& \boldsymbol{c}_\tau^\top \boldsymbol{y}_\tau \leq 0 && \forall \tau \in \{t,\dots,T\} \\
& \boldsymbol{y}_\tau \in \mathbb{R}^{I_\tau} && \forall \tau \in \{t,\dots,T\}.
\end{aligned} \tag{29}$$

For each $t \in [T]$, we denote $\mathcal{I}_t^o \subseteq \mathcal{I}_t$ as the index set of i such that the above optimization problem is feasible and $\boldsymbol{y}_{\tau*}^{t,i}$ as the corresponding optimal solution for any $\tau \in \{t,\dots,T\}$. Similar to the two-stage case, we can further partition the index set $\mathcal{I}_t^o$ as $\mathcal{I}_t^o = \bigcup_{\ell \in [m_t]} \mathcal{I}_{t,\ell}^o$ such that $\boldsymbol{y}_{\tau*}^{t,i_1} = \boldsymbol{y}_{\tau*}^{t,i_2}$ if and only if i_1 and i_2 are in the same $\mathcal{I}_{t,\ell}^o$ and denote $\boldsymbol{y}_{\tau*}^{t,\ell}$ as any $\boldsymbol{y}_{\tau*}^{t,i}$ with $i \in \mathcal{I}_{t,\ell}^o$. Subsequently, we propose the multi-period MLDR

$$\hat{\boldsymbol{y}}_t(\boldsymbol{\xi}_t) \triangleq \bar{\boldsymbol{y}}_t(\boldsymbol{\xi}_t) + \sum_{s\in[t]} \sum_{\ell \in [m_s]} \boldsymbol{y}_{t*}^{s,\ell} \left(\max_{i \in \mathcal{I}_{s,\ell}^o} \left\{ \frac{h_{s,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[s]}(\boldsymbol{\xi}_s), \boldsymbol{\xi}_s)}{\|\boldsymbol{b}_{s,i,s}\|} \right\} \right)^+ \tag{30}$$

where for each $t \in [T]$,

$$\begin{aligned}
\bar{\boldsymbol{y}}_t(\boldsymbol{\xi}_t) &\triangleq \boldsymbol{y}_t^0 + \boldsymbol{Y}_t \boldsymbol{\xi}_t \\
h_{t,i}(\boldsymbol{x}, \boldsymbol{y}_{[t]}, \boldsymbol{\xi}_t) &\triangleq \sum_{\tau \in [t]} \boldsymbol{b}_{t,i,\tau}^\top \boldsymbol{y}_\tau - a_{t,i}^0(\boldsymbol{x}) - \boldsymbol{a}_{t,i}^\top(\boldsymbol{x}) \boldsymbol{\xi}_t ,
\end{aligned}$$

with $\boldsymbol{y}_t^0 \in \mathbb{R}^{I_{yt}}$, $\boldsymbol{Y}_t \triangleq [\boldsymbol{y}_t^1, \dots, \boldsymbol{y}_t^{I_{\xi t}}] \in \mathbb{R}^{I_{yt} \times I_{\xi t}}$, and $\boldsymbol{y}_{[t]}$ is the collection of $\boldsymbol{y}_\tau$ for $\tau \in [t]$.

We establish the feasibility of the multi-period MLDR as follows.

Proposition 8. *For each $t \in [T]$, the multi-period MLDR, $\hat{\boldsymbol{y}}_t$ satisfies the non-anticipativity constraints. Moreover, suppose $\bar{\boldsymbol{y}}_t \in \mathcal{L}^{I_{\xi t}, I_{yt}}$ satisfies $\sum_{\tau \in [t]} \boldsymbol{b}_{t,i,\tau}^\top \bar{\boldsymbol{y}}_\tau(\boldsymbol{\xi}_\tau) \leq a_{t,i}^0(\boldsymbol{x}) + \boldsymbol{a}_{t,i}^\top(\boldsymbol{x}) \boldsymbol{\xi}_t$ for each $i \in \mathcal{I}_t \backslash \mathcal{I}_t^o$, then $\hat{\boldsymbol{y}}_t$ is feasible to Problem (27).*

We construct a tractable approximation of Problem (27) as follows.

Theorem 7. *The multi-period model* (27) *has a practicably solvable lower bound as follows:*

$$
\begin{aligned}
\max\ & \rho \\
\text{s.t.}\ & \sum_{t\in[T]} \theta_t p_t \le \kappa \\
& (p_t, \kappa, \rho - \nu_t) \in \mathcal{K}_{\exp} && \forall t \in [T] \\
& r_0^t - \sum_{s\in[t]} \sum_{\ell\in[m_s]} (\boldsymbol{c}_t^\top \boldsymbol{y}_{t*}^{s,\ell}) r_{s,\ell}^t \ge \nu_t && \forall t \in [T] \\
& \kappa_0^t - \sum_{s\in[t]} \sum_{\ell\in[m_s]} (\boldsymbol{c}_t^\top \boldsymbol{y}_{t*}^{s,\ell}) \kappa_{s,\ell}^t = \kappa && \forall t \in [T] \\
& \boldsymbol{c}_t^\top \boldsymbol{y}_t^0 + \sum_{j\in[I_{\xi_t}]} \phi_j(\kappa_0^t, \boldsymbol{c}_t^\top \boldsymbol{y}_t^j) \ge r_0^t && \forall t \in [T] \\
& \Phi\left(\kappa_{s,\ell}^t,\ \bar{\boldsymbol{\alpha}}_{s,\mathcal{I}_{s,\ell}^o} - \sum_{\tau\in[s]} \bar{\boldsymbol{b}}_{s,\mathcal{I}_{s,\ell}^o,\tau}^\top \boldsymbol{y}_\tau^0,\ \bar{\boldsymbol{\beta}}_{s,\mathcal{I}_{s,\ell}^o} - \sum_{\tau\in[s]} \bar{\boldsymbol{b}}_{s,\mathcal{I}_{s,\ell}^o,\tau}^\top \bar{\boldsymbol{Y}}_{\tau s} \right) \ge r_{s,\ell}^t && \forall t \in [T], s\in[t], \ell\in[m_s] \\
& \bar{\boldsymbol{\lambda}}_{t,i}^\top \bar{\boldsymbol{\xi}}_t - \underline{\boldsymbol{\lambda}}_{t,i}^\top \underline{\boldsymbol{\xi}}_t \le \alpha_{t,i} - \sum_{\tau\in[t]} \boldsymbol{b}_{t,i,\tau}^\top \boldsymbol{y}_\tau^0 && \forall t\in[T], i\in\mathcal{I}_t \backslash \mathcal{I}_t^o \\
& \sum_{\tau\in[t]} \bar{\boldsymbol{Y}}_{\tau t}^\top \boldsymbol{b}_{t,i,\tau} - \boldsymbol{\beta}_{t,i} = \bar{\boldsymbol{\lambda}}_{t,i} - \underline{\boldsymbol{\lambda}}_{t,i} && \forall t\in[T], i\in\mathcal{I}_t \backslash \mathcal{I}_t^o \\
& a_{t,i}^0(\boldsymbol{x}) \ge \alpha_{t,i} && \forall t\in[T], i\in\mathcal{I}_t \\
& a_{t,i}^j(\boldsymbol{x}) \ge \beta_{t,i}^j && \forall t\in[T], i\in\mathcal{I}_t, j\in\mathcal{J}_t^+ \\
& a_{t,i}^j(\boldsymbol{x}) \le \beta_{t,i}^j && \forall t\in[T], i\in\mathcal{I}_t, j\in\mathcal{J}_t^- \\
& a_{t,i}^j(\boldsymbol{x}) = \beta_{t,i}^j && \forall t\in[T], i\in\mathcal{I}_t, j\in\mathcal{J}_t \\
& \rho\in\mathbb{R}, \boldsymbol{x}\in\mathcal{X} \\
& p_t, \nu_t \in \mathbb{R}, \boldsymbol{\alpha}_t \in \mathbb{R}^{|\mathcal{I}_t|}, \boldsymbol{\beta}_t \in \mathbb{R}^{|\mathcal{I}_t|\times I_{\xi_t}}, \boldsymbol{y}_t^0 \in \mathbb{R}^{I_{y_t}}, \boldsymbol{Y}_t \in \mathbb{R}^{I_{y_t}\times I_{\xi_t}} && \forall t\in[T] \\
& \boldsymbol{\kappa}^t \in \mathbb{R}_+^{1+\sum_{s\in[t]} m_s}, \boldsymbol{r}^t \in \mathbb{R}^{1+\sum_{s\in[t]} m_s}, \bar{\boldsymbol{\lambda}}_t, \underline{\boldsymbol{\lambda}}_t \in \mathbb{R}_+^{|\mathcal{I}_t\backslash\mathcal{I}_t^o|\times I_{\xi_t}} && \forall t\in[T]
\end{aligned}
\tag{31}
$$

where $\bar{\boldsymbol{Y}}_{\tau t} \triangleq \begin{bmatrix} \boldsymbol{Y}_\tau & \boldsymbol{0} \end{bmatrix} \in \mathbb{R}^{I_{y_\tau}\times I_{\xi_t}}$ *for* $\tau \le t$, $\bar{\boldsymbol{\alpha}}_{s,\mathcal{I}_{s,\ell}^o} \triangleq \begin{bmatrix} 0 \\ \boldsymbol{\alpha}_{s,\mathcal{I}_{s,\ell}^o} \end{bmatrix} \in \mathbb{R}^{|\mathcal{I}_{s,\ell}^o|+1}$, $\bar{\boldsymbol{\beta}}_{s,\mathcal{I}_{s,\ell}^o} \triangleq \begin{bmatrix} \boldsymbol{0}^\top \\ \boldsymbol{\beta}_{s,\mathcal{I}_{s,\ell}^o} \end{bmatrix} \in \mathbb{R}^{(|\mathcal{I}_{s,\ell}^o|+1)\times I_{\xi_s}}$, $\bar{\boldsymbol{b}}_{s,\mathcal{I}_{s,\ell}^o,\tau} \triangleq \begin{bmatrix} \boldsymbol{0} & \boldsymbol{b}_{s,\mathcal{I}_{s,\ell}^o,\tau} \end{bmatrix} \in \mathbb{R}^{I_{y_\tau}\times(|\mathcal{I}_{s,\ell}^o|+1)}$ *and* $\boldsymbol{\alpha}_{s,\mathcal{I}_{s,\ell}^o}$, $\boldsymbol{\beta}_{s,\mathcal{I}_{s,\ell}^o}$, $\boldsymbol{b}_{s,\mathcal{I}_{s,\ell}^o,\tau}$ *are the stacked vectors or matrices of* $\alpha_{s,i}/\|\boldsymbol{b}_{s,i,s}\|$, $\boldsymbol{\beta}_{s,i}^\top/\|\boldsymbol{b}_{s,i,s}\|$, $\boldsymbol{b}_{s,i,\tau}/\|\boldsymbol{b}_{s,i,s}\|$ *for* $s\in[T]$, $\ell\in[m_s]$, $i\in\mathcal{I}_{s,\ell}^o$, *respectively.*

Note that in practical implementation of the robust optimization solutions, we may ignore the solutions for the MLDR, but only to implement the solutions for the here-and-now decision, $\boldsymbol{x}\in\mathcal{X}$. As information unfolds, the future wait-and-see decision will become here-and-now. In a rolling-horizon implementation, this decision can be obtained by solving a new robust optimization problem with updated priors (see, *e.g.,* Ben-Tal et al. 2004, Bertsimas et al. 2019).

On time consistency

We are aware that time consistency is a *maxim* in many multi-period stochastic programming models mandating that an optimal policy perceived in one time period must be recognized as optimal in another. However, we do not enforce time consistency as a preference in our multi-period decision framework. Apart from the time inconsistency issues that may arise from robust decision making (see, *e.g.,* Delage and Iancu 2015), we also allow arbitrary choice of temporal discounting, such as behaviorally inspired *hyperbolic discounting* (Laibson 1997) that would result in time inconsistent preferences. In reality, time consistency is not a dominant human behavior even in the absence of uncertainty (see, *e.g.*, Loch and Wu 2007, Frederick et al. 2002). Moreover, since dynamic optimization problems are potentially PSPACE-hard (Dyer and Stougie 2006), the consideration of time consistency presupposes an impractical amount of computational resources needed to ensure the optimality of a time consistent policy. We acknowledge that our model may not cater to a fully rational agent with unlimited computational resources.

Our stand to relegate time consistency is not uncommon in the literature, and we refer interested readers to Kydland and Prescott (1977), Bajeux-Besnainou and Portait (1998). A trivial fix would be to adopt the *pre-committed policy* approach by firmly adhering to the optimum policy evaluated at the first period throughout the planning horizon. In one of our numerical studies, we evaluate this approach by comparing the performance of the pre-committed MLDR policy against the time-consistent optimal DP policy, with both policies constructed from an empirical distribution at the beginning of the period. Another common criticism of robust optimization is the perceived over-conservativeness, which may not be true with more sophisticated ambiguity sets and approximation techniques such as those introduced in Goh and Sim (2010), See and Sim (2010), Chen et al. (2020). The proof the pudding should be in its eating. Hence, it is imperative for us to compare the quality of the solutions obtained from the deterministic approximations of our robust CARA optimization models against those obtained from the Monte-Carlo approximations of stochastic CARA optimization models.

5. Numerical studies

In this section, we apply the tractable approximation of robust CARA optimization models to study its numerical performance on solving two adaptive linear optimization problems. In the first experiment, we consider a project management problem, and benchmark our solutions against those obtained from SAA approximations of stochastic optimization. In the second experiment, we study a multi-period inventory management problem and benchmark the multi-period MLDR policy against the policy obtained using dynamic programming (DP). In both problems we show that our tractable approximation yields solutions with better out-of-sample performance when there are insufficient training samples or when the risk tolerance levels are low.

Project management

We consider solving a risk-averse project management problem (*e.g.* Ben-Tal et al. 2009, Chen et al. 2007b) via our tractable approximation and a stochastic optimization model using an empirical distribution to mimic a data-driven setting where the underlying data generating model is not known to the decision maker. We fixed the empirical distribution and vary the risk tolerance level to obtain the solution profiles of both approaches.

We represent the project management problem by a directed acyclic graph with n nodes and m arcs and we denote the set of arcs by $\mathcal{E}$. Each node represents an event of the completion of a subset of activities and each arc represents an activity connecting two events. An event occurs only when all the activities that correspond to all its incoming edges have been completed. By convention, we use node 1 as the start event and the last node n as the end event. We denote y_i as the completion time of event $i \in [n]$. An activity starts being processed only after the event that corresponds to the node from which the activity originated has occurred and each activity $(i,j) \in \mathcal{E}$ is associated with an uncertain processing time $\tilde{t}_{ij}$. We assume that the random processing time $\tilde{t}_{ij}$ can be reduced by allocating additional resources and is represented by $\tilde{t}_{ij} = (1+\tilde{z}_{ij})b_{ij} - a_{ij}x_{ij}$ where $\tilde{z}_{ij}$ is a zero-mean random variable and $x_{ij} \in [0, \bar{x}_{ij}]$ is the amount of resources allocated to activity $(i,j) \in \mathcal{E}$. We assume the random processing time $\tilde{t}_{ij}$ is independent of each other and is non-negative for all realization of $\tilde{z}_{ij}$ and all range of x_{ij}. We denote c_{ij} as the cost of using each unit of resource for the activity on the arc (i,j). Our goal is to seek an optimal resource allocation decision $\boldsymbol{x}$ which minimizes the CARA certainty equivalent of completion time of the project subject to the constraint that the total resource available is no more than a budget C, *i.e.*,

$$\min_{\boldsymbol{x} \in \mathcal{X}} \overline{\mathbb{C}}_{\mathbb{P}}^{\kappa}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right]$$

where

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{z}) = \min_{\boldsymbol{y}} \ & y_n \\ \text{s.t.} \ & y_j \geq y_i + (1+z_{ij})b_{ij} - a_{ij}x_{ij} \ \forall (i,j) \in \mathcal{E} \\ & y_1 = 0 \end{aligned} \tag{32}$$

and

$$\mathcal{X} = \left\{ \boldsymbol{x} : \ \boldsymbol{c}^\top \boldsymbol{x} \leq C, \ \boldsymbol{0} \leq \boldsymbol{x} \leq \bar{\boldsymbol{x}} \right\}.$$

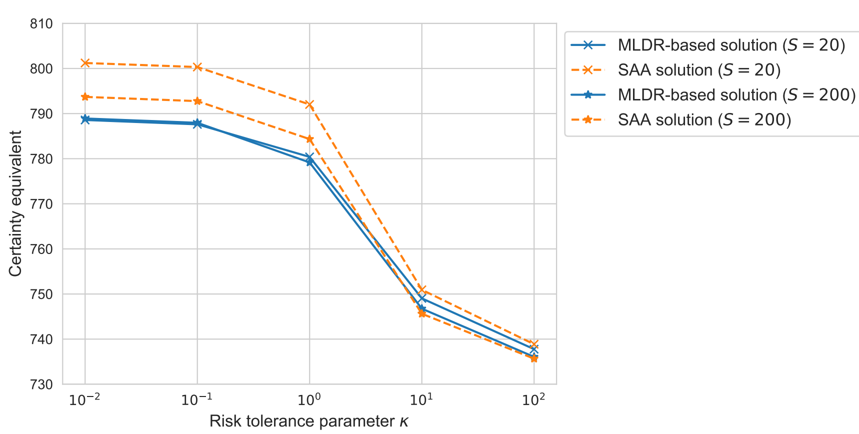
Our computational set up is similar to Chen et al. (2007b); we consider a fictitious activity network represented a H by W grid with a total of $n = H \times W$ nodes and $m = H(W-1) + W(H-1)$ arcs. The first node is at the left bottom corner, while the last node is at the right upper corner. Each arc on the graph proceeds either towards the right node or the upper node. We set $H = 4$, $W = 6$, hence, we have $n = 24$ and $m = 38$. We assume that for all activities, $a_{ij} = c_{ij} = 1$, $\bar{x}_{ij} = 24$

and $b_{ij} = 100$. We assume each $\tilde{z}_{ij}$ is a uniform random variable on $[-\bar{z}_{ij}, \bar{z}_{ij}]$ where $\bar{z}_{ij}$ is generated uniformly at random from $[0, 0.24]$. However, the distribution is unknown to the decision maker. Instead, we generate $S \in \{20, 200\}$ i.i.d. samples of $\tilde{\boldsymbol{z}}$ from the underlying distribution for each problem instance and we denote each sample by $\hat{\boldsymbol{z}}^s$, $s \in [S]$. Then we obtain the here-and-now decisions by an SAA approach and an MLDR-based approximation approach, denoted as $\boldsymbol{x}^S$ and $\boldsymbol{x}^M$, respectively. For SAA, we use the empirical average to replace the expectation in the certainty equivalent, *i.e.*,

$$\min_{\boldsymbol{x} \in \mathcal{X}} \kappa \log \frac{1}{S} \sum_{s \in [S]} \exp\left(\frac{f(\boldsymbol{x}, \hat{\boldsymbol{z}}^s)}{\kappa}\right).$$

For the MLDR-based approximation, we emphasize that MLDR is only used for approximating the certainty equivalent $\overline{\mathbb{C}}_{\mathbb{P}}^{\kappa}[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})]$ so as to obtain the here-and-now decisions, $\boldsymbol{x}^M$. The expectation in the MLDR-based approximation is evaluated with respect to empirical marginal distributions and assuming that the processing times are stochastically independent. After obtaining $\boldsymbol{x}^S$ and $\boldsymbol{x}^M$, we implement them and evaluate the out-of-sample certainty equivalent of $f(\boldsymbol{x}, \tilde{\boldsymbol{z}})$ on 10,000 i.i.d. test samples generated from the underlying distribution. As we have observed in Example 5, although 10,000 samples may not provide accurate estimates of the certainty equivalent of completion time especially when the risk tolerance level is low, it suffices to serve the purpose of comparing the two solutions, since the monotonicity is preserved under the log function. We set risk tolerance parameter $\kappa \in \{0.01, 0.1, 1, 10, 100\}$ and budget $C = 8m$. The results are averaged over 50 random problem instances. In Figure 5, we plot the the out-of-sample CARA certainty equivalents of the completion times under different risk tolerance parameters. We have also tested with $C \in \{4m, 12m\}$ and the results are similar.

Figure 5 Certainty equivalent of completion time under different risk tolerance parameters



We observe that when the training sample size is limited ($S = 20$), the solution $\boldsymbol{x}^M$ outperforms $\boldsymbol{x}^S$. Also, the gap of out-of-sample certainty equivalent between the two solutions becomes larger as κ decreases. As we expect, the out-of-sample certainty equivalent evaluated at SAA solutions improves as S increases from 20 to 200. With larger training size, $\boldsymbol{x}^S$ performs slightly better than $\boldsymbol{x}^M$ when $\kappa \in \{10, 100\}$, however, it is still dominated by $\boldsymbol{x}^M$ when the risk tolerance level is low. Therefore, for a fixed sample size, it will become more challenging for the SAA approach to maintain the quality of the approximation as the risk tolerance level decreases. In contrast, the MLDR-based solution is more robust to limited sample size. As S increases from 20 to 200, the out-of-sample certainty equivalent evaluated for the MLDR-based solutions does not change as much across different risk tolerance levels.

We can conclude in this experiment that our tractable approximation performs reasonably well against the SAA approach. It shines in situations when low risk tolerance level is desired.

Multi-period inventory management with financing

In the second experiment, we apply the multi-period MLDR approximation approach to solve a risk-averse multi-period inventory management with financing problem. We benchmark its performance with the optimum policy obtained via dynamic programming (DP) using a limited sized empirical distribution that is sampled from the true distribution. In contrast to the previous experiment, the MLDR is now implemented as a pre-committed policy in our numerical study, which is less ideal than a rolling-horizon implementation, but will greatly accelerate our computational studies. A similar robust optimization model has also been proposed in See and Sim (2010) to address a risk-neutral multi-period inventory management without the consideration of financing. Unfortunately, their proposed approximations do not naturally extend to the CARA criterion.

Specifically, we consider a multi-period inventory management proposed in Chen et al. (2007a), where the risk-averse firm aims to maximize the expected utility of consumption over a finite time horizon. We assume the demand is exogenous and stochastically independent across periods. At the beginning of each period $t \in [T]$, the inventory level is x_t; the firm makes a replenishment decision $y_t \geq x_t$ at the cost of $c_t(y_t - x_t)$ before the uncertain demand $\tilde{z}_t$ is realized. We assume the unsatisfied demand is backlogged so that the next-period inventory level is $x_{t+1} = y_t - \tilde{z}_t$ and the firm obtains an income $q_t = p_t\tilde{z}_t - h(y_t - \tilde{z}_t)^+ - b(\tilde{z}_t - y_t)^+ - c_t(y_t - x_t)$, where p_t is the unit selling price, h is the unit holding cost, and b is the unit backlogging cost b. Subsequently, the firm determines the consumption level f_t and receives the corresponding utility $\left(1 - e^{-f_t/\kappa}\right)$. We do not consider temporal discounting and set $\theta_t = 1/T$, $t \in [T]$. Meanwhile, its wealth w_t transits according to $w_{t+1} = (1+\beta)(w_t + q_t - f_t)$ where β is the interest rate. We can interpret β as either the saving or borrowing rate depending on whether $(w_t + q_t - f_t)$ is positive or negative respectively. To tractably solve this problem by DP, it is necessary to assume that the saving and borrowing rates are identical (Chen et al. 2007a). We assume the firm aims to maximize the expected time-additive exponential utility function of consumption so the problem can be formulated as

$$
\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{f},\boldsymbol{w},\boldsymbol{q}} \;& \mathbb{E}_{\mathbb{P}}\left[\sum_{t\in[T]} \theta_t(1-e^{-f_t(\tilde{\boldsymbol{\xi}}_t)/\kappa})\right] && \\
\text{s.t.}\;& f_t(\tilde{\boldsymbol{\xi}}_t) = w_t(\tilde{\boldsymbol{\xi}}_{t-1}) - \tfrac{w_{t+1}(\tilde{\boldsymbol{\xi}}_t)}{1+\beta} + q_t(\tilde{\boldsymbol{\xi}}_t) && \forall t\in[T] \\
& q_t(\tilde{\boldsymbol{\xi}}_t) \le p_t\tilde{z}_t - h(y_t(\tilde{\boldsymbol{\xi}}_{t-1}) - \tilde{z}_t) - c_t(y_t(\tilde{\boldsymbol{\xi}}_{t-1}) - x_t(\tilde{\boldsymbol{\xi}}_{t-1})) && \forall t\in[T] \\
& q_t(\tilde{\boldsymbol{\xi}}_t) \le p_t\tilde{z}_t - b(\tilde{z}_t - y_t(\tilde{\boldsymbol{\xi}}_{t-1})) - c_t(y_t(\tilde{\boldsymbol{\xi}}_{t-1}) - x_t(\tilde{\boldsymbol{\xi}}_{t-1})) && \forall t\in[T] \\
& y_t(\tilde{\boldsymbol{\xi}}_{t-1}) \ge x_t(\tilde{\boldsymbol{\xi}}_{t-1}) && \forall t\in[T] \\
& x_{t+1}(\tilde{\boldsymbol{\xi}}_t) = y_t(\tilde{\boldsymbol{\xi}}_{t-1}) - \tilde{z}_t && \forall t\in[T-1] \\
& w_{T+1}(\tilde{\boldsymbol{\xi}}_T) = 0
\end{aligned}
\tag{33}
$$

where the initial wealth w_1 and inventory level x_1 are given. Chen et al. (2007a) shows that a base-stock policy is optimal for all $t\in[T]$. Moreover, the optimal policy of problem (33) can be obtained by solving the DP with Bellman equation

$$
G_t(x) = \max_{y\ge x} \mathbb{C}_{\mathbb{P}}^{R_t}\left[q_t(y,\tilde{z}_t) + \frac{1}{1+\beta}G_{t+1}(y-\tilde{z}_t)\right]
$$

where the *effective risk tolerance* $R_t = \sum_{\tau=t}^{T}\frac{\kappa}{(1+\beta)^{\tau-t}}$ and $G_{T+1}(x)=0$. The optimal consumption is given by

$$
f_t^*(w,y,\tilde{z}) = \frac{\kappa}{R_t}\left(w + q_t(y,\tilde{z}) + \frac{1}{1+\beta}G_{t+1}(y-\tilde{z})\right) + C_t
$$

where $C_t = -\frac{R_{t+1}\kappa}{R_t(1+\beta)}\log\frac{A_{t+1}(1+\beta)\kappa}{\theta_t R_{t+1}}$ and $A_t = \frac{(1+\beta)R_t}{R_{t+1}}A_{t+1}\left(\frac{A_{t+1}(1+\beta)\kappa}{\theta_t R_{t+1}}\right)^{-\kappa/R_t}$.

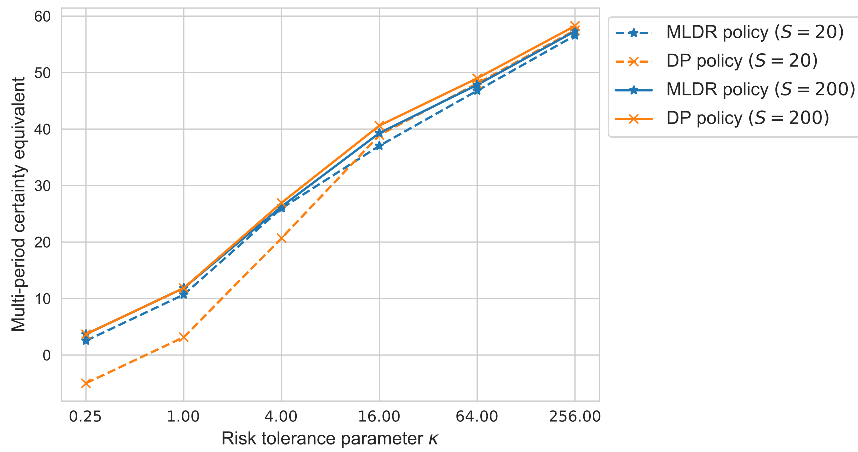
We also solve the problem (33) using our multi-period MLDR approximation approach. To do so, we first have to remove the equality constraints of the problem; we eliminate x_t by substitution and q_t by Fourier-Motzkin elimination. The reformulated problem becomes,

$$
\begin{aligned}
\max_{\boldsymbol{y},\boldsymbol{f},\boldsymbol{w}} \ & \mathbb{C}_{\mathbb{P}}^{\kappa,\boldsymbol{\theta}}\left[\boldsymbol{f}(\tilde{\boldsymbol{z}})\right] \\
\text{s.t. } & y_1 \geq x_1 \\
& y_t(\tilde{\boldsymbol{\xi}}_{t-1}) \geq y_{t-1}(\tilde{\boldsymbol{\xi}}_{t-2}) - \tilde{z}_{t-1} && \forall t \in \{2,...,T\} \\
& f_1(\tilde{\boldsymbol{\xi}}_1) \leq w_1 - \tfrac{w_2(\tilde{\boldsymbol{\xi}}_1)}{1+\beta} + (p_1+h)\tilde{z}_1 - (h+c_1)y_1 + c_1 x_1 \\
& f_1(\tilde{\boldsymbol{\xi}}_1) \leq w_1 - \tfrac{w_2(\tilde{\boldsymbol{\xi}}_1)}{1+\beta} + (p_1-b)\tilde{z}_1 + (b-c_1)y_1 + c_1 x_1 \\
& f_t(\tilde{\boldsymbol{\xi}}_t) \leq w_t(\tilde{\boldsymbol{\xi}}_{t-1}) - \tfrac{w_{t+1}(\tilde{\boldsymbol{\xi}}_t)}{1+\beta} + (p_t+h)\tilde{z}_t - c_t\tilde{z}_{t-1} \\
& \qquad\qquad -(h+c_t)y_t(\tilde{\boldsymbol{\xi}}_{t-1}) + c_t y_{t-1}(\tilde{\boldsymbol{\xi}}_{t-2}) && \forall t \in \{2,...,T-1\} \\
& f_t(\tilde{\boldsymbol{\xi}}_t) \leq w_t(\tilde{\boldsymbol{\xi}}_{t-1}) - \tfrac{w_{t+1}(\tilde{\boldsymbol{\xi}}_t)}{1+\beta} + (p_t-b)\tilde{z}_t - c_t\tilde{z}_{t-1} \\
& \qquad\qquad +(b-c_t)y_t(\tilde{\boldsymbol{\xi}}_{t-1}) + c_t y_{t-1}(\tilde{\boldsymbol{\xi}}_{t-2}) && \forall t \in \{2,...,T-1\} \\
& f_T(\tilde{\boldsymbol{\xi}}_T) \leq w_T(\tilde{\boldsymbol{\xi}}_{T-1}) + (p_T+h)\tilde{z}_T - c_T\tilde{z}_{T-1} \\
& \qquad -(h+c_T)y_T(\tilde{\boldsymbol{\xi}}_{T-1}) + c_T y_{T-1}(\tilde{\boldsymbol{\xi}}_{T-2}) \\
& f_T(\tilde{\boldsymbol{\xi}}_T) \leq w_T(\tilde{\boldsymbol{\xi}}_{T-1}) + (p_T-b)\tilde{z}_T - c_T\tilde{z}_{T-1} \\
& \qquad\qquad +(b-c_T)y_T(\tilde{\boldsymbol{\xi}}_{T-1}) + c_T y_{T-1}(\tilde{\boldsymbol{\xi}}_{T-2}).
\end{aligned}
\tag{34}
$$

We use the similar parameter setting as in Chen et al. (2007a). In particular, we set $h=6$, $b=3$, $\beta=0.1$, $x_1=w_1=0$, and $c_t=1$, $p_t=8$ for all $t\in[T]$. We set $\kappa\in\{0.25,1,4,16,64,256\}$, and the uncertain demand distribution is the empirical distribution of $S\in\{20,200\}$ i.i.d. samples of $\tilde{\boldsymbol{z}}$ from the underlying uniform distribution over $\{0,1,2,...,20\}$ for each problem instance. Then we solve the problem by two approaches: DP and MLDR where the expectation is taken with respect to the empirical distribution. After solving the corresponding problems, we implement the optimal policy solved from DP and MLDR on 10,000 i.i.d. samples generated from the same underlying distribution. The results are averaged over 50 random instances and summarized in Figure 6. We report the out-of-sample multi-period CARA certainty equivalent of consumption profile obtained from the two approaches under different sample size S and risk tolerance parameter κ.

We observe that when the risk tolerance level is high ($\kappa\in\{64,256\}$), both methods can perform well even when the sample size is limited ($S=20$). However, when the risk tolerance level is low ($\kappa\in\{0.25,1,4\}$), the MLDR policy would perform better than the DP policy when the sample size is limited ($S=20$), in which case the performance of DP policy becomes worse as κ decreases. We also observe that when the sample size becomes larger, both policies improve and the performance is comparable, although DP policy is slightly better at high risk tolerance levels. We conclude that the MLDR policy performs reasonably well against the DP policy when the risk tolerance level is high. Moreover, it is less sensitive to sample size and could outperform DP policy when the risk tolerance level is low.

Figure 6 Multi-peirod CARA certainty equivalent under different risk tolerance parameters



It is important to note that this is a relatively simple multi-period model where we could obtain the optimal policy reasonably well using DP. The assumptions needed to obtain a tractable DP formulation can be quite fragile. For instance, if the borrowing and saving rates are different, the state space will significantly be enlarged and it may not be as computationally viable to solve for the optimal policy via DP. In contrast, we can easily incorporate these changes in our framework. The fact that the approximate MLDR policy performs reasonably well against the optimal DP policy is therefore a comforting assurance attesting to the effectiveness of the hierarchy of approximations that we have introduced to solve the multi-period robust CARA optimization problems.

6. Conclusions and future work

In this paper, we motivate and propose robust CARA optimization models as well as their tractable approximations; the numerical studies suggest that our approach can be applied to obtain high-quality solutions that could outperform current stochastic optimization approaches, especially in situations with high levels of risk aversion. In future work, we will be developing new analytics tools by extending the RSOME algebraic modeling language of Chen et al. (2020) to incorporate the robust CARA optimization models. We will also incorporate in the platform other non-utility based criteria such as the *entropic value-at-risk* (Ahmadi-Javid 2012) and the *riskiness index* (Aumann and Serrano 2008), which are derivatives of the CARA criterion. The modeling tool will facilitate testing, evaluating and implementing of the solutions obtained by these robust CARA optimization models.

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Appendix A: Proofs of results

Proof of Proposition 1. The proof is the same as that of Lemma 1 in Jaillet et al. (2016) and thus omitted. □

Proof of Proposition 2. We only prove the super-additivity as follows since the proof of another property can be referred to Lemma 1 in Jaillet et al. (2016). For any $\kappa_1, \kappa_2 > 0$, let $\kappa = \kappa_1 + \kappa_2$, we have

$$\mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}_1 + \tilde{v}_2] = \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\frac{\kappa_1}{\kappa}\frac{\kappa\tilde{v}_1}{\kappa_1} + \frac{\kappa_2}{\kappa}\frac{\kappa\tilde{v}_2}{\kappa_2}\right] \geq \frac{\kappa_1}{\kappa}\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\frac{\kappa\tilde{v}_1}{\kappa_1}\right] + \frac{\kappa_2}{\kappa}\mathbb{C}^{\kappa}_{\mathbb{P}}\left[\frac{\kappa\tilde{v}_2}{\kappa_2}\right] = \mathbb{C}^{\kappa_1}_{\mathbb{P}}[\tilde{v}_1] + \mathbb{C}^{\kappa_2}_{\mathbb{P}}[\tilde{v}_2]$$

where the inequality is from concavity of CARA certainty equivalent in Proposition 1. For the cases of either κ_1 or κ_2 is zero, we assume $\kappa_1 = \kappa$ and $\kappa_2 = 0$ without loss of generality. Then by the last property in Proposition 1, we have

$$\mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}_1 + \tilde{v}_2] = \mathbb{C}^{\kappa}_{\mathbb{P}}\left[\tilde{v}_1 + \tilde{v}_2 - \mathbb{C}^{0}_{\mathbb{P}}[\tilde{v}_2]\right] + \mathbb{C}^{0}_{\mathbb{P}}[\tilde{v}_2] \geq \mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}_1] + \mathbb{C}^{0}_{\mathbb{P}}[\tilde{v}_2].$$

The super-additivity of $\mathbb{C}^{\kappa}_{\mathcal{F}}[\tilde{v}]$ can be proved in the same way. □

Proof of Proposition 3. Note that $\lambda^j = a^j(\boldsymbol{x})$ at optimality in Problem (8). Since $\tilde{z}_j$'s are independent, we have $\mathbb{C}^{\kappa}_{\mathcal{F}}[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})] = a^0(\boldsymbol{x}) + \sum_{j \in [I_z]} \phi_j(\kappa, a^j(\boldsymbol{x}))$ from Proposition 2. □

Proof of Theorem 1. We denote $\bar{\mathcal{U}} = \left\{(\boldsymbol{x}, \kappa, y) \,\middle|\, \exists\, \boldsymbol{q} \in \mathbb{R}^I : \sum_{i \in [I]} p_i q_i \leq \kappa, (q_i, \kappa, y - x_i) \in \mathcal{K}_{\exp}, \forall i \in [I]\right\}$ and $\mathcal{U} = \{(\boldsymbol{x}, \kappa, y) | y \leq g(\boldsymbol{x}, \kappa), \kappa > 0\}$. Observe that

$$\begin{aligned}
\mathcal{U} &= \left\{(\boldsymbol{x}, \kappa, y) \,\middle|\, -\kappa \log \sum_{i \in [I]} p_i e^{-x_i/\kappa} \geq y, \kappa > 0\right\} \\
&= \left\{(\boldsymbol{x}, \kappa, y) \,\middle|\, \sum_{i \in [I]} p_i \kappa e^{(y - x_i)/\kappa} \leq \kappa, \kappa > 0\right\} \\
&= \left\{(\boldsymbol{x}, \kappa, y) \,\middle|\, \exists\, \boldsymbol{q} \in \mathbb{R}^I : \sum_{i \in [I]} p_i q_i \leq \kappa, \kappa > 0, (q_i, \kappa, y - x_i) \in \mathcal{K}_{\exp}, \forall i \in [I]\right\}.
\end{aligned}$$

Clearly $\mathcal{U} \subseteq \bar{\mathcal{U}}$ and the latter is closed. Hence the closure $\mathrm{cl}(\mathcal{U}) \subseteq \bar{\mathcal{U}}$.

Next, we show $\bar{\mathcal{U}} \subseteq \mathrm{cl}(\mathcal{U})$. For any $(\boldsymbol{x}, \kappa, y) \in \bar{\mathcal{U}} \backslash \mathcal{U}$, we have $\kappa = 0$, $x_i \geq y$ for all $i \in [I]$. We denote $\bar{x} = \min_{i \in [I]}\{x_i\}$ and consider the sequence $\{(\boldsymbol{x}^j, \kappa^j, y^j)\}_{j=1}^{\infty} \in \mathcal{U}$ where $\boldsymbol{x}^j = \boldsymbol{x}$, $\kappa^j = 1/j$ and $y^j = \min\{y, g(\boldsymbol{x}^j, \kappa^j)\}$. Since

$$\lim_{j \to \infty} g(\boldsymbol{x}^j, \kappa^j) = \bar{x} - \lim_{j \to \infty} \kappa^j \log \sum_{i \in [I]} p_i e^{(\bar{x} - x_i)/\kappa^j} \leq \bar{x} - \lim_{j \to \infty} \kappa^j \log \sum_{\{i \in [I] : x_i = \bar{x}\}} p_i = \bar{x},$$

and

$$\bar{x} - \lim_{j \to \infty} \kappa^j \log \sum_{i \in [I]} p_i e^{(\bar{x} - x_i)/\kappa^j} \geq \bar{x} - \lim_{j \to \infty} \kappa^j \log \sum_{i \in [I]} p_i = \bar{x},$$

we know

$$\lim_{j \to \infty} y^j = \min\left\{y, g(\boldsymbol{x}^j, \kappa^j)\right\} = \min\{y, \bar{x}\} = y$$

and hence $\lim_{j \to \infty}(\boldsymbol{x}^j, \kappa^j, y^j) = (\boldsymbol{x}, \kappa, y)$. Therefore, $(\boldsymbol{x}, \kappa, y) \in \mathrm{cl}(\mathcal{U})$ and $\mathrm{cl}(\mathcal{U}) = \bar{\mathcal{U}}$. □

Proof of Example 3. Note that $\phi(\kappa,\lambda) = \mathbb{C}_{\mathcal{G}}^{\kappa}[\lambda\tilde{z}] = -\kappa \log \sup_{\mathbb{P}\in\mathcal{G}} \mathbb{E}_{\mathbb{P}}\left[e^{-\lambda\tilde{z}/\kappa}\right]$ and

$$\begin{aligned}
\sup_{\mathbb{P}\in\mathcal{G}} \mathbb{E}_{\mathbb{P}}[e^{-\lambda\tilde{z}}] \leq \; & \inf_{\gamma\geq 0,\alpha,\beta} && \alpha+\beta\mu+\gamma\delta \\
& \quad \text{s.t.} && e^{-\lambda z} \leq \alpha+\beta z+\gamma|z-\mu| \quad \forall z\in[-1,1] \\
= \; & \inf_{\gamma\geq 0,\alpha,\beta} && \alpha+\beta\mu+\gamma\delta \\
& \quad \text{s.t.} && e^{-\lambda z} \leq \alpha+\beta z+\gamma(\mu-z) \quad \forall z\in[-1,\mu] \\
& && e^{-\lambda z} \leq \alpha+\beta z+\gamma(z-\mu) \quad \forall z\in[\mu,1] \\
= \; & \inf_{\gamma\geq 0,\alpha,\beta} && \alpha+\beta\mu+\gamma\delta \\
& \quad \text{s.t.} && e^{\lambda} \leq \alpha-\beta+\gamma(\mu+1) \\
& && e^{-\mu\lambda} \leq \alpha+\beta\mu \\
& && e^{-\lambda} \leq \alpha+\beta+\gamma(1-\mu) \\
= \; & \sup_{p_1,p_2,p_3\geq 0} && p_1e^{\lambda}+p_2e^{-\mu\lambda}+p_3e^{-\lambda} \\
& \quad \text{s.t.} && p_1+p_2+p_3=1 \\
& && -p_1+\mu p_2+p_3=\mu \\
& && (\mu+1)p_1+(1-\mu)p_3 \leq \delta
\end{aligned}$$

where the first inequality is by weak duality, the second equality is because the optimal solution of a convex maximization problem is attained at the boundary, and the third equality is due to linear optimization strong duality. Clearly, the worst-case distribution is attained by a three-point distribution with probability mass p_1,p_2,p_3 on $-1,\mu,1$. Solving the last linear optimization problem in the above bound, we get $p_1=\frac{\delta}{2(1+\mu)}$, $p_3=\frac{\delta}{2(1-\mu)}$ and $p_2=1-p_1-p_3$ and conclude the proof. □

Proof of Theorem 2. We first note that

$$\begin{aligned}
\mathbb{C}_{\mathcal{F}}^{\kappa}[f(\boldsymbol{x},\tilde{\boldsymbol{z}})] = \; & \sup_{\boldsymbol{\alpha},\boldsymbol{\beta}} \; \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\min_{i\in\mathcal{I}}\left\{\alpha_i+\boldsymbol{\beta}_i^{\top}\tilde{\boldsymbol{z}}\right\}\right] \\
& \text{s.t.} \; a_i^0(\boldsymbol{x}) \geq \alpha_i && \forall i\in\mathcal{I} \\
& \quad\; a_i^j(\boldsymbol{x}) \geq \beta_i^j && \forall i\in\mathcal{I}, j\in\mathcal{J}^+ \\
& \quad\; a_i^j(\boldsymbol{x}) \leq \beta_i^j && \forall i\in\mathcal{I}, j\in\mathcal{J}^- \\
& \quad\; a_i^j(\boldsymbol{x}) = \beta_i^j && \forall i\in\mathcal{I}, j\in\mathcal{J}.
\end{aligned} \tag{35}$$

Then for any $\boldsymbol{\gamma}\in\mathbb{R}^{I_z}$, we have

$$\begin{aligned}
& \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\min_{i\in\mathcal{I}}\left\{\alpha_i+\boldsymbol{\beta}_i^{\top}\tilde{\boldsymbol{z}}\right\}\right] \\
= & \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{\gamma}^{\top}\tilde{\boldsymbol{z}}+\min_{i\in\mathcal{I}}\{\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^{\top}\tilde{\boldsymbol{z}}\}\right] \\
\geq & \sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}} \mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^{\top}\tilde{\boldsymbol{z}}\right]+\mathbb{C}_{\mathcal{F}}^{\kappa_1}\left[\min_{i\in\mathcal{I}}\{\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^{\top}\tilde{\boldsymbol{z}}\}\right] \\
= & \sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}} \mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^{\top}\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{\max_{i\in\mathcal{I}}\{-\alpha_i-(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^{\top}\tilde{\boldsymbol{z}}\}}{\kappa_1}\right)\right] \\
\geq & \sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}} \mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^{\top}\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sum_{i\in\mathcal{I}}\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{-\alpha_i+(\boldsymbol{\gamma}-\boldsymbol{\beta}_i)^{\top}\tilde{\boldsymbol{z}}}{\kappa_1}\right)\right] \\
= & \sup_{\kappa_0+\kappa_1=\kappa,\boldsymbol{\kappa}\geq\mathbf{0}} \mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^{\top}\tilde{\boldsymbol{z}}\right]-\kappa_1\log\sum_{i\in\mathcal{I}}\exp\left(-\frac{\mathbb{C}_{\mathcal{F}}^{\kappa_1}\left[\alpha_i+(\boldsymbol{\beta}_i-\boldsymbol{\gamma})^{\top}\tilde{\boldsymbol{z}}\right]}{\kappa_1}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\boldsymbol{\kappa}\geq \mathbf{0},\boldsymbol{r}} r_0 - \kappa_1 \log \sum_{i\in\mathcal{I}} e^{-r_i/\kappa_1} \\
&\qquad \kappa_0 + \kappa_1 = \kappa \\
&\qquad \mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{\gamma}^\top \tilde{\boldsymbol{z}}\right] \geq r_0 \\
&\qquad \mathbb{C}_{\mathcal{F}}^{\kappa_1}\left[\alpha_i + (\boldsymbol{\beta}_i - \boldsymbol{\gamma})^\top \tilde{\boldsymbol{z}}\right] \geq r_i \qquad \forall i\in\mathcal{I} \\
&= \max_{\boldsymbol{\kappa}\geq \mathbf{0},\boldsymbol{r},\rho,\boldsymbol{q}} r_0 + \rho \\
&\qquad\quad \kappa_0 + \kappa_1 = \kappa \\
&\qquad\quad \sum_{i\in\mathcal{I}} q_i \leq \kappa_1 \\
&\qquad\quad (q_i, \kappa_1, \rho - r_i) \in \mathcal{K}_{\exp} \qquad \forall i\in\mathcal{I} \\
&\qquad\quad \sum_{j\in[I_z]} \phi_j(\kappa_0, \gamma^j) \geq r_0 \\
&\qquad\quad \alpha_i + \sum_{j\in[I_z]} \phi_j(\kappa_1, \beta_i^j - \gamma^j) \geq r_i \qquad \forall i\in\mathcal{I}
\end{aligned}
\tag{36}
$$

where the first inequality is due to super-additivity of $\mathbb{C}_{\mathcal{F}}^{\kappa}[\tilde{v}]$ with respect to $(\kappa, \tilde{v})$ in Proposition 2, and the last equality is from Theorem 1. Combine (35) and (36) together and take infimum over all $\boldsymbol{\gamma}\in\mathbb{R}^{I_z}$, we obtain (12). □

Proof of Theorem 3. Without loss of generality, we can focus on proving the properties of the best lower bound (36) over $\boldsymbol{\gamma}\in\mathbb{R}^{I_z}$ of the ambiguity-averse CARA certainty equivalent $\mathbb{C}_{\mathcal{F}}^{\kappa}[\min_{i\in\mathcal{I}}\{\alpha_i + \boldsymbol{\beta}_i^\top \tilde{\boldsymbol{z}}\}]$ for any fixed $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Then all the conclusions in Theorem 3 can be obtained easily from the equivalence (35).

Since $\mathbb{C}_{\mathcal{F}}^{\kappa_0}[\boldsymbol{\gamma}^\top \tilde{\boldsymbol{z}}]$ is non-decreasing in κ_0, the lower bound (36) is non-decreasing in κ as one can fix κ_1 and increase κ_0 when κ becomes larger.

Consider the lower bound (36), note that $\kappa = 0$ implies $\kappa_0 = \kappa_1 = 0$, which further implies $\boldsymbol{q} = \mathbf{0}$ and $r_i \geq \rho$ for all $i\in\mathcal{I}$. Therefore, we must have

$$
\begin{aligned}
r_0 &= \sum_{j\in[I_z]} \phi_j(\kappa_0, \gamma^j) = \mathbb{C}_{\mathcal{F}}^{0}\left[\boldsymbol{\gamma}^\top \tilde{\boldsymbol{z}}\right] = \inf_{\boldsymbol{z}\in\mathcal{Z}} \boldsymbol{\gamma}^\top \boldsymbol{z} \\
r_i &= \alpha_i + \sum_{j\in[I_z]} \phi_j(\kappa_1, \beta_i^j - \gamma^j) = \mathbb{C}_{\mathcal{F}}^{0}\left[\alpha_i + (\boldsymbol{\beta}_i - \boldsymbol{\gamma})^\top \tilde{\boldsymbol{z}}\right] = \alpha_i + \inf_{\boldsymbol{z}\in\mathcal{Z}} (\boldsymbol{\beta}_i - \boldsymbol{\gamma})^\top \boldsymbol{z} \ \forall i\in\mathcal{I} \\
\rho &= \min_{i\in\mathcal{I}}\{r_i\}
\end{aligned}
$$

at optimality so that the best lower bound (36) over $\boldsymbol{\gamma}\in\mathbb{R}^{I_z}$ equals

$$
\sup_{\boldsymbol{\gamma}} \left(\inf_{\boldsymbol{z}\in\mathcal{Z}} \boldsymbol{\gamma}^\top \boldsymbol{z} + \min_{i\in\mathcal{I}} \left\{ \alpha_i + \inf_{\boldsymbol{z}\in\mathcal{Z}} (\boldsymbol{\beta}_i - \boldsymbol{\gamma})^\top \boldsymbol{z} \right\} \right),
$$

which equals $\inf_{\boldsymbol{z}\in\mathcal{Z}} \min_{i\in\mathcal{I}}\{\alpha_i + \boldsymbol{\beta}_i^\top \boldsymbol{z}\}$ since

$$
\inf_{\boldsymbol{z}\in\mathcal{Z}} \min_{i\in\mathcal{I}} \left\{\alpha_i + \boldsymbol{\beta}_i^\top \boldsymbol{z}\right\} \geq \inf_{\boldsymbol{z}\in\mathcal{Z}} \boldsymbol{\gamma}^\top \boldsymbol{z} + \min_{i\in\mathcal{I}} \left\{ \alpha_i + \inf_{\boldsymbol{z}\in\mathcal{Z}} (\boldsymbol{\beta}_i - \boldsymbol{\gamma})^\top \boldsymbol{z} \right\}
$$

for any $\boldsymbol{\gamma}\in\mathbb{R}^{I_z}$ and the equality holds when $\boldsymbol{\gamma} = \mathbf{0}$. Hence, the best lower bound over $\boldsymbol{\gamma}\in\mathbb{R}^{I_z}$ is exactly $\mathbb{C}_{\mathcal{F}}^{0}[\min_{i\in\mathcal{I}}\{\alpha_i + \boldsymbol{\beta}_i^\top \tilde{\boldsymbol{z}}\}]$.

If there is some $i^* \in \mathcal{I}$ such that $\alpha_{i^*} + \boldsymbol{\beta}_{i^*}^\top \boldsymbol{z} = \min_{i\in\mathcal{I}}\{\alpha_i + \boldsymbol{\beta}_i^\top \boldsymbol{z}\}$ for all $\boldsymbol{z} \in \mathcal{Z}$, then we let $\boldsymbol{\gamma} = \boldsymbol{\beta}_{i^*}$, $\kappa_0 = \kappa$ and $\kappa_1 = 0$ so that

$$
\begin{aligned}
r_0 &= \sum_{j\in[I_z]} \phi_j(\kappa, \gamma^j) = \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{\gamma}^\top \tilde{\boldsymbol{z}}\right] = \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{\beta}_{i^*}^\top \tilde{\boldsymbol{z}}\right] \\
r_i &= \alpha_i + \sum_{j\in[I_z]} \phi_j(\kappa_1, \beta_i^j - \gamma^j) = \mathbb{C}_{\mathcal{F}}^{0}\left[\alpha_i + (\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i^*})^\top \tilde{\boldsymbol{z}}\right] = \alpha_i + \inf_{\boldsymbol{z}\in\mathcal{Z}} (\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i^*})^\top \boldsymbol{z} \ \forall i \in \mathcal{I} \\
\rho &= \min_{i\in\mathcal{I}}\{r_i\}
\end{aligned}
$$

at optimality and obtain the lower bound (36) as

$$
\begin{aligned}
r_0 + \rho &= \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{\beta}_{i^*}^\top \tilde{\boldsymbol{z}}\right] + \min_{i\in\mathcal{I}}\left\{\alpha_i + \inf_{\boldsymbol{z}\in\mathcal{Z}} (\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i^*})^\top \boldsymbol{z}\right\} \\
&= \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\alpha_{i^*} + \boldsymbol{\beta}_{i^*}^\top \tilde{\boldsymbol{z}}\right] + \inf_{\boldsymbol{z}\in\mathcal{Z}} \min_{i\in\mathcal{I}}\left\{\alpha_i - \alpha_{i^*} + (\boldsymbol{\beta}_i - \boldsymbol{\beta}_{i^*})^\top \boldsymbol{z}\right\} \\
&\geq \mathbb{C}_{\mathcal{F}}^{\kappa}\left[\alpha_{i^*} + \boldsymbol{\beta}_{i^*}^\top \tilde{\boldsymbol{z}}\right],
\end{aligned}
$$

which implies the lower bound is greater than $\mathbb{C}_{\mathcal{F}}^{\kappa}\left[\min_{i\in\mathcal{I}}\{\alpha_i + \boldsymbol{\beta}_i^\top \tilde{\boldsymbol{z}}\}\right]$ and hence exact. □

Proof of Proposition 4. It follows from Jensen's inequality:

$$
\mathbb{E}_{\mathbb{P}^S}\left[-\kappa \log \frac{1}{S}\sum_{s\in[S]} \exp\left(-\frac{f(\boldsymbol{x}, \tilde{\boldsymbol{z}}^s)}{\kappa}\right)\right] \geq -\kappa \log \frac{1}{S}\sum_{s\in[S]} \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{f(\boldsymbol{x}, \tilde{\boldsymbol{z}}^s)}{\kappa}\right)\right] = \mathbb{C}_{\mathbb{P}}^{\kappa}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}})\right]
$$

as the function $-\kappa\log(\cdot)$ is convex for any $\kappa > 0$. □

Proof of Theorem 4. We claim that $\boldsymbol{c}^\top \hat{\boldsymbol{y}}(\boldsymbol{z}) \geq \boldsymbol{c}^\top \boldsymbol{y}^\dagger(\boldsymbol{z})$ for all $\boldsymbol{z} \in \mathcal{Z}$, which implies the conclusion directly. To show this, we note that for any $\boldsymbol{z} \in \mathcal{Z}$,

$$
\begin{aligned}
&\boldsymbol{c}^\top \hat{\boldsymbol{y}}(\boldsymbol{z}) - \boldsymbol{c}^\top \boldsymbol{y}^\dagger(\boldsymbol{z}) \\
&= \sum_{\ell\in[m]} \boldsymbol{c}^\top \boldsymbol{y}_*^\ell \left(\max_{i\in\mathcal{I}_\ell^o}\left\{\frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|}\right\}\right)^+ - \sum_{i\in\mathcal{I}^o} \boldsymbol{c}^\top \boldsymbol{y}_\diamond^i \left(h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})\right)^+ \\
&= \sum_{\ell\in[m]} \boldsymbol{c}^\top \boldsymbol{y}_*^\ell \left(\left(\max_{i\in\mathcal{I}_\ell^o}\left\{\frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|}\right\}\right)^+ - \sum_{i\in\mathcal{I}_\ell^o}\left(\frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|}\right)^+\right).
\end{aligned}
$$

by noting $\boldsymbol{y}_\diamond^i = \boldsymbol{y}_*^\ell / \|\boldsymbol{b}_i\|$ for any $\ell \in [m]$, $i \in \mathcal{I}_\ell^o$. Hence it suffices to prove $\boldsymbol{c}^\top \boldsymbol{y}_*^\ell \leq 0$ for all $\ell \in [m]$ since

$$
\left(\max_{i\in\mathcal{I}_\ell^o}\left\{\frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|}\right\}\right)^+ \leq \sum_{i\in\mathcal{I}_\ell^o} \max\left\{\frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|}, 0\right\} = \sum_{i\in\mathcal{I}_\ell^o}\left(\frac{h_i(\boldsymbol{x}, \bar{\boldsymbol{y}}(\boldsymbol{z}), \boldsymbol{z})}{\|\boldsymbol{b}_i\|}\right)^+.
$$

Suppose there is some $\ell \in [m]$ such that $\boldsymbol{c}^\top \boldsymbol{y}_*^\ell > 0$ and $\boldsymbol{b}_i^\top \boldsymbol{y}_*^\ell \leq 0$ for all $i \in \mathcal{I}$, then for any $\boldsymbol{x}$ and $\boldsymbol{y}(\boldsymbol{z})$ feasible in Problem (17), the solution $\boldsymbol{y}(\boldsymbol{z}) + \lambda \boldsymbol{y}_*^\ell$ with any $\lambda > 0$ is also feasible. Hence the optimal value of Problem (17) is unbounded above, a contradiction. □

Proof of Theorem 5. Since the two-stage problem (16) has complete recourse with only one recourse decision variable, we must have $b_i > 0$ for all $i \in \mathcal{I}$ or $b_i < 0$ for all $i \in \mathcal{I}$. Observe that the second-stage linear optimization

$$
\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{z}) = \max_{y} \ & b_0 y \\
\text{s.t.} \ & b_i y \leq a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} \quad \forall i \in \mathcal{I}
\end{aligned}
$$

is unbounded above if $b_0 b_i < 0$ for any $i \in \mathcal{I}$. Since the recourse decision y is unconstrained, for the optimal value of the problem to be finite, we can assume without loss of generality that $b_i > 0$ and $b_0 \geq 0$. In which case, the optimal decision rule $y^{OPT}(\boldsymbol{z}) = \min_{i\in\mathcal{I}}\left\{\frac{a_i^0(\boldsymbol{x})+\boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z}}{b_i}\right\}$ and $y_*^i = -1$ for all $i\in\mathcal{I}$. Hence, $m=1$ and $y_*^\ell = -1$ for all $\ell\in[1]$. Hence the MLDR is

$$\begin{aligned}\hat{y}(\boldsymbol{z}) &= y^0 + \boldsymbol{y}^\top\boldsymbol{z} - \left(\max_{i\in\mathcal{I}}\left\{\frac{b_i(y^0+\boldsymbol{y}^\top\boldsymbol{z}) - a_i^0(\boldsymbol{x}) - \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z}}{b_i}\right\}\right)^+\\ &= y^0 + \boldsymbol{y}^\top\boldsymbol{z} + \min\left\{0, \min_{i\in\mathcal{I}}\left\{-y^0 - \boldsymbol{y}^\top\boldsymbol{z} + \frac{a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z}}{b_i}\right\}\right\}\\ &= \min\left\{y^0 + \boldsymbol{y}^\top\boldsymbol{z}, \min_{i\in\mathcal{I}}\left\{\frac{a_i^0(\boldsymbol{x}) + \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z}}{b_i}\right\}\right\}.\end{aligned}$$

Let $y^0 = a_i^0(\boldsymbol{x})/b_i$ and $\boldsymbol{y} = \boldsymbol{a}_i(\boldsymbol{x})/b_i$ for any $i\in\mathcal{I}$ we can recover $\hat{y}(\boldsymbol{z}) = y^{OPT}(\boldsymbol{z})$. □

Proof of Proposition 5. We prove it by case distinction.

(i) For $i\in\mathcal{I}\backslash\mathcal{I}^o$, we have $\boldsymbol{b}_i^\top\hat{\boldsymbol{y}}(\boldsymbol{z}) \le \boldsymbol{b}_i^\top\bar{\boldsymbol{y}}(\boldsymbol{z})$ for all $\boldsymbol{z}\in\mathcal{Z}$ since $\boldsymbol{b}_i^\top\boldsymbol{y}_*^\ell \le 0$ for all $\ell\in[m]$.

(ii) For $i\in\mathcal{I}^o$, let ℓ be the index such that $i\in\mathcal{I}_\ell^o$. For all $\boldsymbol{z}\in\mathcal{Z}$ we have

$$\begin{aligned}\boldsymbol{b}_i^\top\hat{\boldsymbol{y}}(\boldsymbol{z}) - a_i^0(\boldsymbol{x}) - \boldsymbol{a}_i^\top(\boldsymbol{x})\boldsymbol{z} &= h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\boldsymbol{z}),\boldsymbol{z}) + \sum_{\ell\in[m]}\boldsymbol{b}_i^\top\boldsymbol{y}_*^\ell\left(\max_{j\in\mathcal{I}_\ell^o}\left\{\frac{h_j(\boldsymbol{x},\bar{\boldsymbol{y}}(\boldsymbol{z}),\boldsymbol{z})}{\|\boldsymbol{b}_j\|}\right\}\right)^+\\ &\le h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\boldsymbol{z}),\boldsymbol{z}) + \boldsymbol{b}_i^\top\boldsymbol{y}_*^i\frac{(h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\boldsymbol{z}),\boldsymbol{z}))^+}{\|\boldsymbol{b}_i\|}\\ &= \min\{h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\boldsymbol{z}),\boldsymbol{z}),0\}\\ &\le 0\end{aligned}$$

where the first inequality is because $\left(\max_{j\in\mathcal{I}_\ell^o}\left\{\frac{h_j(\boldsymbol{x},\bar{\boldsymbol{y}}(\boldsymbol{z}),\boldsymbol{z})}{\|\boldsymbol{b}_j\|}\right\}\right)^+ \ge \frac{(h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\boldsymbol{z}),\boldsymbol{z}))^+}{\|\boldsymbol{b}_i\|}$ and $\boldsymbol{b}_i^\top\boldsymbol{y}_*^k \le 0$ for all $k\in\mathcal{I}^o\backslash\{i\}$, and the second equality is due to $\boldsymbol{b}_i^\top\boldsymbol{y}_*^i = -\|\boldsymbol{b}_i\|$. □

Proof of Theorem 6. Note Problem (23) has the following lower bound:

$$\begin{aligned}&\mathbb{C}_{\mathcal{F}}^{\kappa}\left[\boldsymbol{c}^\top\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}}) + \sum_{\ell\in[m]}\boldsymbol{c}^\top\boldsymbol{y}_*^\ell\left(\max_{i\in\mathcal{I}_\ell^o}\left\{\frac{h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{z}})}{\|\boldsymbol{b}_i\|}\right\}\right)^+\right]\\ &\ge \sup_{\substack{\kappa_0,\kappa_\ell\ge 0,\forall\ell\in[m]\\ \kappa_0+\sum_{\ell\in[m]}\kappa_\ell=\kappa}} \mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{c}^\top\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}})\right] + \sum_{\ell\in[m]}\mathbb{C}_{\mathcal{F}}^{\kappa_\ell}\left[\boldsymbol{c}^\top\boldsymbol{y}_*^\ell\left(\max_{i\in\mathcal{I}_\ell^o}\left\{\frac{h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{z}})}{\|\boldsymbol{b}_i\|}\right\}\right)^+\right]\\ &= \sup_{\boldsymbol{\kappa}\ge\boldsymbol{0},\boldsymbol{r}} r_0 + \sum_{\ell\in[m]} r_\ell\\ &\quad\text{s.t.}\,\mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{c}^\top\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}})\right] \ge r_0\\ &\qquad\mathbb{C}_{\mathcal{F}}^{\kappa_\ell}\left[\boldsymbol{c}^\top\boldsymbol{y}_*^\ell\left(\max_{i\in\mathcal{I}_\ell^o}\left\{\frac{h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{z}})}{\|\boldsymbol{b}_i\|}\right\}\right)^+\right] \ge r_\ell\\ &\qquad\kappa_0 + \sum_{\ell\in[m]}\kappa_\ell = \kappa\\ &= \sup_{\boldsymbol{\kappa}\ge\boldsymbol{0},\boldsymbol{r}} r_0 - \sum_{\ell\in[m]}(\boldsymbol{c}^\top\boldsymbol{y}_*^\ell)r_\ell\\ &\quad\text{s.t.}\,\mathbb{C}_{\mathcal{F}}^{\kappa_0}\left[\boldsymbol{c}^\top\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}})\right] \ge r_0\\ &\qquad\mathbb{C}_{\mathcal{F}}^{\kappa_\ell}\left[-\left(\max_{i\in\mathcal{I}_\ell^o}\left\{\frac{h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{z}})}{\|\boldsymbol{b}_i\|}\right\}\right)^+\right] \ge r_\ell\\ &\qquad\kappa_0 - \sum_{\ell\in[m]}(\boldsymbol{c}^\top\boldsymbol{y}_*^\ell)\kappa_\ell = \kappa\end{aligned}\tag{37}$$

where the first inequality is from super-additivity of $\mathbb{C}^{\kappa}_{\mathcal{F}}[\tilde{v}]$ in Proposition 2, and the last equality is because $\boldsymbol{c}^\top \boldsymbol{y}^\ell_* \le 0$ for all $\ell \in [m]$ from Theorem 4. Combine (23) and (37), we obtain the following lower bound of Problem (17):

$$
\begin{aligned}
\max_{\substack{\boldsymbol{\kappa}\ge \boldsymbol{0},\\ \boldsymbol{r},\bar{\boldsymbol{y}}}} \quad & r_0 - \sum_{\ell\in[m]} (\boldsymbol{c}^\top \boldsymbol{y}^\ell_*) r_\ell && && \text{(38a)}\\
\text{s.t.} \quad & \kappa_0 - \sum_{\ell\in[m]} (\boldsymbol{c}^\top \boldsymbol{y}^\ell_*)\kappa_\ell = \kappa && && \text{(38b)}\\
& \mathbb{C}^{\kappa_0}_{\mathcal{F}}\left[\boldsymbol{c}^\top \bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}})\right] \ge r_0 && && \text{(38c)}\\
& \mathbb{C}^{\kappa_\ell}_{\mathcal{F}}\left[\min\left\{0, \min_{i\in\mathcal{I}^o_\ell}\left\{\frac{-h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\tilde{\boldsymbol{z}}),\tilde{\boldsymbol{z}})}{\|\boldsymbol{b}_i\|}\right\}\right\}\right] \ge r_\ell && \forall \ell\in[m] && \text{(38d)}\\
& h_i(\boldsymbol{x},\bar{\boldsymbol{y}}(\boldsymbol{z}),\boldsymbol{z}) \le 0 && \forall \boldsymbol{z}\in\mathcal{Z}, \forall i\in\mathcal{I}\backslash\mathcal{I}^o && \text{(38e)}\\
& \bar{\boldsymbol{y}} \in \mathcal{L}^{I_z, I_y} && && \text{(38f)}
\end{aligned}
$$

To derive tractable reformulation of the above problem, we know constraint (38c) can be tractably reformulated by Proposition 3, constraints (38d) have safe $\mathcal{K}_{\exp}$-representable approximations from Theorem 2, and constraints (38e) are robust linear constraints with box uncertainty set $\mathcal{Z}$, which can be easily reformulated as tractable linear constraints by standard robust optimization techniques. The resultant tractable model is exactly Problem (24). □

Proof of Proposition 6. For notation simplicity, let $g_{\boldsymbol{\theta}}(\boldsymbol{\nu},\kappa) = -\kappa \log\left(\sum_{t\in[T]} \theta_t \exp\left(-\frac{\nu_t}{\kappa}\right)\right)$ and note that $g_{\boldsymbol{\theta}}(\boldsymbol{\nu},\kappa)$ can be viewed as the CARA certainty equivalent of a random variable $\tilde{\nu}$ which realizes as ν_t with probability θ_t, $t\in[T]$. We first prove the variational representation (25). For any $\kappa > 0$, as $g_{\boldsymbol{\theta}}(\boldsymbol{\nu},\kappa)$ is non-decreasing in ν_t, the maximum is attained at $\nu_t = \mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}_t]$ for all $t\in[T]$. Hence the right hand side of (25) equals $-\kappa\log\left(\sum_{t\in[T]}\theta_t \exp(-\mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}_t]/\kappa)\right) = \mathbb{C}^{\kappa,\boldsymbol{\theta}}_{\mathbb{P}}[\tilde{\boldsymbol{v}}]$. Moreover, since $\mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}_t]$ is non-decreasing in $\kappa>0$ and $g_{\boldsymbol{\theta}}(\boldsymbol{\nu},\kappa)$ is non-decreasing in $\kappa>0$ and $\boldsymbol{\nu}\in\mathbb{R}^T$, we have $\mathbb{C}^{\kappa,\boldsymbol{\theta}}_{\mathbb{P}}[\tilde{\boldsymbol{v}}]$ is non-decreasing in $\kappa>0$ from representation (25) and the limit cases are exactly $\min_{t\in[T]:\theta_t>0}\{\operatorname{ess\,inf}_{\mathbb{P}}[\tilde{v}_t]\}$ at $\kappa=0$ and $\sum_{t\in[T]}\theta_t\mathbb{E}_{\mathbb{P}}[\tilde{v}_t]$ at $\kappa=\infty$ according to Proposition 1. $\mathbb{C}^{\kappa,\boldsymbol{\theta}}_{\mathbb{P}}[\tilde{\boldsymbol{v}}]$ is jointly concave in $(\tilde{\boldsymbol{v}},\kappa)$ with $\kappa>0$ because its hypograph

$$
\left\{(\tilde{\boldsymbol{v}},\kappa,\rho)\,\middle|\,\exists\,\boldsymbol{\nu}\in\mathbb{R}^T \;:\; \kappa>0,\; g_{\boldsymbol{\theta}}(\boldsymbol{\nu},\kappa)\ge\rho,\; \mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}_t]\ge\nu_t, \forall t\in[T]\right\}
$$

is convex thanks to concavity of $g_{\boldsymbol{\theta}}(\boldsymbol{\nu},\kappa)$ and $\mathbb{C}^{\kappa}_{\mathbb{P}}[\tilde{v}]$ in Proposition 1. Finally, for any $\nu\in\mathbb{R}$, the equality $\mathbb{C}^{\kappa,\boldsymbol{\theta}}_{\mathbb{P}}[\tilde{\boldsymbol{v}}+\nu\boldsymbol{1}] = \mathbb{C}^{\kappa,\boldsymbol{\theta}}_{\mathbb{P}}[\tilde{\boldsymbol{v}}]+\nu$ is straightforward to verify. □

Proof of Proposition 7. The proof of the first five properties is almost the same as that in Proposition 6 and hence omitted. The last property follows from the observation

$$
\inf_{\mathbb{P}\in\mathcal{F}} \mathbb{C}^{\kappa,\boldsymbol{\theta}}_{\mathbb{P}}[\tilde{\boldsymbol{v}}] = -\kappa\log\left(\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\sum_{t\in[T]}\theta_t\exp\left(-\frac{\tilde{v}_t}{\kappa}\right)\right]\right) \ge -\kappa\log\left(\sum_{t\in[T]}\theta_t\sup_{\mathbb{P}\in\mathcal{F}}\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\frac{\tilde{v}_t}{\kappa}\right)\right]\right).
$$

□

Proof of Proposition 8. Clearly $\hat{\boldsymbol{y}}_t$ depends only on $\boldsymbol{\xi}_t$ and hence satisfies non-anticipativity. To show feasibility, note that for each $t \in [T]$, $i \in \mathcal{I}_t^o$, we have

$$\begin{aligned}
&\sum_{\tau\in[t]} \boldsymbol{b}_{t,i,\tau}^\top \hat{\boldsymbol{y}}_\tau(\boldsymbol{\xi}_\tau) - a_{t,i}^0(\boldsymbol{x}) - \boldsymbol{a}_{t,i}^\top(\boldsymbol{x})\boldsymbol{\xi}_t \\
&= h_{t,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[t]}(\boldsymbol{\xi}_t), \boldsymbol{\xi}_t) + \sum_{\tau\in[t]} \boldsymbol{b}_{t,i,\tau}^\top \left(\sum_{s\in[\tau]} \sum_{\ell\in[m_s]} \boldsymbol{y}_{\tau*}^{s,\ell} \left(\max_{k\in\mathcal{I}_{s,\ell}^o} \left\{ \frac{h_{s,k}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[s]}(\boldsymbol{\xi}_s), \boldsymbol{\xi}_s)}{\|\boldsymbol{b}_{s,k,s}\|} \right\} \right)^+ \right) \\
&= h_{t,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[t]}(\boldsymbol{\xi}_t), \boldsymbol{\xi}_t) + \sum_{s\in[t]} \sum_{\ell\in[m_s]} \left(\sum_{\tau=s}^{t} \boldsymbol{b}_{t,i,\tau}^\top \boldsymbol{y}_{\tau*}^{s,\ell} \right) \left(\max_{k\in\mathcal{I}_{s,\ell}^o} \left\{ \frac{h_{s,k}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[s]}(\boldsymbol{\xi}_s), \boldsymbol{\xi}_s)}{\|\boldsymbol{b}_{s,k,s}\|} \right\} \right)^+ . \\
&\leq h_{t,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[t]}(\boldsymbol{\xi}_t), \boldsymbol{\xi}_t) + \left(\boldsymbol{b}_{t,i,t}^\top \boldsymbol{y}_{t*}^{t,i} \right) \left(h_{t,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[t]}(\boldsymbol{\xi}_t), \boldsymbol{\xi}_t) \right)^+ / \|\boldsymbol{b}_{t,i,t}\| \\
&= \min\left\{ h_{t,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[t]}(\boldsymbol{\xi}_t), \boldsymbol{\xi}_t), 0 \right\} \\
&\leq 0
\end{aligned}$$

where the first inequality is because $\sum_{\tau=s}^{t} \boldsymbol{b}_{t,i,\tau}^\top \boldsymbol{y}_{\tau*}^{s,\ell} \leq 0$ for each $s \in [t]$ and $\ell \in [m_s]$ so that we can focus only on the case of $s = t$, ℓ such that $i \in \mathcal{I}_{t,\ell}^o$, and $k = i$, and the last equality is because $\boldsymbol{b}_{t,i,t}^\top \boldsymbol{y}_{t*}^{t,i} = -\|\boldsymbol{b}_{t,i,t}\|$. ☐

Proof of Theorem 7. We first note that Problem (27) is equivalent to:

$$\begin{aligned}
\max_{\boldsymbol{x}\in\mathcal{X}, \boldsymbol{\nu}\in\mathbb{R}^T, \boldsymbol{y}_1,\ldots,\boldsymbol{y}_T} \quad & -\kappa \log \left(\sum_{t\in[T]} \theta_t e^{-\nu_t/\kappa} \right) \\
\text{s.t.} \quad & \mathbb{C}_{\mathcal{F}}^{\kappa} \left[\boldsymbol{c}_t^\top \boldsymbol{y}_t(\tilde{\boldsymbol{\xi}}_t) \right] \geq \nu_t && \forall t \in [T] \\
& h_{t,i}(\boldsymbol{x}, \boldsymbol{y}_{[t]}(\boldsymbol{\xi}_t), \boldsymbol{\xi}_t) \leq 0 && \forall t \in [T],\ \forall i \in \mathcal{I}_t,\ \forall \boldsymbol{z} \in \mathcal{Z} \\
& \boldsymbol{y}_t \in \mathcal{R}^{I_{\xi_t}, I_{y_t}} && \forall t \in [T].
\end{aligned} \tag{39}$$

from representation (26). Then we apply the MLDR (30) to obtain a lower bound of Problem (39):

$$\max_{\boldsymbol{x}\in\mathcal{X}, \boldsymbol{\nu}\in\mathbb{R}^T, \bar{\boldsymbol{y}}_1,\ldots,\bar{\boldsymbol{y}}_T} \ -\kappa \log \left(\sum_{t\in[T]} \theta_t e^{-\nu_t/\kappa} \right) \tag{40a}$$

$$\text{s.t.} \quad \mathbb{C}_{\mathcal{F}}^{\kappa} \left[\boldsymbol{c}_t^\top \bar{\boldsymbol{y}}_t(\tilde{\boldsymbol{\xi}}_t) + \sum_{s\in[t]} \sum_{\ell\in[m_s]} \boldsymbol{c}_t^\top \boldsymbol{y}_{t*}^{s,\ell} \left(\max_{i\in\mathcal{I}_{s,\ell}^o} \left\{ \frac{h_{s,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[s]}(\tilde{\boldsymbol{\xi}}_s), \tilde{\boldsymbol{\xi}}_s)}{\|\boldsymbol{b}_{s,i,s}\|} \right\} \right)^+ \right] \geq \nu_t \quad \forall t \in [T] \tag{40b}$$

$$h_{t,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[t]}(\boldsymbol{\xi}_t), \boldsymbol{\xi}_t) \leq 0 \qquad \forall t \in [T],\ \forall i \in \mathcal{I}_t \backslash \mathcal{I}_t^o,\ \forall \boldsymbol{z} \in \mathcal{Z} \tag{40c}$$

$$\bar{\boldsymbol{y}}_t \in \mathcal{L}^{I_{\xi_t}, I_{y_t}} \qquad \forall t \in [T]. \tag{40d}$$

according to Proposition 8. Note that the objective function (40a) is $\mathcal{K}_{\exp}$-representable by Theorem 1. Similar to the proof in Theorem 6, the constraints (40b) have safe approximations

$$\begin{aligned}
& r_0^t - \sum_{s\in[t]} \sum_{\ell\in[m_s]} (\boldsymbol{c}_t^\top \boldsymbol{y}_{t*}^{s,\ell}) r_{s,\ell}^t \geq \nu_t && \forall t \in [T] \\
& \kappa_0^t - \sum_{s\in[t]} \sum_{\ell\in[m_s]} (\boldsymbol{c}_t^\top \boldsymbol{y}_{t*}^{s,\ell}) \kappa_{s,\ell}^t = \kappa && \forall t \in [T] \\
& \mathbb{C}_{\mathcal{F}}^{\kappa_0^t} \left[\boldsymbol{c}_t^\top \bar{\boldsymbol{y}}_t(\tilde{\boldsymbol{\xi}}_t) \right] \geq r_0^t && \forall t \in [T] \\
& \mathbb{C}_{\mathcal{F}}^{\kappa_{s,\ell}^t} \left[\min\left\{ 0, \min_{i\in\mathcal{I}_{s,\ell}^o} \left\{ -\frac{h_{s,i}(\boldsymbol{x}, \bar{\boldsymbol{y}}_{[s]}(\tilde{\boldsymbol{\xi}}_s), \tilde{\boldsymbol{\xi}}_s)}{\|\boldsymbol{b}_{s,i,s}\|} \right\} \right\} \right] \geq r_{s,\ell}^t && \forall t \in [T], s \in [t], \ell \in [m_s],
\end{aligned}$$

in which the last constraints can be further safely approximated using Theorem 2 and the second-last constraints can be reformulated by Proposition 3. Finally, the robust linear constraints (40c) with box uncertainty set admit tractable robust counterparts as in Theorem 6. The resultant tractable model is exactly Problem (31). ☐