Adjustability in Robust Linear Optimization

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Abstract

Dynamic robust optimization involves sequential decisions over multiple stages against worst-case uncertainty realizations. At each stage, the decision-maker observes the uncertainty realization before committing to decisions, known as adjustable decisions. We focus on adjustability — the difference between objective values of two problems: a static robust optimization problem where all decisions have to be made before uncertainty realization, and a fully adjustable robust optimization problem where all decisions are made after uncertainty realization. In this work, we develop a theoretical framework to quantify adjustability based on the input data of a robust optimization problem with linear objective, linear constraints, and fixed recourse. We make very few additional assumptions. In particular, we do not assume constraint-wise separability or parameter nonnegativity. Based on the discovery of an interesting connection between the reformulations of the static and fully adjustable problems, our analysis gives a necessary and sufficient criterion for adjustability to be zero when the uncertainty set is polyhedral. Then, we develop a constructive approach to quantify adjustability when the uncertainty set is general. We also develop an efficient algorithm to bound adjustability. We exemplify the value of our theoretical framework by applying it to interdiction, supply chain design, and inventory control problems.

1 Introduction

Consider a decision-maker who wants to find optimal decisions y in an environment plagued by some uncertainty represented by parameters ξ. Assume that the decision-maker cares about worst-case performance, either because she is risk-averse or because uncertainty realization is picked by an adversary. If the sequence of events is fixed and known, *i.e.*, the order of decision commitments and uncertain parameter realizations can be specified, then one can formulate this problem into a dynamic robust optimization model. Otherwise, even setting up an optimization model becomes challenging. How should the decision-maker analyze this ill-posed problem?

It turns out that something can still be said by just looking at the two "extreme" cases of the problem, represented by the following two optimization models,

$$\text{(RO)}\ \ \mathrm{I} := \min_{y \in \bigcap\limits_{\xi\in\Xi} \mathcal{Y}_\xi} \max_{\xi\in\Xi} f(\xi, y), \qquad \text{(FARO)}\ \ \mathrm{II} := \min_{y(\cdot) \in \prod\limits_{\xi\in\Xi} \mathcal{Y}_\xi} \max_{\xi\in\Xi} f(\xi, y(\xi)),$$

where $\mathcal{X}, \mathcal{Y}$ are two Euclidean spaces, $\Xi \subseteq \mathcal{X}$ is the uncertainty set, $\mathcal{Y}_\xi := \{y \in \mathcal{Y} \mid g(\xi, y) \le 0\}$ is the decision space of y for a fixed set of parameters ξ, and the mapping $g : \Xi \times \mathcal{Y} \to \mathbb{R}^k$ describes the constraints. Then, $\bigcap_{\xi\in\Xi} \mathcal{Y}_\xi$ consists of solutions that are feasible for every realization of $\xi \in \Xi$, while $\prod_{\xi\in\Xi} \mathcal{Y}_\xi$, called the *policy space*, contains every function $y : \mathcal{X} \to \mathcal{Y}$ that satisfies $y(\xi) \in \mathcal{Y}_\xi$ for all $\xi \in \Xi$ (the product sign is commonly used for the set of dependent functions). Formulations I and II are recognized as (static) robust optimization (RO) and fully adjustable robust optimization

(FARO) [10] models. The former corresponds to the case that all decisions have to be made prior to any uncertainty realization, and the latter corresponds to the case that all uncertainty parameters are realized before any decision commitment.

For instance, in a medical supply chain design example [46], Ξ describes the uncertainty set of possible public health outbreak scenarios, and $\mathcal{Y}_\xi$ contains all the resource production, preposition, and reallocation strategies that are feasible under the given realization of ξ. In reality, the sequence of events is often partially known and may depend on exogenous factors. This sequence might be partially known because the peak of an outbreak in a community may come sooner or later than our expectation. It may depend on exogenous factors in the sense that the possibility and timing of resource preposition and reallocation could be affected by regulatory (*e.g.*, resource sharing across health care networks or state boundaries; FDA approval process for new medications) and technological (*e.g.*, lead times of different manufacturing processes) factors. Thus, the analysis of Formulations I and II can very well inform the decision-maker's actions, even if one does not know the sequence of events to start with. For instance, in the same example, if I and II have a large difference in their objective values, then the decision-maker might want to start medicine manufacturing *after* collecting enough information about the circulating disease. To enable such delay, the decision-maker would have to reduce the manufacturing lead time by various means such as investing in new technologies.

In general, if I and II have a significant difference in their objective values, the decision-maker is inclined to consider investing heavily in (i) observing the current sequence of events and (ii) broadening the decision space to include some regulatory and technological actions in order to change the sequence of events. On the other hand, if this difference is small, then the decision-maker may divert investment to other places, such as analyzing and refining the knowledge about Ξ, or enlarging the decision space $\mathcal{Y}_\xi$ directly.

The growing literature on robust and *adjustable robust optimization* (ARO) thus provides ample opportunities to apply the framework we will introduce in this paper. In addition, comparing Formulations I and II in general can provide many theoretical insights to other related problems. We provide a precise account of these relationships in the following subsection. Many of these relationships have been previously identified in the literature, and some are shown explicitly for the first time. To the authors' knowledge, we are the first to systematically organize them together.

Let the objective values of Formulations I and II be $z(\mathrm{I})$ and $z(\mathrm{II})$ respectively, we measure the difference between $z(\mathrm{I})$ and $z(\mathrm{II})$ using *adjustability gap* and *adjustability ratio* defined as follows,

$$\delta_{\mathrm{abs}} := z(\mathrm{I}) - z(\mathrm{II}), \qquad \delta_{\mathrm{rel}} := |z(\mathrm{I})|/|z(\mathrm{II})|,$$

where the latter is meaningfully defined only if $z(\mathrm{I})$ and $z(\mathrm{II})$ have the same sign. The concept of adjustability ratio δ_{rel} is equivalent to *adaptability gap* defined and studied by Bertsimas and Goyal [12] and Bertsimas et al. [19]. In this paper, we focus on adjustability in robust optimization with a linear objective function, linear constraints, fixed recourse, and an arbitrarily shaped, closed uncertainty set Ξ.

1.1 Related Problems and Concepts

Formulations I and II, along with adjustability, are closely related to many other problems and concepts. Therefore, results from studying adjustability can potentially lead to new findings in a wide range of problems. We describe these related concepts in this subsection.

Conservativeness of Static Robust Optimization. Static robust optimization (Formulation I) is sometimes seen to be too conservative due to its worst-case objective evaluation and simultaneous constraint satisfaction for all uncertainty parameter realizations under fixed decisions. Thus, the difference in objective values of Formulations I and II has been used as an instrument to analyze the conservativeness of static robust optimization [12, 19].

Parameterized Policies in Adjustable Robust Optimization. In general, Formulation II suffers from tractability issues caused by the infinite dimensionality of the policy space $\prod_{\xi\in\Xi}\mathcal{Y}_\xi$ [9]. To address this, a common method is to restrict the decision space to a subset of $\prod_{\xi\in\Xi}\mathcal{Y}_\xi$ that is represented by a policy family (also called *decision rules*) $\theta : \mathcal{P} \to \prod_{\xi\in\Xi}\mathcal{Y}_\xi$ for some Euclidean space $\mathcal{P}$. For instance, the constant and affine policy families can be represented by $\bar{\theta}(p,\xi) = p$ and $\hat{\theta}((P,p),\xi) = P\xi + p$. There are also many other policy families, including piecewise constant, piecewise affine, and polynomial policies. In particular, restricting the policy space of an ARO to $\bar{\theta}$ reduces the problem to a static robust optimization problem. Because almost every commonly used policy family contains the set of constant policies, its performance (optimality gap induced by such a policy family) is bounded by the adjustability gap and ratio.

Interdiction Game & Defender-Attacker-Defender Game. An interdiction game [49] is a problem between two players — an attacker (maximizer) and a defender (minimizer) — that have conflicting objectives, which has the following general form,

$$\text{III} := \max_{\xi\in\Xi}\min_{y\in\mathcal{Y}_\xi} f(\xi, y).$$

It is intuitively true and can be easily shown that $z(\text{II}) = z(\text{III})$ (see Lemma 1). Another version of the interdiction-type problem is the defender-attacker-defender (DAD) problem [1, 24, 37], where some of the defender's decisions have to be made before the attacker's strategies. In both cases, the adjustability gap is an upper bound of the objective value increment if the defender counters the attack proactively.

Minimax Theorems. Clearly, we have $z(\text{I}) \geq z(\text{III})$, which resembles the classical minimax inequality. However, the solution spaces Ξ and $\mathcal{Y}_\xi$ exhibit a dependence that is not present in the classical minimax theorems [39, 47]. Thus, the adjustability gap δ_{abs} can be viewed as a generalization of the minimax gap for problems with dependent decision spaces. Conditions for $\delta_{\text{abs}} = 0$ can be interpreted as the criteria for minimax equality under this more general setup.

Reformulation-Linearization Technique (RLT). Sherali and Alameddine [44] introduced RLT as a reformulation technique that transforms a given bilinear formulation into a linear program to improve tractability. Recently, Ardestani-Jaafari and Delage [3] discovered that the linear reformulation obtained from RLT is equivalent to an ARO restricted by the affine policy family, *i.e.*, the so-called affinely adjustable robust counterpart (AARC) of the ARO. Consider a formulation with a bilinear objective function,

$$\text{IV} := \max_{(x,y)\in\mathcal{Z}} c^\intercal x + d^\intercal y + x^\intercal G y,$$

where $\mathcal{Z}$ is an arbitrary joint space for x and y. If for every fixed x (or y), the problem has a dual representation that satisfies strong duality, then Formulation IV can be reformulated equivalently into a maximin problem as Formulation III. Then, the linear program obtained from RLT is equivalent to the AARC of Formulation II. Therefore, the adjustability gap also bounds the optimality gap induced by RLT in a given bilinear program.

Minimax Control & Online Algorithm. Minimax control and optimal online algorithm design problems [11, 17] can often be modeled as a dynamic robust optimization as follows,

$$\left(\min_{y_1 \in \mathcal{Y}_1} \max_{\xi_1 \in \Xi_1} \min_{y_2 \in \mathcal{Y}_2} \max_{\xi_2 \in \Xi_2} \cdots \min_{y_l \in \mathcal{Y}_l} \max_{\xi_l \in \Xi_l} \cdots \right) f(\xi, y),$$

where $l \in L$ indexes the lth iteration of the min and max operations. For $l \in L$, $\mathcal{Y}_l$ and Ξ_l are subsets of Euclidean spaces, and we have $y = (y_l)_{l \in L}$ and $\xi = (\xi_l)_{l \in L}$. When $|L|$ is finite (infinite), this formulation represents a dynamic robust optimization with a finite (infinite) horizon. We can rewrite this formulation as,

$$\mathrm{V} := \min_{y(\cdot) \in \mathfrak{Y}_\Xi} \max_{\xi \in \Xi} f(\xi, y),$$

where $\Xi = \prod_{l \in L} \Xi_l$ and the policy set $\mathfrak{Y}_\Xi$ contains all the feasible history-dependent policies. As a consequence, the value $z(\mathrm{V})$ of a dynamic robust optimization is sandwiched between $z(\mathrm{I})$ and $z(\mathrm{II})$. Therefore, in the control context, adjustability gap bounds the difference in objective values between open-loop control (the RO formulation) and closed-loop control. In the online algorithm setting, adjustability bounds the difference in objective values between the optimal online algorithm and the oracle (the FARO formulation).

Regret Optimization. A two-stage worst-case regret optimization [41] is modeled as

$$\mathrm{VI} := \min_{y \in \mathcal{Y}_\Xi} \max_{\xi \in \Xi} \left(f(\xi, y) - \min_{y' \in \mathcal{Y}_\xi} f(\xi, y') \right),$$

which searches for a solution y that minimizes the point-wise regret between a realization $f(\xi, y)$ and an oracle. Interestingly, the adjustability gap δ_{abs} is a valid lower bound of $z(\mathrm{VI})$. To see this, we rewrite Formulation VI as the following equivalent formulation,

$$\min_{y \in \mathcal{Y}_\Xi} \max_{\xi \in \Xi} \min_{\xi' \in \Xi : \xi' = \xi} \left(f(\xi, y) - \min_{y' \in \mathcal{Y}_{\xi'}} f(\xi', y') \right).$$

Without the constraint $\xi' = \xi$, this formulation is equivalent to Formulation I minus Formulation III. Thus, for every fixed pair $y \in \mathcal{Y}_\Xi$ and $\xi \in \Xi$, the value of Formulation VI is greater than or equal to the value of Formulation I minus Formulation III, which is also true for the optimal y and ξ. That is, $z(\mathrm{VI}) \geq z(\mathrm{I}) - z(\mathrm{III}) = \delta_{\text{abs}}$.

***A Priori* Optimization & Stochastic Gap.** When the uncertainty set Ξ is coupled with a joint probability distribution $\mathcal{D}$, one may be interested in comparing the average performances between making decisions proactively and the ideal case where reoptimization is conducted for every $\xi \in \Xi$.

$$\mathrm{VII} := \min_{y \in \bigcap_{\xi \in \Xi} \mathcal{Y}_\xi} \mathbb{E}_{\xi \sim \mathcal{D}} \left[f(\xi, y) \right], \qquad \mathrm{VIII} := \mathbb{E}_{\xi \sim \mathcal{D}} \left[\min_{y \in \mathcal{Y}_\xi} f(\xi, y) \right].$$

These two problems are introduced as *a priori optimization* and *reoptimization* methods in [22]. Then, the gap or ratio between $z(\mathrm{VII})$ and $z(\mathrm{VIII})$ is a measure of the quality of *a priori* decisions. From this perspective, adjustability can be viewed as the counterpart of the difference between $z(\mathrm{VII})$ and $z(\mathrm{VIII})$ under the worst-case lens. We note that the relationship between $z(\mathrm{VII}) - z(\mathrm{VIII})$ and δ_{abs} is inconclusive in general since the distribution over Ξ can be arbitrary. However, as pointed out by Bertsimas and Goyal [12], we have $z(\mathrm{VIII}) \leq z(\mathrm{III})$ in general, which implies the relationship $z(\mathrm{VIII}) \leq z(\mathrm{II}) \leq z(\mathrm{I})$. Therefore, given $z(\mathrm{VIII}) > 0$, adjustability ratio δ_{rel} is a valid lower bound of the *stochastic gap* $z(\mathrm{I})/z(\mathrm{VIII})$ studied in Bertsimas and Goyal [12] and Bertsimas et al. [19].

1.2 Literature Review

Robust optimization (RO) is a modeling approach to address parameter uncertainty in various decision problems. This method has been extensively studied [8, 10, 14, 15] and has gained traction in many application areas such as inventory theory [2, 16], supply chain management [6, 46], queuing theory [7], scheduling and transportation [3, 36, 45], portfolio optimization [30], healthcare [34], Markov decision process [40], among others [27, 48]. The core idea of RO is to produce a worst-case optimal solution that is feasible to all possible realizations of uncertainty parameters.

Sometimes robust optimization is considered to be too conservative. Consequently, a handful of results have been developed to measure this conservativeness, which are closely related to the concept of adjustability gap and ratio. When first introducing linear ARO with nonnegative fixed recourse, Ben-Tal et al. [9] showed that constant policy is optimal if the uncertainty set Ξ is a box. Marandi and den Hertog [38] generalized this result to ARO with constraints that are convex-concave on the product space of uncertainty set and decision space. These results heavily depend on the constraint-wise separability condition provided by the box uncertainty set. For non-box uncertainty sets, Bertsimas and Goyal [12] derived several constant policy approximation ratios for uncertainty sets with special properties. Later, Bertsimas et al. [19] generalized this method to provide tighter bounds for various uncertainty sets using an upper bound called the stochastic gap. For a particular class of non-fixed (variable) recourse ARO problems, Bertsimas et al. [20] proved that constant policy is optimal if the constraint set satisfies certain convexity conditions, and a non-convexity measurement can bound the approximation ratio. Awasthi et al. [4] studied the constant optimality gap in a two-stage adjustable robust packing linear optimization problem where the uncertainty set is column-wise and constraint-wise. For these non-box set results, the uncertainty set under consideration is assumed to locate entirely within the nonnegative orthant. Recently, Iancu et al. [34] showed that in a multiperiod problem, the static (constant) policy is optimal if the objective has certain monotonicity properties and the uncertainty set has certain ordering (*e.g.*, lattice) properties.

To address the seemingly conservativeness nature of RO, ARO [9] has been proposed, where some of the decision variables can be determined after uncertainty realization, *i.e.*, are functions (policies) over the uncertainty set. The downside is, solving ARO exactly is intractable in general [9, 10]. Thus, one either has to use heuristics (*e.g.*, parameterized policies) or adopt a column/row generation method (Zhen et al. [50] have shown that one can eliminate adjustable decisions from an ARO problem via Fourier-Motzkin elimination, but at the cost of adding many extra constraints). Parameterized policy families include constant [20], affine [29], piecewise constant (also called K-adaptability [32]), piecewise affine [26], or polynomial policies [5]. An interesting question about ARO is the optimality criteria and optimality gap (or ratio) of various policy families. Besides the constant policy family, *i.e.*, the conservativeness measurement mentioned before, another policy family that has been often studied is the affine policy family. We list some results below and refer readers to İhsan Yanıkoğlu et al. [51] for a comprehensive survey on this topic.

Chen and Zhang [25] discovered that under certain conditions, lifting the uncertainty set Ξ into a high-dimensional space can make the affine policy family optimal. Bertsimas et al. [17] proved that affine policy is also optimal in a one-dimensional multistage robust optimization setting where the decision and uncertainty at each stage is a one-dimensional quantity, and the overall uncertainty set Ξ is a box set. Bertsimas and Goyal [13] showed that under mild assumptions, the affine policy family is optimal when the uncertainty set Ξ is a simplex; using this result, they also provided general lower and upper bounds for optimality gap in their setting. Iancu et al. [33] studied affine optimality criteria in a dynamic robust optimization setting, where the objective function has certain supermodularity and convexity properties simultaneously, and the uncertainty set is a lattice. They provided the most general affine optimality result to date. In the study of the robust optimization of sums of piecewise linear functions, Ardestani-Jaafari and Delage [2] discovered an equivalence between the affine optimality gap and an integrality gap between two reformulations of the main problem. Simchi-Levi et al. [46] applied affine policy family in an ARO problem of supply chain design. Their uncertainty set has a hierarchical structure, and they proved that affine policy is optimal given that the underlying topology of the supply chain is a rooted tree. Recently, El Housni and Goyal [28] showed that affine policy gives $O(\frac{\log n}{\log \log n})$-approximation in a linear ARO with fixed nonnegative recourse for budgeted uncertainty sets.

1.3 Contributions

Most results in the literature about adjustability (*i.e.*, constant optimality gap, adaptability gap) have restrictive assumptions on the uncertainty set and/or optimization model, such as the non-negativity condition in [4, 12, 19, 20], or the constraint-wise separability requirement in [9, 38]. This poses certain restrictions on the application scope of the corresponding results. For instance, in [12] and [19], the optimization model's (uncertain) right hand side has to be nonnegative, and the inequalities can only take the "greater than or equal to" direction. This precludes the opportunities of modeling certain types of natural constraints, such as capacity constraints in network optimization, budget constraints in a resource-limited setting, or problems with equality constraints.

In this work, our results require very few assumptions on the input parameters. For the uncertainty set, we only assume closedness; for other input parameters, we just require that the corresponding problem is feasible and bounded. Such a setting removes the aforementioned restrictions. For instance, in Section 6, we will use our framework to analyze the adjustability ratio of maximum flow interdiction where the uncertainty set is entirely contained in the non-positive orthant, and at the same time, there are equality constraints.

In order to relax the previous restrictive assumptions and enable our analysis, we develop a new set of theoretical tools under a coherent framework to study adjustability. Some techniques we develop along the way may be of theoretical interest themselves and can be generalized to analyze other problems such as affine optimality in ARO. It turns out that the optimality criteria and gap derived using our framework are more general and tighter than existing results in the literature.

We develop this new theoretical framework as follows. First, we discover several equivalent reformulations of RO and FARO problems (Section 2) and an interesting algebraic property (Theorem 1 in Section 3) that bridges them together. This allows us to provide a necessary and sufficient condition (Theorem 2 in Section 4) for the adjustability gap to be zero. This sharp result then allows us to use constructive approaches to characterize (Theorem 3 in Section 5) and efficiently approximate (Algorithm 1 in Section 5) the adjustability ratio. Finally (Section 6), we apply the framework to maximum flow interdiction, supply chain design, and inventory control problems to demonstrate its practical value.

1.4 Notation

For an optimization problem Π, we use $z(\Pi)$ to denote the associated optimal objective value. Given $n \in \mathbb{N}$, $[n]$ is defined as the set $\{1, 2, \cdots, n\}$. For any subset $S \subseteq \mathbb{R}^n$, int(S) is the interior points of S, and we use conv(S) and cone(S) to represent the respective convex hull and conic hull formed by the elements in S. For a scalar $r \in \mathbb{R}$, we use rS to denote the scaled set $\{r\xi \mid \xi \in S\}$. Given any two sets of vectors S_1 and S_2, the Minkowski sum is defined as $S_1 + S_2 := \{v_1 + v_2 \mid v_1 \in S_1, v_2 \in S_2\}$. For a given polyhedron Ξ, ext(Ξ), eray(Ξ) are the sets of extreme points and extreme rays. Slightly abusing the notation, we use conv(Ξ) := conv(ext(Ξ)), cone(Ξ) := cone(eray(Ξ)) to represent the polytope part and cone part of Ξ. It is well known that every polyhedron Ξ can be decomposed into Ξ = conv(Ξ) + cone(Ξ).

We use upper and lower case letters to denote matrices and vectors, respectively. For a matrix A, we take a_i and a_{ij} as the ith row and (i, j)th entry of A. We adopt the convention that all vectors without the transpose sign are column vectors. For instance, the ith row a_i of matrix A or any explicitly constructed vector (v_1, v_2) are all considered as column vectors, of which the row vector counterparts are denoted as $a_i^\intercal$ and $(v_1, v_2)^\intercal = (v_1^\intercal, v_2^\intercal)$. Given v_1, v_2 with the same size, we use the notation $[v_1, v_2]$ (separated by comma) to horizontally concatenate them into a two-column matrix. Similarly, vertical stacking is done by $[v_1^\intercal; v_2^\intercal]$ (separated by semi-colon). These two stacking operations naturally extend to multiple matrices and/or vectors with compatible shapes. We also use I for the identity matrix, $\mathbb{1}$ and $\mathbb{0}$ for (column) vectors filled with value ones and zeros. We also view a matrix as the set of its rows, *i.e.*, $a \in A$ is some row in A (viewed as a column vector).

We define the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ in the common sense, *i.e.*, $\langle x_1, x_2 \rangle$ (or $\langle X_1, X_2 \rangle$ for two matrices) is the sum of the product of all the entries. It is well-known that $\langle v_1, Av_2 \rangle = \langle A^\intercal v_1, v_2 \rangle$ and $\langle A, BC \rangle = \langle B^\intercal A, C \rangle = \langle AC^\intercal, B \rangle$.

The rest of the paper is organized as follows. In Section 2, we give a detailed problem setting and introduce two reformulations that are crucial for later development. In Section 3, we prove a fundamental theorem that equates the adjustability gap with a measurement called symmetry gap. Then, in Section 4, we derive a sufficient and necessary condition for the adjustability gap to be zero. In Section 5, we explore this intuition to prove a theorem for bounding the adjustability ratio and develop the corresponding evaluation algorithm. In Section 6, we apply our framework to several instance problems and discuss some observations. Finally, we conclude the paper in Section 7. To better streamline the discussion of the paper, except for Lemma 2 and all the theorems, we provide the mathematical proofs in Appendix A.

2 Problem Setting and Equivalent Reformulations

Throughout the paper, we focus on RO (Formulation I) and FARO (Formulation II) problems restricted to a linear objective function, linear constraints, fixed-recourse, and an arbitrarily shaped closed uncertainty set. We name these restricted versions the *constant policy problem* $\bar{\Pi}$ and *policy problem* Π, respectively. In this section, we first give a rigorous account of Π and $\bar{\Pi}$ along with some mild assumptions. Then, we introduce a reformulation of Π called the *value problem* and a reformulation of $\bar{\Pi}$ call the *bidual problem*, which will be used to derive a characterization of the adjustability gap in Section 3.

A policy problem Π with input parameters (Ξ, a, A, c, C) is defined as follows,

$$\Pi := \min_{y \in \mathcal{Y}^{\mathcal{X}}} \max_{\xi \in \Xi} \quad \langle c, \xi \rangle + \langle a, y(\xi) \rangle \tag{1a}$$

$$\text{s.t.} \quad C\xi - Ay(\xi) \leq 0, \quad \forall \xi \in \Xi \tag{1b}$$

where $\mathcal{X} := \mathbb{R}^n$ and $\mathcal{Y} := \mathbb{R}^m$ are two Euclidean spaces, $\mathcal{Y}^{\mathcal{X}}$ is the *policy space* that contains all the functions from $\mathcal{X}$ to $\mathcal{Y}$, and $\Xi \subseteq \mathcal{X}$ is a given *uncertainty set*. When Ξ is a polyhedron, we represent it as $\Xi := \{\xi \in \mathcal{X} \mid B\xi \leq b\}$. Matrices $C \in \mathbb{R}^{k \times n}$ and $A \in \mathbb{R}^{k \times m}$ are used to restrict the policy space. We also define the augmented matrix $\bar{C} := [c^\intercal; C]$. We have the following main technical assumption.

Assumption 1. Throughout the paper, we assume the following,

1. Ξ is a nonempty closed set.

2. Π is feasible and its optimal objective value $z(\Pi)$ is bounded.

Assumption 1.1 is a very mild condition on uncertainty sets. Before Section 5, we mainly focus on polyhedral uncertainty set. For Assumption 1.2, feasibility of Π implies that there exists a feasible policy $y : \mathcal{X} \to \mathcal{Y}$ that satisfies (1b) for all $\xi \in \Xi$, while boundedness means for some feasible strategy pair (ξ, y), the objective value $\langle c, \xi \rangle + \langle a, y(\xi) \rangle$ is finite.

With the same input parameters, the corresponding constant policy problem $\bar{\Pi}$ is the following,

$$\bar{\Pi} := \min_{y \in \mathcal{Y}} \max_{\xi \in \Xi} \quad \langle c, \xi \rangle + \langle a, y \rangle \tag{2a}$$

$$\text{s.t.} \quad C\xi - Ay \leq 0, \quad \forall \xi \in \Xi. \tag{2b}$$

Clearly, we have $z(\Pi) \leq z(\bar{\Pi})$ as $\mathcal{Y}$ can be viewed as the set of constant functions from $\mathcal{X}$ to $\mathcal{Y}$, which is a subset of $\mathcal{Y}^{\mathcal{X}}$. Then, adjustability is defined as follows,

Definition 1 (Adjustability)**.** Given a policy problem Π and its constant policy problem $\bar{\Pi}$, we define the following two metrics,

$$\begin{aligned} &\text{Adjustability gap} \quad \delta_{\text{abs}}(\Pi) &&:= z(\bar{\Pi}) - z(\Pi). \\ &\text{Adjustability ratio} \quad \delta_{\text{rel}}(\Pi) &&:= |z(\bar{\Pi})|/|z(\Pi)|, \ \text{given } z(\Pi) > 0 \text{ or } z(\bar{\Pi}) < 0. \end{aligned}$$

When $\delta_{\text{abs}}(\Pi) = 0$, we say Π is *zero-adjustable*.

In this paper, we study the conditions for zero-adjustability and estimate a tight bound for adjustability ratio when these conditions are violated. As the first step, we introduce the two reformulations — the value problem and the bidual problem. This will allow us to define what we call *symmetry gap* in Section 3 for characterizing the adjustability gap $\delta_{\text{abs}}(\Pi)$.

2.1 Value Problem

We start with the following definition,

Definition 2 (Value Problem)**.** For a given policy problem Π, the associated value problem Π^v is defined as follows,

$$\Pi^v := \max_{\xi \in \Xi} \min_{y \in \mathcal{Y}} \quad \langle c, \xi \rangle + \langle a, y \rangle \tag{3a}$$

$$\text{s.t.} \quad C\xi - Ay \leq 0. \tag{3b}$$

The connection between Π and its value problem Π^v is stated in the following lemma.

Lemma 1. *For a given policy problem Π and its value problem Π^v, we have $z(\Pi) = z(\Pi^v)$.*

Notice that, when $C = 0$, Π^v can be separated into two independent optimization problems over solution spaces Ξ and $\mathcal{Y}$, respectively. Under Lemma 1, this implies Π is zero-adjustable. For subsequent analysis, we exclude this trivial case via the following assumption.

Assumption 2. The input matrix C is nonzero.

Another consequence of Lemma 1 is that we can obtain further implications from Assumption 1 since it can be equivalently stated from the perspective of Formulation (3). In particular, we have the following proposition.

Proposition 1. *In Assumption 1.2, feasibility implies (i) for every $\xi \in \Xi$ and $u \in \mathbb{R}^k_+$, if $A^\intercal u = 0$ then $\langle C\xi, u\rangle \leq 0$; (ii) for every $\xi \in cone(\Xi)$, $\langle c_i, \xi\rangle \leq 0$. Boundedness entails (i) there exists some $u \in \mathbb{R}^k_+$ such that $A^\intercal u = a$; (ii) for every $\xi \in cone(\Xi)$, $\langle c, \xi\rangle \leq 0$.*

Remark 1. This proposition implies that there always exists an optimal solution ξ in conv(Ξ), even though we do not require Ξ to be bounded. In this case, we say Ξ is *effectively compact*.

This proposition also implies that $\{u \in \mathbb{R}^k_+ \mid A^\intercal u = a\}$ is nonempty. We define it formally as,

Definition 3 (Dual Polyhedron)**.** For a policy problem Π, the dual polyhedron is defined as

$$\mathcal{U} := \{u \in \mathbb{R}^k_+ \mid A^\intercal u = a\}.$$

Thus, a policy problem Π with parameters (Ξ, a, A, c, C) can also be represented as $\Pi = (\Xi, \mathcal{U}, \bar{C})$.

2.2 Bidual Problem

In this subsection, we introduce another formulation that is important for subsequent analysis.

Definition 4 (Bidual Problem)**.** Given a constant policy problem $\bar{\Pi}$ with a polyhedral uncertanty set Ξ, the corresponding *bidual problem* is defined as

$$\bar{\Delta} := \max_{\xi\in\Xi, u\in\mathcal{U}, V} \quad \langle c, \xi\rangle + \langle C, V\rangle \tag{4a}$$

$$\text{s.t.} \quad BV^\intercal \leq bu^\intercal. \tag{4b}$$

As suggested by the word *bidual*, we obtain $\bar{\Delta}$ by first dualizing all the constraints in (2b), then dualizing the resulting formulation for a fixed $\xi \in \Xi$. We call this the *bidualiztion procedure*, which is depicted in the proof of Lemma 2. It has been used to study the performance of affine policies in Ardestani-Jaafari and Delage [2]. We have the following relationship between $\bar{\Pi}$ and $\bar{\Delta}$.

Lemma 2. *Let $\bar{\Pi}$ and $\bar{\Delta}$ be the constant policy problem and its bidual, we have $z(\bar{\Pi}) = z(\bar{\Delta})$.*

Proof. We prove this by showing that $\bar{\Delta}$ is obtained by performing a sequence of linear programming dualizations and minimax theorem. Together with feasibility and boundedness in Assumption 1, it implies that the optimal objective value is preserved during this process. The first dualization is a standard procedure in obtaining the robust counterpart of a (static) robust optimization problem. It begins with rewriting Formulation (2) as follows,

$$\min_y \max_{\xi\in\Xi} \quad \langle c, \xi\rangle + \langle a, y\rangle \tag{5a}$$

$$\text{s.t.} \quad \max_{\xi\in\Xi} \langle c_i, \xi\rangle - \langle a_i, y\rangle \leq 0, \quad \forall i \in [k]. \qquad w_i \tag{5b}$$

For any constraint i in (5b), we take vector $w_i \geq 0$ as the dual variables of the left-hand side (LHS). Then the dual formulation of the LHS of constraint i is,

$$\min_{w_i \geq 0} \quad \langle b, w_i\rangle - \langle a_i, y\rangle$$

$$\text{s.t.} \quad -B^\intercal w_i + c_i = 0.$$

After substituting this back to the LHS of constraint i in (5b), and defining W to be the matrix whose ith row is w_i, Formulation (5) can be further written as $\min_{y\in\mathcal{Y}'}\max_{\xi\in\Xi}\langle c,\xi\rangle+\langle a,y\rangle$ where $\mathcal{Y}'$ is the projection of the following polyhedron onto the $\mathcal{Y}$ space,

$$\left\{(y,W)\;\middle|\;\begin{array}{rcl} Wb-Ay & \le & 0,\\ WB & = & C,\\ W & \ge & 0.\end{array}\right\}$$

Clearly, the objective function is linear in y and ξ, Ξ is effectively compact (see Remark 1), and $\mathcal{Y}'$ is compact due to $C\neq 0$ by Assumption 2 (and independent of Ξ). Thus, minimization and maximization can be swapped by the minimax theorem [39], resulting in the following

$$\begin{aligned}\max_{\xi\in\Xi}\ \min_{y,W\ge 0}\quad & \langle c,\xi\rangle+\langle a,y\rangle\\ \text{s.t.}\quad & Wb-Ay\le 0, && u\\ & WB=C. && V\end{aligned}$$

Now, we dualize the inner problem of this formulation with a fixed ξ. Let vector u and matrix V be the corresponding dual variables, we get the following formulation,

$$\max_{\xi\in\Xi,u\ge 0,V}\quad \langle c,\xi\rangle+\langle C,V\rangle \tag{6a}$$

$$\text{s.t.}\quad A^\intercal u=a, \qquad y \tag{6b}$$

$$BV^\intercal\le bu^\intercal. \qquad W^\intercal \tag{6c}$$

Notice (6b) and $u\ge 0$ form the dual polyhedron $\mathcal{U}$. Thus, we obtain the bidual problem $\bar{\Delta}$. □

This theorem requires the uncertainty set Ξ to be a polyhedron. When this is violated, we can still derive a relaxed inequality statement according to the following.

Corollary 1. *Suppose for some closed set $\Xi'\supseteq\Xi$, the problem $\max_{\xi'\in\Xi'}\langle c_i,\xi'\rangle-\langle a_i,y\rangle$ has a dual representation that satisfies weak duality, then, we have $z(\bar{\Pi})\le z(\bar{\Delta}_{\Xi'})$, where $\bar{\Delta}_{\Xi'}$ is the problem obtained using bidualization procedure with uncertainty set Ξ'.*

In this section, we established the two equalities $z(\Pi)=z(\Pi^v)$ and $z(\bar{\Pi})=z(\bar{\Delta})$. In the next section, we characterize the relationship between Π and $\bar{\Pi}$ via the analysis on Π^v and $\bar{\Delta}$.

3 Symmetry Gap and Symmetric Optimality

Recall that our main goal is to quantify $\delta_{\text{abs}}=z(\bar{\Pi})-z(\Pi)$ and $\delta_{\text{rel}}=|z(\bar{\Pi})|/|z(\Pi)|$. By the equivalent formulations we derived in the previous section, we immediate have an alternative characterization of the two quantities: $\delta_{\text{abs}}=z(\bar{\Delta})-z(\Pi^v)$ and $\delta_{\text{rel}}=|z(\bar{\Delta})|/|z(\Pi^v)|$. In this section, we provide some additional insights on this characterization. In subsequent sections, we show how and why such characterization leads to new machinery to analyze δ_{abs} and δ_{rel}.

Definition 5 (Symmetric Bidual)**.** We define the *symmetric bidual problem* $\bar{\Delta}^{\text{sym}}$ as Formulation (4) with an additional set of constraints $V=u\xi^\intercal$, called the *symmetry constraint*.

$$\bar{\Delta}^{\text{sym}}:=\max_{\xi\in\Xi,u\in\mathcal{U},V}\quad \langle c,\xi\rangle+\langle C,V\rangle \tag{7a}$$

$$\text{s.t.}\quad BV^\intercal\le bu^\intercal, \tag{7b}$$

$$V=u\xi^\intercal. \tag{7c}$$

Definition 6 (Symmetric Solution/Optimality & Symmetry Gap)**.** A feasible solution (ξ, u, V) of the bidual $\bar{\Delta}$ is said to be symmetric if $V = u\xi^\intercal$. The *symmetry gap* is defined as $\delta_{\text{sym}}(\bar{\Delta}) := z(\bar{\Delta}) - z(\bar{\Delta}^{\text{sym}})$. We say $\bar{\Delta}$ is *symmetrically optimal* if its symmetry gap is zero.

The following theorem shows the equivalence between the adjustability gap $\delta_{\text{abs}}(\Pi)$ and the symmetry gap of the bidual $\bar{\Delta}$.

Theorem 1. *Given a policy problem* Π *with polyhedral uncertainty set and its bidual problem* $\bar{\Delta}$*, we have* $\delta_{abs}(\Pi) = \delta_{sym}(\bar{\Delta})$*.*

Proof. First, we show that the dual of the value problem Π^v, denoted by Δ^v, is actually the symmetric bidual problem, *i.e.*, $\Delta^v = \bar{\Delta}^{\text{sym}}$. The formulation for Π^v is (3). We fix the uncertainty variables in the outer problem at ξ and let u be the dual variables for all the constraints. Then, dualizing the inner problem gives the following,

$$\Delta^v := \max_{\xi \in \Xi, u \in \mathcal{U}} \quad \langle c, \xi \rangle + \langle C\xi, u \rangle.$$

On the other hand, the formulation of $\bar{\Delta}^{\text{sym}}$ is the bidual (4) with the symmetry constraint $V = u\xi^\intercal$, which gives the following,

$$\begin{aligned} \max_{\xi \in \Xi, u \in \mathcal{U}} \quad & \langle c, \xi \rangle + \langle C, u\xi^\intercal \rangle \\ \text{s.t.} \quad & (B\xi - b)u^\intercal \leq 0. \end{aligned}$$

Notice that the constraint set $(B\xi - b)u^\intercal \leq 0$ becomes redundant since ξ and u are chosen from Ξ and $\mathcal{U}$, respectively. Hence, $\bar{\Delta}^{\text{sym}} = \Delta^v$. Then, we have the following chain of relations,

$$z(\Pi) = z(\Pi^v) = z(\Delta^v) = z(\bar{\Delta}^{\text{sym}}) \leq z(\bar{\Delta}) = z(\bar{\Pi}).$$

The first equality is due to Lemma 1, the second is due to strong duality, the third is by the equivalence $\Delta^v = \bar{\Delta}^{\text{sym}}$ we have just shown, the inequality is because $\bar{\Delta}^{\text{sym}}$ has a more restricted feasible region than $\bar{\Delta}$, and the last equality is by Lemma 2. Therefore, the gap between $z(\Pi)$ and $z(\bar{\Pi})$ is entirely captured by the symmetry gap of $\bar{\Delta}$. □

Remark 2. We note that the dualization of Π^v is done by first fixing some $\xi \in \Xi$. Thus, we still have $z(\Pi) = z(\Pi^v) = z(\Delta^v)$ even if Ξ is not a polyhedron or not a convex set.

The following optimality criterion is a direct consequence of Theorem 1, so we omit its proof.

Corollary 2. Π *is zero-adjustable if and only if there exists an optimal solution of the bidual* $\bar{\Delta}$ *that is also symmetric.*

We will call these *symmetric-optimal solutions* hereafter. The following corollary allows us to restrict our attention to the extreme points of $\mathcal{U}$ and Ξ.

Corollary 3. Π *is zero-adjustable if and only if there exists a symmetric-optimal solution* $(\xi^\star, u^\star, u^\star\xi^{\star\intercal})$ *of the bidual* $\bar{\Delta}$ *such that* $\xi^\star \in ext(\Xi)$ *and* $u^\star \in ext(\mathcal{U})$*.*

Theorem 1 translates the adjustability gap δ_{abs} into the symmetry gap δ_{sym} that is defined upon the bidual problem $\bar{\Delta}$. Compared to the former, the latter gap is more advantageous for analytic purposes since it reveals an interesting structure: the symmetry constraint $V = u\xi^\intercal$ dictates the optimality gap of the constant policy family. In later sections, we use this characterization to derive zero-adjustability criteria and adjustability ratio bounds.

4 When is Adjustability Gap Zero?

With the characterization and insights derived in the preceding section, we now study the conditions under which $\delta_{\text{abs}} = 0$, *i.e.*, the zero-adjustability criteria of Π. One interesting observation about Formulation (4) is that the constraint set (4b) is similar to the definition of Ξ. The following proposition formalizes this observation, which provides a geometric interpretation of (4b).

Proposition 2. *In the bidual formulation $\bar{\Delta}$, constraint set (4b) is equivalent to the following,*

$$v_i \in u_i \mathit{conv}(\Xi) + \mathit{cone}(\Xi),\ \forall i \in [k]. \tag{8}$$

Geometrically, this constraint says that the feasible region of the ith row of matrix V is the scaled polyhedron $u_i\text{conv}(\Xi) + \text{cone}(\Xi)$. Thus, constraint set (8) can be viewed as a *constraint propagator* that propagates each constraint of Ξ to the feasible space of each v_i with a scaling factor u_i.

This result allows us to represent each row v_i of matrix V as $u_i\xi_i + \xi_i'$ for some $\xi_i \in \text{conv}(\Xi)$ and $\xi_i' \in \text{cone}(\Xi)$. Then, the bidual formulation has the following alternative form,

$$\bar{\Delta} = \max_{\xi' \in \text{cone}(\Xi)} \langle c, \xi' \rangle + \max_{\xi \in \text{conv}(\Xi)} \langle c, \xi \rangle + \sum_{i \in [k]} \max_{\xi' \in \text{cone}(\Xi)} \langle c_i, \xi_i' \rangle + \max_{u \in \mathcal{U}} \sum_{i \in [k]} u_i \left(\max_{\xi_i \in \text{conv}(\Xi)} \langle c_i, \xi_i \rangle \right).$$

Note that, by Assumption 1 and its implications in Proposition 1, the first and third terms in the above formulation are equal to zero. Thus, we obtain

$$\bar{\Delta} = \max_{\xi \in \text{conv}(\Xi)} \langle c, \xi \rangle + \max_{u \in \mathcal{U}} \sum_{i \in [k]} u_i \left(\max_{\xi_i \in \text{conv}(\Xi)} \langle c_i, \xi_i \rangle \right). \tag{9}$$

This means, under Assumption 1, we can further replace (8) with $v_i \in u_i\text{conv}(\Xi)$ in the bidual $\bar{\Delta}$. With such intuitive interpretation of the constraints, it is expected that the feasible region of (4), denoted by $\mathfrak{P}$, has the following nice properties.

Proposition 3. *Any solution $(\xi, u, V) \in \mathfrak{P}$ is an extreme point if and only if $\xi \in \mathit{ext}(\Xi)$, $u \in \mathit{ext}(\mathcal{U})$, and for each row $v_i = u_i\xi_i$ of matrix V, either $u_i = 0$ or $\xi_i \in \mathit{ext}(\Xi)$.*

4.1 Zero-Adjustability Criteria

We use $\bar{\Delta}(u)$ to denote Formulation (9) with a fixed u. Then, $\mathcal{U}^\star := \arg\max_{u \in \mathcal{U}} \bar{\Delta}(u)$ is the set of optimal solutions in $\mathcal{U}$. Interestingly, for a fixed u, $\bar{\Delta}(u)$ can be viewed as $k+1$ independent optimization problems, each of which is the *support function* [42, p. 28] over conv(Ξ) with cost vector c or c_i. Based on this observation, we can use the following definition to characterize the zero-adjustability criterion,

Definition 7 (Normal Cone [42, p. 15])**.** Vector c' is *normal* to a closed convex set Ξ at $\xi \in \Xi$ if $\langle c', \xi' - \xi \rangle \leq 0$ for all $\xi' \in \Xi$. The set of all such vectors, $N_\Xi(\xi)$, is the *normal cone* to Ξ at ξ.

When a set of cost vectors $\{c_i\}_{i \in L}$ belong to the same normal cone $N_\Xi(\xi^\star)$ for some $\xi^\star \in \Xi$, the optimization problems in $\{\max_{\xi \in \Xi} \langle c_i, \xi \rangle\}_{i \in L}$ share the same optimal solution $\xi^\star$. In this case, we call this family of problems *co-optimal*. It should be clear that a family of maximization problems over the same Ξ is co-optimal if and only if the cost vectors belong to the same normal cone. With these definitions, we present the zero-adjustability theorem as follows,

Theorem 2. $\Pi = (\Xi, \mathcal{U}, C)$ *is zero-adjustable if and only if there exists some* $u \in \mathcal{U}^\star \cap ext(\mathcal{U})$ *with nonzero entries labeled by* L *such that* $\{c\} \cup \{c_i\}_{i \in L} \subseteq N_\Xi(\xi)$ *for some* $\xi \in ext(\Xi)$.

Proof. By Proposition 2, Formulation (9) is equivalent to the bidual formulation (4) with the identities $v_i = u_i \xi_i$ for all $i \in [k]$. Then, every symmetric solution of (4) corresponds to a feasible solution $(\xi, u, \{\xi_i\}_{i \in [k]})$ of Formulation (9) that satifies $\xi = \xi_i$ for every index i such that $u_i > 0$. For sufficiency, we take u and ξ that satisfy the premise, and construct the solution $(\xi, u, \{\xi_i = \xi\}_{i \in [k]})$. This solution is optimal for (9) by the choice of u and ξ, thus corresponds to an optimal and symmetric solution for (4) since $\xi_i = \xi$ for all i. Then, by Corollary 2, Π is zero-adjustable. For necessity, according to Corollary 3, Π being zero-adjustable implies there exists a symmetric-optimal solution $(\xi^\star, u^\star, u^\star \xi^{\star\intercal})$ for (4) where $\xi^\star \in \text{ext}(\Xi)$ and $u^\star \in \text{ext}(\mathcal{U})$. Therefore, $(\xi^\star, u^\star, \{\xi_i = \xi^\star\}_{i \in [k]})$ is an optimal solution for (9). This further implies $\xi^\star$ is an optimal solution of the problems $\max_{\xi \in \Xi} \langle c, \xi \rangle$ and $\max_{\xi_i \in \Xi} \langle c_i, \xi_i \rangle$ for every index i such that $u_i > 0$, *i.e.*, these problems are co-optimal. Thus, c and $\{c_i\}_{i \in L}$ belong to the same normal cone $N_\Xi(\xi^\star)$. □

The following corollary provides two sufficient criteria.

Corollary 4. $\Pi = (\Xi, \mathcal{U}, C)$ *is zero-adjustable if it satisfies either of the following conditions:*

1. $\{c\} \cup \{c_i\}_{i \in [k]} \subseteq N_\Xi(\xi)$ *for some* $\xi \in ext(\Xi)$;
2. *for every* $u \in ext(\mathcal{U})$ *with nonzero entries labeled by* L_u, *we have* $\{c\} \cup \{c_i\}_{i \in L_u} \subseteq N_\Xi(\xi_u)$ *for some* $\xi_u \in ext(\Xi)$.

In general, the second statement cannot imply the first since being in the same normal cone is not an equivalence relation. For instance, for any polyhedron Ξ, every vector c_i is in the same normal cone as the zero vector, but this does not imply these vectors are in the same normal cone.

4.2 Examples

In this subsection, we introduce several special cases of Theorem 2 and Corollary 4, emphasizing geometric intuition. Some of these examples have been discovered before. We unify them under our framework. In some cases, we also extend previous results.

In the first example, a special $\bar{C}$ can guarantee zero-adjustability for arbitrary input Ξ and $\mathcal{U}$.

Example 1 (Coupled Uncertainty)**.** Suppose all rows of the augmented matrix $\bar{C}$ are in the same direction, *i.e.*, all c and c_i's can be represented by $\alpha_i d$ for some nonnegative scalar $\alpha_i \geq 0$ and a direction d, then Π is zero-adjustable by Corollary 4. In this case, the effective uncertainty $\{\langle c_i, \xi \rangle\}_i$ and $\langle c, \xi \rangle$ are coupled or "comonotone". △

For some special uncertainty sets Ξ and arbitrary $\mathcal{U}$, Π is zero-adjustable.

Example 2 (Box Uncertainty Set)**.** A box uncertainty set [12, 18, 38] is defined as $\Xi = \{\xi \in \mathbb{R}^n \mid \xi_i \in [s_i, t_i] \ \forall i \in [n]\}$ where $[s_i, t_i]$ is some nonempty interval. In particular, a unit L_∞ ball is a special box set with $s_i = -1$ and $t_i = 1$ for all $i \in [n]$. It is clear that, when $\{c_i\}_{i \in [k]}$ and c are in the same orthant (*e.g.*, c_i's and c are nonnegative), they are in the same normal cone. Thus, Π is zero-adjustable by Corollary 4. This echoes the results in Bertsimas and Goyal [12] (Table 1), where the uncertainty set is a hypercube, c is the zero vector, and C is the identity matrix. It can also be seen as a special case of Theorem 1 in Marandi and den Hertog [38]. △

Box uncertainty sets may seem too conservative, but they do appear in important problem classes. For example, in dynamic programming, perturbations are usually assumed to be independent across time periods, leading to box or rectangular uncertainty sets [35]. For the next example, we need the following definition.

Definition 8 (Dominant Column)**.** Let c_i and c_{*j} be the ith row and jth column of the matrix C. A column c_{*j} is called dominant if, for every row index i, $|c_{ij}| \geq |c_{ij'}|$ for all j'. Moreover, if a dominant column $c_{*j} \geq 0$ (or ≤ 0), we call it a positive (or negative) dominant column.

Example 3 (L_1 Ball Uncertainty Set)**.** A unit L_1 ball in space $\mathbb{R}^n$ is defined as the convex hull of the set of points $\{\pm e_j\}_{j\in[n]}$ where $\{e_j\}_{j\in[n]}$ is the standard basis of $\mathbb{R}^n$. Suppose the jth column of matrix $\bar{C}$ is a positive (negative) dominant column, then all the rows in $\bar{C}$ are contained in $N_\Xi(e_j)$ ($N_\Xi(-e_j)$). By Corollary 4, Π is zero-adjustable. △

For some special dual polyhedron $\mathcal{U}$ and arbitrary uncertainty set Ξ, Π is zero-adjustable.

Example 4 (Simplex Dual Polyhedron)**.** A simplex is described by $\{u \in \mathbb{R}^k_+ \mid \sum_{i\in[k]} \alpha_i u_i = \alpha_0\}$ with positive coefficients α_i's. Suppose $\mathcal{U}$ is a simplex and the input parameter c is a zero vector, then Π is always zero-adjustable regardless of shape of the Ξ because every extreme point of $\mathcal{U}$ only has exactly one nonzero entry, say $u_i > 0$, which means the corresponding vector c_i and $c = 0$ trivially belong to the same normal cone. Then, Π is zero-adjustable according to the second statement of Corollary 4. △

This example presents a simple but interesting mirror result to the case where Ξ is a simplex — it is well known in the literature that affine policy is optimal for Π when Ξ is a simplex [13], and we just showed constant policy is optimal for Π is $\mathcal{U}$ is a simplex.

Conditions required by the above examples are somewhat restrictive since they allow the total flexibility of Ξ or/and $\mathcal{U}$. For specific problems where the exact descriptions of both ext(Ξ) and ext($\mathcal{U}$) are accessible, Theorem 2 can often be used to produce more interesting optimality conditions for the specific problem at hand. We provide an example in Appendix C.1, in which the zero-adjustability criteria has been derived for a policy problem where Ξ is a budgeted uncertainty set and $\mathcal{U}$ is the product of simplices. The same problem has been studied in [2] for analyzing the performance of affine policies.

4.3 Zero-Adjustability Under Affine Transformations

Zero-adjustability results can be preserved under certain affine transformations. Using this idea, for instance, we can extend the aforementioned examples to additional zero-adjustability cases, such as parallelotope uncertainty sets.

Our main result for zero-adjustability under affine transformation is the following corollary. Recall that we also view a matrix as the set of its rows; thus, cone($\bar{D}$) is the conic hull of the rows in $\bar{D}$.

Corollary 5. *Given a policy problem $\Pi = (\Xi, \mathcal{U}, \bar{C})$ that is zero-adjustable and two matrices $\bar{D}$ and R that satisfy $\bar{D}R \subseteq$ cone($\bar{C}$). Let ϕ be the affine transformation $\phi(\xi) = R\xi + \beta$ for any β, then, the transformed problem $\Pi' = (\phi(\Xi), \mathcal{U}, \bar{D})$ is also zero-adjustable if any of the following is satisfied:*

1. *Π satisfies Corollary 4;*
2. *there exists a symmetric-optimal solution $[\xi, u, u\xi^\intercal]$ of $\bar{\Delta}$ such that $u > 0$;*
3. *$\beta = 0$ and $DR = \lambda C$ for some $\lambda \geq 0$, where C and D are $\bar{C}$ and $\bar{D}$ without the first row.*

The proof is in Appendix B, where we also use this corollary to study zero-adjustability with respect to other uncertainty sets, such as parallelotopes and simplices.

Remark 3. In particular, for any scalar $\lambda > 0$, both $(\lambda I)\bar{C}$ and $\bar{C}(\lambda I)$ (viewed as the set of their rows) are contained in cone($\bar{C}$). Thus, when any of the three condition is satisfied for a zero-adjustable policy problem $\Pi = (\Xi, \mathcal{U}, \bar{C})$, the problem $\Pi' = (\phi'(\Xi), \mathcal{U}, \lambda I)$ with $\phi'(\xi) = \bar{C}\xi + \beta$ and $\Pi'' = (\phi''(\Xi), \mathcal{U}, \bar{C})$ with $\phi''(\xi) = \lambda\xi + \beta$ are also zero-adjustable for any translation vector β. The former implies "absorbing" $\bar{C}$ into the definition of the uncertainty set will not affect the zero-adjustability criterion, while the latter says zero-adjustability is preserved under any scaling and translation on the uncertainty set Ξ.

5 Adjustability Ratio

In this section, we examine the cases where the adjustability gap may be non-zero, *i.e.*, the adjustability ratio may not be one. In 5.1, we provide a constructive approach to characterize the adjustability ratio. In 5.2, we present an algorithmic procedure to estimate the adjustability ratio.

Recall that in the proof of Theorem 1, we have shown $z(\Pi) = z(\Delta^v) \le z(\bar{\Delta}) = z(\bar{\Pi})$. Thus, the objective value of the original problem $z(\Pi)$ can be lower bounded by solving Δ^v restricted to some subset of Ξ as the uncertainty set; the value of the constant policy problem $z(\bar{\Pi})$ can be upper bounded by solving $\bar{\Delta}_{\Xi'}$ with some polyhedron $\Xi' \supseteq \Xi$. When this superset is properly constructed to satisfy the condition of Theorem 2 or Corrolary 4, it becomes possible to estimate the adjustability ratio $\delta_{\text{rel}}(\Pi)$. In this section, Ξ is only assumed to be a closed set.

5.1 Bound on Adjustability Ratio

The following result provides a constructive way to bound $\delta_{\text{rel}}(\Pi)$.

Theorem 3. *Given a policy problem $\Pi = (\Xi, \mathcal{U}, \bar{C})$ where $z(\Pi) > 0$ (or $z(\Pi) < 0$), if there exist some polyhedron $\Xi' \supseteq \Xi$, some $\xi' \in \Xi'$, and a scalar $K \ge 1$ (or $0 < K \le 1$) that satisfy $\bar{C} \subseteq N_{\Xi'}(\xi')$ and $\xi'/K \in \Xi$, then, we have the bound $1 \le \delta_{rel}(\Pi) \le K$ (or $1 \ge \delta_{rel}(\Pi) \ge K$).*

Proof. Let Π' be the policy problem with input parameters $(\Xi', \mathcal{U}, \bar{C})$, and $\bar{\Pi}'$ be its corresponding value problem. Clearly, we have $z(\bar{\Pi}') \ge z(\bar{\Pi}) \ge z(\Pi)$, where the first inequality is due to $\Xi \subseteq \Xi'$. The premise also states that all rows of $\bar{C}$ belong to the same normal cone of Ξ', then according to Corrolary 4, constant policy family is optimal for Π'. That is, the corresponding bidual is symmetrically optimal. Thus, the optimal value is $z(\Pi') = z(\bar{\Pi}') = \max_{u \in \mathcal{U}} \langle c, \xi' \rangle + \langle C, u\xi'^{\intercal} \rangle$. On the other hand, we have $\xi'/K \in \Xi$. Thus, for any $u \in \mathcal{U}$, the solution $(\xi'/K, u, u\xi'^{\intercal}/K)$ is feasible to Δ^v — the dual of the value problem, which gives the following lower bound,

$$z(\Pi) = z(\Pi^v) = z(\Delta^v) \ge \max_{u \in \mathcal{U}} \langle c, \xi'/K \rangle + \langle C, u\xi'^{\intercal}/K \rangle = z(\bar{\Pi}')/K.$$

The first two equalities have been shown in the proof of Theorem 1 and are true for any closed set Ξ by Remark 2. Combining all these inequalities and $K > 0$, we get $Kz(\Pi) \ge z(\bar{\Pi}') \ge z(\bar{\Pi}) \ge z(\Pi)$. Finally, when $z(\Pi) > 0$ and $K \ge 1$, we have $Kz(\Pi) \ge z(\bar{\Pi}) \ge z(\Pi) > 0$; when $z(\Pi) < 0$ and $0 < K \le 1$, we also have $z(\Pi) \le z(\bar{\Pi}) \le Kz(\Pi) < 0$. In both cases, the adjustability ratio $\delta_{\text{rel}}(\Pi)$ is well-defined and can be computed directly, which gives to the desired bound K. □

Remark 4. Note that Theorem 3 does not require Ξ to be a polyhedron. In fact, Ξ can be an arbitrary closed set, such as a discrete, scenario-based uncertainty set, which can arise from practical/data-driven robust optimization problems [21, 43].

Theorem 3 provides a new perspective to understand similar results on adaptability gap [12, 19]. For example, we provide the following corollary that is consistent with the bounds in [12], followed by an example that is consistent with results in [19]. The following definition is from [12, 19]. ("Symmetry" used for defining uncertainty sets in [12, 19] is unrelated to symmetry-related definitions in the current paper.)

Definition 9 (Definition 1.2 and 1.3 in [12])**.** A set Ξ is symmetric, if there exists some $\xi_0 \in \Xi$, such that, for all $z \in \mathbb{R}^n$, $(\xi_0 + z) \in \Xi \Leftrightarrow (\xi_0 - z) \in \Xi$, where ξ_0 is called the point of symmetry of Ξ. A set $\Xi \subseteq \mathbb{R}^n_+$ is positive, if there exists some symmetric set $\Xi' \subseteq \mathbb{R}^n_+$ with point of symmetry ξ_0', such that $\xi_0' \in \Xi \subseteq \Xi'$.

Corollary 6. *Given a policy problem $\Pi = (\Xi, \mathcal{U}, \bar{C})$ that satisfies $z(\Pi) > 0$, $\Xi \subseteq \mathbb{R}^n_+$, and $\bar{C} \geq 0$,*

1. *$\delta_{rel}(\Pi) = 1$ when Ξ is a hypercube,*
2. *$\delta_{rel}(\Pi) \leq 2$ when Ξ is symmetric or positive,*
3. *$\delta_{rel}(\Pi) \leq n$ when Ξ is an arbitrary convex set.*

Corollary 6 demonstrates one way to extend Theorem 3 for cases where $\bar{C} \geq 0$, and has a tighter bound n than the $O(n)$ bound in [12]. Other extensions can be shown for different $\bar{C}$. The main idea is that, for a certain $\bar{C}$, we first construct a polyhedron Ξ' such that all rows of $\bar{C}$ lead to the same extreme point in Ξ'. Then, we use some scaled-translated version of Ξ' to contain the set Ξ. By Corollary 5 and Remark 3, zero-adjustability is invariant under scaling and translation. Then, we can use such Ξ' to estimate $\delta_{\text{rel}}(\Pi)$ with the aid of Theorem 3. We use the next example to illustrate this idea.

Example 5 (Bounds for Ellipsoidal Uncertainty Sets; $\bar{C} \geq 0$)**.** An ellipsoidal uncertainty set can be defined as $\Xi = \{\xi \mid \sum_{i\in[n]} \xi_i^2/l_i^2 \leq 1\}$ for some $l = (l_i)_{i\in[n]} > 0$. We take Ξ' as the box set that circumscribes Ξ, *i.e.*, $\Xi' = \prod_{i\in[n]}[-l_i, l_i]$. Because $\bar{C} \geq 0$, all the row vectors of $\bar{C}$ belong to the normal cone $N_{\Xi'}(l)$. Then, the intersection point between the line segment $[0, l]$ and the b undary $\partial\Xi = \{\xi \mid \sum_{i\in[n]} \xi_i^2/l_i^2 = 1\}$ can be directly computed as $l/\sqrt{n}$. Applying Theorem 3, we get $\delta_{\text{rel}}(\Pi) \leq \sqrt{n}$ for ellipsoids that are centered at the origin.

The same technique can be used to derive closed-form bounds for translated and/or rotated ellipsoidal uncertainty sets. For instance, for a given ellipsoidal set $\Xi = \{\xi \mid \sum_{i\in[n]} \xi_i^2/l_i^2 \leq 1\}$, let x be the intersection point of the line segment $[0, l]$ and the boundary of Ξ, we can easily derive a tight bound of δ_{rel} for the translated ellipsoidal set $\Xi' = \{\xi \mid \sum_{i\in[n]}(\xi_i - \lambda l_i)^2/l_i^2 \leq 1\}$ for some $\lambda \geq 0$. We construct the tightest box set whose maximal point is $\lambda l + l$ to enclose Ξ'. Then, we get the following bound

$$\delta_{\text{rel}} \leq \frac{\|\lambda l\|_2 + \|l\|_2}{\|\lambda l\|_2 + \|x\|_2} \leq \frac{\|\lambda l\|_2 + \|l\|_2}{\|\lambda l\|_2 + \frac{\|l\|_2}{\sqrt{n}}} = \frac{(\lambda+1)\sqrt{n}}{\lambda\sqrt{n}+1} = 1 + \frac{\sqrt{n}-1}{\lambda\sqrt{n}+1},$$

where the first inequality is obtained by direct computation (notice λl, l, and x are in the same direction), the second is due to the result $\|l\|_2/\|x\|_2 \leq \sqrt{n}$ from the above example. △

In Appendix C.2, we also apply Theorem 3 to other uncertainty sets and input matrix $\bar{C}$. In particular, when Ξ is ellipsoidal and the jth column of $\bar{C}$ is a dominant column, δ_{rel} is bounded by $\|l\|_2/(2l_j)$; when Ξ is the budgeted set $\{\xi \in [-1, 1]^m \mid \|\xi\|_1 \leq \Gamma\}$, δ_{rel} is bounded by n/Γ given $\bar{C} \geq 0$, and is bounded by Γ given $\bar{C}$ has a dominant column.

5.2 Algorithmic Estimation of Adjustability Ratio

In the previous examples, we have shown that Theorem 3 can be used to analytically study the bounds for the adjustability ratio δ_{rel}. In this subsection, we formalize this idea into a numerical method, called the *anchor cone algorithm*, that produces the tightest bound for δ_{rel} accordingly. Throughout this subsection, we assume that the problem $\max_{\xi\in\Xi}\langle c, \xi\rangle$ can be efficiently solved for every possible vector c or there exists an oracle.

Anchor cone is a special class of polyhedrons that we choose to construct Ξ' in Theorem 3 for producing a valid bound. We define it as follows.

Definition 10 (Anchor Cone)**.** Given a finite set of vectors $\mathcal{C} = \{c_i\}_{i\in L}$ and a point x_0, we define the corresponding anchor cone as $\mathfrak{A}_{\mathcal{C},x_0} := \{x \mid \langle c_i, x\rangle \le \langle c_i, x_0\rangle,\ \forall i \in L\}$ where x_0 is called the *anchor* of $\mathfrak{A}_{\mathcal{C},x_0}$.

By this definition, an anchor cone is a convex set constructed by anchoring a cone at x_0. This constitutes a more liberal use of the concept of *cone* than conventionally done, since our "cone" may not be anchored at the origin. We nonetheless keep this name for its geometric intuition. It has several interesting properties that will be used later. For cone($\mathcal{C}$), we use cone$^*(\mathcal{C})$ and cone$^\circ(\mathcal{C})$ to denote the corresponding dual and polar cones.

Proposition 4. *Every anchor cone $\mathfrak{A}_{\mathcal{C},x_0}$ has the following properties:*

1. $\mathfrak{A}_{\mathcal{C},x_0} = \{x_0\} + cone^\circ(\mathcal{C})$;
2. $N_{\mathfrak{A}_{\mathcal{C},x_0}}(x_0) = cone(\mathcal{C})$;
3. *constraints of $\mathfrak{A}_{\mathcal{C},x_0}$ that correspond to vectors in eray($\mathcal{C}$) are sufficient to define $\mathfrak{A}_{\mathcal{C},x_0}$.*

Take J as the index set for eray($\bar{C}$) and let $\omega_j := \max_{\xi\in\Xi}\langle c_j, \xi\rangle$ for every $j \in J$, then, the *anchor cone formulation* is defined as,

$$\min_{\gamma,x,\xi\in\Xi} (\text{or } \max) \quad \gamma \tag{10a}$$
$$\text{s.t.} \quad \langle c_j, x_k\rangle \le \langle c_j, \gamma\xi\rangle, \quad \forall j, k \in J, \tag{10b}$$
$$\langle c_j, x_j\rangle \ge \omega_j, \quad \forall j \in J, \tag{10c}$$
$$\gamma \ge 1 \ (\text{or } \gamma \le 1), \ \ x \text{ free}. \tag{10d}$$

The minimization with $\gamma \ge 1$ and maximization with $\gamma \le 1$ are designed for the two cases $z(\Pi) > 0$ and $z(\Pi) < 0$, respectively. The main idea of this formulation is to search for an element $\xi \in \Xi$ and an optimized positive scalar γ such that the anchor cone $\mathfrak{A}_{\bar{C},\gamma\xi}$ contains Ξ. The first constraint set says that every x_k belongs to the anchor cone $\mathfrak{A}_{\bar{C},\gamma\xi}$. The second constraint set ensures that $\mathfrak{A}_{\bar{C},\gamma\xi} \supseteq \Xi$, since, according to the last property in Proposition 4, the constraints labeled by J are sufficient to define $\mathfrak{A}_{\bar{C},\gamma\xi}$. Constraint set (10c) lets every such constraint contain Ξ, so the anchor cone $\mathfrak{A}_{\bar{C},\gamma\xi}$ also contains Ξ.

The following two theorems indicate that solving the anchor cone formulation produces not only a valid bound but also the tightest bound of δ_{rel} under Theorem 3.

Theorem 4. *Given a policy problem $\Pi = (\Xi, \mathcal{U}, \bar{C})$, let γ be any feasible solution of the corresponding anchor cone formulation. We have $\delta_{rel}(\Pi) \le \gamma$ when $z(\Pi) > 0$ and $\delta_{rel}(\Pi) \ge \gamma$ when $z(\Pi) < 0$.*

Algorithm 1: Calculating optimal bound for $\delta_{\text{rel}}(\Pi)$ given $z(\Pi) > 0$.

Data: Problem data $\Xi, \mathcal{U}, \bar{C}$; solution tolerance ϵ; M; number of steps T
initialization: $t \leftarrow 1$, $\underline{\gamma} \leftarrow 1$, $\overline{\gamma} \leftarrow M$, `hasSolution` = FALSE;
while $t < T$ *and* $\overline{\gamma} - \underline{\gamma} > \epsilon$ **do**
  $\gamma \leftarrow (\overline{\gamma} + \underline{\gamma})/2$;
  if *Anchor Cone Formulation is feasible with* γ **then**
    $\overline{\gamma} \leftarrow \gamma$; `hasSolution` $\leftarrow$ TRUE;
  else
    $\underline{\gamma} \leftarrow \gamma$
  $t \leftarrow t + 1$;
return γ, `hasSolution`;

Proof. When $\bar{C}$ is full-rank, the anchor cone $\mathfrak{A}_{\bar{C},\gamma\xi}$ has the unique extreme point $\gamma\xi$. By the second property in Proposition 4, all vectors in $\bar{C}$ lead to $\gamma\xi$. Then, a direct application of Theorem 3 proves the claim. When $\bar{C}$ is not full-rank, the anchor cone $\mathfrak{A}_{\bar{C},\gamma\xi}$ does not have any extreme point. However, the uncertain sets of both Π and $\bar{\Pi}'$ can be projected without changing the corresponding optimal objective values. That is, $z(\Pi)$ and $z(\bar{\Pi}')$ will not change if we replace Ξ and $\Xi' = \mathfrak{A}_{\bar{C},\gamma\xi}$ with $\text{proj}_{\bar{C}}(\Xi)$ and $\text{proj}_{\bar{C}}(\Xi')$ where $\text{proj}_{\bar{C}}(\cdot)$ projects the input set onto the subspace spanned by $\bar{C}$. Then, it is easy to see that $\text{proj}_{\bar{C}}(\Xi')$ has the unique extreme $\gamma\xi$ where all the vectors in $\bar{C}$ are leading to. Thus, we can still apply Theorem 3 to prove the bound γ. □

Theorem 5. *Given a policy problem $\Pi = (\Xi, \mathcal{U}, \bar{C})$, let γ be the optimal value of* (10) *and let K be any bound calculated using Theorem 3 with a polyhedron Ξ', then, γ is a tighter bound than K.*

Proof. We will show that every such Ξ' corresponds to a feasible solution $(\gamma_0, \xi, \{x_j\}_{j\in J})$ for Formulation (10). Because $\Xi' \supseteq \Xi$, we can find $\{x_j\}_{j\in J} \subseteq \Xi'$ that satisfy constraint set (10c). Let $\xi' \in \Xi'$ be the extreme point that all vectors in $\bar{C}$ lead to. According to Theorem 3, $\xi'/K \in \Xi$. Let $\gamma_0 = K$ and $\xi = \xi'/K$, we have $\xi' = \gamma_0\xi$. Since all vectors in $\bar{C}$ are in the normal cone $N_{\Xi'}(\gamma_0\xi)$ and $\{x_j\}_{j\in J} \subseteq \Xi'$ by selection, we have $\langle c_j, x_k - \gamma_0\xi\rangle \le 0$ for all $j, k \in J$ by the definition of normal cone, which is the same as constraint set (10b). Thus, $(K, \xi'/K, \{x_j\}_{j\in J})$ is a feasible solution of Formulation (10). This concludes the proof. □

Notice that Formulation (10) is nonlinear due to the term $\gamma\xi$. However, we can still solve it efficiently using a line search algorithm on γ (see Algorithm 1) where at each iteration, only a feasibility check is required. This linear search algorithm is justified since given $z(\Pi) > 0$ ($z(\Pi) < 0$), the anchor cone $\mathfrak{A}_{\bar{C},\gamma\xi}$ is increasing (decreasing) on γ under the inclusion relation $\subseteq$. Therefore, the complexity of solving Formulation (10) mainly depends on the property of Ξ. For instance, when Ξ is a polyhedron, Formulation (10) with a fixed γ is a linear program with $O(n \times |\text{eray}(\bar{C})|)$ variables and $O(|\text{eray}(\bar{C})|^2)$ constraints; when Ξ is convex, it is a convex optimization with the same size.

The following two propositions provide the feasibility/infeasibility criteria for Formulation (10).

Proposition 5. *Given Ξ is bounded, Formulation* (10) *is feasible if either $\Xi \cap int(cone^*(\bar{C})) \neq \varnothing$ or $\Xi \subseteq int(cone^\circ(\bar{C}))$.*

Implicitly, the first condition is associated with the case $z(\Pi) > 0$, while the second is for $z(\Pi) < 0$. These two conditions are not necessary. Hence, even this proposition is violated, Formulation (10) may still be feasible, in which case it can produce a valid bound for the adjustability ratio.

Proposition 6. *Formulation* (10) *is infeasible if* $\dim(\Xi) + \dim(-\mathit{cone}(\bar{C}) \cap \mathit{cone}(\bar{C})) > n$.

The main tool for this proof is the equality $\dim(\text{cone}^\circ(\bar{C})) = n - \dim(\hat{\mathcal{C}})$ where $\hat{\mathcal{C}}$ denotes $-\text{cone}(\bar{C}) \cap \text{cone}(\bar{C})$. Thus, when vectors in $\bar{C}$ cannot lie in some halfspace, we have $\text{cone}^\circ = \{0\}$, which means the anchor cone is a single point. On the other hand, when vectors in $\bar{C}$ lie in the interior of some halfspace, $\hat{\mathcal{C}} = \{0\}$, which implies the anchor cone is full-dimensional.

Finally, we finish the section with a special case where the anchor cone formulation has an analytical solution to illustrate its usage.

Example 6 ($\text{cone}(\bar{C}) = \mathbb{R}^n_+$ and $z(\Pi) > 0$)**.** In this case, $\text{cone}(\bar{C})$ is self-dual. Thus, for any $\xi \geq 0$, the anchor cone $\mathfrak{A}_{\bar{C},\xi}$ is simply obtained by removing all the lower bounds from a box set, which leaves the only extreme point ξ. Then, Formulation (10) reduces to

$$\min_{\gamma \geq 1, x, \xi \in \Xi} \quad \gamma \tag{11a}$$

$$\text{s.t.} \quad x^k_j \leq \gamma \xi_j \quad \forall j, k \in J, \tag{11b}$$

$$x^j_j \geq \bar{\xi}_j \quad \forall j \in J. \tag{11c}$$

where $\bar{\xi}_j = \max_{\xi \in \Xi} \xi_j$ is the maximum value of the jth entry of ξ. Clearly, the variables x can be further reduced. The feasibility of the resulting formulation depends on Ξ. Suppose Ξ contains some element $\xi > 0$, this formulation is always feasible and bounded, which can be further reduced to

$$\min_{\xi \in \Xi} \max_j \frac{\bar{\xi}_j}{\xi_j} \iff \left(\max_{\xi \in \Xi} \min_j \frac{\xi_j}{\bar{\xi}_j} \right)^{-1}.$$

Depending on the uncertainty set Ξ, this can be calculated directly (see examples in Section 6). Or, we can simply conduct a line search on γ as before. △

6 Applications in Dynamic Robust Optimization

Having derived sharp results for adjustability gap and ratio in settings more general than previously studied, we provide several applications to demonstrate new ways that the concept of adjustability can be applied in practice. In this section, we study a maximum flow interdiction game and a robust supply chain design problem. An additional example regarding minimax inventory control is provided in Appendix C.3.

6.1 Maximum Flow Interdiction

Given a network $G = (V, E)$ with a source node $s \in V$ and a sink node $t \in V$, the maximum flow problem seeks to push the maximum amount of s-t flow through G with flow capacity $\{\bar{\alpha}_e\}_{e \in E}$ defined on the edges. In turn, the maximum flow interdiction game is an adversarial Stackelberg game between two players: an attacker that interdicts the edges to inflict deterioration on the network (flow capacity is reduced by $\alpha_e \xi_e$, for every $e \in E$ and some attack strength α), and a defender who then constructs the maximum flow on the resulting network [49].

We change the defender's objective to minimization with a negative sign on the cost vector to be consistent with the convention in this paper. Thus, $z(\Pi)$ is non-positive. We assume $z(\Pi) < 0$.

$$\max_{\xi \in [0,1]^{|E|}} \min_{y \geq 0} \quad \langle -w, y \rangle$$

$$
\begin{aligned}
\text{s.t.} \quad & My = 0, \\
& y + \text{diag}(\alpha)\xi \leq \bar{\alpha}, \\
& \langle 1, \xi \rangle \leq \Gamma.
\end{aligned}
$$

Vector $w \geq 0$ denotes the weights or rewards assigned on the flow, M is the directed incidence matrix of the underlying graph where entry $M_{i,e} = 1$ (or -1) if edge e enters (or leaves) the node i, diag(α) is a diagonal matrix generated from the attack strength vector α, vector $\bar{\alpha}$ represents the flow capacities, and $\Gamma \in \mathbb{N}$ is the budget for the attacker. This formulation can be viewed as a value problem in the form of Formulation (3). Hence, adjustability of the corresponding policy problem Π measures the difference between deciding the optimal flow strategy before and after the interdiction. We can extract the following input parameters (Ξ, a, A, c, C, β),

$$
\begin{aligned}
&\Xi := \{\xi \in [0,1]^{|E|} \mid \langle 1, \xi \rangle \leq \Gamma\}, && a := -w, && A := [-M; M; -I; I], \\
&c := 0, && C := [0; 0; \text{diag}(\alpha); 0], && \beta := (0, 0, -\bar{\alpha}, 0),
\end{aligned}
$$

where β represents the constant part in the constraint set that is independent of ξ. This extra input parameter β is not included in the main Formulation (1). To obtain the desired form, we can "absorb" both $\bar{C}$ and β into the definition of the uncertainty set, *i.e.*, we define $\bar{C}_0 := I$ and $\Xi_0 := \{\bar{C}\xi + \bar{\beta} \mid \xi \in \Xi\}$ where $\bar{\beta} = (0, \beta)$ is obtained by stacking the constant term (zero in this case) in the objective function to β. It can be easily verified that this will not affect $z(\Pi)$ or $z(\bar{\Pi})$. Because of the translation vector β, the uncertainty set Ξ_0 is not contained in $\mathbb{R}^n_+$. Thus, previous results from the literature [12, 19] cannot inform adjustability analysis for this problem.

First, we study when this gap is zero. According to Corollary 5 and Remark 3, the above transformation on Ξ and $\bar{C}$ preserves the zero-adjustability. Thus, we can still focus on the optimality conditions of the original input parameters Ξ and $\bar{C}$. In the trivial cases, when the budget $\Gamma = 0$ or $\Gamma \geq |E|$, the uncertainty set Ξ is a box set, in which case $\delta_{\text{abs}}(\Pi) = 0$ by Corollary 6. Otherwise, it can be shown that the extreme points of Ξ form the set $\{\xi \in \{0,1\}^{|E|} \mid \langle 1, \xi \rangle \leq \Gamma\}$. Clearly, in this case, all cost vectors in $\bar{C}$ lead to the same extreme point of Ξ if the vector α has at most Γ nonzero entries. Thus, we have the following conclusion.

Proposition 7. *For the maximum flow interdiction, $\delta_{abs}(\Pi) = 0$ if either (i) $\Gamma = 0$ or $\Gamma \geq |E|$, or (ii) there are at most Γ nonzero entries in vector α.*

Other than such special cases, the adjustability gap might be nonzero. Theorem 3 gives us a way to estimate the adjustability ratio. Since $\bar{C}_0 = I$, we can take a box set Ξ' to enclose Ξ_0. However, because Ξ_0 is transformed from Ξ using $\bar{C}$ and $\bar{\beta}$, it is difficult to obtain a closed-form expression directly from Theorem 3. Instead, we resort to the anchor cone algorithm. We can still use Formulation (10) by replacing ξ in constraint (10b) with $(\bar{C}\xi + \bar{\beta})$ and all the c_j's with the standard basis e_j's. In particular, since $\bar{C}_0 = I$ and Ξ' is a box set, and we know $z(\Pi) < 0$, we can solve for this value using the maximization version of Formulation (11) that was designed for the case $z(\Pi) < 0$. After replacing all the parameters and simplication, we get

$$
\begin{aligned}
\max_{\gamma \in [0,1], \xi \in \Xi} \quad & \gamma \\
\text{s.t.} \quad & \gamma(\alpha_e \xi_e - \bar{\alpha}_e) \geq \alpha_e - \bar{\alpha}_e, \quad \forall e \in E.
\end{aligned}
$$

For any $e \in E$ such that $\alpha_e = 0$, the corresponding constraint reduces to $\gamma \leq 1$, which is redundant. We use $E_0 \subseteq E$ to denote all edges with $\alpha_e = 0$. For any $e \in E$ such that $\alpha_e = \bar{\alpha}_e > 0$, the associated constraint becomes $\gamma(\xi_e - 1) \geq 0$, which is feasible if and only if either $\lambda = 0$ or $\xi_e = 1$. Because the objective function is to maximize λ, the formulation will try to adjust ξ to avoid λ being zero. Let $\bar{E} := \{e \in E \mid \alpha_e = \bar{\alpha}_e \text{ and } \bar{\alpha}_e > 0\}$. Then, when $|\bar{E}| > \Gamma$, the budget is not sufficient for satisfying all the constraints with a nonzero γ, so γ has to be zero; when $|\bar{E}| \leq \Gamma$, for each $e \in \bar{E}$, ξ_e is assigned a unit and the corresponding constraint becomes redundant, which is equivalent to the case where the budget equals $\Gamma - |\bar{E}|$ and the constraints are labeled by the edge set $E \setminus (E_0 \cup \bar{E})$. Then, the above formulation can be reduced to the following,

$$\gamma = \left(\min_{\xi \in \Xi_{\Gamma - |\bar{E}|}} \max_{e \in E \setminus (E_0 \cup \bar{E})} \frac{\bar{\alpha}_e - \alpha_e \xi_e}{\bar{\alpha}_e - \alpha_e} \right)^{-1},$$

where $\Xi_{\Gamma - |\bar{E}|}$ is the budgeted set with a budget of $\Gamma - |\bar{E}|$. The problem in brackets can be solved as a linear program since $\Xi_{\Gamma - |\bar{E}|}$ is a polytope.

6.2 Preposition-Reallocation in Supply Chain

Simchi-Levi et al. [46] studied a multi-stage resource preposition-reallocation problem on a network $G = (V, E)$. At the first stage, the defender determines a preposition plan b and a reallocation policy $y(\cdot)$ (a function over the uncertainty set Ξ). Then, an attacker executes an attack strategy ξ to cause the surge of demands over the supply chain network. In the final stage, the defender executes the policy $y(\xi)$ to reallocate resources. If one allows the prepositioning decisions to be adjustable as well, we arrive at the following fully adjustable formulation.

$$\min_{s(\cdot) \geq 0, y(\cdot) \geq 0, b(\cdot) \geq 0} \max_{\xi \in \Xi} \quad \sum_{i \in V} w_i^1 s_i(\xi) + w_i^2 y_{\mathcal{P}(i)i}(\xi) + w_i^3 b_i(\xi) \tag{12a}$$

$$\text{s.t.} \quad s_i(\xi) + y_{\mathcal{P}(i)i}(\xi) + b_i(\xi) \geq d_i \xi_i + y_{i\mathcal{C}(i)}(\xi), \quad \forall \xi \in \Xi, \ \forall i \in V, \tag{12b}$$

$$y_{\mathcal{P}(i)i}(\xi) + b_i(\xi) \geq y_{i\mathcal{C}(i)}(\xi), \quad \forall \xi \in \Xi, \ \forall i \in V. \tag{12c}$$

where b is the resource preposition policy, y is the resource reallocation policy, and s measures demand loss. On each node $i \in V$, $d_i \xi_i$ is the demand inflicted on node i given attacker's strategy is ξ_i. $\mathcal{P}(i)$ and $\mathcal{C}(i)$ are the sets of parents and children for node i respectively; $y_{\mathcal{P}(i)i}$ is the sum of resources coming from parents of i into i, and $y_{i\mathcal{C}(i)}$ is the sum of resources allocated away from i. The first constraint set says the resources transferred to node i and the prepositioned ones plus the deficit have to cover the demand of node i plus the resources allocated away from i. The second constraint set is flow balance at node i. Finally, the objective is to minimize the sum of deficit cost, preposition cost, and reallocation cost.

For uncertainty, Simchi-Levi et al. [46] used a generalization of budgeted uncertainty set,

$$\Xi := \left\{ \xi \in \mathbb{R}^{|V|} \mid \xi_0 = 1, \ 0 \leq \xi_i \leq \xi_{p(i)} \ \forall i \in V \setminus \{0\}, \ \sum_{j \in \mathcal{C}(i)} \xi_j \leq \Gamma_i \xi_i \ \forall i \in V \right\},$$

where Γ_i is a positive integer budget for the attacker at node i, $p(i)$ is the main parent of i (*e.g.*, from a tree structure that can represent the federal-state-city hierarchy). So, this generalizes the traditional budgeted uncertainty set into a hierarchical budgeted uncertainty set. One can easily check that if there is only one level, this set reduces to a traditional budgeted uncertainty set. In [46], the authors proved that ext(Ξ) are all binary. In fact, it can be verified that ext(Ξ) consists of all the indicator vectors of rooted trees (with node 0 as the root) in this tree that satisfy the hierarchical budget requirement. We will call these the *budgeted trees* hereafter.

Formulation (12) fits the general policy problem (1). We can extract the input parameters $c := 0$ and $C := [\text{diag}(d); 0]$. With this and the description of Ξ, we can study adjustability of this problem. According to Corollary 4, $\delta_{\text{abs}}(\Pi) = 0$ if all rows in $\bar{C}$ are in the same normal cone $N_\Xi(\xi)$ for some $\xi \in \text{ext}(\Xi)$. In this case, every row of $\bar{C}$ is either a zero vector or equal to $d_i e_i$ where $d_i \geq 0$ and e_i is the ith standard unit vector. We have two trivial cases: $\Gamma_0 = 0$ or $\Gamma_i \geq |\mathcal{C}(i)|$ for all $i \in V$. For the former, Ξ only contains a single point; for the latter, $\xi = 1$ is feasible to Ξ. In both cases, it is clear that constant policy is optimal. Otherwise, all the vectors in $\bar{C}$ lead to the same extreme point if the set of nodes $V(d)$ that corresponds to the nonzero entries of d is contained in some budgeted tree. Thus, we have the following,

Proposition 8. *In Formulation (12), $\delta_{abs}(\Pi) = 0$ if either (i) $\Gamma_0 = 0$ or $\Gamma_i \geq |\mathcal{C}(i)|$ for all $i \in V$, or (ii) there is a budgeted tree that contains all the nodes in $V(d)$.*

This means, in practice, suppose we know that the nodes facing threats are within the attacker's budget, then implementing the optimal preposition-reallocation plan before the attack is as good as acting afterward (in terms of the worst attack).

When none of the above situations is satisfied, the adjustability gap may be nonzero, and we can use Theorem 3 to estimate $\delta_{\text{rel}}(\Pi)$. First, notice that Ξ is a convex set in the nonnegative orthant; thus, a trivial bound $|V|$ can be obtained directly from Corollary 6. To obtain the tightest bound under Theorem 3, we use the anchor cone method. According to Example 6, we can compute this bound by computing $(\max_{\xi \in \Xi} \min_j \xi_j)^{-1}$. Because Ξ is defined based on a rooted tree structure, the denominator can be calculated recursively. We provide the resulting bound γ in the following proposition.

Proposition 9. *Given a preposition-reallocation problem formulated by (12) that satisfies $0 < \Gamma_i < |\mathcal{C}(i)|$ for all $i \in V$, let node 0 and set $\mathcal{L}$ denote the root node and the set of leaf nodes in the underlying rooted tree of Ξ. Then, the adjustability ratio obtained from the anchor cone algorithm equals $\gamma = 1/f_0$ where f_0 is computed recursively as $f_i = 1$ for all $i \in \mathcal{L}$ and $f_i = H\left(f_{\mathcal{C}(i)}\right) \frac{\Gamma_i}{|\mathcal{C}(i)|}$ for all $i \notin \mathcal{L}$, where $f_{\mathcal{C}(i)}$ is the set $\{f_j\}_{j \in \mathcal{C}(i)}$ and $H(\cdot)$ is the harmonic mean of the input values.*

Thus, for each non-leaf node i, value f_i is the harmonic mean of values of the child nodes weighted by $\Gamma_i/|\mathcal{C}(i)|$, which we call the *vulnerability ratio* of node i. There are several implications of this proposition. First, by its properties, the harmonic mean tends strongly toward the smallest elements in the input. If the decision-maker prefers a supply chain with a low degree of adjustability, it is beneficial to spread the children nodes so that the vulnerability ratio $\Gamma_i/|\mathcal{C}(i)|$ is the same across all nodes on the same level. By this design, the adjustability ratio is bounded by $\prod_{l \in L} |\mathcal{C}_l|/\Gamma_l$ where L contains all the levels except the root node and $|\mathcal{C}_l|/\Gamma_l$ is the associated ratio at level l. Second, suppose the decision-maker can change the level of nodes without affecting the value of Γ_l's, then making the ratio $|\mathcal{C}_l|/\Gamma_l$ the same for all levels also decreases the bound of δ_{rel}. Finally, a tree with a smaller depth has a smaller value of δ_{rel}. In conclusion, in a shallow hierarchy with evenly spread vulnerability ratios, the objective cost of committing to a fixed preposition-reallocation plan is close to the cost from fully adjustable decisions.

7 Conclusion

In this paper, we study two related problems: the static robust optimization problem where decisions have to be made before observing uncertainty realization, and the fully adjustable robust optimization version where decisions can be made after uncertainty realization. For every pair of such problems, we use the concepts of adjustability gap and ratio to quantify the difference in their objective values.

Previous literature on this topic provided interesting but restrictive findings. The most general work to date had some key assumptions, for example, requiring the right hand side to be positive and the constraints to have only the "greater than or equal to" direction. These assumptions precluded the study of many problems, such as network optimization with arc capacities and resource allocation problems with budget constraints.

In this work, we drop such assumptions and develop a new theoretical framework to analyze and quantify the adjustability gap and ratio, enabled by our discovery of several equivalent reformulations of the static and adjustable problems (Section 2) and an interesting algebraic property (Theorem 1 in Section 3) that bridges them together. This allows us to provide a necessary and sufficient condition (Theorem 2 in Section 4) for the adjustability gap to be zero. This sharp result allows us to use constructive approaches to characterize (Theorem 3 in Section 5) and efficiently approximate (Algorithm 1 in Section 5) the adjustability ratio. The ability to efficiently analyze arbitrary problem setups proves to be practically valuable. For example, Section 6 provides two network optimization problems where positive/negative coefficients and equality constraints are present.

For future work, it would be interesting to examine if Theorem 3 is, in general, the "optimal" way to bound adjustability ratio given an arbitrary problem setup. If the answer is affirmative, then one could examine more specialized settings (such as the network examples in Section 6) to derive context-specific managerial and policy insights. In addition, the theoretical framework and new proof techniques may be of interest for researchers that work on areas mentioned in Section 1.

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A Mathematical Proofs

Lemma 1. *For a given policy problem Π and its value problem Π^v, we have $z(\Pi) = z(\Pi^v)$.*

Proof. For notational simplicity, we define

$$\begin{aligned} f(\xi, y) &= \langle c, \xi \rangle + \langle a, y \rangle, \\ \mathcal{Y}_\xi &= \{y \in \mathcal{Y} \mid C\xi - Ay \le 0\}, \\ \mathcal{Y}_\Xi &= \{y \in \mathcal{X}^{\mathcal{Y}} \mid C\xi - Ay(\xi) \le 0,\ \forall \xi \in \Xi\}. \end{aligned}$$

With these notations, the problems Π and Π^v can be written as $\min_{y \in \mathcal{Y}_\Xi} \max_{\xi \in \Xi} f(\xi, y(\xi))$ and $\max_{\xi \in \Xi} \min_{y \in \mathcal{Y}_\xi} f(\xi, y)$, respectively.

First, the policy $\xi \mapsto \arg\min_{y \in \mathcal{Y}_\xi} f(\xi, y)$ belongs to the space $\mathcal{Y}^{\mathcal{X}}$. Thus, $z(\Pi) \le z(\Pi^v)$. On the other hand, suppose $z(\Pi) < z(\Pi^v)$, we use $y_1^\star \in \mathcal{Y}_\Xi, \xi_1^\star \in \Xi$ and $\xi_2^\star \in \Xi, y_2^\star \in \mathcal{Y}_{\xi_2^\star}$ for some optimal solutions of Π and Π^v, respectively. Then, we have

$$f(\xi_2^\star, y_1^\star(\xi_2^\star)) \le f(\xi_1^\star, y_1^\star(\xi_1^\star)) < f(\xi_2^\star, y_2^\star),$$

where the first inequality is from the fact that $\xi_1^\star$ is a maximizer in Π for the fixed $y_1^\star$, and the second is due to the assumption $z(\Pi) < z(\Pi^v)$. By definition of $\mathcal{Y}_\Xi$, we know that $y_1^\star(\xi_2^\star) \in \mathcal{Y}_{\xi_2^\star}$. Thus, $y_1^\star(\xi_2^\star)$ is a better choice than $y_1^\star$, a contradiction. □

Proposition 1. *In Assumption 1.2, feasibility implies (i) for every $\xi \in \Xi$ and $u \in \mathbb{R}_+^k$, if $A^\intercal u = 0$ then $\langle C\xi, u \rangle \le 0$; (ii) for every $\xi \in$ cone(Ξ), $\langle c_i, \xi \rangle \le 0$. Boundedness entails (i) there exists some $u \in \mathbb{R}_+^k$ such that $A^\intercal u = a$; (ii) for every $\xi \in$ cone(Ξ), $\langle c, \xi \rangle \le 0$.*

Proof. For simplicity, we use $\mathcal{Y}_\xi := \{y \in \mathcal{Y} \mid Ay \ge C\xi\}$ to denote the solution space relative to a fixed $\xi \in \Xi$. From the perspective of the value problem Π^v, feasibility means that for every $\xi \in \Xi$, there is a feasible y. This implies two things. First, the dual of the inner problem $\min_{y \in \mathcal{Y}_\xi} f(\xi, y)$ has a bounded objective value for every $\xi \in \Xi$. The rays of the dual problem is the set $\{u \in \mathbb{R}_+^k \mid A^\intercal u = 0\}$, which proves the first statement for feasibility. Second, suppose $\langle c_i, \xi \rangle > 0$ for some $i \in [k]$ and some $\xi \in$ cone(Ξ), then, for every possible y, there always exists a sufficiently large $\lambda > 0$ such that $\lambda\xi \in$ cone(Ξ) violates the corresponding constraint in (3b). This contracdicts the feasibility assumption. This proves the second statement regarding feasibility.

The boundedness of Π^v also implies two things. First, there exists some $\xi \in \Xi$ such that the optimal objective value of the inner problem is bounded. Then, the corresponding dual feasibility yields the first statement of boundedness. Second, suppose $\langle c, \xi \rangle > 0$ for some $\xi \in$ cone(Ξ), then the term in the objective function (3a) is unbounded. This concludes the proof. □

Corollary 1. *Suppose for some closed set $\Xi' \supseteq \Xi$, the problem $\max_{\xi' \in \Xi'} \langle c_i, \xi' \rangle - \langle a_i, y \rangle$ has a dual representation that satisfies weak duality, then, we have $z(\bar{\Pi}) \le z(\bar{\Delta}_{\Xi'})$, where $\bar{\Delta}_{\Xi'}$ is the problem obtained using bidualization procedure with uncertainty set Ξ'.*

Proof. Notice that both the relaxation $\Xi' \supseteq \Xi$ and the optimality gap induced by the dualization impose a set of constraints that are stronger than (5b) for the space of y, which may increase the objective value. On the other hand, $\max_{\xi' \in \Xi'} \langle c, \xi' \rangle$ in the objective function (5a) also affects the optimal value in the same direction. Moreover, strong duality holds in the second dualization step under the feasibility and boundedness assumption. Then, by following a similar argument as the proof of Lemma 2, we obtain the desired inequality. □

Corollary 3. Π *is zero-adjustable if and only if there exists a symmetric-optimal solution* $(\xi^\star, u^\star, u^\star\xi^{\star\intercal})$ *of the bidual* $\bar{\Delta}$ *such that* $\xi^\star \in ext(\Xi)$ *and* $u^\star \in ext(\mathcal{U})$.

Proof. Corollary 2 establishes the "if" part. We only need to show the "only if" part. Given a symmetric-optimal solution $(\xi^\star, u^\star, u^\star\xi^{\star\intercal})$, suppose $\xi^\star \notin \text{ext}(\Xi)$ or $u^\star \notin \text{ext}(\mathcal{U})$, we can construct a new solution $(\xi', u', u'\xi'^\intercal)$ in the desired form. With loss of generality, suppose $\xi^\star \notin \text{ext}(\Xi)$, we can fix the variables u at $u^\star$ in the symmetric bidual $\bar{\Delta}^{\text{sym}}$, then the resulting problem is a linear program with decision variables ξ. Because the original problem Π is feasible and bounded according to Assumption 1, there is an optimal extreme point ξ' such that the new solution $(\xi', u^\star, u^\star\xi'^\intercal)$ preserves the objective value. Now, suppose $u^\star \in \text{ext}(\mathcal{U})$, we simply set $u' = u^\star$. Otherwise, we fix ξ at ξ' and use the same method to identify some $u' \in \text{ext}(\mathcal{U})$ that preserves the objective value. Then, the constructed solution is a symmetric-optimal solution of $\bar{\Delta}$ where both parts u' and ξ' are extreme points. □

Proposition 2. *In the bidual formulation* $\bar{\Delta}$*, constraint set* (4b) *is equivalent to the following,*

$$v_i \in u_i conv(\Xi) + cone(\Xi),\ \forall i \in [k]. \tag{8}$$

Proof. Constraint set (4b) can be written as $Bv_i \le u_i b$ for all i. Every $v_i \in u_i\text{conv}(\Xi) + \text{cone}(\Xi)$ can be written as $u_i\xi_i + \xi_i'$ for some $\xi_i \in \text{conv}(\Xi)$ and $\xi_i' \in \text{cone}(\Xi)$. $Bv_i \le u_i b$ is then satisfied, because u_i is nonnegative, $B\xi_i \le b$, and $B\xi_i' \le 0$. Conversely, when $u_i = 0$, v_i belongs to $\text{cone}(\Xi) = 0\text{conv}(\Xi) + \text{cone}(\Xi)$; when $u_i > 0$, $v_i/u_i \in \Xi$, which is the same as $v_i \in u_i\text{conv}(\Xi) + \text{cone}(\Xi)$. □

Proposition 3. *Any solution* $(\xi, u, V) \in \mathfrak{P}$ *is an extreme point if and only if* $\xi \in ext(\Xi)$, $u \in ext(\mathcal{U})$, *and for each row* $v_i = u_i\xi_i$ *of matrix* V*, either* $u_i = 0$ *or* $\xi_i \in ext(\Xi)$.

Proof. For sufficiency, we take a solution $(\xi, u, V) \in \mathfrak{P}$ that satisfies the requirement. Towards a contradiction, we assume there exists two distinct feasible solutions (ξ', u', V') and (ξ'', u'', V'') such that

$$0.5(\xi', u', V') + 0.5(\xi'', u', V'') = (\xi, u, V).$$

Focusing on the equalities that correspond to u, we have $0.5u' + 0.5u'' = u$. Because u is an extreme point by selection, we have $u = u' = u''$. Similarly, we also have $\xi = \xi' = \xi''$. Next, we focus on the equalities corresponding to each row of V. By Proposition 2, each row v_i, v_i', v_i'' can be represented as $u_i\xi_i, u_i'\xi_i', u_i''\xi_i''$ with some $\xi_i, \xi_i', \xi_i'' \in \Xi$. Thus, for a fixed row index i, we have

$$0.5u_i'\xi_i' + 0.5u_i''\xi_i'' = u_i\xi_i.$$

Suppose $u_i = 0$, then $v_i = v_i' = v_i'' = 0$. Otherwise, we can cancel u_i from the above equalities since $u = u' = u''$ by the previous argument, which gives $0.5\xi_i' + 0.5\xi_i'' = \xi_i$. This also implies $v_i = v_i' = v_i''$ since ξ_i is an extreme point by choice. In either case, we reach the identity $(\xi', u', V') = (\xi'', u'', V'')$, which leads to the desired contradiction.

For necessity, we prove the contrapositive. Take any (ξ, u, V) that does not satisfy the requirement, then either $\xi \notin \text{ext}(\Xi)$, $u \notin \text{ext}(\mathcal{U})$, or at some row $v_i = u_i\xi_i$ of matrix V such that $u_i \ne 0$, we have $\xi_i \notin \text{ext}(\Xi)$. Every possible case implies that the associated vector x (x represents ξ, u, or some v_i) can be written as $x = 0.5x' + 0.5x''$ for some distinct, feasible x' and x''. Then, replacing the vector x in (ξ, u, V) with x' and x'' respectively gives two distinct feasible solutions in $\mathfrak{P}$ whose convex combination contains the solution (ξ, u, V). Thus, (ξ, u, V) cannot be an extreme point of $\mathfrak{P}$. □

Corollary 4. $\Pi = (\Xi, \mathcal{U}, C)$ *is zero-adjustable if it satisfies either of the following conditions:*

1. $\{c\} \cup \{c_i\}_{i\in[k]} \subseteq N_\Xi(\xi)$ *for some* $\xi \in$ *ext*(Ξ)*;*

2. *for every* $u \in$ *ext*$(\mathcal{U})$ *with nonzero entries labeled by* L_u*, we have* $\{c\} \cup \{c_i\}_{i\in L_u} \subseteq N_\Xi(\xi_u)$ *for some* $\xi_u \in$ *ext*(Ξ)*.*

Proof. The first statement is a trivial consequence of Theorem 2. For the second, Proposition 3 implies there must exist some $u^\star \in \mathcal{U}^\star$ such that $u^\star$ is also an extreme point of $\mathcal{U}$. Then, by Theorem 2, the second statement also implies zero-adjustability. □

Corollary 6. *Given a policy problem* $\Pi = (\Xi, \mathcal{U}, \bar{C})$ *that satisfies* $z(\Pi) > 0$*,* $\Xi \subseteq \mathbb{R}^n_+$*, and* $\bar{C} \geq 0$*,*

1. $\delta_{rel}(\Pi) = 1$ *when* Ξ *is a hypercube,*

2. $\delta_{rel}(\Pi) \leq 2$ *when* Ξ *is symmetric or positive,*

3. $\delta_{rel}(\Pi) \leq n$ *when* Ξ *is an arbitrary convex set.*

Proof. For all the cases, we construct a box set Ξ' to contain Ξ. Because $\bar{C} \geq 0$, *i.e.*, the row vectors are in the same orthant, thus lead to the same extreme point in Ξ'. For case 1, we can simply let $\Xi' = \Xi$, $K = 1$, and pick any $\xi' \in \Xi'$. Clearly, $\xi'/1 \in \Xi$, thus Theorem 3 proves the bound. For case 2, let $\xi_0 \in \Xi$ be the point of symmetry, we can set Ξ' as the box set uniquely determined by using the origin and $2\xi_0$ as the minimal and maximal points of the box. Being a symmetric or positive set in $\mathbb{R}^n_+$ implies that Ξ is entirely contained in Ξ'. Then, $\xi' = \xi_0$ and $K = 2$ give the desired bound. For the last case, we define $\bar{\xi} = (\max_{\xi\in\Xi} \xi_i)_{i\in[n]}$ and $v_i \in \arg\max_{\xi\in\Xi} \xi_i$ where ξ_i is the ith entry of ξ. Then, the convex combination $v_0 = \sum_{i\in[n]} v_i/n$ belongs to Ξ as it is convex. Consider the box set Ξ' uniquely determined by nv_0 and the origin. It should be clear that $\Xi \subseteq \Xi'$ because $\Xi \subseteq \mathbb{R}^n_+$ and $nv_0 = \sum_{i\in[n]} v_i \geq \bar{\xi}$. Taking $\xi' = nv_0$ and $K = n$, we reach the desired bound. □

Proposition 4. *Every anchor cone* $\mathfrak{A}_{\mathcal{C},x_0}$ *has the following properties:*

1. $\mathfrak{A}_{\mathcal{C},x_0} = \{x_0\} +$ *cone*$^\circ(\mathcal{C})$*;*

2. $N_{\mathfrak{A}_{\mathcal{C},x_0}}(x_0) =$ *cone*$(\mathcal{C})$*;*

3. *constraints of* $\mathfrak{A}_{\mathcal{C},x_0}$ *that correspond to vectors in eray*$(\mathcal{C})$ *are sufficient to define* $\mathfrak{A}_{\mathcal{C},x_0}$*.*

Proof. Both 1 and 2 can be verified directly from the definitions. For 3, every $c_i \in \mathcal{C} \setminus \text{eray}(\mathcal{C})$ can be written as a conic combination of the extreme rays from eray$(\mathcal{C})$. Then, combining the constraints $\{\langle c_j, x\rangle \leq \langle c_j, x_0\rangle\}_{c_j\in\text{eray}(\mathcal{C})}$ with the same coefficients produces the constraint associated with c_i. □

Proposition 5. *Given* Ξ *is bounded, Formulation* (10) *is feasible if either* $\Xi \cap$ *int*$($*cone*$^*(\bar{C})) \neq \varnothing$ *or* $\Xi \subseteq$ *int*$($*cone*$^\circ(\bar{C}))$*.*

Proof. For the first case, pick any $\xi \in \Xi \cap \text{int}(\text{cone}^*(\bar{C})) \neq \varnothing$. By the choice of ξ, we have $\langle c_j, \gamma\xi\rangle > 0 = \langle c_j, 0\rangle$ for all $j \in J$ and $\gamma > 0$. This implies $0 \in \text{int}(\mathfrak{A}_{\bar{C},\gamma\xi})$, which further implies $\lim_{\gamma\to\infty} \mathfrak{A}_{\bar{C},\gamma\xi} = \mathbb{R}^n$. Thus, for any bounded Ξ, there exists a sufficiently large scalar γ_1 so that $\mathfrak{A}_{\bar{C},\gamma_1\xi} \supseteq \Xi$, which implies that γ_1 is a feasible solution to Formulation (10). For the second case, notice that for any ξ, $\lim_{\lambda\to 0} \mathfrak{A}_{\bar{C},\lambda\xi} = \text{cone}^\circ(\bar{C})$. Thus, if $\Xi \subseteq \text{int}(\text{cone}^\circ(\bar{C}))$ and Ξ is bounded, there always exists some $\lambda_2 > 0$ so that $\mathfrak{A}_{\bar{C},\lambda_2\xi} \supseteq \Xi$. □

Proposition 6. *Formulation* (10) *is infeasible if* $\dim(\Xi) + \dim(-$*cone*$(\bar{C}) \cap$ *cone*$(\bar{C})) > n$*.*

Proof. We use $\hat{\mathcal{C}}$ to denote the set $-\text{cone}(\bar{C}) \cap \text{cone}(\bar{C})$. First, notice that $\hat{\mathcal{C}}$ is a subspace. Moreover, the dimension of the polar cone $\dim(\text{cone}^\circ(\bar{C})) = n - \dim \hat{\mathcal{C}}$, which can be proved using induction. Finally, the first property in Proposition 4 implies $\dim(\mathfrak{A}_{\bar{C},x_0}) = \dim(\text{cone}^\circ(\bar{C}))$, which along with the inequality in the claim imply that $\dim(\Xi) > \dim(\mathfrak{A}_{\bar{C},x_0})$. Therefore, the anchor cone $\mathfrak{A}_{\bar{C},x_0}$ can never enclose the target space Ξ. ☐

Proposition 9. *Given a preposition-reallocation problem formulated by* (12) *that satisfies* $0 < \Gamma_i < |\mathcal{C}(i)|$ *for all* $i \in V$*, let node* 0 *and set* $\mathcal{L}$ *denote the root node and the set of leaf nodes in the underlying rooted tree of* Ξ*. Then, the adjustability ratio obtained from the anchor cone algorithm equals* $\gamma = 1/f_0$ *where* f_0 *is computed recursively as* $f_i = 1$ *for all* $i \in \mathcal{L}$ *and* $f_i = H\left(f_{\mathcal{C}(i)}\right) \frac{\Gamma_i}{|\mathcal{C}(i)|}$ *for all* $i \notin \mathcal{L}$*, where* $f_{\mathcal{C}(i)}$ *is the set* $\{f_j\}_{j \in \mathcal{C}(i)}$ *and* $H(\cdot)$ *is the harmonic mean of the input values.*

Proof. It suffices to show that $\max_{\xi \in \Xi} \min_j \xi_j = f_0$. We start with several observations. First, for any feasible solution ξ, $\xi_i \geq \xi_j$ for every $i \in \mathcal{A}(j)$ where $\mathcal{A}(j)$ contains all the ancestors of j in the rooted tree. Thus, maximizing the minimum of ξ is equivalent to maximizing the minimum among all the leaf nodes. Second, for every node i, let $\Xi(\xi_i)$ be the projection of Ξ onto the descendant nodes of i with a fixed budget ξ_i on node i, then $\max_{\xi \in \Xi(\xi_i)} \min_{j \in \mathcal{L}(i)} \xi_j$ is an increasing function on ξ_i. These two observations indicate an optimal solution where all available budgets are utilized at every level and all the leaf nodes share the same amount of budgets, which we call the leaf value λ. For the recursion, we use $\{f_i\}_{i \in V}$ to denote the percentage of budget that is converted to the leaf value, *i.e.*, $f_i = \lambda / \xi_i^\star$ for some optimal $\xi^\star$. For the base case, $f_i = 1$ for every leaf node $i \in \mathcal{L}$. For every non-leaf node i, assume it receives a unit budget from the parent, then it has $\Gamma_i > 0$ budget for $|\mathcal{C}(i)|$ child nodes where $\Gamma_i < |\mathcal{C}(i)|$. Let h_j be the amount of budget assigned to the child $j \in \mathcal{C}(i)$, then, we have the following equations,

$$\sum_{j \in \mathcal{C}(i)} h_j = \Gamma_i$$
$$f_j h_j = f_k h_k, \quad \forall j, k \in \mathcal{C}(i),$$

where the first says all budget at node i has been used for the child nodes and the second implies the value of the leaf nodes at each branch is the same (equal to λ). Solving this system for any fixed h_j gives

$$h_j = \frac{\Gamma_i}{f_j \sum_{k \in \mathcal{C}(i)} f_k^{-1}}.$$

Notice h_j is just the amount of budget transfer to the child node j, the amount transfer to the leaf node is

$$f_j h_j = \frac{\Gamma_i}{\sum_{k \in \mathcal{C}(i)} f_k^{-1}} = H(f_{\mathcal{C}(i)}) \frac{\Gamma_i}{|\mathcal{C}(i)|}.$$

This result is obtained based on the assumption that node i received a unit budget from the parent node; thus, it is also the value of f_i. This concludes the proof. ☐

B Zero-Adjustability under Affine Transformations

According to Theorem 2 and Corollary 4, the relation between the rows in $\bar{C}$ and Ξ dictates whether constant policy family is optimal, *i.e.*, adjustability gap δ_{abs} is zero. Indeed, in many examples from Section 4.2, we derived sufficient conditions on $\bar{C}$ for various uncertainty sets Ξ to ensure zero-adjustability. In order to extend these known results to a larger class of policy problems, it is interesting to study the conditions for affine transformations to preserve co-optimality. Recall that we also view any matrix as the set of its rows. Thus, cone(C) is the conic combination of the rows in C. Then, we have the following main theorem.

Proposition 10. *Suppose $C \subseteq N_\Xi(\xi^\star)$, given two matrices D and R such that $DR \subseteq$ cone(C), let ϕ be the affine transformation $\phi(\xi) = R\xi + \beta$ for any vector β. Then, we have $D \subseteq N_{\phi(\Xi)}(\phi(\xi^\star))$.*

Proof. $C \subseteq N_\Xi(\xi^\star)$ means $\langle c_i, (\xi - \xi^\star)\rangle \leq 0$ for all $c_i \in C$ and $\xi \in \Xi$. Then, we need to show that $\langle d_i, \phi(\xi) - \phi(\xi^\star)\rangle \leq 0$ for all $\xi \in \Xi$ and all $d_i \in D$. For every fixed i and ξ, we have

$$\langle d_i, \phi(\xi) - \phi(\xi^\star)\rangle = \langle d_i, R\xi + \beta - (R\xi^\star + \beta)\rangle = \langle R^\intercal d_i, \xi - \xi^\star\rangle.$$

Notice that $DR \subseteq$ cone(C) implies $R^\intercal d_i \in$ cone(C), *i.e.*, $R^\intercal d_i = \sum \lambda_i c_i$ for some $\lambda \geq 0$. Thus, we further have

$$\langle d_i, \phi(\xi) - \phi(\xi^\star)\rangle = \sum \lambda_i \langle c_i, \xi - \xi^\star\rangle \leq 0.$$

This concludes the proof. □

This proposition gives a general condition for preserving co-optimality of a set of problems: if $\{\max_{\xi\in\Xi}\langle c_i, \xi\rangle\}_{i\in L}$ is co-optimal at some $\xi^\star \in \Xi$, then the class of problems $\{\max_{\xi\in\Xi}\langle d_i, R\xi + \beta\rangle\}_{i\in L}$ is also co-optimal at $R\xi^\star + \beta$ given that D and R satisfy $DR \subseteq$ cone(C).

In Theorem 2, given a policy problem Π that is zero-adjustable, we only know that $\bar{C}_L \subseteq N_\Xi(\xi)$ for some $\xi \in \Xi$ (L labels the nonzero entry of some optimal $u^\star$). Hence, we need further conditions to ensure the preservation of zero-adjustability. Several sufficient conditions are listed in the following corollary.

Corollary 5. *Given a policy problem $\Pi = (\Xi, \mathcal{U}, \bar{C})$ that is zero-adjustable and two matrices $\bar{D}$ and R that satisfy $\bar{D}R \subseteq$ cone$(\bar{C})$. Let ϕ be the affine transformation $\phi(\xi) = R\xi + \beta$ for any β, then, the transformed problem $\Pi' = (\phi(\Xi), \mathcal{U}, \bar{D})$ is also zero-adjustable if any of the following is satisfied:*

1. *Π satisfies Corollary 4;*
2. *there exists a symmetric-optimal solution $[\xi, u, u\xi^\intercal]$ of $\bar{\Delta}$ such that $u > 0$;*
3. *$\beta = 0$ and $DR = \lambda C$ for some $\lambda \geq 0$, where C and D are $\bar{C}$ and $\bar{D}$ without the first row.*

Proof. Given Π is zero-adjustable, we only know that, for some optimal $u^\star$ with nonzero entries labeled by L, the rows in $\bar{C}_L$ belong to the same normal cone. To prove the claim, we have to show that, in the following transformed problem (13), there exists some optimal solution $u^{\star\prime}$ with nonzero entries labeled by L' such that the rows in $\bar{C}'_{L'}$ also belong to the same normal cone.

$$\max_{\xi\in\Xi}\langle d, \phi(\xi)\rangle + \max_{u\in\mathcal{U}} \sum_{i\in[k]} u_i \max_{\xi_i\in\Xi}\langle d_i, \phi(\xi_i)\rangle. \tag{13}$$

For Case 1, the first statement of Corollary 4 states all rows are in the same normal cone, thus the claim is trivially true. For the second statement of Corollary 4, we know that there must exist an optimal $u^{\star\prime}$ of (13) with nonzero entries labeled by L' that is also an extreme point of $\mathcal{U}$. By the premise of the second statement of Corollary 4, the rows in $\bar{C}_{L'}$ are from the same normal cone. Then, by Proposition 10, so are the vectors in $\bar{D}_{L'}$. Case 2 implies that the set L contains the entire index set, thus the claim is trivially true. The last condition implies

$$\max_{\xi\in\Xi}\langle d_i, \phi(\xi)\rangle = \max_{\xi_i\in\Xi}\langle R^\intercal d_i, \xi_i\rangle = \lambda \max_{\xi_i\in\Xi}\langle c_i, \xi_i\rangle,\ \forall i \in [k].$$

Thus, the cost vector of u is scaled by $\lambda > 0$, which will lead to the same optimal $u^\star$ as before. This concludes the proof. □

Using these results, we can extend the class of zero-adjustable policy problems using known instances. We introduce several of them in the following.

Example 7 (Parallelotope Uncertainty Set)**.** For any full-dimensional parallelotope Ξ, there is a unique bijective affine transformation $\phi(\xi) = R\xi + \beta$ that maps the unit L_∞ ball (a special box set) onto Ξ. According to Example 2, Π is zero-adjustable with respect to the uncertainty set L_∞ if $\bar{C} \subseteq \mathbb{R}^n_+$. Then, as long as $\bar{D} = \bar{C}R^{-1}$ for any $\bar{C} \subseteq \mathbb{R}^n_+$, we have $\bar{D}R = \bar{C}$, in which case the constant policy is optimal with respect to the parallelotope Ξ. △

Example 8 (Simplex Uncertainty Set)**.** A simplex uncertainty set Ξ is defined as $\{\xi \mid \langle b, \xi\rangle \leq b_0\}$ for some positive vector b and positive scalar b_0. In particular, $\Xi_1 = \{\xi \mid \langle 1, \xi\rangle \leq 1\}$ is called the standard orthogonal simplex. Clearly, a cost vector c belong to $N_{\Xi_1}(e_i)$ if and only if the largest value of c is at the ith entry. Let $\phi(\xi) = R\xi + \beta$ be the bijective affine transformation that maps Ξ_1 to Ξ. Then, $\Pi' = (\Xi, \mathcal{U}, \bar{D})$ is zero-adjustable if $\bar{D} = \bar{C}R^{-1}$ for some matrix $\bar{C}$ where the maximum value of each row is at the same column. △

C Additional Examples

C.1 Zero-Adjustability: Budgeted Ξ and Product Simplices $\mathcal{U}$

In this setting, we have

$$\Xi_\lambda = \{\xi \in [-1, 1]^n \mid \|\xi\|_1 \leq \lambda\} \text{ and } \mathcal{U} = \{U \in \mathbb{R}^{l\times k}_+ \mid \|u_i\|_1 = 1\},$$

where λ is a number from $[n]$ and u_i is the ith row of matrix U. The corresponding policy problem is essentially the *robust optimization of sums of piecewise linear functions* that has been extensively studied by Ardestani-Jaafari and Delage [2] and Gorissen and den Hertog [31]. The former examined optimality condition and optimality gap for *affine* policies in this class problems. In this example, we will use Theorem 2 to produce optimality criterion for the constant policy family.

Clearly, $\mathcal{U}$ is the product space of l copies of $(k-1)$-dimensional standard simplices. Thus, the effect of each $U \in \text{ext}(\mathcal{U})$ on the bidual Formulation (9) can be considered as an extreme point selector that picks exactly one extreme point from the set of vectors $\{\xi_{ij}\}_{j\in[k]}$ for every $i \in [l]$. On the other hand, the budgeted uncertainty set Ξ has the following set of extreme points,

$$\text{ext}(\Xi_\lambda) = \{\xi \in \{-1, 0, 1\}^n \mid \|\xi\|_1 = \lambda\}.$$

In particular, when $\lambda = 1$, the uncertainty set Ξ_1 is the L_1-norm ball, each extreme point of which is a standard unit vector e_i or its additive inverse $-e_i$; when $\lambda = n$, the uncertainty set Ξ_n is simply the L_∞ ball $[-1, 1]^n$ where each extreme point is a vector full of ± 1 entries. We can also study Ξ_λ by its effect on any cost vector c. For a given $\lambda \in [n]$, we define λ-sorted 1-norm (*cf.* [23]) of a vector c as

$$\|c\|_{1,\lambda} := \max_{\xi\in\Xi_\lambda} \langle c, \xi\rangle.$$

This equals the sum of the first λ largest absolute values in the vector c. When $\lambda = 1$ and n, $\|\cdot\|_{1,\lambda}$ is simply the one-norm and infinity norm, respectively. Therefore, the maximization of $\langle c, \xi\rangle$ over the space Ξ_λ simply calculates the λ-sorted 1-norm of c. Then, we have the following exact zero-adjustability criterion.

Proposition 11. *Given the above defined polyhedrons Ξ_λ and $\mathcal{U}$, Π is zero-adjustable if and only if for each $i \in [l]$, we can find an index*

$$j_i \in \arg\max_{j\in[k]} \|c_{ij}\|_{1,\lambda}$$

such that the first λ largest (absolute-value) entries of each vector in $\{c_{ij_i}\}_{i\in[l]}$ and c can be indexed by the same set $L \subseteq [n]$ with $|L| = \lambda$, such that the projected vectors $\{[c_{ij_i}]_L\}_{i\in[l]}$ are in the same orthant of $\mathbb{R}^\lambda$.

Proof. By definition, $u \in \mathcal{U}^\star \cap \text{ext}(\mathcal{U})$ if and only if for each $i \in [l]$, $u_{ij} = 1$ at some entry with the maximum value of $\max_{\xi_{ij}\in\Xi_\lambda}\langle c_{ij}, \xi_{ij}\rangle$ among all $j \in [k]$. As discussed before, the first term $\max_{\xi_{ij}\in\Xi_\lambda}\langle c_{ij}, \xi_{ij}\rangle$ is the λ-sorted 1-norm $\|c_{ij}\|_{1,\lambda}$. According to Theorem 2, the constant policy is optimal if and only if the cost vectors that correspond to the nonzero u_{ij}'s are in the same normal cone $N_{\Xi_\lambda}(\xi_0)$ for some $\xi_0 \in \text{ext}(\Xi_\lambda)$. In this specific case, a cost vector $c \in N_{\Xi_\lambda}(\xi_0)$ if and only if the λ largest absolute-value entries in c are indexed by the λ largest (absolute value) entries (labeled by L) of ξ_0 and the projected vector $[c]_L$ is in the same orthant as $[\xi_0]_L$. This finishes the proof. □

Similarly, in other problems where we can explicitly describe the extreme points of both Ξ and $\mathcal{U}$, Theorem 2 can be used to generate zero-adjustability criteria.

C.2 Adjustability Ratio Computation

Example 9 (Ellipsoidal Set Ξ; Positive Dominant Column in $\bar{C}$)**.** In Example 3, we have shown that, when $\bar{C}$ has a positive dominant column and the uncertainty set is an L_1 ball, the adjustability gap is zero. Hence, we can bound the ellipsoidal Ξ with an L_1 ball Ξ', which can be uniquely determined by radius $\lambda > 0$. To ensure Ξ' contains Ξ, we can set $\lambda := \max_{\xi\in\Xi}\langle 1, \xi\rangle$. Clearly, Slater's condition is satisfied, so we can use KKT condition to obtain a closed form solution $\lambda = \|l\|_2/2$ where $\|\cdot\|_2$ is the L_2 norm. Now, suppose the jth column of $\bar{C}$ is a positive dominant column, then, all the row vectors in $\bar{C}$ lead to the extreme point $\lambda e_j \in \Xi'$. We also know that $l_j e_j \in \Xi$. Thus, applying Theorem 3, $K = \|l\|_2/(2l_j)$ is a valid bound. For instance, when Ξ is an L_2 ball, adjustability is $K = \sqrt{n}/2$. △

Example 10 (Budgeted Uncertainty Set Ξ)**.** We first consider the case $\bar{C} \geq 0$. Let $\Xi' = [-1, 1]^n$, we have $\Xi \subseteq \Xi'$. Then, solving for the minimum K such that $1_n/K \in \Xi$ is the same as finding the intersection point between the line segment $[0_n, 1_n]$ and the boundary of Ξ. This point can be directly computed as $(\Gamma/n)1_n$. Thus, n/Γ bounds the adjustability ratio. Now, we assume $\bar{C}$ has a positive dominant column j. In this case, a valid choice of Ξ' is the scaled unit L_1 ball ΓL_1. Again, to compute the ratio, we calculate the intersection point between the line segment $[0_n, \Gamma e_j]$ and the boundary of Ξ. When Γ is within the assumed range, this intersection point is e_j. Thus, the adjustability ratio is bounded by Γ. △

C.3 Case Study: Dynamic Inventory Problem

As mentioned before, the adjustability of a multistage dynamic robust optimization is bounded by the adjustability of its fully adjustable counterpart, *i.e.*, the policy problem Π where all the decisions can be made after observing the values of all the uncertainty variables. In this example, we study a classic multistage robust inventory problem that falls into this category. Let τ be the total number of stages, we consider the following problem,

$$
\begin{aligned}
\Pi_0 := \min_{w_1}\max_{\xi_1}\cdots\min_{w_\tau}\max_{\xi_\tau} \quad & \sum_{t\in[\tau]} c_t w_t + h_t(x_{t+1}) \\
\text{s.t.} \quad & x_{t+1} = x_t + w_t - \xi_t, \quad \forall t \in [\tau], \\
& w_t \in [\underline{w}_t, \bar{w}_t], \quad \forall t \in [\tau], \\
& \xi_t \in [\underline{\xi}_t, \bar{\xi}_t], \quad \forall t \in [\tau].
\end{aligned}
$$

The vectors x, w, and ξ indicate the numbers of inventory, orders, and demands at every stage $t \in [\tau]$. In particular, x_1 represents the initial inventory. The vectors $\underline{w}, \bar{w}$ and $\underline{\xi}, \bar{\xi}$ are the lower and upper bounds for the order vector w and demand vector ξ. Finally, vector c is the ordering cost and h represents the holding and backlog costs ($c^+ \geq 0$ and $c^- \leq 0$) defined as

$$h_t(x_{t+1}) = \max\{c_t^+ x_{t+1}, c_t^- x_{t+1}\}.$$

This formulation represents a one-commodity multistage robust inventory problem, where at each stage t, the user decides the order amount of commodities to minimize the worst-case total costs that the future uncertain demands could inflict. As mentioned before, the gap between the value of an open-loop control and $z(\Pi_0)$ as well as the gap between the $z(\Pi_0)$ and an oracle are both bounded by the corresponding adjustability.

To analyze the adjustability, we first reduce all the state variables $\{x_{t+1}\}_{t\in[\tau]}$ using $x_{t+1} = x_1 + \langle \bar{1}_t, w - \xi \rangle$ where $\bar{1}_t := (1_t, 0_{\tau-t})$ and push the convex objective function h_t into the constraint set. This gives the fully-adjustable counterpart Π as

$$\begin{aligned} \Pi := \min_{w(\cdot),h(\cdot)} \max_{\xi\in[\underline{\xi},\bar{\xi}]} \quad & \langle (c,1),(w,h) \rangle \\ \text{s.t.} \quad & 1_2 h_t \geq c_t^{\pm}(x_1 + \langle \bar{1}_t, w-\xi \rangle), && \forall t \in [\tau], \forall \xi \in \Xi \\ & \underline{w} \leq w \leq \bar{w}, && \forall \xi \in \Xi. \end{aligned}$$

where 1_2 is the all-ones vector of size two and $c_t^{\pm}$ is the vector (c_t^+, c_t^-). We can further rewrite this into the matrix form. We use $I_\triangle$ for the matrix where each *row* is $\bar{1}_t$, *i.e.*, $I_\triangle$ is a lower triangular matrix filled with ones and use $C^{\pm}$ to denote the matrix where each *column* is $c_t^{\pm}$. We also use vec($C^{\pm}$) and diag($C^{\pm}$) to denote the column-wise vectorization (stacking columns into a vector) and diagonalization (replace the tth diagonal entry of the identity matrix I with the tth column of $C^{\pm}$). We also use $\otimes$ for the tensor product. Then, we have,

$$\begin{aligned} \Pi := \min_{w(\cdot),h(\cdot)} \max_{\xi\in[\underline{\xi},\bar{\xi}]} \quad & \langle (c,1),(w,h) \rangle \\ \text{s.t.} \quad & (I_\tau \otimes 1_2)h \geq \text{vec}(C^{\pm})x_1 + \text{diag}(C^{\pm}) I_\triangle (w-\xi), && \forall \xi \in \Xi \\ & \underline{w} \leq w \leq \bar{w}, && \forall \xi \in \Xi. \end{aligned}$$

Thus, we can extract the input parameters as

$$\begin{aligned} &\Xi := \{\xi \mid \underline{\xi} \leq \xi \leq \bar{\xi}\}, \quad a := (c,1), && A := \left[[-\text{diag}(C^{\pm})I_\triangle, I_\tau \otimes 1_2]; [-I, 0]; [I, 0]\right], \\ &c := 0, \qquad C := [-\text{diag}(C^{\pm})I_\triangle; 0; 0], && \beta := (\text{vec}(C^{\pm})x_1, -\bar{w}, \underline{w}). \end{aligned}$$

In this problem, the uncertainty set is quite simple. It is a box set that belongs to the nonnegative orthant (demands are assumed nonnegative). On the other hand, the complexity of matrix C mainly depends on the part diag($C^{\pm}$)$I_\triangle$, which can be expressed as

$$\left[c_1^{\pm}\bar{1}_1^\intercal; c_2^{\pm}\bar{1}_2^\intercal; \cdots; c_\tau^{\pm}\bar{1}_\tau^\intercal\right].$$

An observation is that the two rows of every submatrix $c_t^{\pm}\bar{1}_t^\intercal$ are at the opposite directions that belong to the nonnegative and non-positive orthants, respectively. Thus, we analyze several cases according to the value of these vectors.

- Suppose either there is no holding costs ($c^+ = 0$) or no backlog costs ($c^- = 0$), then all the cost vectors lead to the same extreme point of Ξ. By Theorem 2, we can directly conclude that $\delta_{\text{abs}}(\Pi) = 0$, which also implies the adjustability gap of Π_0 is zero.

- Suppose for some $t \in [\tau]$, we have $c^+_{>t} = 0$ or $c^-_{>t} = 0$, then applying constant policy after stage t will not induce any optimality gap, since all the terms related to the fixed decisions $w_{\leq t}$ and $\xi_{\leq t}$ will go to the constant part β, which leaves the same $\bar{C}$ and a box uncertainty set with a lower dimension.

- Suppose for every $t \in [\tau]$, at most one of the two costs is nonzero. Let $[\tau]_+$ and $[\tau]_-$ index the stages where the holding and backlog costs are nonzero, respectively. Then, we can focus on the following polyhedron,

$$\Omega := \left\{ \omega \in W \,\middle|\, \begin{array}{ll} \langle \bar{1}_t, \omega \rangle \geq 0, & \forall t \in [\tau]_+, \\ \langle \bar{1}_t, \omega \rangle \leq 0, & \forall t \in [\tau]_-. \end{array} \right\}$$

  If Ω is full-dimensional, then any interior point of Ω induces a hyperplane that contains all the nonzero vectors in $\bar{C}$ as interior points. This implies $-\text{cone}(\bar{C}) \cap \text{cone}(\bar{C}) = \{0\}$. By the discussion in Proposition 6, this shows that any constructed anchor cone is full dimensional. Thus, it is possible to use the anchor cone formulation (10) to obtain a bound for $\delta_{\text{rel}}(\Pi)$. Otherwise, suppose Ω is not full-dimensional, we can directly conclude that the anchor cone formulation is infeasible according to Proposition 6.

- Suppose $c^+ > 0$ and $c^- < 0$. This implies $\text{cone}(\bar{C}) = \mathbb{R}^\tau$. By Proposition 6, we know that the anchor cone method (also Theorem 3) cannot be used. To estimate this gap, we resort to the direct computation method. An upper bound and a lower bound can be derived separately as follows.

  – To evaluate the upper bound, we solve the constant policy problem $\bar{\Pi}$. It is a robust optimization. Thus, the attacker's decision is easy to identity: for each c^+_t, the attacker will choose the strategy $\underline{\xi}$ in order to inflict a maximum amount of holding inventory; for each c^-_t, they will apply $\bar{\xi}$ to cause the maximum amount of backlog. Then, $z(\bar{\Pi})$ can be obtained by solving the following linear program,

$$\begin{aligned} \bar{\Pi} := \min_{w,h} \quad & \langle (c, 1), (w, h) \rangle \\ \text{s.t.} \quad & h_t \geq c^+_t (x_1 + \langle \bar{1}_t, w - \underline{\xi} \rangle), \quad \forall t \in [\tau], \\ & h_t \geq c^-_t (x_1 + \langle \bar{1}_t, w - \bar{\xi} \rangle), \quad \forall t \in [\tau], \\ & \underline{w} \leq w \leq \bar{w}, \quad \forall \xi \in \Xi. \end{aligned}$$

  Moreover, any feasible solution of this formulation is a valid upper bound. A heuristic solution of $\bar{\Pi}$ is that at every stage t, with previous decisions given as $w_{<t}$, the decision-maker optimize the following problem,

$$\min_{w_t} \max \left\{ \begin{array}{l} \left(c_t + c^+_t\right) w_t + c^+_t \left(x_1 + \|w_{<t}\|_1 - \left\|\underline{\xi}_{\leq t}\right\|_1\right) \\ \left(c_t + c^-_t\right) w_t + c^-_t \left(x_1 + \|w_{<t}\|_1 - \left\|\bar{\xi}_{\leq t}\right\|_1\right) \end{array} \right\}.$$

  Suppose $c_t + c^-_t \geq 0$, this objective value is increasing on w_t, we simply pick $w_t = 0$; otherwise, it is a convex function where the minimum is obtained by equating the two pieces. This gives the following inductive solution for every $t \in [\tau]$,

$$w_t = \begin{cases} \text{clip}\left(\frac{c^-_t \left(x_1 + \|w_{<t}\|_1 - \left\|\bar{\xi}_{\leq t}\right\|_1\right) - c^+_t \left(x_1 + \|w_{<t}\|_1 - \left\|\underline{\xi}_{\leq t}\right\|_1\right)}{c^+_t - c^-_t}, \underline{w}_t, \bar{w}_t \right), & \text{if } c_t + c^-_t \leq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\text{clip}(w, \underline{w}, \bar{w})$ clips the value of w using lower bound $\underline{w}$ and upper bound $\bar{w}$. Then, this solution can be used to produce an upper bound of $z(\bar{\Pi})$.

– For the lower bound, we can pick an arbitrary element $\xi_0 \in \Xi$, then solve the defender's problem,

$$\begin{aligned} \min_{w,h} \quad & \langle (c, 1), (w, h) \rangle \\ \text{s.t.} \quad & h_t \geq c_t^+(x_1 + \langle \bar{1}_t, w - \xi_0 \rangle), \quad \forall t \in [\tau], \\ & h_t \geq c_t^-(x_1 + \langle \bar{1}_t, w - \xi_0 \rangle), \quad \forall t \in [\tau], \\ & \underline{w} \leq w \leq \bar{w}, \quad \forall \xi \in \Xi. \end{aligned}$$

In particular, suppose $x_1 = 0$, w has sufficient bounds, and $c_t \leq c_{t+1}$ for all $t \in [\tau]$. Then, we can always satisfy the demand at each stage, in which case no holding nor backlog costs will be induced. This gives a lower bound $\max_{\xi \in \Xi} \langle c, \xi \rangle = \langle c, \bar{\xi} \rangle$. In the general case, to obtain a better bound, we may generate multiple elements from Ξ, and pick the one with the maximum objective value.

In this example, because of the generality on the matrix $\bar{C}$, we can not provide an exact analytical expression for the bound of $\delta_{\text{rel}}(\Pi)$. However, using the tools developed before, we can still have a detailed analysis of the problem based on different cases of $\bar{C}$.