Strong duality of a conic optimization problem with a single hyperplane and two cone constraints

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Abstract

Our target conic optimization problem (COP) is minimizing a linear function in a vector variable \( x \) subject to a single hyperplane constraint \( x \in H \) and two cone constraints \( x \in K_1, x \in K_2 \). This COP is geometrically representable and extensive enough to study strong (Lagrangian) duality of more general COPs with cone and multiple linear inequality constraints. It can also be identically reformulated as a simpler COP with the single hyperplane constraint \( x \in H \) and the single cone constraint \( x \in K_1 \cap K_2 \), which has no duality gap without any constraint qualification. The dual of the original target COP is equivalent to the dual of the reformulated COP if the Minkowski sum of the duals of the two cones \( K_1 \) and \( K_2 \) is closed or if the dual of the reformulated COP satisfies a Slater condition. Thus, these two conditions make it possible to transfer all duality results, including the existence and/or boundedness of optimal solutions, on the reformulated COP to the ones on the original target COP, and further to the ones on a standard primal-dual pair of COPs with symmetry.

Key words. Duality, conic optimization problems, simple conic optimization problems, generalizing the Slater condition, closedness of the Minkowski sum of two cones

AMS Classification. 90C20, 90C22, 90C25, 90C26, 90C27.

1 Introduction

It is well-known that strong duality of conic optimization problems (COPs), including semidefinite programs (SDPs) [3, 10, 15], second order programs [2, 7], doubly nonnegative programs [4, 16] and copositive programs [5, 6] depends on their representation. In

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this paper, we consider a primal COP of the following form:

\[ \eta_p = \inf \left\{ \langle q, x \rangle : x \in K_1, \ x \in K_2, \ \langle h, x \rangle = 1 \right\}. \] (1)

Here,

\[ V = \text{a finite dimensional vector space endowed with an inner product } \langle x, y \rangle \text{ and a norm } \|x\| = \sqrt{\langle x, x \rangle} \text{ for every } x, y \in V, \]

\[ K_1, K_2 = \text{a nonempty closed convex cone in } V, \ q \in V, \ 0 \neq h \in K_1^*, \]

\[ K^* = \text{the dual } \{ y \in V : \langle x, y \rangle \geq 0 \text{ for every } x \in K \} \text{ of a cone } K \subset V. \] (2)

A general form of COP with inequality constraints can be converted into COP (1) where the requirement \( 0 \neq h \in K_1^* \) above is naturally fulfilled. The conversion is described in Section 4.

The (Lagrangian) dual of COP (1) is written as

\[ \eta_d = \sup \{ t_0 : q - h t_0 \in K_1^* + K_2^* \} \]

\[ = \sup \{ t_0 : q - h t_0 - y_2 \in K_1^*, \ y_2 \in K_2^* \}, \] (3)

where \( K_1^* + K_2^* \) denotes the Minkowski sum \( \{ y_1 + y_2 : y_1 \in K_1^*, y_2 \in K_2^* \} \) of \( K_1^* \) and \( K_2^* \). If primal COP (1) (or dual COP (3)) is feasible, \( \eta_p \) (or \( \eta_d \)) takes either a finite value or \(-\infty \) (or \( \infty \)). If they are infeasible, we set \( \eta_p = \infty \) and \( \eta_d = -\infty \). Thus, the well-known weak duality \( \eta_d \leq \eta_p \) holds even if one of them is infeasible. We simply write \(-\infty < \eta_p < \infty \) (or \( -\infty < \eta_d < \infty \)) for the case where primal COP (1) (or dual COP (3)) is feasible and has a finite optimal value \( \eta_p \) (or \( \eta_d \)). We say that strong duality holds if \(-\infty < \eta_d = \eta_p < \infty \), and primal COP (1) (of dual COP (3)) is solvable if \( \eta_p \) (or \( \eta_d \)) is attained by a feasible solution.

Given a closed convex cone \( K \subset V \), primal COP (1) remains identical regardless of how \( K \) is decomposed into the intersection \( K = K_1 \cap K_2 \) of two closed convex cones \( K_1 \) and \( K_2 \). Dual COPs (3) with different decompositions, however, may yield different duality results. For instance, when we take \( K_1 = K \) and \( K_2 = V \), there is no duality gap as we will see in Section 2. In the case where \( K_1 \neq V \) and \( K_2 \neq V \), \( \eta_p < \eta_d \) can occur.

Slater conditions to guarantee strong duality between COPs (1) and (3) can be written as:

Sp: There exists a feasible solution \( \bar{x} \) of COP (1) which lies in the interior of \( K_1 \), denoted by \( \text{int} K_1 \).

Sd: There exists a \( (\bar{t}, \bar{y}_2) \in \mathbb{R} \times K_2^* \) such that \( \bar{y}_1 = q - h \bar{t} - \bar{y}_2 \in \text{int} K_1^* \).

If \( K_2 \) is described by only linear equalities, \( K_2 \) forms a linear subspace of \( V \), and Conditions Sp and Sd become the standard Slater conditions. To generalize the Slater conditions above, we introduce the following conditions:

Tp: \( K_1^* + K_2^* \) is closed.

Td: There exists a \( \bar{t} \in \mathbb{R} \) such that \( \bar{y} = q - h \bar{t} \in \text{int}(K_1^* + K_2^*) \).
It is known that the closure of \( K_1^* + K_2^* \) coincides with \( (K_1 \cap K_2)^* \) (see Lemma 3.1). Hence Condition Tp can be restated as \( K_1^* + K_2^* = (K_1 \cap K_2)^* \). Also \( \text{int}(K_1^* + K_2^*) \) can be replaced by \( \text{int}((K_1 \cap K_2)^*) \) in Condition Td. Condition Tp is weaker than the Slater condition Sp. In fact, we can prove that if Condition Sp holds or both \( K_1 \) and \( K_2 \) are polyhedral as in the LP case, then Condition Tp is satisfied (see Lemma 3.3). Moreover, it is easy to see that Condition Td is weaker than Condition Sd (see Lemma 3.4). Therefore, Conditions Tp and Td may be regarded as generalizations of Sp and Sd.

The main purpose of this paper is to establish the following strong duality results between COPs (1) and (3).

**Theorem 1.1.** Assume that \(-\infty < \eta_p < \infty \) or \(-\infty < \eta_d < \infty \) holds.

(i) If Condition Tp or Condition Td is satisfied, then \(-\infty < \eta_p = \eta_d < \infty \) holds.

(ii) If Condition Tp is satisfied, then dual COP (3) is solvable.

(iii) Condition Td is satisfied if and only if the set of optimal solutions of primal COP (1) is nonempty and bounded.

Assertion (i) indicates that both Conditions Tp and Td are equivalently important to guarantee strong duality between primal COP (1) and dual COP (3). The most significant (and straightforward) consequence of assertions (iii) and (i) of the theorem is:

**Corollary 1.2.** If the set of optimal solutions (or the feasible region) of primal COP (1) is nonempty and bounded, then \(-\infty < \eta_p = \eta_d < \infty \).

**Existing work and contribution of the paper**

Strong duality of COPs including SDPs has been widely studied [1, 10, 12, 13, 14]. Nesterov and Nemirovskii gave a comprehensive discussion on duality for general COPs in their book [10] among others, and presented a strong duality result under a Slater condition (see Theorem 4.2.1 of [10]). Shapiro proposed the closedness of a certain cone, instead of Slater condition, under which strong duality was established for a fairly general COPs [14, Propositions 2.6 and 2.8]. As far as the authors are aware, his closedness condition had been the weakest one among the conditions to ensure strong duality. More recently, Ajayi, Gupte, Khademi and Schaefer [1, Theorem 3.1] also utilized a similar condition to characterize strong duality. The results presented in this paper are extensions of the results in [1, 14].

A unique feature of the proposed approach in this paper is that strong duality of general COPs is obtained with a very simple form of COPs described by a cone, a hyperplane and a line. More precisely, we introduce the following two simple COPs:

\[
\begin{align*}
P(K) & : \quad \zeta_p(K) = \inf \{\langle q, x \rangle : x \in K, \langle h, x \rangle = 1 \}, \\
D(J) & : \quad \zeta_d(J) = \sup \{t : \langle q - ht \rangle \in J \}.
\end{align*}
\]

Here \( K \) and \( J \) are closed convex cone in a finite dimensional vector space \( V \), \( q \in V \) and \( 0 \neq h \in J \). If we take \( J = K^* \), then COPs \( P(K) \) and \( D(K^*) \) serves as a primal-dual pair. The feasible region of primal COP \( P(K) \) can be geometrically represented as the intersection of the hyperplane \( H \equiv \{x \in V : \langle h, x \rangle = 1 \} \) and the cone \( K \). Also the
feasible region of dual COP \( D(K^*) \) can be viewed as the intersection of the 1-dimension line \( \{ q - ht : t \in \mathbb{R} \} \) with the dual cone \( K^* \) of \( K \), where \( h \) is a normal vector of the hyperplane \( H \). This geometrical representation is an essential feature of the primal-dual pair of COPs \( P(K) \) and \( D(K^*) \), which makes it possible to geometrically interpret not only strong duality relation, but also duality gap on the pair. See Figures 1 and 2. In particular, strong duality \( -\infty < \zeta_p(K) = \zeta_d(K^*) < \infty \) holds whenever either primal COP \( P(K) \) or dual COP \( D(K^*) \) has a finite optimal value without any condition (Theorem 2.1, (i)). If we choose \( K = K_1 \cap K_2 \) and \( J = K_1^* + K_2^* \), then the pair of COPs \( P(K_1 \cap K_2) \) and \( D(K_1^* + K_2^*) \) represent the primal-dual pair of COPs (1) and (3). All assertions (i), (ii) and (iii) of Theorem 1.1 on COPs (1) and (3) follow from duality relations between COPs \( P(K) \) and \( D(K^*) \).

We emphasize that assertions (i), (ii) and (iii) of Theorem 1.1 are extensions of the general strong duality results given in [14, Propositions 2.6 and 2.8] when they are applied to a well-known primal-dual pair of COPs with symmetry as shown in Section 4.

The simple primal-dual pair of COPs \( P(K) \) and \( D(K^*) \) was originally introduced in [8] as a Lagrangian-doubly nonnegative (DNN) relaxation of a class of quadratic optimization problems (QOPs), and their strong duality was studied in [4, Lemma 2.5]. Some relation of their results and Theorem 1.1 will be discussed in Section 5.

Outline of the paper

In Section 2, we discuss strong duality of the primal-dual pair of simple COPs \( P(K) \) and \( D(K^*) \), and establish Theorem 2.1. This theorem may be regarded as a special case of Theorem 1.1 for \( K_1 = K \) and \( K_2 = V \), and makes it easier to directly handle Theorem 1.1. Assertions of Theorem 2.1 are illustrated in Figure 1. Based on Theorem 2.1, the main theorem of the paper, Theorem 1.1, is proved in Section 3. Geometric implication of Conditions Tp and Td is illustrated in Figure 2. In addition, some lemmas are presented to relate Conditions Tp and Td to Conditions Sp and Sd. In Section 4, we apply Theorem 1.1 to a standard primal-dual pair of COPs with symmetry, and present an extension of the strong duality results given in [14, Propositions 2.6 and 2.8]. Some remarks on [1, Theorem 3.1] related to Theorem 1.1 are also provided. We conclude in Section 5 with remarks on the strong duality result in [8, 4] for the pair of COPs \( P(K) \) and \( D(K^*) \) which was originally proposed as a Lagrangian-DNN relaxation of a class of QOPs. Section 6 Appendix includes supplementary materials on technical details of some lemmas.

2 Strong duality between COPs \( P(K) \) and \( D(K^*) \)

In this section, we establish Theorem 2.1 which leads to assertions (i), (ii) and (iii) of our main theorem, Theorem 1.1. Throughout this section we assume that \( K \) is a closed convex cone in a finite dimensional vector space \( V \), \( q \in V \) and \( 0 \neq h \in K^* \). To facilitate connecting assertions of Theorem 2.1 with those of Theorem 1.1, we modify Condition Td for dual COP \( D(J) \).

Td’(J): There exists a \( \bar{t} \in \mathbb{R} \) such that \( \bar{y} \equiv q - h\bar{t} \in \text{int} J \).
We will substitute $\mathbb{K}^*$ for $\mathbb{J}$ in (iii) of Theorem 2.1 and Lemma 2.5. Note that the original condition $T_d$ can be rewritten as $T_d'(\mathbb{K}_1^* + \mathbb{K}_2^*)$.

**Theorem 2.1.** Assume that $-\infty < \zeta_p(\mathbb{K}) < \infty$ or $-\infty < \zeta_d(\mathbb{K}^*) < \infty$ holds.

(i) $-\infty < \zeta_p(\mathbb{K}) = \zeta_d(\mathbb{K}^*) < \infty$ holds.

(ii) Dual COP $D(\mathbb{K}^*)$ is solvable.

(iii) Condition $T_d'(\mathbb{K}^*)$ is satisfied if and only if the set of optimal solutions of primal COP $P(\mathbb{K})$ is nonempty and bounded.

Geometric interpretation of assertions (i), (ii) and (iii) is illustrate with 7 two-dimensional examples in Figure 1 and Table 1 to help the reader understand them. A proof of Theorem 2.1 is presented after a series of lemmas below.

**Lemma 2.2.** Let $\hat{\mathbb{L}}$ be the linear subspace of $\mathbb{V}$ generated by $\mathbb{K}$, and let $\hat{\mathbf{q}}$ and $\hat{\mathbf{h}}$ be the metric projections of $\mathbf{q}$ and $\mathbf{h}$ onto $\hat{\mathbb{L}}$, respectively. Then, $\hat{\mathbf{h}} \in \mathbb{K}^* \cap \hat{\mathbb{L}}$, and $P(\mathbb{K})$ and $D(\mathbb{K}^*)$ are equivalent to

$$\hat{P}(\mathbb{K}) : \quad \hat{\zeta}_p(\mathbb{K}) = \inf \left\{ \langle \hat{\mathbf{q}}, \mathbf{x} \rangle : \mathbf{x} \in \mathbb{K} (= \mathbb{K} \cap \hat{\mathbb{L}}), \langle \hat{\mathbf{h}}, \mathbf{x} \rangle = 1 \right\},$$

$$\hat{D}(\mathbb{K}^*) : \quad \hat{\zeta}_d(\mathbb{K}^*) = \sup \left\{ t \in \mathbb{R} : \hat{\mathbf{q}} - \hat{\mathbf{h}}t \in \mathbb{K}^* \cap \hat{\mathbb{L}} \right\},$$

respectively. More precisely, $P(\mathbb{K})$ (or $D(\mathbb{K}^*)$) and $\hat{P}(\mathbb{K})$ (or $\hat{D}(\mathbb{K}^*)$, respectively) share a common feasible region and a common objective value at each feasible solution.

**Proof.** By definition, we see that

$$\mathbb{K} = \mathbb{K} \cap \hat{\mathbb{L}}, \quad \langle \hat{\mathbf{q}}, \mathbf{x} \rangle = \langle \mathbf{q}, \mathbf{x} \rangle \text{ and } \langle \hat{\mathbf{h}}, \mathbf{x} \rangle = \langle \mathbf{h}, \mathbf{x} \rangle \text{ for every } \mathbf{x} \in \mathbb{K}.$$ 

Hence, $P(\mathbb{K})$ is equivalent to $\hat{P}(\mathbb{K})$. It follows from $\mathbf{h} \in \mathbb{K}^*$ that $\langle \hat{\mathbf{h}}, \mathbf{x} \rangle = \langle \mathbf{h}, \mathbf{x} \rangle \geq 0$ for every $\mathbf{x} \in \mathbb{K}$; hence $\hat{\mathbf{h}} \in \mathbb{K}^* \cap \hat{\mathbb{L}}$. For the equivalence between $D(\mathbb{K}^*)$ and $\hat{D}(\mathbb{K}^*)$, it suffices to show that $t \in \mathbb{R}$ is a feasible solution of $\hat{D}(\mathbb{K}^*)$ if and only if it is a feasible solution of $D(\mathbb{K}^*)$. Assume first that $t$ is a feasible solution of $\hat{D}(\mathbb{K}^*)$. Let $\hat{\mathbf{q}} = \mathbf{q} - \hat{\mathbf{q}} \in \hat{\mathbb{L}}^\perp$ and $\hat{\mathbf{h}} = \mathbf{h} - \hat{\mathbf{h}} \in \hat{\mathbb{L}}^\perp$. Then,

$$\mathbf{q} - \mathbf{h}t = \hat{\mathbf{q}} - \hat{\mathbf{h}}t + \hat{\mathbf{q}} - \hat{\mathbf{h}}t \in \mathbb{K}^* \cap \hat{\mathbb{L}} + \hat{\mathbb{L}}^\perp \subset \mathbb{K}^*,$$

where the last inclusion relation follows from $\mathbb{K} \subset \hat{\mathbb{L}}$. Therefore, $t$ is a feasible solution of $D(\mathbb{K}^*)$. Now, assume that $t \in \mathbb{R}$ is a feasible solution of $D(\mathbb{K}^*)$. Then

$$\hat{\mathbf{q}} - \hat{\mathbf{h}}t \in \hat{\mathbb{L}} \quad \text{and} \quad \langle \hat{\mathbf{q}} - \hat{\mathbf{h}}t, \mathbf{x} \rangle = \langle \mathbf{q} - \mathbf{h}t, \mathbf{x} \rangle \geq 0 \text{ for every } \mathbf{x} \in \mathbb{K}.$$ 

Consequently, $\hat{\mathbf{q}} - \hat{\mathbf{h}}t \in \mathbb{K}^* \cap \hat{\mathbb{L}}$, and $t$ is a feasible solution of $\hat{D}(\mathbb{K}^*)$. \hfill \square

We note that $\hat{\mathbb{L}}$ remains to be the 2-dimensional space (since $\mathbb{K}$ is 2-dimensional) in cases (a), (b), (c) and (d) of Figure 1, while $\hat{\mathbb{L}}$ becomes a 1-dimensional (vertical) linear space generated by the 1-dimensional cone $\mathbb{K}$ in cases (e), (f) and (g). In cases (f) and (g), the projection $\hat{\mathbf{h}}$ of $\mathbf{h}$ onto $\hat{\mathbb{L}}$ becomes the zero vector; hence $\hat{P}(\mathbb{K})$ is infeasible, and
Figure 1: Cases (a) and (e) satisfy Condition $T_d'(\mathbb{K}^*)$; hence the set of optimal solutions of $P(\mathbb{K})$ is nonempty and bounded. In cases (b) and (c), either of $P(\mathbb{K})$ and $D(\mathbb{K}^*)$ has an optimal solution, but the set of optimal solutions of $P(\mathbb{K})$ is unbounded because Condition $T_d'(\mathbb{K}^*)$ is violated. We note that case (b) is sensitive to the change in $q$: if the direction of $q$ is perturbed slightly, then either $T_d'(\mathbb{K}^*)$ is satisfied or $D(\mathbb{K}^*)$ gets infeasible (or, either $P(\mathbb{K})$ has a unique optimal solution at $x^*$ or $\zeta_p(\mathbb{K}) = -\infty$). Case (d) is an example for $-\infty = \zeta_d(\mathbb{K}^*) = \zeta_p(\mathbb{K})$, and case (f) for $\zeta_d(\mathbb{K}^*) = \zeta_p(\mathbb{K}) = \infty$. In case (g), both $P(\mathbb{K})$ and $D(\mathbb{K}^*)$ are infeasible; hence $-\infty = \zeta_d(\mathbb{K}^*) < \zeta_p(\mathbb{K}) = \infty$. Table 1 summarizes all cases, where cases designated by $\times$ cannot occur by weak duality or (i) of Theorem 2.1.

$\hat{D}(\mathbb{K}^*)$ is unbounded as in case (f) or infeasible as in case (g). In case (e), the projection $\hat{h}$ remains nonzero and both $\hat{P}(\mathbb{K})$ and $\hat{D}(\mathbb{K}^*)$ are feasible. In case (d), $\hat{D}(\mathbb{K}^*) = D(\mathbb{K}^*)$ is infeasible. Lemma 2.3 proves that $P(\mathbb{K}) = \hat{P}(\mathbb{K})$ has a finite optimal value in cases (a), (b), (c) and (e), where $\hat{D}(\mathbb{K}^*) = D(\mathbb{K}^*)$ has a finite optimal value. Lemma 2.4 proves that $-\infty < \hat{\zeta}_d(\mathbb{K}^*) = \hat{\zeta}_p(\mathbb{K}) < \infty$ holds in these cases.

**Lemma 2.3.** Assume that $-\infty < \zeta_d(\mathbb{K}^*) < \infty$. Then $-\infty < \zeta_p(\mathbb{K}) < \infty$. 
Infeasible (a), (b), (c), (e)\
\(×\)\
\(×\)\
\(×\)\
\(−∞\)\
We see that \(\hat{−∞}\) the assumption respectively. Therefore, it suffices to show that \(\hat{−∞}\) Lemma 2.4.

Table 1: Cases (a) through (g) correspond to those of Figure 1, respectively. Cases designated by \(×\) cannot occur by weak duality and (i) of Theorem 2.1.

<table>
<thead>
<tr>
<th>(\mathbb{D}(\mathbb{K}^*))</th>
<th>Feasible (\zeta_d(\mathbb{K}^*) &lt; \infty)</th>
<th>(\mathbb{P}(\mathbb{K})) Feasible (−\infty &lt; \zeta_p(\mathbb{K}))</th>
<th>(\mathbb{P}(\mathbb{K})) Infeasible (\zeta_p(\mathbb{K}) = \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feasible (\zeta_d(\mathbb{K}^*) = \infty)</td>
<td>((a),(b),(c),(e))</td>
<td>((\text{by Th.2.1 (i)}))</td>
<td>((\text{by Th.2.1 (i)}))</td>
</tr>
<tr>
<td>Feasible (\zeta_d(\mathbb{K}^*) = \infty)</td>
<td>(\times(\text{by Th.2.1 (i)}))</td>
<td>((\text{by w.duality}))</td>
<td>((f))</td>
</tr>
<tr>
<td>Infeasible (\zeta_d(\mathbb{K}^*) = −\infty)</td>
<td>(\times(\text{by Th.2.1 (i)}))</td>
<td>((d))</td>
<td>((g))</td>
</tr>
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</table>

Proof. Let \(\hat{\mathbb{L}}\) denote the subspace of \(\mathbb{V}\) generated by \(\mathbb{K}\). Then \(\mathbb{K}\) has a relative interior point with respect to \(\hat{\mathbb{L}}\), and \(\mathbb{K}^* \cap \hat{\mathbb{L}}\) is pointed, i.e.,

\[
\text{there is no nonzero } y \in \mathbb{K}^* \cap \hat{\mathbb{L}} \text{ such that } -y \in \mathbb{K}^* \cap \hat{\mathbb{L}},
\]

(see Section 6.1 and Figure 1) and

\[
\langle x, y \rangle > 0 \text{ if } x \in \text{rel.int}\mathbb{K} \text{ w.r.t. } \hat{\mathbb{L}} \text{ and } 0 \neq y \in \mathbb{K}^* \cap \hat{\mathbb{L}},
\]

where \(\text{rel.int}\mathbb{K} \text{ w.r.t. } \hat{\mathbb{L}}\) denotes the set of relative interior points of \(\mathbb{K}^*\) with respect to \(\hat{\mathbb{L}}\) (see Section 6.2 and Figure 1). Let \(\hat{q}\) and \(\hat{h}\) be the metric projections of \(q\) and \(h\) onto \(\hat{\mathbb{L}}\), respectively. By Lemma 2.2, \(\mathbb{P}(\mathbb{K})\) and \(\mathbb{D}(\mathbb{K}^*)\) are equivalent to \(\hat{\mathbb{P}}(\mathbb{K})\) and \(\hat{\mathbb{D}}(\mathbb{K}^*)\), respectively. Therefore, it suffices to show that \(\hat{\zeta}_p(\mathbb{K}) < \infty\), i.e., \(\hat{\mathbb{P}}(\mathbb{K})\) is feasible under the assumption \(-\infty < \hat{\zeta}_d(\mathbb{K}^*) < \infty\). (Note that \(-\infty < \hat{\zeta}_p(\mathbb{K})\) follows from weak duality.)

We see that \(\hat{h} \in \mathbb{K}^* \cap \hat{\mathbb{L}}\) is nonzero, since otherwise \(\hat{\zeta}_d(\mathbb{K}^*) = \infty\). Choose an arbitrary \(x \in \text{rel.int}\mathbb{K} \text{ w.r.t. } \hat{\mathbb{L}}\). By (5), \(\langle \hat{h}, x \rangle > 0\). Then \(\hat{x} = x/\langle \hat{h}, x \rangle\) is a feasible solution of \(\hat{\mathbb{P}}(\mathbb{K})\).

Lemma 2.4. Assume that \(-\infty < \zeta_p(\mathbb{K}) < \infty\). Then \(\zeta_p(\mathbb{K}) = \zeta_d(\mathbb{K}^*)\) and \(\mathbb{D}(\mathbb{K}^*)\) is solvable.

Proof. This lemma follows from [4, Lemma 2.3] whose proof relies on the duality theorem for the pair of general COPs [10, Theorem 4.2]. We present a more direct proof using the separation theorem on convex sets for the theorem and proof to be self-contained.

As in the proof of the previous lemma, let \(\hat{\mathbb{L}}\) be the subspace of \(\mathbb{V}\) generated by \(\mathbb{K}\), and \(\hat{q}\) and \(\hat{h}\) be the metric projections of \(q\) and \(h\) onto \(\hat{\mathbb{L}}\), respectively. By Lemma 2.2, \(\mathbb{P}(\mathbb{K})\) is equivalent to \(\hat{\mathbb{P}}(\mathbb{K})\) and dual COP \(\mathbb{D}(\mathbb{K}^*)\) is equivalent to \(\hat{\mathbb{D}}(\mathbb{K}^*)\). Therefore, it suffices to show that \(\hat{\zeta}_p(\mathbb{K}) = \hat{\zeta}_d(\mathbb{K}^*)\) and \(\hat{\mathbb{D}}(\mathbb{K}^*)\) is solvable under the assumption \(-\infty < \hat{\zeta}_p(\mathbb{K}) < \infty\). Since \(\hat{\mathbb{P}}(\mathbb{K})\) is feasible, we know that \(0 \neq \hat{h} \in \mathbb{K}^* \cap \hat{\mathbb{L}}\). Let

\[
S = \left\{ (\langle \hat{q}, x \rangle - \hat{\zeta}_p(\mathbb{K})), (\hat{h}, x) - 1) : x \in \mathbb{K} \right\} \quad \text{and} \quad T = \{(s, 0) : s < 0\}.
\]

Then both \(S\) and \(T\) are convex subsets in \(\mathbb{R}^2\) and \(S \cap T = \emptyset\). By the separation theorem on convex sets (see for example [11, Theorem 11.3]), there exist an \((\alpha, \beta) \neq (0, 0)\) and a \(\gamma\) such that

\[
\alpha(\langle \hat{q}, x \rangle - \hat{\zeta}_p(\mathbb{K}))+ \beta(\langle \hat{h}, x \rangle - 1) \geq \gamma \geq \alpha s + \beta \times 0 \quad \text{for every } x \in \mathbb{K} \text{ and } s < 0; \quad \text{hence,} \quad \langle \alpha \hat{q} + \beta \hat{h}, x \rangle - (\alpha \hat{\zeta}_p(\mathbb{K}) + \beta) \geq \gamma \geq \alpha s \quad \text{for every } x \in \mathbb{K} \text{ and } s < 0,
\]
which implies that
\[
\langle \alpha \hat{q} + \beta \hat{h}, x \rangle \geq 0 \quad \text{(since } K \text{ is a cone)}, \quad \gamma \geq 0, \quad \alpha \geq 0, \quad 0 \geq \alpha \hat{\zeta}_p(K) + \beta.
\]

Assume that \( \alpha = 0 \). Then \( \beta < 0 \) and \( \langle \beta \hat{h}, x \rangle \geq 0 \) for every \( x \in K \); hence \( -\hat{h} \in K^* \).

Since \( \hat{h} \) also lies \( \hat{L} \), we see that \( -\hat{h} \in K^* \cap \hat{L} \) and \( \hat{h} \in K^* \cap \hat{L} \). This contradicts to (4) and \( \hat{h} \neq 0 \). Therefore, we have shown that \( \alpha > 0 \). Letting \( t^* = -\beta/\alpha \), we obtain that
\[
\langle \hat{q} - \hat{h}t^*, x \rangle \geq 0 \quad \text{for every } x \in K \quad \text{and } 0 \geq \hat{\zeta}_p(K) - t^* \quad \text{or equivalently},
\]
\[
\hat{q} - \hat{h}t^* \in K^* \cap \hat{L} \quad \text{(hence } t^* \text{ is a feasible solution of } \hat{D}(K^*) \text{ and } t^* \geq \hat{\zeta}_p(K),
\]
which together with the weak duality inequality \( \hat{\zeta}_d(K^*) \leq \hat{\zeta}_p(K) \) implies \( t^* = \hat{\zeta}_d(K^*) = \hat{\zeta}_p(K) \).

**Lemma 2.5.** Assume that \( -\infty < \zeta_p(K) < \infty \) or \( -\infty < \zeta_d(K^*) < \infty \). Condition Td'(K*) is satisfied if and only if the set of optimal solutions of primal COP P(K) is nonempty and bounded.

**Proof.** “only if part”: By Lemma 2.3, we have already seen that \( -\infty < \zeta_p(K) < \infty \) is implied by \( -\infty < \zeta_d(K) < \infty \). So we assume the former. Assume on the contrary that the set of optimal solutions of P(K) is either empty or unbounded. Since \( -\infty < \zeta_p(K) < \infty \) by the assumption, there exist an \( \epsilon \geq 0 \) and a sequence \( \{x^k\} \) such that
\[
x^k \in K, \quad \langle q, x^k \rangle \leq \zeta_p(K) + \epsilon, \quad \langle h, x^k \rangle = 1 \tag{6}
\]
for \( k = 1, 2, \ldots \), and \( \|x^k\| \to \infty \) as \( k \to \infty \). (If the set of optimal solutions is nonempty, we can take \( \epsilon = 0 \) and \( \langle q, x^k \rangle = \zeta_p(K) \). We may assume without loss of generality that \( x^k/\|x^k\| \in K \) converges to some \( \Delta x \in K \) with \( \|\Delta x\| = 1 \). Dividing the relation (6) by \( \|x^k\| \) and taking the limit as \( k \to \infty \), we see that
\[
\Delta x \in K, \quad \|\Delta x\| = 1, \quad \langle q, \Delta x \rangle \leq 0, \quad \langle h, \Delta x \rangle = 0.
\]

By Condition Td'(K*), there exists a \( \tilde{t} \in \mathbb{R} \) such that \( \tilde{y} = q - h\tilde{t} \in \text{int}(K^*) \). If we take a sufficiently small \( \delta > 0 \) such that \( \tilde{y} - \delta \Delta x = q - h\tilde{t} - \delta \Delta x \in K^* \), then \( 0 \leq \langle q - h\tilde{t} - \delta \Delta x, \Delta x \rangle \leq -\delta < 0 \), which is a contradiction.

“if part”: Assume on the contrary that the line \( S = \{z = q - ht : t \in \mathbb{R}\} \) does not intersect with \( \text{int}(K^*) \). By the separation theorem on convex sets (see, for example, [11, Theorem 11.3]), there exist an \( \alpha \in \mathbb{R} \) and a nonzero \( \Delta x \in V \) such that
\[
\langle y, \Delta x \rangle \geq \alpha \geq \langle z, \Delta x \rangle \quad \text{if } y \in K^* \text{ and } z \in S.
\]
Since \( K^* \) is a closed cone containing \( 0 \in V \), we see that
\[
\langle y, \Delta x \rangle \geq 0 \quad \text{for every } y \in K^*, \quad \text{hence } 0 \neq \Delta x \in K,
\]
\[
0 \geq \alpha \geq \langle q - ht, \Delta x \rangle \quad \text{for every } t \in \mathbb{R}; \quad \text{hence } \langle h, \Delta x \rangle = 0 \text{ and } 0 \geq \langle q, \Delta x \rangle.
\]

Let \( x^* \) be an optimal solution of P(K) whose existence is guaranteed by the assumption of the “if” part. Then,
\[
x^* + \mu \Delta x \in K, \quad \langle h, x^* + \mu \Delta x \rangle = 1;
\]
(hence \( x^* + \mu \Delta x \) is a feasible solution of P(K)),
\[
\zeta_p(K) = \langle q, x^* \rangle \geq \langle q, x^* + \mu \Delta x \rangle \geq \langle q, x^* \rangle = \zeta_p(K)
\]

hold for all \( \mu \geq 0 \). This implies that \( \{ x^* + \mu \Delta x : \mu \geq 0 \} \) forms an unbounded ray in the set of optimal solutions of \( P(K) \). This contradicts to the assumption of “if” part. \( \square \)

**Proof of Theorem 2.1:** (i) \( -\infty < \zeta_d(K) < \infty \Rightarrow -\infty < \zeta_p(K) < \infty \) is shown by Lemma 2.3 and \( -\infty < \zeta_p(K) < \infty \Rightarrow -\infty < \zeta_p(K) = \zeta_d(K^*) < \infty \) by Lemma 2.4. Assertion (ii) follows from Lemmas 2.3 and Lemmas 2.4, and assertion (iii) from Lemma 2.5.

### 3 Proofs of Theorem 1.1 and related results

Throughout this section, we use the notation and symbols given in (2). We need the following lemmas to prove Theorem 1.1.

**Lemma 3.1.** Let \( K_1 \) and \( K_2 \) be nonempty closed convex cones in \( V \). Then \( (K_1 \cap K_2)^\ast\) coincides with the closure of \( K_1^\ast + K_2^\ast\), denoted by \( cl(K_1^\ast + K_2^\ast) \), and \( int((K_1 \cap K_2)^\ast) = int(K_1^\ast + K_2^\ast) \).

**Proof.** The assertion of the lemma may be well-known, but we present a proof for the completeness of the paper.

(i) \( cl(K_1^\ast + K_2^\ast) \subset (K_1 \cap K_2)^\ast\): Suppose that \( y = y_1 + y_2 \), \( y_1 \in K_1^\ast \) and \( y_2 \in K_2^\ast \). Then \( \langle x, y_1 \rangle \geq 0 \) for every \( x \in K_1 \cap K_2 \subset K_1 \) and \( \langle x, y_2 \rangle \geq 0 \) for every \( x \in K_1 \cap K_2 \subset K_2 \). Hence, \( \langle x, y \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle \geq 0 \) for every \( x \in K_1 \cap K_2 \), which implies \( y \in (K_1 \cap K_2)^\ast \). Since \( (K_1 \cap K_2)^\ast \) is closed, we obtain \( cl(K_1^\ast + K_2^\ast) \subset (K_1 \cap K_2)^\ast \).

(ii) \( (K_1 \cap K_2)^\ast \subset cl(K_1^\ast + K_2^\ast) \): By assuming that \( \tilde{y} \not\in cl(K_1^\ast + K_2^\ast) \), we show that \( \tilde{y} \not\in (K_1 \cap K_2)^\ast \). By the separation theorem on convex sets (See for example [11, Corollary 11.4.2]), there exist a nonzero \( \tilde{x} \in V \) and an \( \alpha \in \mathbb{R} \) such that

\[
\langle \tilde{x}, y \rangle > \alpha > \langle \tilde{x}, \tilde{y} \rangle \quad \text{for every} \quad y \in cl(K_1^\ast + K_2^\ast).
\]

Since \( cl(K_1^\ast + K_2^\ast) \) is a closed convex cone such that \( cl(K_1^\ast + K_2^\ast) \supset (K_1^\ast + K_2^\ast) \ni 0 \), we then obtain that

\[
\langle \tilde{x}, y_1 + y_2 \rangle \geq 0 > \alpha > \langle \tilde{x}, \tilde{y} \rangle \quad \text{for every} \quad y_1 \in K_1^\ast \quad \text{and} \quad y_2 \in K_2^\ast.
\]

This implies that \( \tilde{x} \in (K_1^\ast)^\ast \cap (K_2^\ast)^\ast = K_1 \cap K_2 \) and \( \tilde{y} \not\in (K_1 \cap K_2)^\ast \).

Assertion \( int((K_1 \cap K_2)^\ast) = int(K_1^\ast + K_2^\ast) \) follows from \( cl(K_1^\ast + K_2^\ast) = (K_1 \cap K_2)^\ast \) and the convexity of \( K_1^\ast + K_2^\ast \). See Section 6.3 for details. \( \square \)

**Lemma 3.2.** Let \( K = K_1 \cap K_2 \). Assume that \( -\infty < \eta_d < \infty \) and Condition \( Td'(K_1^\ast + K_2^\ast) \) is satisfied. Then \( \eta_d = \zeta_d(K_1^\ast + K_2^\ast) = \zeta_d(K^\ast) \).

**Proof.** The identity \( \eta_d = \zeta_d(K_1^\ast + K_2^\ast) \) holds by definition. Since \( \zeta_d(K_1^\ast + K_2^\ast) \leq \zeta_d(K^\ast) \) holds by \( K_1^\ast + K_2^\ast \subset K^\ast \), it suffices to show that \( \zeta_d(K^\ast) \leq \zeta_d(K_1^\ast + K_2^\ast) \). Let \( t \) be a feasible solution of dual COP \( D(K^\ast); \quad q - h \tilde{t} \in \{ q - h \tilde{t} \mid \forall \tilde{t} \in K_1^\ast + K_2^\ast \} \) by Lemma 3.1. By Condition \( Td'(K_1^\ast + K_2^\ast) \), we know that \( q - h \tilde{t} \in \text{int}(K_1^\ast + K_2^\ast) \). Hence the convex combination

\[
(1 - \epsilon)(q - h \tilde{t}) + \epsilon(q - h \tilde{t}) \quad \text{with any small} \quad \epsilon \in (0, 1] \quad \text{lies in} \quad \text{int}(K_1^\ast + K_2^\ast).
\]

It follows that

\[
K_1^\ast + K_2^\ast \supset \text{int}(K_1^\ast + K_2^\ast) \ni (1 - \epsilon)(q - h \tilde{t}) + \epsilon((q - h \tilde{t}) = q - h(t + \epsilon(\tilde{t} - t))
\]

9
holds for any small \( \epsilon \in (0, 1] \). This implies that \( (t + \epsilon(\tilde{t} - t)) \) is a feasible solution of dual COP \( D(K_1^* + K_2^*) \) for any small \( \epsilon \in (0, 1] \). Therefore we conclude that \( \zeta_d(K^*) \leq \zeta_d(K_1^* + K_2^*) \).

**Proof of Theorem 1.1 (i)**: Let \( K = K_1 \cap K_2 \). Then primal COP (1) and \( P(K) \) are identical (hence \( \eta_p = \zeta_p(K) \)). If Condition \( T_p \) is satisfied, then dual COP (3) and \( D(K^*) \) are identical (hence \( \eta_d = \zeta_d(K^*) \)), and if Condition \( T_d \) is satisfied, then \( \eta_d = \zeta_d(K^*) \) holds by Lemma 3.2. Therefore, assertion (i) follows from assertion (i) of Theorem 2.1.

**Proof of Theorem 1.1 (ii)**: Let \( K = K_1 \cap K_2 \). Then primal COP (1) and \( P(K) \) are identical to \( P(K) \) and \( D(K^*) \), respectively. Hence assertion (ii) follows from assertion (ii) of Theorem 2.1.

**Proof of Theorem 1.1 (iii)**: Let \( K = K_1 \cap K_2 \). Then we can replace primal COP (1) with \( P(K) \) in assertion (iii) since they are identical. Since \( \text{int}(K_1^* + K_2^*) = \text{int}((K^*)) \) by Lemma 3.1, Condition \( T_d'(K_1^* + K_2^*) \) (= Condition \( T_d \)) holds if and only if Condition \( T_d'(K^*) \) holds. If Condition \( T_d'(K^*) \) is satisfied, then \( \zeta_d(K_1^* + K_2^*) = \zeta_d(K^*) \). Thus, \( -\infty < \zeta_p(K) < \infty \) or \( -\infty < \zeta_d(K^*) < \infty \) by the assumption of the theorem. Therefore, the set of optimal solutions of \( P(K) \), which coincides with the set of optimal solutions of primal COP \( P(K_1 \cap K_2) \), is nonempty and bounded by (iii) of Theorem 2.1. Conversely, if the set of optimal solutions of \( P(K_1 \cap K_2) = P(K) \) is nonempty bounded, which implies \( -\infty < \zeta_p(K) < \infty \), then Condition \( T_d'(K^*) = T_d'(K_1^* + K_2^*) \) (= Condition \( T_d \)) holds by (iii) of Theorem 2.1.

Figure 2 illustrates assertion (i) of Theorem 1.1. In case (a), neither \( T_p \) nor \( T_d'(K_1^* + K_2^*) \) is satisfied. The cone \( K_1^* + K_2^* \) is almost open except the solid green line that forms an extremal ray of the cone. The line \( \{q - ht : t \in \mathbb{R} \} \) touches the cone at the single point \( q - h t^* \), but its line segment between \( q - h t^* \) and \( q - h \tilde{t} \) is included in a face of its closure \( \text{cl}(K_1^* + K_2^*) = (K_1 \cap K_2)^* \). Hence \( \zeta_d(K_1^* + K_2^*) = t^* < \tilde{t} = \zeta_d((K_1 \cap K_2)^*) = \zeta_p(K_1 \cap K_2) \), which implies that a duality gap \( \tilde{t} - t^* > 0 \) exists between primal COP \( P(K_1 \cap K_2) \) and its dual \( D(K_1^* + K_2^*) \).

The cone \( K_1^* + K_2^* \) in case (b) is the same as in case (a), but the line \( \{q - ht : t \in \mathbb{R} \} \) penetrates the interior of the cone, so that \( T_d'(K_1^* + K_2^*) \) is satisfied, where \( \tilde{y} = q - h \tilde{t} \in \text{int}(K_1^* + K_2^*) \). Both \( D(K_1^* + K_2^*) \) and \( D((K_1 \cap K_2)^*) \) share the common optimal value \( t^* \).
although \( q - h t^* \not\in K_1^* + K_2^* \) and \( D(K_1^* + K_2^*) \) has no feasible solution that attains the optimal value \( t^* \). In this case, there is no duality gap between primal COP \( P(K_1 \cap K_2) \) and its dual \( D(K_1^* + K_2^*) \).

The lemma below shows two important cases where \((K_1 \cap K_2)^* = K_1^* + K_2^*\) holds. The first case corresponds to LPs, and the second case is an implication of Condition Sp.

**Lemma 3.3.** \((K_1 \cap K_2)^* = K_1^* + K_2^*\) holds if either of the following conditions (i) and (ii) is satisfied.

(i) \( K_1 \) and \( K_2 \) are polyhedral, i.e., \( K_i = \{ x \in V : \langle a_i^j, x \rangle \geq 0 \ (j = 1, \ldots, \ell_i) \} \) for some \( a_i^j \in V \ (j = 1, 2, \ldots, \ell_i, i = 1, 2) \).

(ii) There exists an \( \bar{x} \in K_1 \cap K_2 \) which lies in \( \text{int} K_1 \).

**Proof.** (i) In this case, we see that

\[
(K_1 \cap K_2)^* = \left\{ y = \sum_{j=1}^{\ell_1} a_1^j u_j + \sum_{k=1}^{\ell_2} a_2^k v_k : u_j \geq 0 \ (j = 1, \ldots, \ell_1), v_k \geq 0 \ (k = 1, \ldots, \ell_2) \right\} = K_1^* + K_2^*.
\]

(ii) We prove that \( K_1^* + K_2^* \) is closed. Then the desired result follows from Lemma 3.1. We consider a sequence \( \{ y^k \in K_1^* + K_2^* \} \) which converges to some \( y \in V \), and show that \( y \in K_1^* + K_2^* \). Each \( y^k \) can be represented as \( y^k = u^k + v^k \) with some \( u^k \in K_1^* \) and \( v^k \in K_2^* \). Since the sequence \( \{ y^k \} \) is bounded, we can take a positive number \( \rho \) such that

\[
\langle \bar{x}, v^k \rangle \geq 0 \quad \text{and} \quad \rho \geq \langle \bar{x}, u^k + v^k \rangle \geq \langle \bar{x}, y^k \rangle \quad \text{for all} \quad k.
\]

Since \( \bar{x} \in \text{int} K_1 \), we know that the set \( U = \{ u \in K_1^* : \langle \bar{x}, u \rangle \leq \rho \} \) is bounded. (See Section 6.4 and also cases (a) and (c) of Figure 1.) Hence \( \{ u^k \} \) converges to some \( u \in K_1^* \) along a subsequence. Therefore, \( \{ v^k = y^k - u^k \} \) converges to some \( v \in K_2^* \) along the subsequence, and we obtain \( y = \bar{u} + \bar{v} \in K_1^* + K_2^* \). Thus, we have shown that \( K_1^* + K_2^* \) is closed.

The lemma below shows that \( \text{Td}'(K_1^* + K_2^*) \) is weaker than Condition Sd.

**Lemma 3.4.** Assume that Condition Sd hold. Then Condition \( \text{Td}'(K_1^* + K_2^*) \) holds.

**Proof.** Assuming Condition Sd, we show that \( \tilde{y} = \tilde{y}_1 + \tilde{y}_2 = q - h \tilde{t} \in \text{int}(K_1^* + K_2^*) \). Since \( \tilde{y}_1 \in \text{int} K_1^* \), we can take a positive number \( \epsilon \) such that \( y_1 \in K_1^* \) if \( \| y_1 - \tilde{y}_1 \| \leq \epsilon \). Assume that \( y \in V \) and \( \| y - \tilde{y} \| \leq \epsilon \). Let \( y_1 = \tilde{y}_1 + (y - \tilde{y}) \). Then \( y = y_1 + y_2, \quad \| y_1 - \tilde{y}_1 \| = \| y - \tilde{y} \| \leq \epsilon \) (hence \( y_1 \in K_1^* \)), and \( y_2 \in K_2^* \) by Condition Sd. Therefore, \( y \in K_1^* + K_2^* \) and we have shown that \( \tilde{y} \in \text{int}(K_1^* + K_2^*) \). 

\[\square\]
4 Strong duality in a symmetric primal-dual pair of COPs

Let $E_p$ and $E_d$ be finite dimensional vector spaces with inner products, which are both denoted by $\langle \cdot, \cdot \rangle$, and $J_p$ and $J_d$ closed convex cones of $E_p$ and $E_d$, respectively. Let $b \in E_d$, $c \in E_p$, $A$ a linear map from $E_p$ into $E_d$. We denote the adjoint of $A$ by $A^\ast$. We consider the following primal-dual pair of COPs:

\[
\theta_p = \inf \{ \langle c, u \rangle : u \in J_p, \ A u - b \in J_d \} . \tag{7}
\]
\[
\theta_d = \sup \{ \langle b, v \rangle : v \in J_d^\ast, \ c - A^\ast v \in J_p^\ast \} . \tag{8}
\]

The following two convex cones play essential roles in characterizing strong duality of the primal-dual pair above in the subsequent discussion.

\[
M_p = \{ (\alpha, v - A u) : \alpha \geq \langle c, u \rangle, \ u \in J_p, \ v \in J_d \} , \]
\[
N_p = \{ v - A u : u \in J_p, \ v \in J_d \} .
\]

These cones were originally introduced by Shapiro [14] for the following result.

**Theorem 4.1.** [14, Proposition 2.6, Proposition 2.8] Assume that $-\infty < \theta_p < \infty$.

(i) If $M_p$ is closed or $-b \in \text{int} N_p$, then $-\infty < \theta_p = \theta_d < \infty$ holds.

(ii) If $M_p$ is closed, then primal COP (7) is solvable.

(iii) $-b \in \text{int} N_p$ holds if and only if the set of optimal solutions of dual COP (8) is nonempty and bounded.

**Remark 4.2.** Shapiro in [14] dealt with the case where the spaces $E_p$ and $E_d$ can be a general vector (not necessarily finite dimensional) space. Assertions above are stated when $E_d$ is the finite dimensional case. See [14, Proposition 2.8]

The purpose of this section is to derive the following result from Theorem 1.1.

**Theorem 4.3.** Assume that $-\infty < \theta_p < \infty$ or $-\infty < \theta_d < \infty$. Then assertions (i), (ii) and (iii) of Theorem 4.1 hold.

In Theorems 4.1 and 4.3, the same characterizations of strong duality $-\infty < \theta_p = \theta_d < \infty$, solvability of primal COP (7) and solvability of dual COP (8) are discussed. The underlying assumption in Theorem 4.3 is, however, weaker than that in Theorems 4.1, and Theorem 4.3 clearly covers Theorem 4.1. In particular, we obtain:

**Corollary 4.4.** If the set of optimal solutions (or the feasible region) of primal COP (7) or dual COP (8) is nonempty and bounded, then $-\infty < \theta_p = \theta_d < \infty$.

**Proof.** Assume that the set of optimal solutions of dual COP (8) is nonempty and bounded. Then, by the assumption and (iii) of Theorem 4.3, we see that $-\infty < \theta_d < \infty$ and $-b \in \text{int} N_p$ hold. By (i) of Theorem 4.3, we obtain that $-\infty < \theta_p = \theta_d < \infty$. Since the primal-dual pair of COPs (7) and (8) is symmetric, we can derive the same conclusion even if we regard COP (8) as primal and COP (7) as dual. See the discussions in Sections 4.1 and 4.2. This corollary could also be obtained from Corollary 1.2
It should be noted this result cannot be derived from Theorem 4.1 since the relaxed part “or \(-\infty < \theta_d < \infty\)” of the assumption plays a critical role in the proof above.

We convert primal COP (7) to COP (1) in Sections 4.1, and dual COP (8) to COP (1) in Section 4.2. The proof of Theorem 4.3 is stated in Section 4.3 where the latter conversion given in Section 4.2 is utilized. Although Sections 4.1 is not relevant to the proof in Section 4.2, the discussion in Section 4.1 will help the reader to derive such counter part. We also mention some remarks on Theorem 3.1 of [1], which is related to (i) of Theorem 4.3, in Section 4.4.

4.1 Conversion of the primal-dual pair of COP (7) and COP (8) to the primal-dual pair of COP (1) and COP (3)

Let

\[
\begin{align*}
V &= \mathbb{R} \times \mathbb{E}_p = \{(x_0, u) : x_0 \in \mathbb{R}, \ u \in \mathbb{E}_p\}, \\
K_1 &= \mathbb{R}_+ \times \mathbb{J}_p, \ K_2 = \{(x_0, u) \in V : x_0 \in \mathbb{R}, \ \mathcal{A}u - bx_0 \in \mathbb{J}_d\}, \\
q &= (0, c) \in V, \ h = (1, 0) \in V.
\end{align*}
\]

(9)

It should be noted that the symbols \(V, K_1, K_2, q\) and \(h\) above are defined locally in this section. The same symbols will be also defined locally in Section 4.2 with different meanings. As the inner product of \(x = (x_0, u) \in V, \ y = (t_0, w) \in V\), we define \(\langle x, y \rangle = x_0t_0 + \langle u, w \rangle\). Then,

\[
\begin{align*}
K_1^* &= \mathbb{R}_+ \times \mathbb{J}_p^*, \ K_2^* = \{\langle -b, v \rangle, \mathcal{A}^*v : v \in \mathbb{J}_d^*\}, \\
& \quad u \in \mathbb{J}_p, \ \mathcal{A}u - b \in \mathbb{J}_d \text{ if and only if } x = (x_0, u) \in K_1, \ x \in K_2, \ \langle h, x \rangle = 1.
\end{align*}
\]

Thus, we can rewrite COP (7) as COPs (1). We also see that

\[
q - ht - y_2 = (0, c) - (t, 0) - \langle -b, v \rangle, \mathcal{A}^*v
\]

\[
= \langle b, v \rangle - t + c - \mathcal{A}^*v
\]

if \(y_2 = \langle -b, v \rangle, \mathcal{A}^*v \in K_2^*\) for some \(v \in \mathbb{J}_d^*\); hence

\[
\langle b, v \rangle \geq t \text{ and } c - \mathcal{A}^*v \in \mathbb{J}_p^* \text{ if and only if } q - ht \in K_1^* + K_2^*.
\]

Therefore, COP (8) can be rewritten as COP (3). In this case, \(K_1^* + K_2^*\) is represented as

\[
\{(\langle -b, v \rangle + s, \mathcal{A}^*v + u) : s \in \mathbb{R}_+, \ u \in \mathbb{J}_p^*, v \in \mathbb{J}_d^*\}.
\]

Obviously, \(K_1^* + K_2^*\) is closed if and only if

\[
\mathcal{M}_d = \{\langle \beta, \mathcal{A}^*v + u \rangle : \beta \leq \langle b, v \rangle, \ u \in \mathbb{J}_p^*, v \in \mathbb{J}_d^*\}
\]

is closed. The lemma below relates the existence of an interior feasible solution of dual COP (8) to Condition Td.

Lemma 4.5. Condition Td with \(K_1, K_2, q\) and \(h\) given in (9) holds if and only if \(c \in \text{int}N_d\), where \(N_d = \{u + \mathcal{A}^*v : u \in \mathbb{J}_p^*, v \in \mathbb{J}_d^*\}\).
Proof. “only if part”: Assume that \( q - h\tilde{t} = (-\tilde{t}, c) \) lies in \( \text{int}(K_1^* + K_2^*) \). Then there exists \((\tilde{s}, \tilde{u}, \tilde{v}) \in \mathbb{R}_+ \times J_p^* \times J_d^* \) such that

\[
(-\tilde{t}, c) = \langle -b, v \rangle + \tilde{s}, \tilde{u} + A^*\tilde{v} \in \text{int}(K_1^* + K_2^*)
\]

which implies \( c = \tilde{u} + A^*\tilde{v} \in \text{int}N_d \).

“if part”: Assume that \( c = \tilde{u} + A^*\tilde{v} \in \text{int}N_d \). If we take \( \tilde{s} > 0 \) and let \( \tilde{t} = -\langle -b, \tilde{v} \rangle + \tilde{s} \), then \( q - h\tilde{t} = (-\tilde{t}, c) = \langle -b, \tilde{v} \rangle + \tilde{s}, \tilde{u} + A^*\tilde{v} \in \text{int}(K_1^* + K_2^*) \).

**4.2 Conversion of the primal-dual pair of COP (7) and COP (8) to the dual-primal pair of COP (3) and COP (1)**

Since the primal-dual pair of COPs (7) and (8) is symmetric, we can interchange their role as follows:

\[
\eta_p = -\theta_d = \inf \left\{ \langle -b, v \rangle : v \in J_d^*, \ c - A^*v \in J_p^* \right\} \\
= \inf \left\{ \langle \tilde{b}, v \rangle : v \in J_d^*, \ -\tilde{c} - A^*v \in J_p^* \right\} \quad (10)
\]

\[
\eta_d = -\theta_p = \sup \left\{ \langle -c, u \rangle : u \in J_p^*, \ Au - b \in J_d^* \right\} \\
= \sup \left\{ \langle \tilde{c}, u \rangle : u \in J_p^*, \ Au + \tilde{b} \in J_d^* \right\} \quad (11)
\]

Here \( \tilde{b} = -b \) and \( \tilde{c} = -c \). Now we regard COP (10) induced from dual COP (8) as primal, and COP (11) induced from primal COP (7) as dual. Let

\[
\begin{align*}
V & = \mathbb{R} \times E_d = \{(x_0, v) : x_0 \in \mathbb{R}, \ v \in E_d \}, \\
K_1 & = \mathbb{R}_+ \times J_d^*, \ K_2 = \{(x_0, v) \in V : -\tilde{c}x_0 - A^*v \in J_p^* \}, \\
q & = (0, \tilde{b}) \in V, \ h = (1, 0) \in V.
\end{align*}
\]

Then we can similarly show as in Section 4.1 that COPs (10) and (11) are equivalent to COPs (1) and (3), respectively. As a result, the pair of dual COP (8) and primal COPs (7) is equivalently reformulated as the primal-dual pair of COPs (1) and COPs (3) with \( V, K_1, K_2, q \) and \( h \) given in (12). We also see that

\[
K_1^* + K_2^* = \left\{ (\alpha, v - Au) : \alpha \geq \langle c, u \rangle, \ u \in J_p^*, \ v \in J_d^* \right\} = M_p.
\]

The following lemma can be proved similarly to Lemma 4.5.

**Lemma 4.6.** Condition Td with \( K_1, K_2, q \) and \( h \) given in (12) holds if and only if \(-b \in \text{int}N_p\).

**4.3 Proof of Theorem 4.3**

We reformulate the pair of dual COP (8) and primal COP (7) as the pair of primal COP (1) and dual COP (3) with \( V, K_1, K_2, q \) and \( h \) given in (12) as discussed in
the previous section, and apply Theorem 1.1 to the reformulated pair. We know that $\eta_p = -\theta_d$, $\eta_d = -\theta_p$, Condition Tp is equivalent to the closedness of $M_p$, and Condition Td is equivalent to $-b \in \text{int}N_p$. Therefore, assertions (i), (ii) and (iii) of Theorem 4.3 follow from assertions (i), (ii) and (iii) of Theorem 1.1, respectively.

4.4 Remarks on Theorem 3.1 of [1]

In this section, we make some remarks on the following result, which is closely related to assertions (i) and (ii) of Theorems 4.1 and 4.3.

Theorem 4.7. ([1, Theorem 3.1]) Assume that $-\infty < \theta_p < \infty$ and $-\infty < \theta_d < \infty$. If $\mathbb{M}_p = \{(\langle c, u \rangle, v - Au) : u \in \mathbb{J}_p, v \in \mathbb{J}_d\}$ is closed. Then $-\infty < \theta_p = \theta_d < \infty$ and COP (7) is solvable.

Obviously, $\mathbb{M}_p$ is a convex cone contained in $\mathbb{M}_p$, which is used in Theorems 4.3. It is clear that if $M_p$ is closed, then so is $\mathbb{M}_p$. Hence, the closedness of $\mathbb{M}_p$ assumed in Theorem 4.7 may be weaker than the closedness of $M_p$ assumed in Theorems 4.1 and 4.3. On the one hand, Theorem 4.7 imposes the assumption that $-\infty < \theta_p < \infty$ and $-\infty < \theta_d < \infty$, which is stronger than the assumption $-\infty < \theta_p < \infty$ or $-\infty < \theta_d < \infty$ of Theorem 4.3. The following lemma exhibits their relation more precisely.

Lemma 4.8. Assume that $-\infty < \theta_p < \infty$. If $\mathbb{M}_p$ is closed, then $M_p$ is closed.

Proof. To show the closedness of $M_p$, let $\{(a^k, d^k)\}$ be a sequence in $M_p$ which converges to some $(\bar{a}, \bar{d}) \in \mathbb{R} \times \mathbb{E}_p$. We will show that $(\bar{a}, \bar{d}) \in M_p$. Since each $(a^k, d^k) \in M_p$, there exist $u^k \in \mathbb{J}_p$, $v^k \in \mathbb{J}_d$ and $s^k \in \mathbb{R}_+$ such that

$$\langle c, u^k \rangle = a^k - s^k \quad \text{and} \quad v^k - Au^k = d^k.$$  \(13\)

We consider the two cases: the first case where $\{s^k\}$ is bounded and the second case where $s^k \to \infty$ as $k \to \infty$. First, assume that $\{s^k\}$ is bounded. Then $\{s^k\}$ converges to some $\bar{s} \in \mathbb{R}_+$ along a subsequence. Hence, we may regard that the subsequence $\{(a^k - s^k, d^k)\}$ is a sequence in $\mathbb{M}_p$ which converges to $(\bar{a} - \bar{s}, \bar{d})$. Since $\mathbb{M}_p$ is closed, there exist $\bar{u} \in \mathbb{J}_p$ and $\bar{v} \in \mathbb{J}_d$ such that $(\bar{a} - \bar{s}, \bar{d}) = (\langle c, \bar{u} \rangle, \bar{v} - A\bar{u})$. It follows that $(\bar{a}, \bar{d}) = (\langle c, \bar{u} \rangle + \bar{s}, \bar{v} - A\bar{u}) \in \mathbb{M}_p$.

Now, assume that $s^k \to \infty$ as $k \to \infty$. We may assume that each $s^k$ is positive. Dividing the two identities by $s^k$ in (13), we obtain that

$$\langle c, u^k/s^k \rangle = a^k/s^k - 1 \quad \text{and} \quad v^k/s^k - Au^k/s^k = d^k/s^k.$$ 

Since $u^k/s^k \in \mathbb{J}_p$ and $v^k/s^k \in \mathbb{J}_d$, the sequence $\{(a^k/s^k - 1, d^k/s^k)\}$ is included in $\mathbb{M}_p$ and converges to $(-1, 0)$. Since $\mathbb{M}_p$ is closed, there exist $\Delta u \in \mathbb{J}_p$ and $\Delta v \in \mathbb{J}_d$ such that $(\langle c, \Delta u \rangle, \Delta v - A\Delta u) = (-1, 0)$, which implies that $A\Delta u = \Delta v \in \mathbb{J}_d$ and $\langle c, \Delta u \rangle = -1$. Let $\tilde{u}$ be a feasible solution of primal COP (7) whose existence is guaranteed by the assumption. Then we have that

$$\tilde{u} + \mu \Delta u \in \mathbb{J}_p, \quad A(\tilde{u} + \mu \Delta u) \in \mathbb{J}_d \quad \text{for every} \quad \mu \geq 0 \quad \text{and} \quad \langle c, \tilde{u} + \mu \Delta u \rangle = \langle c, \tilde{u} \rangle - \mu \to -\infty \quad \text{as} \quad \mu \to \infty,$$

which implies that $\theta_p = -\infty$. This contradicts the assumption, and this case cannot occur. \(\square\)
5 Concluding remarks

In [8], Kim, Kojima and Toh formulated a DNN relaxation of a class of linearly constrained QOPs in nonnegative and binary variables as a COP of the form

$$\varphi_p = \inf \{ \langle q, x \rangle : x \in K_1, \langle h, x \rangle = 1, \langle h_1, x \rangle = 0 \},$$

(14)

where $K_1$ is a closed convex cone in the space of symmetric matrices, $0 \neq h \in K_1^*$ and $h_1 \in K_1^*$. The dual of COP (14) can be written as

$$\varphi_d = \sup \{ t : q - ht - hs \in K_1^* \}.$$  

(15)

Under the assumption that ensures the feasible region of COP (14) is nonempty and bounded, they proved the strong duality that $-\infty < \varphi_p = \varphi_d < \infty$ [8, Lemma 3] (see also [4, Theorem 2.6] for the same assertion under a weaker assumption). By applying a Lagrangian relaxation to COPs (14) and (15), they induced the primal-dual pair of COPs of the form (1) and the form (3), which has no duality gap, with a Lagrangian multiplier parameter $\lambda$ associated with the constraint $\langle h_1, x \rangle = 1$. The strong duality equality $\varphi_p = \varphi_d$ was derived by taking the limit of their optimal values with no gap as $\lambda \to \infty$. This result was used in the development of the Newton-bracketing method whose convergence is quadratic for solving the Lagrangian-DNN relaxation of the aforementioned class of QOPs in [9]. See [4, 9] for more details.

The primal-dual pair of COPs (14) and (15) can be reformulated as the primal-dual pair of COPs (1) and (3) with $K_2 = \{ x \in V : \langle h_1, x \rangle = 0 \}$. Under their assumption, the set of optimal solutions of primal COP (1) is nonempty and bounded. Hence, their strong duality can be derived directly from Corollary 1.2 (or Theorem 1.1) without relying on the Lagrangian relaxation of COP (14). Since $h_1 \in K_1^*$, $K_2$ forms a supporting hyperplane of $K_1$ and $K_1 \cap K_2$ an exposed face of $K_1$ in their formulation. Any $x \in K_2$ cannot be an interior point of $K_1$. Therefore, the identity $(K_1 \cap K_2)^* = K_1^* + K_1^*$ is not guaranteed by (ii) of Lemma 3.3. However, the authors conjecture that the identity holds.

6 Appendix — Supplements for the proofs of Lemmas 2.3, 3.2 and 3.3

6.1 Proof of (4): “$K^* \cap \hat{L}$ is pointed”

Here $\hat{L}$ is the linear subspace of $V$ generated by $K$. Assume on the contrary that $K^* \cap \hat{L}$ is not pointed. Then, there is a nonzero $\overline{Y} \in K^* \cap \hat{L} \subset \hat{L}$ such that $-\overline{Y} \not\in K^* \cap \hat{L} \subset \hat{L}$. Hence $\langle \overline{Y}, X \rangle = 0$ for every $X \in K$, which implies $\overline{Y} \in \hat{L}^\perp$ since $\hat{L}$ is the linear space generate by $K$. Therefore, $\overline{Y} \in \hat{L} \cap \hat{L}^\perp = \{ O \}$. This contradicts to $\overline{Y} \neq O$. □

6.2 Proof of (5): “$\langle x, y \rangle > 0$ if $x \in \text{rel.int}\,K$ w.r.t. $\hat{L}$ and $0 \neq y \in K^* \cap \hat{L}$”
Here the linear subspace of generated by . Assume on the contrary that . Then, there exists an such that if . Hence, . Since , which implies that . Thus we have shown that , a contradiction to .

6.3 Proof of “int( + ) = int( )”

Since int( + ) ⊂ int( ) follows from cl( + ) = ( ), we only show that int( ) ⊂ int( + ). Let . Then there exists an such that such that . Since the closure of coincides with ( ), we can choose a simplex in such that . Since belong to + , Since + is convex, we see that . Therefore, .

6.4 Proof of “if then is bounded”

Assume on the contrary that is unbounded. Then there exists a sequence such that 

\[ u^k \in K_1^*, \langle \bar{x}, u^k \rangle \leq \rho \text{ and } \|u^k\| \to \infty \text{ as } k \to \infty. \]

We may assume without loss of generality that converges to some in as . Dividing the above relation by and taking the limit as , we have . Since there exists a such that \[ \bar{x} - \delta \Delta u \in K_1, \] we see that \[ 0 \leq \langle \bar{x} - \delta \Delta u, \Delta u \rangle = -\delta < 0, \] a contradiction.

References


