Inexact Restoration for Minimization with Inexact Evaluation both of the Objective Function and the Constraints *

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Abstract

In a recent paper an Inexact Restoration method for solving continuous constrained optimization problems was analyzed from the point of view of worst-case functional complexity and convergence. On the other hand, the Inexact Restoration methodology was employed, in a different research, to handle minimization problems with inexact evaluation and simple constraints. These two methodologies are combined in the present report, for constrained minimization problems in which both the objective function and the constraints, as well as their derivatives, are subject to evaluation errors. Together with a complete description of the method, complexity and convergence results will be proved.

Key words: Inexact Restoration, Inexact Evaluations, Constrained Optimization .

AMS subject classifications: 90C30, 65K05, 49M37, 90C60, 68Q25.

1 Introduction

Consider an optimization problem given by

\[
\begin{align*}
\text{Minimize} & \quad F(x) \\
\text{subject to} & \quad H(x) = 0 \\
& \quad x \in \Omega,
\end{align*}
\]

where \( F : \mathbb{R}^n \to \mathbb{R}, H : \mathbb{R}^n \to \mathbb{R}^m \) and \( \Omega \) is a nonempty compact polytope. As usually, if inequality constraints \( G(x) \leq 0 \) are present, we reduce the problem to the standard form (1) by means of the addition of slack variables. Assume that exact evaluation of \( F(x), H(x) \) and

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their derivatives is not always possible. Instead, each evaluation of $F(x)$ (or $H(x)$) is, according to availability or convenience, replaced with $f(x, y)$ (or $h(x, y)$, respectively) where $y$ lies in an abstract set $Y$ and determines the degree of precision in which the objective function or the constraints are evaluated. We will assume that $g_f : Y \rightarrow \mathbb{R}_+$ is such that $f(x, y) = F(x)$ when $g_f(y) = 0$, $g_h : Y \rightarrow \mathbb{R}_+$ is such that $h(x, y) = H(x)$ when $g_h(y) = 0$ and that, roughly speaking, the precision in the evaluations improves when $g_f(y)$ and $g_h(y)$ decrease. Therefore, writing $g(y) = \max\{g_f(y), g_h(y)\}$, problem (1) is equivalent to:

$$\begin{align*}
\text{Minimize (with respect to } x & ) & f(x, y) \\
\text{subject to} & h(x, y) = 0, \\
& g(y) = 0, \\
& x \in \Omega, \\
& y \in Y.
\end{align*}$$

A solution of (2) could be obtained fixing $y \in \Omega$ in such a way that $g_f(y) = g_h(y) = 0$ and handling the resulting problem as a standard constrained optimization problem. However, we are interested in problems in which such procedure is not affordable because solving (2) fixing $g_f(y) = g_h(y) = 0$ is overwhelmingly expensive or even impossible.

The definition (2) makes sense independently of the meaning of $y, Y$, or $g(y)$. We have especially in mind the case in which $f(x, y)$ represents $F(x)$ with an error governed by $y \in Y$, $h(x, y)$ is $H(x)$ computed with an error that depends of $y$, $g_f(y) = 0$ if and only if $f(x, y) = F(x)$, and $g_h(y) = 0$ if and only if $h(x, y) = H(x)$ for all $x \in \Omega$. However, the results of this paper can be read without reference to this meaning.

In this paper we extend the results of [12] and [24]. In [12], the problem (2) is considered without the presence of the constraints $h(x, y) = 0$. In [24], an Inexact Restoration method with worst-case complexity results is introduced for solving the classical constrained optimization problem. The techniques of [12] and [24] are merged in the present paper in order to handle the constrained optimization problem with inexactness both in the objective function and the constraints.

Inexact Restoration (IR) methods for constrained continuous optimization were introduced in [42], inspired in several classical papers by Rosen [45] and Miele [44], among others. At each iteration of an IR algorithm feasibility is firstly improved and, then, optimality is improved along a tangent approximation of the feasible region. The so far generated trial point is accepted or not as new iterate according to the decrease of a merit function or using filter criteria [32, 46, 23, 26, 46]. Theoretical papers concerning Inexact Restoration methods for constrained optimization include [10, 24]. Algorithmic variations are discussed in [41, 27, 4, 22, 25, 26, 41], and applications may be found in [43, 1, 2, 37, 31, 30, 6, 39, 16, 47].

The idea of using the IR framework to deal with optimization problems in which the objective function is subject to evaluation errors comes from [40], where inexactness came from the fact that the evaluation was the result of an iterative process. Evaluating the function with additional precision was considered in [40] as a sort of inexact restoration. This basic principle was developed in [11] and [12], where complexity results were also proved. Moreover, in [13] the case in which derivatives are not available was considered. Inexactness of the objective function in optimization problems was addressed in several additional papers in recent years [7, 8, 9, 31, 35, 39, 33]. The objective of the present paper is to use the ideas of [40, 11, 12]
to handle the constrained optimization problem in which the evaluation of the objective functions and the constraints is subject to error. We will show that, although the main ideas are applicable, a number of technical difficulties appear whose solution offer additional insight in the problem. From the theoretical point of view we will prove convergence to feasible points (whenever possible) and asymptotic fulfillment of optimality conditions.

This paper is organized as follows. In Section 2 we describe BIRA, the main algorithm for solving 2. In Section 3 we state our final goal in terms of complexity and convergence of BIRA. In Section 4 we state general assumptions on the problem that will be used throughout the paper. In Section 5 we prove several theoretical results with respect to the Restoration Algorithm RESTA that will be useful in forthcoming sections. In Section 6 we prove that every iteration of BIRA is well defined. In Section 7 we prove convergence towards feasible points and convergence to zero of the difference between consecutive iterates. In Section 8 we finish up proving complexity and convergence of the main algorithm. Conclusions are stated in Section 9.

Notation

1. All along this paper $\| \cdot \|$ represents the Euclidean norm.
2. We define
   
   $$e(x, y) = \frac{1}{2} \| h(x, y) \|^2.$$  

3. $P_\Omega(z)$ denotes the Euclidean projection of $z$ onto $\Omega$.

2 Algorithms

In this section we define the Basic Inexact Restoration Algorithm (BIRA) for solving our main problem and two additional algorithms, ACCEL and RESTA, that are called at each iteration of BIRA. RESTA is a restoration procedure and ACCEL is an acceleration algorithm that allows us to progress without increasing precision of evaluations.

All along the paper we will use the merit function that combines objective function and constraints defined by means of the penalty parameter $\theta$ according to:

$$\Phi(x, y, \theta) = \theta f(x, y) + (1 - \theta) [\| h(x, y) \| + g(y)]$$

for all $x \in \Omega$, $y \in Y$, and $\theta \in [0, 1]$.

2.1 Basic Inexact Restoration Algorithm (BIRA)

The iterative Algorithm BIRA has four main steps. Each iteration begins with a Restoration Phase, at which, starting from the current iteration $x^k$ and the current precision variable $y^k$, one computes an inexactly restored $x^k_R$ and a better precision parameter $y^k_R$. At the second step, the penalty parameter that defines the merit function is conveniently updated. At the Acceleration Step 3 we try to improve the overall quality of our approximation without increasing the precision of evaluations. If the Acceleration Step is successful, Step 4 is skipped. Otherwise, at Step 4 (Optimization Phase) we try to improve the merit function by approximate minimization of a
quadratic approximation of the objective function with an adaptive regularization parameter.

The description of Algorithm BIRA begins reporting all the algorithmic parameters that will be used in the calculations. The parameter \( r \in (0, 1) \) is used in the Restoration Phase. At Step 2 we use \( \theta_0 \in (0, 1) \), the initial penalty parameter. Bounds for the regularization parameter used in the Optimization Phase are given by \( \mu_{\text{min}} \) and \( \mu_{\text{max}} \). Finally, the parameter \( \alpha > 0 \) is used at Step 4 to decide acceptance or rejection of the trial point obtained at this step, \( M \) is a bound for Hessian approximations, and \( \beta_{\text{PDP}} > 0 \) is a parameter related to the possible existence of a cheap problem-dependent procedure for restoration.

Other parameters \((\alpha, \sigma_{\text{max}}, \sigma_{\text{min}}, \sigma, \beta, r_{\text{feas}}, \bar{\epsilon}_{\text{prec}}, N_{\text{prec}}, N_{\text{acce}})\) are used in Algorithms RESTA or ACCEL and will be commented later.

Algorithm 2.1 (BIRA)

Given \( \alpha_R, \alpha > 0 \), \( M \geq 1 \), \( \sigma_{\text{max}} \geq \sigma_{\text{min}} > 0 \), \( \mu_{\text{max}} \geq \mu_{\text{min}} > 0 \), \( \beta_{c} > 0 \), \( \beta_{\text{PDP}} > 0 \), \( r \in (0, 1) \), \( r_{\text{feas}} \in (0, r) \), \( \bar{\epsilon}_{\text{prec}} \geq 0 \), \( N_{\text{prec}} \geq 0 \), and \( N_{\text{acce}} \geq 0 \), choose \( \mu_{-1} \in [\mu_{\text{min}}, \mu_{\text{max}}] \), \( x^0 \in \Omega \), \( y^0 \in Y \), set \( k \leftarrow 0 \) and \( \theta_0 \in (0, 1) \).

**Step 1. Restoration Phase**

Using an optional inexpensive problem-dependent procedure (PDP) (if available), try to compute \( y^k_R \in Y \) and \( x^k_R \in \Omega \) such that

\[
g_f(x^k_R) \leq rg_f(y^k), \ g_h(x^k_R) \leq rg_h(y^k), \ ||h(x^k_R, y^k_R)|| \leq r||h(x^k, y^k_R)||, \tag{5}\]

and

\[
\max\{||x^k_R - x^k||, ||y^k_R - y^k||\} \leq \beta_{\text{PDP}}||h(x^k_R, y^k_R)||. \tag{6}\]

If such procedure is activated and (5) and (6) hold, go to Step 2.

Otherwise, compute \((x^k_R, y^k_R)\) using Algorithm RESTA.

If

\[
||h(x^k_R, y^k_R)|| > r ||h(x^k, y^k_R)||, \tag{7}\]

stop the execution of BIRA declaring Restoration Failure.

**Step 2. Update penalty parameter**

Test the inequality

\[
\Phi(x^k_R, y^k_R, \theta_k) - \Phi(x^k, y^k_R, \theta_k) \leq \frac{1 - r}{2} \left[ ||h(x^k_R, y^k_R)|| - ||h(x^k, y^k_R)|| + g(y^k_R) - g(y^k) \right]. \tag{8}\]

If (8) holds, define \( \theta_{k+1} = \theta_k \). Else, compute

\[
\theta_{k+1} = \frac{(1 + r) \left[ ||h(x^k_R, y^k_R)|| - ||h(x^k, y^k_R)|| + g(y^k) - g(y^k_R) \right]}{2 \left[ f(x^k_R, y^k_R) - f(x^k, y^k_R) + ||h(x^k_R, y^k_R)|| - ||h(x^k, y^k_R)|| + g(y^k) - g(y^k_R) \right]}. \tag{9}\]

**Step 3. Acceleration Phase**

Using Algorithm ACCEL, try to obtain \((x^{k+1}, y^{k+1}) \in \Omega \times Y\) (possibly with \(g(y^{k+1}) > g(y^k_R)\)) satisfying the following conditions:
\[
f(x^{k+1}, y^{k+1}) \leq f(x_R^k, y_R^k) - \alpha \|x^{k+1} - x_R^k\|^2 \quad \text{and} \quad (10)
\]

\[
\Phi(x^{k+1}, y^{k+1}, \theta_{k+1}) \leq \Phi(x^k, y^{k+1}, \theta_{k+1}) + \frac{1 - r}{2} \left[ \|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| + g(y_R^k) - g(y^k) \right]. \quad (11)
\]

In the case of success (so that (10) and (11) hold) update \( k \leftarrow k + 1 \) and go to Step 1. Else, continue at Step 4.

**Step 4. Optimization Phase**

Define \( y^{k+1} = y_R^k \). Choose \( \mu \in [\mu_{\text{min}}, \mu_{k-1}] \) and \( H_k \in \mathbb{R}^{n \times n} \) symmetric such that \( \|H_k\| \leq M \).

**Step 4.1**

Compute \( x \in \Omega \) as an approximate solution of the following subproblem:

\[
\begin{align*}
\text{Minimize} & \quad \nabla_x f(x_R^k, y^{k+1})^T(x - x_R^k) + \frac{1}{2} (x - x_R^k)^T H_k(x - x_R^k) + \mu \|x - x_R^k\|^2 \\
\text{subject to} & \quad \nabla_x h(x_R^k, y^{k+1})^T(x - x_R^k) = 0 \\
& \quad x \in \Omega.
\end{align*}
\]  \quad (12)

**Step 4.2**

Test the conditions

\[
f(x, y^{k+1}) \leq f(x_R^k, y_R^k) - \alpha \|x - x_R^k\|^2 \quad (13)
\]

and

\[
\Phi(x, y^{k+1}, \theta_{k+1}) \leq \Phi(x^k, y^{k+1}, \theta_{k+1}) + \frac{1 - r}{2} \left[ \|h(x_R^k, y_R^k)\| - \|h(x^k, y_R^k)\| + g(y_R^k) - g(y^k) \right]. \quad (14)
\]

If (13) and (14) are fulfilled, define \( \mu_k = \mu \), \( x^{k+1} = x \), \( k \leftarrow k + 1 \), and go to Step 1. Otherwise, set

\[
\mu_{\text{new}} \in [2\mu, 10\mu],
\]  \quad (15)

update \( \mu \leftarrow \mu_{\text{new}} \) and go to Step 4.1.

**Remarks** In Assumption A6 we will explain what we mean by an “inexpensive problem-dependent restoration procedure”.

In Assumption A9 we will define in which sense, at Step 4.1, \( x \) should be an approximate solution of (12).

**2.2 Algorithm for the Restoration Phase**

The objective of the restoration algorithm RESTA is to find \( x_R^k \) and \( y_R^k \) such that

\[
g_f(y_R^k) \leq r g_f(y^k), \quad g_h(y_R^k) \leq r g_h(y^k), \quad (16)
\]

and

\[
\|h(x_R^k, y_R^k)\| \leq r \|h(x^k, y_R^k)\|. \quad (17)
\]
In general, the fulfillment of (16) is easy to obtain as, under the usual interpretation, (16) merely imposes that the precision with which $F$ and $H$ are evaluated at $x^k_R$ should be better than the precision with which $F$ and $H$ were evaluated at $x^k$. However, (17) could be difficult to achieve. In critical cases, where the original problem is infeasible, the fulfillment of (17) could be even impossible. Therefore, “Restoration Failure” is a possible diagnostic that needs to appear at any algorithm that aims the fulfillment of (16) and (17). In these cases the required precision in relation to $g_h$ is given by the parameter $\bar{\epsilon}_{prec}$.

Algorithm 2.2 (RESTA)

Assume that $x^k \in \Omega$, $y^k \in Y$, and the parameters that define BIRA are given.

Step 1  If $\|h(x^k, y^k)\| + g(y^k) = 0$, return defining $(x^k_R, y^k_R) = (x^k, y^k)$.

Else, set $i \leftarrow 0$ and $w^0 \leftarrow y^k$.

Step 2  If $i \leq N_{prec}$, set $\bar{g}h \leftarrow rgh(w^i)$, else $\bar{g}h \leftarrow \min\{\bar{\epsilon}_{prec}, rgh(w^i)\}$. If $gf(w^i) = 0$ and $gh(w^i) = 0$ define $w^{i+1} = w^i$, else choose $w^{i+1} \in Y$ such that

$$gf(w^{i+1}) \leq rgf(y^k) \quad \text{and} \quad gh(w^{i+1}) \leq \bar{g}h.$$  

(This choice of $w^{i+1}$ will be assumed to be possible and inexpensive since, in general, merely represents increasing the precision of forthcoming evaluations.)

Step 3  Compute $z^0 \in \Omega$ such that

$$c(z^0, w^{i+1}) \leq c(x^k, w^{i+1})$$  

(see (3) for the definition of $c$) and

$$\|z^0 - x^k\| \leq \beta_c\|h(x^k, w^{i+1})\|.$$  

Set $\ell \leftarrow 0$ and define

$$c_{target} = r^2 c(x^k, w^{i+1}) \quad \text{and} \quad \epsilon_c = rfeas\|h(x^k, w^{i+1})\|$$

(Note that the choice of $z^0$ satisfying (19) and (20) is always possible because the trivial choice $z^0 = x^k$ is admissible.)

Step 4  Test the stopping criteria

$$c(z^\ell, w^{i+1}) \leq c_{target}$$

and

$$\|P_\Omega(z^\ell - \nabla_x c(z^\ell, w^{i+1}) - z^\ell\| \leq \epsilon_c \quad \text{and} \quad gh(w^{i+1}) \leq \bar{\epsilon}_{prec}.$$  

If (22) holds or (23) holds, return to BIRA defining $x^k_R = z^\ell$ and $y^k_R = w^{i+1}$.

(Although both (22) and (23) are reasons for returning, these inequalities have quite different meanings since (22) indicates success of restoration whereas (23) indicates possible failure. In any case, the final success restoration test is made in BIRA.)

If

$$\|P_\Omega(z^\ell - \nabla_x c(z^\ell, w^{i+1}) - z^\ell\| \leq \epsilon_c \quad \text{and} \quad gh(w^{i+1}) > \bar{\epsilon}_{prec},$$


set $i \leftarrow i + 1$ and go to Step 2.

**Step 5** Choose $\sigma \in [\sigma_{\text{min}}, \sigma_{\text{max}}]$ and $B_\ell \in \mathbb{R}^{n \times n}$ such that $B_\ell + \sigma_{\text{min}} I$ be symmetric and positive definite with $\|B_\ell\| \leq M$ and $\|(B_\ell + \sigma_{\text{min}} I)^{-1}\| \leq M$.

**Step 5.1** Compute $z^{\text{trial}} \in \Omega$ as an approximate solution of

\[
\begin{align*}
\text{Minimize} & \quad \nabla_x c(z^\ell, w^{i+1})^T (z - z^\ell) + \frac{1}{2} (z - z^\ell)^T (B_\ell + \sigma I)(z - z^\ell) \\
\text{subject to} & \quad z \in \Omega.
\end{align*}
\]

(25)

**Step 5.2** Test the condition

\[
c(z^{\text{trial}}, w^{i+1}) \leq c(z^\ell, w^{i+1}) - \alpha_R \|z^{\text{trial}} - z^\ell\|^2.
\]

(26)

If (26) is fulfilled, define $z^{\ell+1} = z^{\text{trial}}$, set $\ell \leftarrow \ell + 1$, and go to Step 4. Otherwise, choose $\sigma_{\text{new}} \in [2\sigma, 10\sigma]$, set $\sigma \leftarrow \sigma_{\text{new}}$, and go to Step 5.1.

**Remark** In Assumption A2 we will specify the way in which we choose $z^{\text{trial}}$ in (25).

### 2.3 Algorithm for the Acceleration Phase (ACCEL)

At the Acceleration Phase we aim to improve the optimality of the approximate solution at a low cost. This means that we will try to satisfy the requirements on the objective function and the constraints that are necessary to guarantee convergence, without necessarily improving the precision in the evaluations. As we will see, the attempt made by the acceleration algorithm ACCEL may fail. In that case, we will return with “Acceleration Failure”, we will define $y^{k+1} = y^k_R$ at Step 4 of BIRA and we will proceed to the Optimization Phase. The Optimization Phase will be skipped if the case of success of ACCEL.

**Algorithm 2.2 (ACCEL)**

Given $x^k, x^k_R \in \Omega$, $y^k_R \in Y$, and the parameters that define Algorithm BIRA, set $\ell = 0$ and execute the steps below.

**Step 1** Choose $y^{k+1} \in Y$ (perhaps $g(y^{k+1})$ bigger than $g(y^k_R)$). Choose $\mu \in [\mu_{\text{min}}, \mu_{\text{max}}]$ and $H_k \in \mathbb{R}^{n \times n}$, symmetric such that $\|H_k\| \leq M$.

**Step 2** If $\ell \geq N_{\text{acce}}$, return declaring acceleration failure.

**Step 3** Compute $x \in \Omega$ an approximate solution of

\[
\begin{align*}
\text{Minimize} & \quad \nabla_x f(x^k_R, y^{k+1})^T (x - x^k_R) + \frac{1}{2} (x - x^k_R)^T H_k (x - x^k_R) + \mu \|x - x^k_R\|^2 \\
\text{subject to} & \quad \nabla_x h(x^k_R, y^{k+1})^T (x - x^k_R) = 0 \\
& \quad x \in \Omega.
\end{align*}
\]

(28)

**Step 4** Test the conditions

\[
f(x, y^{k+1}) \leq f(x^k_R, y^k_R) - \alpha \|x - x^k_R\|^2
\]

(29)
and
\[ \Phi(x, y^{k+1}, \theta_{k+1}) \leq \Phi(x^k, y^{k+1}, \theta_{k+1}) + \frac{1 - r}{2} \left[ \|h(x^R_k, y^R_k)\| - \|h(x^k, y^k)\| + g(y^k) - g(y^k) \right]. \] (30)

If (29) and (30) are fulfilled, return with \( y^{k+1}, x^{k+1} \leftarrow x \) and \( \mu_k = \mu \).
Otherwise, choose \( \mu_{\text{new}} \in [2\mu, 10\mu] \), \( \mu \leftarrow \mu_{\text{new}} \), set \( \ell \leftarrow \ell + 1 \) and go to Step 2.

Remark The sense in which \( x \) should be an approximate solution of (28) will be given in Assumption [A9].

3 Purpose

The goal of the present research is to show that, using BIRA and under suitable assumptions, convergence to feasible and optimal solutions takes place and worst-case complexity results, that provide bounds on the evaluation computer work used by the algorithm in terms of given small tolerances, can be proved. These results will be stated in Theorems 8.1 and 8.2.

The main assumption in these theorems is that the algorithm does not stop by Restoration Failure. Note that the possibility of stopping by Restoration Failure is unavoidable in any algorithm for constrained optimization as, in some cases, feasible solutions may not exist at all. In our approach optimality will be measured by means of the Euclidean projection of the gradient of the objective function onto the tangent approximation to the constraints. This is related to using the Sequential Optimality Condition called L-AGP in [2]. Such condition holds at a local minimizer of constrained optimization problems without invoking constraint qualifications. Under weak constraint qualifications, the fulfillment of L-AGP implies KKT conditions [3].

4 General Assumptions

The assumptions stated in this section are supposed to hold all along this paper without specific mention. These assumptions state regularity and boundedness of the functions involved in the definition of the problem.

Assumption G1 Differenciability of \( f \): The function \( f(x, y) \) is continuously differentiable with respect to \( x \) for all \( x \in \Omega \) and all \( y \in Y \).

Assumption G2 Boundedness: There exists \( C_f > 0 \) such that, for all \( x \in \Omega \) and for all \( y \in Y \), we have that
\[ |f(x, y)| \leq C_f. \] (31)

Assumption G3 Lipschitz-continuity: There exists \( L_f \geq 0 \) such that, for all \( x_1, x_2 \in \Omega \) and all \( y \in Y \), we have that:
\[ |f(x_1, y) - f(x_2, y)| \leq L_f \|x_1 - x_2\| \] (32)
and
\[ \|\nabla_x f(x_1, y) - \nabla_x f(x_2, y)\| \leq L_f \|x_1 - x_2\|. \] (33)
**Assumption G4** Upper bound for $f$: For all $x_1, x_2 \in \Omega$ and all $y \in Y$ we have that
\[
f(x_2, y) \leq f(x_1, y) + \nabla_x f(x_1)^T (x_2 - x_1) + L_f \|x_2 - x_1\|^2.
\] (34)

**Assumption G5** Differentiability of $h$: The function $h(x, y)$ is continuously differentiable with respect to $x$ for all $x \in \Omega$ and all $y \in Y$.

**Assumption G6** Boundedness of $\|h\|$ and $\|\nabla_x h\|$ : There exists $C_h \geq 0$ such that, for all $x \in \Omega$ and all $y \in Y$, we have that
\[
\|h(x, y)\| \leq C_h
\] and
\[
\|\nabla_x h(x, y)\| \leq C_h.
\] (35) (36)

**Assumption G7** Lipschitz-continuity of $h$ and $\nabla_x h$: There exists $L_h \geq 0$ such that, for all $x_1, x_2 \in \Omega$ and all $y \in Y$, we have that
\[
\|h(x_1, y) - h(x_2, y)\| \leq L_h \|x_1 - x_2\|,
\] (37)
and
\[
\|\nabla_x h(x_1, y)^T - \nabla_x h(x_2, y)^T\| \leq L_h \|x_1 - x_2\|.
\] (38)

**Assumption G8** Upper bound of $\|h\|$ : For all $x_1, x_2 \in \Omega$ and all $y \in Y$ we have that
\[
\|h(x_2, y)\| \leq \|h(x_1, y) + \nabla_x h(x_1, y)^T (x_2 - x_1)\| + L_h \|x_2 - x_1\|^2.
\] (39)

**Assumption G9** Boundedness of $g_f$ and $g_h$: There exists $C_g \geq 1$ such that
\[
g_f(y) \leq C_g \text{ and } g_h(y) \leq C_g
\] (40)
for all $y \in Y$.

**Assumption G10** Differentiability of $c(x, y)$: The function $c(x, y)$, defined by [3], is continuously differentiable with respect to $x$ for all $x \in \mathbb{R}^n$ and $y \in Y$.

**Assumption G11** Lipschitz continuity of $\nabla_x c$: There exists $L_c \geq 0$ such that for all $x_1, x_2 \in \Omega$ and all $y \in Y$, we have that
\[
\|\nabla_x c(x_1, y) - \nabla_x c(x_2, y)\| \leq L_c \|x_1 - x_2\|.
\] (41)

**Assumption G12** Upper bound of $c(x, y)$: For all $x_1, x_2 \in \Omega$ and all $y \in Y$ we have that
\[
c(x_2, y) \leq c(x_1, y) + \nabla_x c(x_1, y)^T (x_2 - x_1) + L_c \|x_2 - x_1\|^2.
\] (42)

5 Theoretical Results Concerning the Restoration Phase

The Restoration Phase is the subject of Step 1 of BIRA. This phase begins acknowledging the possibility that, using some problem-dependent procedure (PDP), one may be able to compute $x^k_R$ and $y^k_R$, fulfilling the conditions [5] and [6]. In the positive case it is not necessary to call Algorithm RESTA to complete Step 1 of BIRA and the algorithm jumps to Step 3, where the penalty parameter is updated.
If there is no problem-dependent procedure that computes \( x^k_R \) and \( y^k_R \) satisfying (5) and (6), we try to satisfy these conditions using Algorithm RESTA. However, even Algorithm RESTA may fail in that purpose, and in this case we declare “Restoration Failure” and Algorithm BIRA stops. Note that every algorithm for constrained optimization may fail to find feasible points, unless special conditions are imposed to the problem. The main reason is that, in extreme cases, feasible points could not exist at all.

Assumption A1 states that finding \( w^{i+1} \) at Step 2 of RESTA is always inexpensive. The reason is that, in general, (18) merely represent increasing the precision in which the objective function and the constraints will be evaluated.

Assumption A1 Step 2 of Algorithm RESTA, leading to the definition of \( w^{i+1} \) satisfying (18), can be computed in finite time for all \( k \) and \( i \), without evaluations of \( f \) or \( h \).

At Step 5.1 of RESTA we stated that \( z_{\text{trial}} \) is an approximate solution of the problem (25). Assumption A2 states a simple condition that such approximate solution must satisfy. According to this very mild assumption, the trial point \( z_{\text{trial}} \) should not be worse than \( z^\ell \) in terms of functional value. Note that even \( z_{\text{trial}} = z^\ell \) satisfies this assumption.

Assumption A2 For all \( z^\ell \) and \( w^{i+1} \), the point \( z_{\text{trial}} \) found at Step 5.1 of Algorithm RESTA satisfies:

\[
\nabla_x c(z^\ell, w^{i+1})^T(z_{\text{trial}} - z^\ell) + \frac{1}{2}(z_{\text{trial}} - z^\ell)^T(B^\ell + \sigma I)(z_{\text{trial}} - z^\ell) \leq 0.
\]

(43)

In Lemma 5.1 we prove that, taking the regularization parameter \( \sigma \) large enough when solving (25) we obtain sufficient reduction of the sum of squares infeasibility at the approximate solution \( z_{\text{trial}} \). In other words, the loop at Step 5.1 of RESTA necessarily finishes with the fulfillment of (26).

Lemma 5.1 Suppose that Assumptions A1 and A2 hold. Define \( \bar{\sigma} = 2\left(L_c + \frac{M}{2} + \alpha_R\right) \). Then, if \( z_{\text{trial}} \) is computed at Step 5.1 with \( \sigma \geq \bar{\sigma} \), we have that

\[
c(z_{\text{trial}}, w^{i+1}) \leq c(z^\ell, w^{i+1}) - \alpha_R\|z_{\text{trial}} - z^\ell\|^2.
\]

As a consequence, for all \( k, i, \) and \( \ell \) we have that \( \sigma \leq \max\{10\bar{\sigma}, \sigma_{\text{max}}\} \).

Proof: Using (42) for \( x_2 = z_{\text{trial}} \) and \( x_1 = z^\ell \) we have that

\[
c(z_{\text{trial}}, w^{i+1}) \leq c(z^\ell, w^{i+1}) + \nabla_x c(z^\ell, w^{i+1})^T(z_{\text{trial}} - z^\ell) + L_c\|z_{\text{trial}} - z^\ell\|^2
\]

Then, taking

\[
v = \frac{1}{2}(z_{\text{trial}} - z^\ell)^TB^\ell(z_{\text{trial}} - z^\ell) + \frac{\sigma}{2}\|z_{\text{trial}} - z^\ell\|^2,
\]

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we obtain:

\[
c(z_{\text{trial}}, w^{i+1}) - c(z^\ell, w^{i+1}) \leq \nabla_x c(z^\ell, w^{i+1})^T (z_{\text{trial}} - z^\ell) + L_c \|z_{\text{trial}} - z^\ell\|^2 \\
= \nabla_x c(z^\ell, w^{i+1})^T (z_{\text{trial}} - z^\ell) + v - v + L_c \|z_{\text{trial}} - z^\ell\|^2 \\
= \nabla_x c(z^\ell, w^{i+1})^T (z_{\text{trial}} - z^\ell) + v + (L_c - \sigma) \|z_{\text{trial}} - z^\ell\|^2 \\
- \frac{1}{2} (z_{\text{trial}} - z^\ell)^T B_\ell (z_{\text{trial}} - z^\ell).
\]

Since \(\|B_\ell\| \leq M\), we have that \(\|(z_{\text{trial}} - z^\ell)^T B_\ell (z_{\text{trial}} - z^\ell)\| \leq M \|z_{\text{trial}} - z^\ell\|^2\), so:

\[
c(z_{\text{trial}}, w^{i+1}) - c(z^\ell, w^{i+1}) \leq \nabla_x c(z^\ell, w^{i+1})^T (z_{\text{trial}} - z^\ell) + v + \left(L_c + \frac{M}{2} - \frac{\sigma}{2}\right) \|z_{\text{trial}} - z^\ell\|^2. \tag{44}
\]

By (43),

\[
c(z_{\text{trial}}, w^{i+1}) - c(z^\ell, w^{i+1}) \leq \left(L_c + \frac{M}{2} - \frac{\sigma}{2}\right) \|z_{\text{trial}} - z^\ell\|^2. \tag{45}
\]

Therefore, we obtain that, if \(\sigma \geq \bar{\sigma}\), (26) is fulfilled. \(\square\)

Assumption A3 adds an additional condition that the approximate solution of the subproblem (25) must satisfy. It will be required that an approximate optimality condition, expressed in terms of the projected gradient of the objective function of (25), should be fulfilled with tolerance proportional to \(\|z^{\ell+1} - z^\ell\|\).

**Assumption A3** There exists \(\kappa_R > 0\) such that, whenever \(z^{\ell+1}\) is defined at Step 5.2 of RESTA, we have that:

\[
\|P_\Omega \left(z^{\ell+1} - \left[\nabla_x c(z^\ell, w^{i+1}) + B_\ell(z^{\ell+1} - z^\ell) + \sigma(z^{\ell+1} - z^\ell)\right]\right) - z^{\ell+1}\| \leq \kappa_R \|z^{\ell+1} - z^\ell\|. \tag{46}
\]

As a consequence of the previous assumptions, Lemma 5.2 proves that the projected gradient of the linear approximation of the sum of squares at \(z^\ell\), computed at the subproblem solution \(z^{\ell+1}\), is proportional to the norm of the difference between \(z^\ell\) and \(z^{\ell+1}\).

**Lemma 5.2** Suppose that Assumptions A1, A2, and A3 hold. Define \(c_{P_\Omega} = L_c + M + \kappa_R + \max\{10\bar{\sigma}, \sigma_{\max}\}\), where \(\bar{\sigma}\) was defined in Lemma 5.1 Then, whenever \(z^{\ell+1}\) is defined at Step 5.2 of RESTA, we have:

\[
\|P_\Omega \left(z^{\ell+1} - \nabla_x c(z^\ell, w^{i+1})\right) - z^{\ell+1}\| \leq c_{P_\Omega} \|z^{\ell+1} - z^\ell\|. \tag{47}
\]

**Proof:** Define

\[
u = z^{\ell+1} - \left[\nabla_x c(z^\ell, w^{i+1}) + B_\ell(z^{\ell+1} - z^\ell)\right] \quad \text{and} \quad w = u - \sigma(z^{\ell+1} - z^\ell).
\]

By (46),

\[
\|P_\Omega(u) - z^{\ell+1}\| = \|P_\Omega(u) - P_\Omega(w) + P_\Omega(w) - z^{\ell+1}\| \leq \|P_\Omega(u) - P_\Omega(w)\| + \kappa_R \|z^{\ell+1} - z^\ell\|. \tag{48}
\]
By the non-expansivity projections, we have that

\[ \| P_\Omega(u) - P_\Omega(w) \| \leq \| u - w \| = \sigma \| z^{\ell+1} - z^\ell \|. \]

So, (48) implies that

\[ \| P_\Omega(u) - z^{\ell+1} \| \leq (\sigma + \kappa_R) \| z^{\ell+1} - z^\ell \|. \] (49)

Now, define \( v = z^{\ell+1} - \nabla_x c(z^{\ell+1}, w^{i+1}) \). Using (49) and, again, the non-expansivity of projections, we obtain:

\[ \| P_\Omega(v) - z^{\ell+1} \| \leq \| P_\Omega(v) - P_\Omega(u) \| + \| P_\Omega(u) - z^{\ell+1} \| \]
\[ \leq \| v - u \| + (\sigma + \kappa_R) \| z^{\ell+1} - z^\ell \| \]
\[ \leq \| \nabla_x c(z^\ell, w^{i+1}) - \nabla_x c(z^{\ell+1}, w^{i+1}) + B_\ell(z^{\ell+1} - z^\ell) \| + (\sigma + \kappa_R) \| z^{\ell+1} - z^\ell \|. \] (50)

By (41), we have that \( \| \nabla_x c(z^{\ell+1}, w^{i+1}) - \nabla_x c(z^\ell, w^{i+1}) \| \leq L_c \| z^{\ell+1} - z^\ell \|. \) So, since \( \| B_\ell \| \leq M \),

\[ \| P_\Omega(v) - z^{\ell+1} \| \leq \| P_\Omega(v) - P_\Omega(u) \| + \| P_\Omega(u) - z^{\ell+1} \| + \| B_\ell(z^{\ell+1} - z^\ell) \| + (\sigma + \kappa_R) \| z^{\ell+1} - z^\ell \| \]
\[ \leq L_c \| z^{\ell+1} - z^\ell \| + M \| z^{\ell+1} - z^\ell \| + (\sigma + \kappa_R) \| z^{\ell+1} - z^\ell \| \]
\[ = (L_c + M + \sigma + \kappa_R) \| z^{\ell+1} - z^\ell \|. \] (51)

Therefore, recalling that, by Lemma 5.1 we have that \( \sigma \leq \max \{ 10\bar{\sigma}, \sigma_{\text{max}} \} \), we deduce (47), as desired.

Recall the definition of \( c_{\text{target}} \) and \( \epsilon_c \) in (21). Lemma 5.3 establishes that, in a bounded finite number of steps, the sum of squares of infeasibilities is smaller than \( c_{\text{target}} \) or its projected gradient at \( z^\ell \) is smaller than \( \epsilon_c \).

**Lemma 5.3** Suppose that Assumptions A1, A2, and A3 hold. Define \( C_{\text{rest}} = \frac{c_{P_\Omega}(1-r^2)}{2r_\Omega^2 f_{\text{cas}}} + 1 \), where \( c_{P_\Omega} \) is defined in Lemma 5.2. Then, at every call to Algorithm RESTA, there exists \( \ell \leq C_{\text{rest}} \) such that

\[ c(z^\ell, w^{i+1}) \leq c_{\text{target}} \] (52)

or

\[ \| P_\Omega(z^\ell - \nabla_x c(z^\ell, w^{i+1})) - z^\ell \| \leq \epsilon_c. \] (53)

**Proof:** Assume that (53) is not true for the first \( \ell \) iterations of Step 4 of RESTA. Then, for all \( j \in \{0, 1, \ldots, \ell\} \) we have that

\[ \| P_\Omega(z^j - \nabla_x c(z^j, w^{i+1})) - z^j \| > \epsilon_c. \] (54)

By (47), for all \( j \in \{0, 1, \ldots, \ell - 1\} \),

\[ \| P_\Omega(z^{j+1} - \nabla_x c(z^{j+1}, w^{i+1})) - z^{j+1} \| \leq c_{P_\Omega} \| z^{j+1} - z^j \|. \] (55)
Then, by (54) and (55),
\[ \ell \epsilon_c^2 = \sum_{j=0}^{\ell-1} \epsilon_c^2 \leq \sum_{j=0}^{\ell-1} \| P_{\Omega} (z^{j+1} - \nabla_x c(z^{j+1}, w^{i+1})) - z^{j+1} \|^2 \leq c_{\Omega}^2 \sum_{j=0}^{\ell-1} \| z^{j+1} - z^j \|^2. \tag{56} \]

On the other hand,
\[ c(z^\ell, w^{i+1}) = \sum_{j=0}^{\ell-1} [c(z^{j+1}, w^{i+1}) - c(z^j, w^{i+1})] + c(z^0, w^{i+1}). \]

By (26) at Step 5.2 of Algorithm RESTA, we have that
\[ c(z^j, w^{i+1}) = c(z, w^{i+1}) - \alpha_R \| z^j - z \|^2, \] for all \( j \in \{0, 1, \ldots, \ell - 1\} \). Therefore,
\[ c(z^\ell, w^{i+1}) - c(z^0, w^{i+1}) = \sum_{j=0}^{\ell-1} [c(z^{j+1}, w^{i+1}) - c(z^j, w^{i+1})] \leq -\alpha_R \sum_{j=0}^{\ell-1} \| z^{j+1} - z^j \|^2. \]

By (56) and the fact that, at Step 3 of RESTA, we choose \( z^0 \) such that
\[ c(z^0, w^{i+1}) \leq c(x^k, w^{i+1}), \]
we have that
\[ c(z^\ell, w^{i+1}) \leq c(z^0, w^{i+1}) - \alpha_R \frac{\ell \epsilon_c^2}{c_{\Omega}^2} \leq c(x^k, w^{i+1}) - \alpha_R \frac{\ell \epsilon_c^2}{c_{\Omega}^2}. \]

Therefore, if
\[ c(x^k, w^{i+1}) - \alpha_R \frac{\ell \epsilon_c^2}{c_{\Omega}^2} \leq c_{\text{target}} \]
we would have that \( c(z^\ell, w^{i+1}) \leq c_{\text{target}} \) and the stopping condition at Step 4.1 would be fulfilled. Moreover, (57) occurs if and only if
\[ \frac{c_{\Omega}^2}{\alpha_R \ell \epsilon_c^2} \left[ c(x^k, w^{i+1}) - c_{\text{target}} \right] \leq \ell. \tag{58} \]

Using the definitions of \( \epsilon_c \) and \( c_{\text{target}} \), given in (21), we obtain that (58) is equivalent to
\[ \ell \geq \frac{c_{\Omega}^2}{\alpha_R |r_{feas}||h(x^k, w^{i+1})|^2} \left[ c(x^k, w^{i+1}) - r^2 c(x^k, w^{i+1}) \right] = \frac{c_{\Omega}^2 (1 - r^2)}{2 \alpha_R r_{feas}^2}. \]

Therefore, if \( \ell \geq \frac{c_{\Omega}^2 (1 - r^2)}{2 \alpha_R r_{feas}^2} \) and (53) has not been fulfilled before, we have that \( c(z^\ell, w^{i+1}) \leq c_{\text{target}} \). Therefore, we have that in at most \( C_{rest} \) sub-iterations of RESTA either (52) holds or (53) would have been obtained before. This completes the proof. \( \Box \)

The following is a technical assumption that involves \( z^{\ell+1} \) obtained at Step 5 of RESTA. It states that, if we add to (25) the constraint that \( z - z^\ell \) is a multiple of \( z^{j+1} - z^\ell \), the corresponding solution is close to \( z^{j+1} \). Clearly, this assumption holds if \( z_{\text{trial}} \) is the global solution of (25) and very plausibly holds for approximate solutions.
Assumption A4 There exists \( \kappa_\varphi > 0 \) such that, whenever \( z^{t+1} \) is the approximate solution obtained in RESTA and \( z^\star \) is an exact solution to the problem that has the same objective function and constraints as \( z^t \) and, in addition a constraint saying that \( z - z^\star \) is a multiple of \( z^{t+1} - z^t \), we have:

\[
\|z^{t+1} - z^t\| \leq \kappa_\varphi \|z^{t+1} - z^\star\|. \tag{59}
\]

In the following lemma we prove that the difference between consecutive internal iterations in RESTA is proportional to the infeasibility at \( x^k \).

Lemma 5.4 Suppose that Assumptions A1–A4 hold. Define \( C_s = \kappa_\varphi MC_h \), where \( C_h \) is defined in Assumption G6. Then, for all \( k, i \) and \( \ell \), the iterates generated in RESTA satisfy

\[
\|z^{\ell+1} - z^\ell\| \leq C_s \|h(x^k, w^{i+1})\|. \tag{60}
\]

Proof: If \( z^{\ell+1} = z^\ell \), \( 60 \) is trivial. If \( z^{\ell+1} \neq z^\ell \), define \( v = \frac{z^{\ell+1} - z^\ell}{\|z^{\ell+1} - z^\ell\|} \).

Since \( z^{\ell+1} \) is an approximate minimizer of \( 25 \), then, by Assumption A2

\[
\nabla_x c(z^\ell, w^{i+1})^T (z^{\ell+1} - z^\ell) \leq -\frac{1}{2}(z^{\ell+1} - z^\ell)^T (B_\ell + \sigma I)(z^{\ell+1} - z^\ell).
\]

By Step 5 of RESTA, we have that \( \sigma \geq \sigma_{\text{min}} \) and \( B_\ell + \sigma_{\text{min}} I \) is positive definite, so \( \nabla_x c(z^\ell, w^{i+1})^T v < 0 \).

Now, consider the function \( \varphi : \mathbb{R}_+ \to \mathbb{R} \) defined by

\[
\varphi(t) = t \nabla_x c(z^\ell, w^{i+1})^T v + \frac{t^2}{2} v^T (B_\ell + \sigma I)v. \tag{61}
\]

The unconstrained minimizer of \( \varphi(t) \) is

\[
t^* = -\frac{\nabla_x c(z^\ell, w^{i+1})^T v}{v^T (B_\ell + \sigma I)v} \leq -\frac{\nabla_x c(z^\ell, w^{i+1})^T v}{v^T (B_\ell + \sigma_{\text{min}} I)v} \leq \frac{\|\nabla_x c(z^\ell, w^{i+1})\| \|v\|}{\lambda_1 (B_\ell + \sigma_{\text{min}} I) \|v\|^2},
\]

where \( \lambda_1 (B_\ell + \sigma_{\text{min}} I) > 0 \) is the smaller eigenvalue of \( B_\ell + \sigma_{\text{min}} I \). As \( \|v\| = 1 \), by Step 5 of RESTA,

\[
t^* \leq \|(B_\ell + \sigma_{\text{min}} I)^{-1} \|\|\nabla_x c(z^\ell, w^{i+1})\| \leq M \|\nabla_x c(z^\ell, w^{i+1})\|. \tag{62}
\]

Let \( \tilde{t} \) be the minimizer of \( \varphi(t) \) subject to \( z^\ell + tv \in \Omega \). By the convexity of \( \Omega \) we have that \( \tilde{t} \leq t^* \). Moreover, by construction, \( z^\ell + \tilde{t} v = z^{\ell+1} \). So, by \( \|v\| = 1 \), Assumption A4 and \( 62 \), we have that

\[
\|z^{\ell+1} - z^\ell\| \leq \kappa_\varphi \|z^{\ell+1} - z^\star\| = \kappa_\varphi \tilde{t} \leq \kappa_\varphi t^* \leq \kappa_\varphi M \|\nabla_x c(z^\ell, w^{i+1})\|.
\]

Now, by \( 36 \), \( \|\nabla_x h(z^\ell, w^{i+1})^T\| = \|\nabla_x h(z^\ell, w^{i+1})\| \leq C_h \), thus:

\[
\|z^{\ell+1} - z^\ell\| \leq \kappa_\varphi M \|\nabla_x h(z^\ell, w^{i+1})^T h(z^\ell, w^{i+1})\| \leq \kappa_\varphi M \|\nabla_x h(z^\ell, w^{i+1})^T \| \|h(z^\ell, w^{i+1})\| \leq \kappa_\varphi MC_h \|h(z^\ell, w^{i+1})\|.
\]
By (26) we have that \( \|h(z^{\ell+1}, w^{i+1}) \leq \|h(z^{\ell}, w^{i+1})\| \) and, by the choice of \( z^0 \) at Step 3 of RESTA, we have that \( \|h(z^0, w^{i+1}) \leq \|h(x^0, w^{i+1})\| \). Then, \( \|h(z^{\ell}, w^{i+1})\| \leq \|h(x^k, w^{i+1})\| \) for all \( \ell \). Therefore, (60) holds.

\[
\Box
\]

In Lemma 5.5 we prove that, at every call of RESTA, the descent condition on the sum of squares of infeasibilities (26) is tested a finite number of times.

**Lemma 5.5** Suppose that Assumptions A1–A4 hold. Define \( n_\sigma = \lfloor \log_2(\sigma) - \log_2(\sigma_{\min}) \rfloor + 1 \) and \( N_{\text{RESTA}} = (C_{\text{rest}} n_\sigma + 1) N_{\text{prec}} \). Then, at every call to RESTA, the number of tests of the condition (26) and the number of evaluations of \( h \) and \( \nabla_x h \) is bounded by \( N_{\text{RESTA}} \).

**Proof:** For a fixed \( i \), by Lemma 5.3 after at most \( C_{\text{rest}} \) steps we find \( z^\ell \) satisfying (52) or (53). At each of these steps, \( \nabla_x c \) is evaluated only once, therefore, the total number of evaluations of \( \nabla_x c \) is bounded by \( C_{\text{rest}} \).

By Lemma 5.1, \( \sigma \geq \bar{\sigma} \) implies that (26) is fulfilled and, so, \( z^{\ell+1} \) is well defined. By (27), \( \sigma \) is increased according to \( \sigma \in [2\sigma, 10\sigma] \). Therefore, as the initial value of \( \sigma \) is not smaller than \( \sigma_{\min} \), we have that after \( n_\sigma \) trials we will have that \( \sigma \geq 2^{n_\sigma} \sigma_{\min} \). Therefore, we have that \( \sigma \geq 2^{n_\sigma} \sigma_{\min} \geq \bar{\sigma} \) and, so, \( z^{\ell+1} \) is obtained.

Therefore, the descent condition (26) is tested at most \( n_\sigma \) times for each value of \( \ell \). Consequently, \( h \) is evaluated at most \( n_\sigma \) times for every \( \ell \). So, the condition (26) is tested at most \( C_{\text{rest}} n_\sigma \) times for all fixed \( w^{i+1} \).

Finally, observe that, by Step 2, after at most \( N_{\text{prec}} \) trials we have that \( g_h(w^{i+1}) \leq \epsilon_{\text{prec}} \). In this case, the process would finish at Steps 4.1 or 4.2 and, so, the Restoration Phase would be finished. Moreover, only one additional evaluation of \( h \) is performed at each update of \( w^{i+1} \). Then, we obtain the desired result. \( \Box \)

In the following lemma we prove that the norm of the difference between the restored point \( x_R^k \) and the current point \( x^k \) is bounded by a multiple of \( \|h(x^k, y_R^k)\| \).

**Lemma 5.6** Suppose that Assumptions A1–A4 hold. Define \( \beta_R = \max\{\beta_{\text{DP}}, \beta_c + N_{\text{RESTA}} C_s\} \). Then, for every iteration \( k \) of BIRA, \( (x_R^k, y_R^k) \) satisfies

\[
\|x_R^k - x^k\| \leq \beta_R\|h(x^k, y_R^k)\|. \tag{63}
\]

**Proof:** If \( (x_R^k, y_R^k) \) is computed by a problem-dependent procedure, the result is true by (6). Now, let us consider that \( (x_R^k, y_R^k) \) is computed by RESTA. For given \( k \), let \( i \) be such that \( w^{i+1} = y_R^k \) and let \( N_{Rk} \) be the number of sub-iterations performed for the minimization of \( c(z, w^{i+1}) \). By Lemma 5.5 we have that \( N_{Rk} \leq N_{\text{RESTA}} \). Then, by Lemma 5.4 and the choice of \( z^0 \) at Step 3 of RESTA, we have that

\[
\|x_R^k - x^k\| \leq \|z^0 - x^k\| + \sum_{l=1}^{N_{Rk}} \|z^\ell - z^{\ell-1}\| \\
\leq \beta_c\|h(x^k, y_R^k)\| + \sum_{l=1}^{N_{Rk}} C_s\|h(x^k, y_R^k)\| \\
\leq \beta_c\|h(x^k, y_R^k)\| + N_{\text{RESTA}} C_s\|h(x^k, y_R^k)\|.
\]

Therefore, we obtain the desired result. \( \Box \)
In Lemma 5.7 we prove that the deterioration in the objective function in $x_R^k$ with respect to the objective function at $x^k$ is bounded by a quantity that is proportional to the infeasibilities $h$ and $g$. For proving that result we need a final assumption that states that fixing $x^k$ and restoring $y^k$ the deterioration in $f$ is smaller than a multiple of $g(y^k)$.

**Assumption A5** There exists $\beta > 0$ such that, for all iteration $k$,

$$f(x^k, y_R^k) \leq f(x^k, y^k) + \beta g(y^k).$$  \hspace{1cm} (64)

**Lemma 5.7** Suppose that Assumptions A1–A5 hold. Define $\beta_f = L_f \beta_R + \beta$. Then, for every iteration $k$ of Algorithm BIRA, the point $(x_R^k, y_R^k)$ computed at the Restoration Phase, satisfies

$$f(x_R^k, y_R^k) \leq f(x^k, y^k) + \beta f(||h(x^k, y_R^k)|| + g(y^k)).$$

**Proof:** By (32), we have that $|f(x_R^k, y_R^k) - f(x^k, y_R^k)| \leq L_f ||x_R^k - x^k||$. Then, by (64),

$$f(x_R^k, y_R^k) - f(x^k, y^k) \leq f(x_R^k, y_R^k) - f(x^k, y_R^k) + f(x^k, y_R^k) - f(x^k, y^k) \leq L_f ||x_R^k - x^k|| + \beta g(y^k).$$

By Lemma 5.6 we have that $||x_R^k - x^k|| \leq \beta_R ||h(x^k, y_R^k)||$. Then,

$$f(x_R^k, y_R^k) - f(x^k, y^k) \leq L_f \beta_R ||h(x^k, y_R^k)|| + \beta g(y^k) \leq [L_f \beta_R + \beta] ||h(x^k, y_R^k)|| + g(y^k).$$

Thus, we have the desired result. \hspace{1cm} $\Box$

Finally, in Assumption A6 we state the sense in which the problem-dependent restoration procedure PDP is considered to be inexpensive. Then, in Theorem 5.1 the main results of the present section are condensed.

**Assumption A6** There exists $N_{PDP}$, independent of $\epsilon_{prec}$, such that if the problem-dependent restoration procedure is used at Step 1 of BIRA, it employs at most $N_{PDP}$ evaluations of $h$ and $\nabla_x h$ and no evaluation of $f$ and $\nabla_x f$.

**Theorem 5.1** Suppose that the General Assumptions and Assumptions A1–A6 hold. There are $N_R$ and $\beta_f$, independent of $\epsilon_{prec}$, such that, for every iteration $k$ of Algorithm BIRA, the point $(x_R^k, y_R^k)$ is computed employing at most $N_R$ evaluations of $h$ and $\nabla_x h$, no evaluation of $f$ and $\nabla_x f$, and satisfying

$$||x_R^k - x^k|| \leq \beta_R ||h(x^k, y_R^k)||.$$ \hspace{1cm} (65)

and

$$f(x_R^k, y_R^k) \leq f(x^k, y^k) + \beta f(||h(x^k, y_R^k)|| + g(y^k)).$$ \hspace{1cm} (66)

**Proof:** Conditions (65) and (66) follow direct from Lemmas 5.6 and 5.7, respectively. Now observe that no evaluation of $f$ and $\nabla_x f$ is made when calling Algorithm RESTA. So, defining $N_R = N_{RESTA} + N_{PDP}$, by Lemma 5.5 and Assumption A6 we have the desired result. \hspace{1cm} $\Box$
6 BIRA is well defined

All along this section we will assume, without specific mention, that the General Assumptions G1–G12 and the Restoration Assumptions A1–A6 are fulfilled. Assumption $A_7$ will be added when needed to prove specific results and its fulfillment will be mentioned whenever necessary.

As the title of this section indicates, the objective will be that Algorithm BIRA is well defined, that is, that for any iteration of BIRA, either the algorithm stops or it is possible to compute the next iterate.

We begin showing that the penalty parameter is well defined and satisfies the inequality (67), that states that, from the point of view of the merit function, the restored point $x^k_R$ is better than the current iterate $x^k$.

**Lemma 6.1** At every iteration $k$ of BIRA, the penalty parameter $\theta_{k+1} \leq \theta_k$, and

$$\Phi(x^k_R, y^k_R, \theta_{k+1}) - \Phi(x^k_R, y^k_R, \theta_k) \leq \frac{1 - r}{2} \left[ ||h(x^k_R, y^k_R)|| - ||h(x^k, y^k_R)|| + g(y^k_R) - g(y^k) \right].$$

**Proof:** At each iteration $k$ of BIRA we have two options, according to the fulfillment of (8). If (8) holds, we define $\theta_{k+1} = \theta_k$, therefore $\theta_{k+1}$ is well defined and does not increase with respect to $\theta_k$. Moreover, in this case (8) is equivalent to (8), so it is fulfilled.

In the second case, $\theta_{k+1}$ is defined by (9) at Step 2, according to:

$$\theta_{k+1} = \frac{(1 + r)[||h(x^k, y^k_R)|| - ||h(x^k, y^k_R)|| + g(y^k) - g(y^k)]}{2[f(x^k_R, y^k_R) - f(x^k_R, y^k_R) + ||h(x^k, y^k_R)|| - ||h(x^k, y^k_R)|| + g(y^k) - g(y^k)]}.$$

Let us show that both the numerator and the denominator of this expression are positive and that the quotient is smaller than $\theta_k$.

By the restoration step and the assumptions G1–G12, we have that $g(y^k_R) \leq rg(y^k)$, so $g(y^k_R) - g(y^k) \leq 0$. Therefore, as $\frac{1 - r}{2} \in (0, 1)$, we have that

$$g(y^k_R) - g(y^k) \leq \frac{1 - r}{2} [g(y^k_R) - g(y^k)].$$

Moreover, if the execution of BIRA is not stopped declaring Restoration Failure, the restoration always guarantees that $||h(x^k_R, y^k_R)|| \leq r ||h(x^k, y^k_R)||$. Therefore,

$$||h(x^k_R, y^k_R)|| - ||h(x^k, y^k_R)|| \leq \frac{1 - r}{2} [||h(x^k_R, y^k_R)|| - ||h(x^k, y^k_R)||].$$

Now, the equalities in (68) and (69) only take place if $||h(x^k, y^k_R)|| = ||h(x^k, y^k_R)|| = g(y^k_R) = g(y^k) = 0$. In this case, if $(x^k_R, y^k_R)$ is computed by the PDP, by (6), we have that $(x^k_R, y^k_R) = (x^k_R, y^k)$. On the other hand, if RESTA is used, since $g(y^k) = 0$, by Step 2, we would have that $w_i = y^k$ for all $i$, implying that $y^k_R = y^k$. So, $||h(x^k, y^k)|| = ||h(x^k, y^k)|| = 0$ and, by Step 1 of RESTA, we also have that $(x^k_R, y^k_R) = (x^k_R, y^k)$. In this case, (8) would be trivially fulfilled.
and we would have that \( \theta_{k+1} = \theta_k \). Then, at least one of the conditions \((68)\) or \((69)\) is strictly satisfied. This proves that, when \( \theta_{k+1} \neq \theta_k \), the numerator of \((9)\) is positive.

Now let us analyze the expression \( \Phi(x^k_R, y^k_R, \theta) - \Phi(x^k, y^k_R, \theta) \) as a function of \( \theta \). By the definition of the merit function in \((4)\), we have that

\[
\Phi(x^k_R, y^k_R, \theta) - \Phi(x^k, y^k_R, \theta) = \theta \left( f(x^k_R, y^k_R) - f(x^k_R, y^k_R) + \|h(x^k, y^k_R)\| - \|h(x^k_R, y^k_R)\| + g(y^k) - g(y^k_R) \right) - \left[ \|h(x^k, y^k_R)\| - \|h(x^k_R, y^k_R)\| + g(y^k) - g(y^k_R) \right],
\]

which is linear with respect to \( \theta \) and its slope is half the denominator of \((9)\). Moreover, this slope must be positive, otherwise \((8)\) would hold for all non-negative \( \theta \). So the expression of \( \theta_{k+1} \) is well defined and \( \Phi(x^k_R, y^k_R, \theta) - \Phi(x^k, y^k_R, \theta) \) is an increasing bijection from \( \mathbb{R} \) to \( \mathbb{R} \).

When \( \theta = 0 \) we have that

\[
\Phi(x^k_R, y^k_R, 0) - \Phi(x^k, y^k_R, 0) = \left[ \|h(x^k_R, y^k_R)\| - \|h(x^k, y^k_R)\| \right] + \left[ g(y^k_R) - g(y^k) \right].
\]

Since one of the inequalities in \((68)\) or \((69)\) is strict, we have that

\[
\Phi(x^k_R, y^k_R, 0) - \Phi(x^k, y^k_R, 0) < \frac{1-r}{2} \left[ \|h(x^k_R, y^k_R)\| - \|h(x^k, y^k_R)\| + g(y^k_R) - g(y^k) \right]. \quad (71)
\]

However, if \((8)\) does not hold, we have that

\[
\Phi(x^k_R, y^k_R, \theta) - \Phi(x^k, y^k_R, \theta) > \frac{1-r}{2} \left[ \|h(x^k_R, y^k_R)\| - \|h(x^k, y^k_R)\| + g(y^k_R) - g(y^k) \right]. \quad (72)
\]

So there exists only one value of \( \theta \in (0, \theta_k) \) verifying

\[
\Phi(x^k_R, y^k_R, \theta) - \Phi(x^k, y^k_R, \theta) = \frac{1-r}{2} \left[ \|h(x^k_R, y^k_R)\| - \|h(x^k, y^k_R)\| + g(y^k_R) - g(y^k) \right].
\]

By \((70)\), this value of \( \theta \) coincides with \( \theta_{k+1} \) computed in \((9)\). Therefore we also have that \((67)\) holds. So, the proof is complete.

In Lemma \(6.2\) we prove that the penalty parameters are bounded away from zero.

**Lemma 6.2** Define \( \bar{\theta} = \min \left\{ \theta_0, \frac{2}{1+r} \left( \frac{L_f y_R}{1-r} + 1 \right)^{-1} \right\} \). Then, for every iteration \( k \) in Algorithm BIRA we have that

\[
\theta_k \geq \bar{\theta} > 0. \quad (73)
\]

**Proof:** It is enough to prove that \( \theta_{k+1} \) is bounded below by \( \bar{\theta} \) when it is defined by \((9)\).

Equivalently, we need to show that \( \frac{1}{\theta_{k+1}} \) is bounded above in this situation. In fact,

\[
\frac{1}{\theta_{k+1}} = \frac{2[f(x^k_R, y^k_R) - f(x^k_R, y^k_R) + \|h(x^k_R, y^k_R)\| - \|h(x^k_R, y^k_R)\| + g(y^k) - g(y^k_R)]}{(1+r)[\|h(x^k_R, y^k_R)\| - \|h(x^k_R, y^k_R)\| + g(y^k) - g(y^k_R)] - f(x^k_R, y^k_R) - f(x^k_R, y^k_R)}
\]

\[
= \frac{2}{1+r} \left[ \|h(x^k_R, y^k_R)\| - \|h(x^k_R, y^k_R)\| + g(y^k) - g(y^k_R) + 1 \right]. \quad (74)
\]
By Step 1 of RESTA or \([5]\) when using a PDP,
\[-\|h(x_R^k, y_R^k)\| - g(y_R^k) \geq -r\|h(x_R^k, y_R^k)\| - rg(y^k),\]
therefore
\[\|h(x_R^k, y_R^k)\| + g(y_R^k) - \|h(x_R^k, y_R^k)\| - g(y_R^k) \geq \|h(x_R^k, y_R^k)\| + g(y_R^k) - r\|h(x_R^k, y_R^k)\| - rg(y^k) = (1 - r)(\|h(x_R^k, y_R^k)\| + g(y^k)) > 0.\]

Positivity necessarily takes place, otherwise we would have that \((x_R^k, y_R^k) = (x_k, y_k)\) and \(\theta_{k+1} = \theta_k\). Thus,
\[0 < \frac{1}{\|h(x_R^k, y_R^k)\| + g(y_R^k) - \|h(x_R^k, y_R^k)\| - g(y_R^k)} \leq \frac{1}{(1 - r)(\|h(x_R^k, y_R^k)\| + g(y^k))}.\]

On the other hand, by \([32]\), we have that \(|f(x_R^k, y_R^k) - f(x_k, y_R^k)| \leq L_f\|x_R^k - x^k\|\). Then, by \([74]\),
\[\frac{1}{\theta_{k+1}} \leq \frac{2}{1 + r} \left[ \frac{L_f\|x_R^k - x^k\|}{\|h(x_R^k, y_R^k)\| + g(y^k)} + 1 \right].\]

By \([65]\) in Theorem 5.1, there exists a positive constant \(\beta_R = O(1)\) such that \(\|x_R^k - x^k\| \leq \beta_R\|h(x_R^k, y_R^k)\|\). Then, since \(g(y^k) \geq 0\),
\[\frac{1}{\theta_{k+1}} \leq \frac{2}{1 + r} \left[ \frac{L_f\beta_R\|h(x_R^k, y_R^k)\|}{(1 - r)(\|h(x_R^k, y_R^k)\| + g(y^k))} + 1 \right] \leq \frac{2}{1 + r} \left[ \frac{L_f\beta_R(\|h(x_R^k, y_R^k)\| + g(y^k))}{(1 - r)(\|h(x_R^k, y_R^k)\| + g(y^k))} + 1 \right] = \frac{2}{1 + r} \left[ \frac{L_f\beta_R}{1 - r} + 1 \right].\]

The inequality above implies that, when \(\theta_k\) is updated, \(\{\frac{1}{\theta_{k}}\}\) is bounded, so \(\{\theta_k\}\) is bounded away from zero, with \(\bar{\theta} > 0\) as lower bound.

The following assumption establishes the conditions that must be satisfied by an approximate solution of \([12]\).

**Assumption A7** There exists \(\kappa_T > 0\) such that, at every iteration \(k\) of Algorithm BIRA, the approximate solution of the quadratic programming problem \([12]\) satisfies
\[\nabla_x f(x_R^k, y_R^{k+1})^T(x - x_R^k) + \frac{1}{2}(x - x_R^k)^T H_k(x - x_R^k) + \mu \|x - x_R^k\|^2 \leq 0\] (75)
and
\[\|\nabla_x h(x_R^k, y_R^{k+1})^T(x - x_R^k)\| \leq \kappa_T \|x - x_R^k\|^2.\] (76)

In the following lemma we prove that, when in the Optimization Phase, for a sufficiently large regularization parameter \(\mu\) the descent conditions for the objective function and the merit function are satisfied. As a consequence, in the subsequent corollary we establish the maximal number of iterations that could be needed to fulfill those conditions.
Lemma 6.3 Suppose that Assumption $A^7$ holds. Define $C_\mu = M + \bar{\alpha} + L_f$, where
\[
\bar{\alpha} = \max \left\{ \alpha, \frac{1-\bar{\theta}}{\theta} (\kappa_T + L_h) \right\}.
\] (77)

Then, if $\mu \geq C_\mu$ and $x$ is the solution of (12), the conditions (13) e (14) are fulfilled.

Proof: By (34), we have that $f(x, y^{k+1}) \leq f(x_R^k, y^{k+1}) + \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) + L_f \|x - x_R^k\|^2$.

Then, since $\|H_k\| \leq M$, we have that
\[
f(x, y^{k+1}) \leq f(x_R^k, y^{k+1}) + \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k)
\]
\[
- \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) + \bar{\alpha} \|x - x_R^k\|^2 - \bar{\alpha} \|x - x_R^k\|^2 + L_f \|x - x_R^k\|^2
\]
\[
\leq f(x_R^k, y^{k+1}) + \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k)
\]
\[
+ M \|x - x_R^k\|^2 + \bar{\alpha} \|x - x_R^k\|^2 - \bar{\alpha} \|x - x_R^k\|^2 + L_f \|x - x_R^k\|^2
\]
\[
\leq f(x_R^k, y^{k+1}) + \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k)
\]
\[
+ (M + \bar{\alpha} + L_f) \|x - x_R^k\|^2 - \bar{\alpha} \|x - x_R^k\|^2.
\]

Taking $\mu \geq C_\mu$, by Assumption $A^7$
\[
f(x, y^{k+1}) \leq f(x_R^k, y^{k+1}) - \bar{\alpha} \|x - x_R^k\|^2
\]
\[
+ \left[ \nabla_x f(x_R^k, y^{k+1})^T (x - x_R^k) + \frac{1}{2} (x - x_R^k)^T H_k (x - x_R^k) + \mu \|x - x_R^k\|^2 \right]
\]
\[
\leq f(x_R^k, y^{k+1}) - \bar{\alpha} \|x - x_R^k\|^2.
\]

Since $\alpha \leq \bar{\alpha}$ and, at Step 4, $y^{k+1} = y_R^k$, (13) necessarily holds. Moreover, by (77), we have that
\[
f(x, y^{k+1}) - f(x_R^k, y^{k+1}) \leq - \frac{1-\bar{\theta}}{\theta} (\kappa_T + L_h) \|x - x_R^k\|^2.
\] (78)

Let us prove that (14) also holds when $\mu \geq C_\mu$. Note that
\[
\Phi(x, y^{k+1}, \theta_{k+1}) - \Phi(x, y^{k+1}, \theta_{k+1}) = \left[ \Phi(x, y^{k+1}, \theta_{k+1}) - \Phi(x_R^k, y^{k+1}, \theta_{k+1}) \right]
\]
\[
+ \left[ \Phi(x_R^k, y^{k+1}, \theta_{k+1}) - \Phi(x, y^{k+1}, \theta_{k+1}) \right].
\] (79)

Define $v = \Phi(x, y^{k+1}, \theta_{k+1}) - \Phi(x_R^k, y^{k+1}, \theta_{k+1})$. By the definition of $\Phi$ and (78), the first term in the right-hand side of the equality above we have that
\[
v = \theta_{k+1} [f(x, y^{k+1}) - f(x_R^k, y^{k+1})] + (1 - \theta_{k+1}) \left[ \|h(x, y^{k+1})\| - \|h(x_R^k, y^{k+1})\| \right]
\]
\[
\leq \theta_{k+1} \left[ - \frac{1-\bar{\theta}}{\theta} (\kappa_T + L_h) \|x - x_R^k\|^2 \right] + (1 - \theta_{k+1}) \left[ \|h(x, y^{k+1})\| - \|h(x_R^k, y^{k+1})\| \right].
\]

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By (39) and (76),
\[ v \leq \theta_{k+1} \left[ -\frac{1-\bar{\theta}}{\theta} (\kappa_T + L_h) \| x - x^R_k \|^2 \right] + (1-\theta_{k+1}) \left[ \| \nabla_x h(x^R_k, y^{k+1})^T(x - x^R_k) \| + L_h \| x - x^R_k \|^2 \right] \]
\[ \leq \theta_{k+1} \left[ -\frac{1-\bar{\theta}}{\theta} (\kappa_T + L_h) \| x - x^R_k \|^2 \right] + (1-\theta_{k+1}) \left[ \kappa_T \| x - x^R_k \|^2 + L_h \| x - x^R_k \|^2 \right]. \]

Since \( \{\theta_k\} \) is bounded below by \( \bar{\theta} \), we have that
\[ v \leq \bar{\theta} \left[ -\frac{1-\bar{\theta}}{\theta} (\kappa_T + L_h) \| x - x^R_k \|^2 \right] + (1-\bar{\theta})(\kappa_T + L_h) \| x - x^R_k \|^2 = 0. \]

Then, the first term in (79) is not positive. On the other hand, as \( y^{k+1} = y^R_k \), (67) is equivalent to
\[ \Phi(x^R_k, y^{k+1}, \theta_{k+1}) - \Phi(x^R_k, y^{k+1}, \theta_{k+1}) \leq \frac{1-r}{2} \left[ \| h(x^R_k, y^{k+1}) \| - \| h(x^R_k, y^{k+1}) \| + g(y^{k+1}) - g(y^k) \right]. \]

Then, by (79),
\[ \Phi(x, y^{k+1}, \theta_{k+1}) - \Phi(x, y^{k+1}, \theta_{k+1}) \leq \frac{1-r}{2} \left[ \| h(x^R_k, y^{k+1}) \| - \| h(x^R_k, y^{k+1}) \| + g(y^{k+1}) - g(y^k) \right]. \]

Therefore, if \( \mu \geq C_\mu \), both (13) e (14), are fulfilled, guaranteeing that \( x^{k+1} \) is well defined. \( \square \)

**Corollary 6.1** Suppose that Assumption A7 holds. Define \( N_{\text{reg}} = \lfloor \log_2(C_\mu) - \log_2(\mu_{\text{min}}) \rfloor + 1 \) and \( \bar{\mu} = \max\{10C_\mu, 10^{N_{\text{acce}}\mu_{\text{max}}}\} \). Then, after at most \( N_{\text{reg}} \) sub-iterations at Step 4 of BIRA, the conditions (13) and (14) are fulfilled. Moreover, \( \mu_k \leq \bar{\mu} \) for all \( k \).

**Proof:** If \( x^{k+1} \) is computed by ACCEL, we have that \( \mu \) is increased at most \( N_{\text{acce}} \mu_{\text{max}} \) times, starting from a value limited above by \( \mu_{\text{max}} \). Therefore \( \mu_k \leq 10^{N_{\text{acce}}\mu_{\text{max}}} \). On the other hand, if \( x^{k+1} \) is computed at the Optimization Phase, by Lemma 6.3 if \( \mu \geq C_\mu \), the decrease conditions at Step 4.2 are satisfied. So, if the initial value of \( \mu \) is greater than \( C_\mu \), \( \mu_k = \mu \). Otherwise, by (15), we would have that \( \mu_k \leq 10C_\mu \). Since \( 10^{N_{\text{acce}}\mu_{\text{max}}} \geq \mu_{\text{max}} \), we have that \( \mu_k \leq \max\{10C_\mu, 10^{N_{\text{acce}}\mu_{\text{max}}} \} \).

Moreover, as the initial value of \( \mu \) is not smaller than \( \mu_{\text{min}} \), after \( N_{\text{reg}} \) updatings we have that \( \mu \geq 2^{N_{\text{reg}}} \mu_{\text{min}} \). So, if \( 2^{N_{\text{reg}}} \mu_{\text{min}} \geq C_\mu \), or, equivalently, \( N_{\text{reg}} + \log_2(\mu_{\text{min}}) \geq \log_2(C_\mu) \), (13) and (14) are fulfilled. \( \square \)

### 7 Convergence to feasibility and convergence of stepsize

In this section we will prove that, when executing BIRA, the infeasibility measure tends to zero. Moreover, we will prove that the norm of the difference between consecutive iterates also tend to zero.

For all the proofs of this section we will assume, without specific mention, that all the General Assumptions, the Assumptions A1–A7, and the following Assumption A8 take place. Assumption A8 states that bounded deterioration of objective function and also \( h \)-feasibility occurs in
a restricted way, depending of a possibly small parameter that depends of $\bar{\theta}$. This means that, in the worst case, bounded deterioration with respect to precision does not occur at all. Note that, however, the new bounded deterioration condition needs to hold only for $k$ large enough.

**Assumption A8** Let $\bar{\theta}$ be as defined in Lemma 6.2. Then, there exist $k_R$, and $\gamma \in (0, 1)$ such that, for $\bar{\beta} = \frac{\bar{\theta}(1-\gamma)(1-r)^2}{2}$ and all $k \geq k_R$,

$$f(x^k, y^{k+1}) \leq f(x^k, y^k) + \beta g(y^k) \quad \text{and} \quad \|h(x^k, y^{k+1})\| \leq \|h(x^k, y^k)\| + \beta g(y^k). \quad (80)$$

Theorem 7.1 states the summability of all infeasibilities.

**Theorem 7.1** Define

$$C_{\text{feas}} = \frac{2}{\gamma(1-r)^2} \left[k_R(2C_f + C_h) + C_\rho + C_h + C_g\right]. \quad (81)$$

Then,

$$\sum_{j=0}^{k-1} \|h(x^j, y^{j+1}) + g(y^j)\| \leq C_{\text{feas}}. \quad (82)$$

**Proof:** Let us define

$$\rho_j = \frac{1 - \theta_j}{\theta_j} = \frac{1}{\theta_j} - 1, \text{ for all } j \leq k. \quad (83)$$

By Lemma 6.2, we know that $\theta_j \in (0, 1), \{\theta_j\}$ is non-increasing and bounded below by $\bar{\theta}$. Then, the sequence $\{\rho_j\}$ is positive, non-decreasing an bounded above by $\bar{\rho} = \frac{1}{\theta} - 1$. Then, since $\rho_0 > 0$,

$$\sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j) = \rho_k - \rho_0 < \rho_k = \frac{1}{\theta_k} - 1 \leq \frac{1}{\theta} - 1 < \frac{1}{\bar{\theta}} < \infty. \quad (84)$$

By (35), we have that $\|h(x^j, y^{j+1})\| \leq C_h$ for all $j$. Since $\rho_{j+1} - \rho_j \geq 0$, taking $C_\rho \equiv \frac{C_h}{\bar{\theta}}$, thanks to (84), we have that

$$\sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j)\|h(x^j, y^{j+1})\| \leq \sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j)C_h \leq \frac{C_h}{\bar{\theta}} = C_\rho < \infty. \quad (85)$$

Independently of the fact that $x^{j+1}$ could have been computed at Step 3 (Acceleration) or Step 4 (Optimization Phase) we have that

$$\Phi(x^{j+1}, y^{j+1}, \theta_{j+1}) - \Phi(x^j, y^{j+1}, \theta_{j+1}) \leq \frac{1-r}{2} \left[\|h(x^j, y_R^j)\| - \|h(x^j, y_R^j)\| + g(y_R^j) - g(y^j)\right]$$

$$\leq \frac{1-r}{2} \left[\|h(x^j, y_R^j)\| + g(y^j)\right], \quad (86)$$
where the second inequality comes from \( \|h(x^j, y^j_R)\| \leq r\|h(x^j, y^j)\| \) and \( g(y^j_R) \leq rg(y^j) \).

By the definition of \( \Phi \), dividing \( (86) \) by \( \theta_j+1 \), we have that, for all \( j \leq k-1 \),

\[
f(x^{j+1}, y^{j+1}) + \frac{1-\theta_{j+1}}{\theta_{j+1}} \left[ \|h(x^{j+1}, y^{j+1})\| + g(y^{j+1}) \right] - f(x^j, y^{j+1}) - \frac{1-\theta_{j+1}}{\theta_{j+1}} \left[ \|h(x^j, y^{j+1})\| + g(y^{j+1}) \right] \\
\leq -\frac{(1-r)^2}{2\theta_{j+1}} \left[ \|h(x^j, y^j_R)\| + g(y^j) \right].
\]

By the definition of \( \rho_j \) in \( (83) \), using that \( \theta_j \in (0, 1) \) and simplifying, we deduce that

\[
\frac{(1-r)^2}{2} \left[ \|h(x^j, y^j_R)\| + g(y^j) \right] \leq f(x^j, y^{j+1}) - f(x^{j+1}, y^{j+1}) + (\rho_{j+1} - \rho_j)\|h(x^j, y^{j+1})\| \\
+ \rho_j\|h(x^j, y^{j+1})\| - \rho_{j+1}\|h(x^{j+1}, y^{j+1})\|.
\]

Adding and subtracting \( \rho_j\|h(x^j, y^{j+1})\| \) on the right-hand side of \( (87) \), and arranging terms, we have:

\[
\frac{(1-r)^2}{2} \sum_{j=0}^{k-1} \left[ \|h(x^j, y^j_R)\| + g(y^j) \right] \leq f(x^0, y^1) - f(x^k, y^k) + \sum_{j=1}^{k-1} [f(x^j, y^{j+1}) - f(x^j, y^j)] \\
+ \sum_{j=k}^{k-1} [f(x^j, y^{j+1}) - f(x^j, y^j)] + \sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j)\|h(x^j, y^{j+1})\| \\
+ \rho_0\|h(x^0, y^1)\| - \rho_k\|h(x^k, y^k)\| + \sum_{j=1}^{k-1} \rho_j\|h(x^j, y^{j+1})\| - \|h(x^j, y^j)\| \\
+ \sum_{j=k}^{k-1} \rho_j\|h(x^j, y^{j+1})\| - \|h(x^j, y^j)\|.
\]

By \( (85) \), \( \sum_{j=0}^{k-1} (\rho_{j+1} - \rho_j)\|h(x^j, y^{j+1})\| \leq C_\rho \). Moreover, since \( \rho_j \leq \bar{\rho} \), by Assumption \( A8 \) and disregarding the certainly non-positive terms, \( (88) \) implies that

\[
\frac{(1-r)^2}{2} \sum_{j=0}^{k-1} \left[ \|h(x^j, y^j_R)\| + g(y^j) \right] \leq |f(x^0, y^1)| + |f(x^k, y^k)| + \sum_{j=1}^{k-1} |f(x^j, y^{j+1}) - f(x^j, y^j)| + \sum_{j=k}^{k-1} \bar{\rho}g(y^j) \\
+ C_\rho + \rho_0\|h(x^0, y^1)\| + \sum_{j=1}^{k-1} \bar{\rho}\|h(x^j, y^{j+1})\| + \sum_{j=k}^{k-1} \bar{\rho}\bar{\beta}g(y^j).
\]

By \( (31) \), \( (35) \), and \( (40) \) we have that \( f, \|h\|, \) and \( g \) are bounded above by \( C_f \), \( C_h \), and \( C_g \) respectively. Then, as \( \bar{\rho} + 1 = \frac{1}{\theta} \), we obtain that

\[
\frac{(1-r)^2}{2} \sum_{j=0}^{k-1} \left[ \|h(x^j, y^j_R)\| + g(y^j) \right] \leq kR(2C_f + C_h) + C_\rho + \frac{\bar{\rho}}{\theta} \sum_{j=k}^{k-1} g(y^j).
\]
Therefore, using that \(0 \leq g(y^j) \leq g(y^i) + \|h(x^j, y^j_R)\|\) and \(\frac{\gamma}{2} = \frac{(1-\gamma)(1-r)^2}{2}\) we obtain:

\[
(1-r)^2 \sum_{j=0}^{k} \left( \|h(x^j, y^j_R)\| + g(y^j) \right) \leq \|x^k_R\| + C_h + C_\rho + \frac{(1-\gamma)(1-r)^2}{2} \sum_{j=k}^{k-1} g(y^j) \\
+ \|x^k_R\| + g(y^k) \\
. \leq \|x^k_R\| + C_h + C_\rho + \frac{(1-\gamma)(1-r)^2}{2} \sum_{j=0}^{k} \left( \|h(x^j, y^j_R)\| + g(y^j) \right) \\
+ \|x^k_R\| + g(y^k)
\]

Thus,

\[
\gamma(1-r)^2 \sum_{j=0}^{k} \left( \|h(x^j, y^j_R)\| + g(y^j) \right) \leq \|x^k_R\| + C_h + C_\rho + C_h + C_g.
\]

So, by (81), we obtain (82), as desired. □

Theorem 7.2 states the summability of the squared norms of consecutive iterations. The obvious consequence is that the norm of the difference between consecutive iterations tends to zero.

**Theorem 7.2** Define \(C_d = \frac{1}{\alpha}[(\beta_f + \beta)C_{feas} + 2C_f]\). Then,

\[
\sum_{j=0}^{k} \|x^{j+1} - x^j_R\|^2 \leq C_d.
\]

**Proof:** By (10) or (13), we have that

\[
\alpha \|x^{j+1} - x^j_R\|^2 \leq f(x^j_R, y^j_R) - f(x^{j+1}, y^{j+1}) \\
= f(x^j_R, y^{j+1}_R) - f(x^{j+1}, y^{j+1}) - f(x^j, y^j) - f(x^j, y^{j+1}) + f(x^j, y^j) - f(x^{j+1}, y^{j+1}).
\]

For all \(j \leq k - 1\), by (60), we have that \(f(x^j_R, y^j_R) - f(x^j, y^j) \leq \beta_f \left[ \|h(x^j, y^j_R)\| + g(y^j) \right].\)

On the other hand, (64) implies that \(f(x^j_R, y^j_R) - f(x^j, y^j) \leq \beta g(y^j).\) So,

\[
\alpha \|x^{j+1} - x^j_R\|^2 \leq \beta_f \left[ \|h(x^j, y^j_R)\| + g(y^j) \right] + \beta g(y^j) + f(x^j, y^j) - f(x^{j+1}, y^{j+1}).
\]

Using that \(\|h(x^j, y^j_R)\| \geq 0\) and adding terms from \(j\) to \(k - 1\), we obtain:

\[
\alpha \sum_{j=0}^{k-1} \|x^{j+1} - x^j_R\|^2 \leq (\beta_f + \beta) \sum_{j=0}^{k-1} \left[ \|h(x^j, y^j_R)\| + g(y^j) \right] + \sum_{j=0}^{k-1} \left[ f(x^j, y^j) - f(x^{j+1}, y^{j+1}) \right].
\]

Therefore, by (82),

\[
\alpha \sum_{j=0}^{k-1} \|x^{j+1} - x^j_R\|^2 \leq (\beta_f + \beta)C_{feas} + f(x^0, y^0) - f(x^k, y^k).
\]

Finally, by (31) and (92), the desired result is obtained. □
8 Complexity and Convergence

In this section we suppose, without specific mention, that the General Assumptions, Assumptions A1–A8, and the following Assumption A9 hold. Assumption A9 merely states the approximate optimality conditions that the approximate solutions mentioned in (12) and (28) must fulfill.

**Assumption A9** There exists $\kappa > 0$ such that, for every iteration $k$ at Algorithm BIRA, the approximate solutions of (12) and (28) satisfy

$$\|P_{D^{k+1}}(x^{k+1} - \nabla f(x^k, y^{k+1}) - H_k(x^{k+1} - x^k) - 2\mu_k(x^{k+1} - x^k)) - x^{k+1}\| \leq \kappa \|x^{k+1} - x^k\|, \quad (93)$$

where $D^{k+1}$ is defined by

$$D^{k+1} = \{ x \in \Omega | \nabla x h(x^k, y^{k+1})^T (x - x^k) = 0 \}. \quad (94)$$

In Lemma 8.1 we prove that the projected gradient of the objective function onto the tangent set to the constraints tends to zero proportionally to the norm of the difference between $x^{k+1}$ and the restored point $x^k$.

**Lemma 8.1** Define

$$C_p = M + \kappa + 2\bar{\mu} + 2, \quad (95)$$

where $\bar{\mu}$ is defined in Corollary 6.1. Then, independently of the fact of $x^{k+1}$ having been obtained at the Acceleration Phase or at the Optimization Phase, we have:

$$\|P_{D^{k+1}}(x^k - \nabla f(x^k, y^{k+1})) - x^k\| \leq C_p \|x^{k+1} - x^k\|. \quad (96)$$

**Proof:** Define $v = P_{D^{k+1}}(x^{k+1} - \nabla f(x^k, y^{k+1}) - H_k(x^{k+1} - x^k) - 2\mu_k(x^{k+1} - x^k))$.

By (93),

$$\|P_{D^{k+1}}(x^k - \nabla f(x^k, y^{k+1})) - x^k\| = \|P_{D^{k+1}}(x^k - \nabla f(x^k, y^{k+1})) - v + v + x^{k+1} + x^{k+1} - x^k\|$$

$$\leq \|P_{D^{k+1}}(x^k - \nabla f(x^k, y^{k+1})) - v\| + \|v - x^{k+1}\| + \|x^{k+1} - x^k\|$$

$$\leq \|P_{D^{k+1}}(x^k - \nabla f(x^k, y^{k+1})) - v\| + \|v - x^{k+1}\| + \|x^{k+1} - x^k\| \quad (97)$$

By Step 4 of BIRA or Step 1 of ACCEL, we have that $\|H_k\| \leq M$. By the non-expansive property of projections,

$$\|P_{D^{k+1}}(x^k - \nabla f(x^k, y^{k+1})) - v\| \leq \|x^k - x^{k+1} + H_k(x^{k+1} - x^k) + 2\mu_k(x^{k+1} - x^k)\|$$

$$\leq \|H_k(x^{k+1} - x^k)\| + (2\mu_k + 1)\|(x^{k+1} - x^k)\|$$

$$\leq (M + 2\mu_k + 1)\|x^{k+1} - x^k\|. \quad (98)$$

By Corollary 6.1 we have that $\mu_k \leq \bar{\mu}$. Then, by (97) and (98):

$$\|P_{D^{k+1}}(x^k - \nabla f(x^k, y^{k+1})) - x^k\| \leq \|P_{D^{k+1}}(x^k - \nabla f(x^k, y^{k+1})) - v\| + (\kappa + 1)\|x^{k+1} - x^k\|$$

$$\leq (M + 2\mu_k + 1)\|x^{k+1} - x^k\| + (\kappa + 1)\|x^{k+1} - x^k\|$$

$$= (M + \kappa + 2\bar{\mu} + 2)\|x^{k+1} - x^k\|. \quad (99)$$

\[\text{Page 25}\]
Lemma 8.2 establishes the summability of squared norms of the projected gradients of the objective function computed as the restored iterates.

**Lemma 8.2** Define $C_{proj} = C_p^2 C_d$. Then, for every iteration $k$ of BIRA, we have that

$$
\sum_{j=0}^{k} \| P_{D_j+1}(x^j_R - \nabla_x f(x^j, y^{j+1})) - x^j_R \|^2 \leq C_{proj}.
$$

**Proof:** By Lemma 8.1

$$
\| P_{D_{j+1}}(x^j_R - \nabla_x f(x^j, y^{j+1})) - x^j_R \| \leq C_p \| x^{j+1} - x^j_R \|,
$$

for all $j$.

Adding the first $k$ squared terms of (100), by (89), we have that

$$
\sum_{j=0}^{k} \| P_{D_{j+1}}(x^j_R - \nabla_x f(x^j, y^{j+1})) - x^j_R \|^2 \leq \sum_{j=0}^{k} \left[ C_p \| x^{j+1} - x^j_R \| \right]^2
$$

$$
= C_p^2 \sum_{j=0}^{k} \| x^{j+1} - x^j_R \|^2
$$

$$
\leq C_p^2 C_d.
$$

Lemma 8.3 is a complexity result establishing that the number of iterations at which infeasibility takes place with respect to given precisions is, in the worst case, proportional to the multiplicative inverse of the precisions required. From a practical point of view, to be consistent with the Restoration Failure criterion, the accuracy with respect to $g$ should be less demanding than the one used in RESTA. However this is not a mathematical requirement and is not used in the following lemma.

**Lemma 8.3** Let $\epsilon_{feas} > 0$ and $\epsilon_{prec} > 0$ be given. Let $N_{hinfeas}$ be the number of iterations of BIRA at which $\| h(x^k_R, y^k_R) \| > \epsilon_{feas}$, $N_{ginfeas}$ the number of iterations of BIRA at which $g(y^k) > \epsilon_{prec}$, and $N_{inffeas}$ the number of iterations of BIRA such that $h(x^k_R, y^k_R) > \epsilon_{feas}$ or $g(y^k) > \epsilon_{prec}$. Then,

$$
N_{hinfeas} \leq \left[ \frac{r C_{feas}}{\epsilon_{feas}} \right], \quad N_{ginfeas} \leq \left[ \frac{C_{feas}}{\epsilon_{prec}} \right] \quad e \quad N_{inffeas} \leq \left[ \max \left\{ \frac{r C_{feas}}{\epsilon_{feas}}, \frac{r C_{feas}}{\epsilon_{prec}} \right\} \right].
$$

**Proof:** By (82),

$$
\sum_{j=0}^{k} \| h(x^j, y^j_R) \| + g(y^j) \leq C_{feas}.
$$
Then, as $0 \leq \|h(x^j_R, y^j_R)\| \leq r\|h(x^j, y^j_R)\|$ and $g(y^j) \geq 0$, 

$$rC_{\text{feas}} \geq \sum_{j=0}^{k} r\|h(x^j, y^j_R)\| + r g(y^j) \geq \sum_{j=0}^{k} \|h(x^j_R, y^j_R)\| \geq N_{\text{infeas}} \epsilon_{\text{feas}}.$$ 

So, \( \frac{rC_{\text{feas}}}{\epsilon_{\text{feas}}} \geq N_{\text{infeas}} \). Analogously, for $g(y^j) > \epsilon_{\text{prec}}$, we have that 

$$C_{\text{feas}} \geq \sum_{j=0}^{k} \{\|h(x^j, y^j_R)\| + g(y^j)\} \geq \sum_{j=0}^{k} g(y^j) \geq N_{\text{infeas}} \epsilon_{\text{prec}},$$ 

therefore \( \frac{C_{\text{feas}}}{\epsilon_{\text{prec}}} \geq N_{\text{infeas}} \).

Finally, if $\|h(x^j_R, y^k_R)\| > \epsilon_{\text{feas}}$ or $g(y^k_R) > \epsilon_{\text{prec}}$ we have that $\|h(x^j_R, y^j_R)\| + g(y^j_R) > \min\{\epsilon_{\text{feas}}, \epsilon_{\text{prec}}\}$. Thus, as $\|h(x^j_R, y^j_R)\| + g(y^j_R) \leq r(\|h(x^j, y^j_R)\| + g(y^j))$,

$$rC_{\text{feas}} \geq \sum_{j=0}^{k} r(\|h(x^j, y^j_R)\| + g(y^j)) \geq \sum_{j=0}^{k} \|h(x^j, y^j_R)\| + g(y^j_R) \geq N_{\text{infeas}} \epsilon_{\text{feas}},$$

so $N_{\text{infeas}} \leq \max\left\{ \frac{rC_{\text{feas}}}{\epsilon_{\text{feas}}}, \frac{rC_{\text{feas}}}{\epsilon_{\text{prec}}} \right\}$. \( \square \)

Lemma 8.4 is a complexity result that states that the number of iterations at which the projected gradient of the objective function at the restored points is bigger than a given tolerance $\epsilon_{\text{opt}}$ is proportional, in the worst case, to $\epsilon_{\text{opt}}^{-2}$.

**Lemma 8.4** Suppose that $\epsilon_{\text{opt}} > 0$. Let $N_{\text{opt}}$ be the number of iterations such that $\|P_{D^{j+1}}(x^j_R - \nabla_x f(x^j_R, y^{k+1})) - x^j_R\| > \epsilon_{\text{opt}}$. Then,

$$N_{\text{opt}} \leq \left| \frac{C_\text{proj}}{\epsilon_{\text{opt}}^2} \right|. \quad (102)$$

**Proof:** If during $N_{\text{opt}}$ iterations we have $\|P_{D^{j+1}}(x^j_R - \nabla_x f(x^j_R, y^{k+1})) - x^j_R\| > \epsilon_{\text{opt}}$, by (99), we have that 

$$C_\text{proj} \geq \sum_{j=0}^{k} \|P_{D^{j+1}}(x^j_R - \nabla_x f(x^j_R, y^{k+1})) - x^j_R\| \geq N_{\text{opt}} \epsilon_{\text{opt}}^2$$

Therefore, \( \left| \frac{C_\text{proj}}{\epsilon_{\text{opt}}^2} \right| \geq N_{\text{opt}} \). \( \square \)

**Theorem 8.1** Suppose that the General Assumptions and Assumptions A1–A8 hold. Given $\epsilon_{\text{prec}} > 0$, $\epsilon_{\text{feas}} > 0$, and $\epsilon_{\text{opt}} > 0$, then:
• If RESTA does not stop by Restoration Failure and \( N_{\text{max}} \) is the maximum number of iterations \( j \) of BIRA such that \( g(y_R^j) > \epsilon_{\text{prec}} \), or \( g(y_R^{j+1}) > \epsilon_{\text{prec}} \), or \( \|h(x_R^j, y_R^j)\| > \epsilon_{\text{feas}} \) then \( N_{\text{max}} \leq \left[ \max \left\{ \frac{rC_{\text{feas}}}{\epsilon_{\text{feas}}}, \frac{rC_{\text{feas}}}{\epsilon_{\text{prec}}} \right\} + \frac{C_{\text{feas}}}{\epsilon_{\text{prec}}} \right] + \frac{C_{\text{proj}}}{\epsilon_{\text{opt}}} \right] \). 

\( \text{(103)} \)

• The total number of evaluations of \( h, \nabla_x h, f, \) and \( \nabla_x f \) in BIRA until declaring Resoration Failure or finding \( x_R^j \) such that

\[
\|h(x_R^j, y_R^j)\| \leq \epsilon_{\text{feas}}, \ g(y_R^j) \leq \epsilon_{\text{prec}}, \ g(y_R^{j+1}) \leq \epsilon_{\text{prec}} \text{ and } \|P_D(x_R^j - \nabla_x f(x_R^j, y_R^{j+1}))-x_R^j\| \leq \epsilon_{\text{opt}}
\]

is bounded by \( N_{av} \), where

\[
N_{av} = O(\min\{\epsilon_{\text{prec}}, \epsilon_{\text{feas}}\}^{-1} + \epsilon_{\text{prec}}^{-1} + \epsilon_{\text{opt}}^{-2}).
\]

\( \text{(105)} \)

Proof: Assume firstly that BIRA does not stop with Restoration Failure. By Lemma 8.3, the inequalities \( \|h(x_R^j, y_R^j)\| > \epsilon_{\text{feas}} \) or \( g(y_R^j) > \epsilon_{\text{prec}} \) may occur at most during \( \left[ \max \left\{ \frac{rC_{\text{feas}}}{\epsilon_{\text{feas}}}, \frac{rC_{\text{feas}}}{\epsilon_{\text{prec}}} \right\} \right] \) iterations. Therefore, after \( \left[ \max \left\{ \frac{rC_{\text{feas}}}{\epsilon_{\text{feas}}}, \frac{rC_{\text{feas}}}{\epsilon_{\text{prec}}} \right\} \right] + \left[ \frac{C_{\text{feas}}}{\epsilon_{\text{prec}}} \right] + \frac{C_{\text{proj}}}{\epsilon_{\text{opt}}} \right] + 1 \) iterations, we know that at least \( \frac{C_{\text{feas}}}{\epsilon_{\text{prec}}} \) or \( \frac{C_{\text{proj}}}{\epsilon_{\text{opt}}} \) of these iterations, the inequalities \( g(y_R^j) \leq \epsilon_{\text{prec}} \) and \( h(x_R^j, y_R^j) \leq \epsilon_{\text{feas}} \) are fulfilled.

By Lemma 8.4, the inequality \( \|P_D(x_R^j - \nabla_x f(x_R^j, y_R^{j+1}))-x_R^j\| > \epsilon_{\text{opt}} \) may occur at most in \( \left[ \frac{C_{\text{proj}}}{\epsilon_{\text{opt}}} \right] + 1 \) iterations. Thus, at least in \( \left[ \frac{C_{\text{feas}}}{\epsilon_{\text{prec}}} \right] + 1 \) iterations we should have that \( \|h(x_R^j, y_R^j)\| \leq \epsilon_{\text{feas}}, \ g(y_R^j) \leq \epsilon_{\text{prec}}, \text{ and } \|P_D(x_R^j - \nabla_x f(x_R^j, y_R^{j+1}))-x_R^j\| \leq \epsilon_{\text{opt}} \).

Analogously, by Lemma 8.3 the number of iterations at which \( g(y_R^{j+1}) \leq \epsilon_{\text{prec}} \) is bounded by \( \frac{C_{\text{feas}}}{\epsilon_{\text{prec}}} \), then at least in one over the \( \frac{C_{\text{feas}}}{\epsilon_{\text{prec}}} \) + 1 iterations (104) takes place. So, (103) is proved.

Thinking (104) as a “stopping criterion” for BIRA, the total number of iterations would be at most \( N_{\text{max}} + 1 \), since we would have stopped by Restoration Failure or the conditions (104) would be satisfied. Let us now analyze the number of functions evaluations in each iteration.

For every iteration of BIRA, by Lemma 5.5 the Restoration Phase finishes after at most \( N_R \) evaluations of \( h \) and \( \nabla_x h \). At Step 2 of BIRA, there are no evaluations of \( \nabla_x h \) and additional evaluations of \( h \) are not necessary, since \( h(x^k, y_R^j) \) and \( h(x_R^j, y_R^k) \) have been already computed in RESTA or to check (5).

Let us see what happens when the ACCEL algorithm is called at each iteration of BIRA. Not that we only use one evaluation of \( \nabla_x h(x_R^k, y^{k+1}) \), for building subproblem (28). Moreover, \( h(x^k, y^{k+1}) \) is computed once in order to verify (30). In addition, \( h(x, y^{k+1}) \) is computed each time that (30) needs to be verified, which amounts \( N_{\text{acce}} \) evaluations. Conditions (10) and (11) are already tested as the stopping criterion of ACCEL, so they do not entail further evaluations. Therefore, at the Acceleration Phase, just one evaluation of \( \nabla_x h \) is made and at most \( N_{\text{acce}} + 1 \) evaluations of \( h \).
Similarly, in the Optimization Phase we employ one evaluation of \( \nabla_x h(x^k, y^{k+1}) \) to define \( h(x^k, y^{k+1}) \). We do not need to evaluate \( h(x^k, y^{k+1}) \), because \( y^{k+1} = y^R \), so it has already been computed at the Restoration Phase. On the other hand, Corollary \[6.1\] guarantees that at most \( N_{\text{reg}} \) evaluations of \( h(x, y^{k+1}) \) are necessary to obtain \( \{\parallel \} \).

Therefore, the number of evaluations of \( h \) and \( \nabla_x h \), at each iteration of BIRA is, respectively, \( N_R + N_{\text{acce}} + 1 + N_{\text{reg}} \) and \( N_R + 1 + 1 \). Thus, since the total number of iterations before declaring Restoration Failure or obtaining \( \{\parallel \} \) is \( O(\min\{\epsilon_{\text{prec}}, \epsilon_{\text{feas}}\})^{-1} + \epsilon_{\text{opt}}^{-2} \), we have that analogous bound corresponds to the number of evaluations of \( h \) and \( \nabla_x h \).

Let us now consider the number of evaluations of \( f \). In the Restoration Phase it is not necessary to compute \( f \) nor \( \nabla_x f \). At Step 2 of BIRA we have two evaluations of \( f \). In ACCEL we need to compute \( \nabla_x f(x^k_R, y^{k+1}) \). At Step 4 of ACCEL, there is no need to calculate \( f(x_R, y^{k+1}) \), which has already been evaluated in the Restoration Phase, but we need to compute \( f(x^k, y^{k+1}) \), which can be used for every verification of \( \{\parallel \} \). Finally, \( f(x, y^{k+1}) \) needs to be computed at every loop of the main process, which amounts \( N_{\text{acce}} \) evaluations.

On the Optimization Phase, we need to compute \( \nabla_x f(x_R, y^{k+1}) \). Since \( f(x^k, y^{k+1}) = f(x^k, y^R) \) has been already computed at Step 2, we do not need an additional evaluation at Step 4. Finally, by Corollary \[6.1\] the Optimization Phase finishes after at most \( N_{\text{reg}} \) calls to Step 4. Then, \( f \) is evaluated at most \( N_{\text{reg}} \) times at the Optimization Phase of every iteration of BIRA. So, at each iteration of BIRA, \( f \) is computed at most \( N_{\text{acce}} + N_{\text{reg}} + 3 \) times and, at most, two evaluations of \( \nabla_x f \) are necessary. Since \( N_{\text{acce}} \) and \( N_{\text{reg}} \) do not depend on \( \epsilon_{\text{prec}}, \epsilon_{\text{feas}} \), and \( \epsilon_{\text{opt}} \), this finishes the proof of the desired result.

Our last theorem concerns the asymptotic convergence of BIRA. For this, it is natural to consider that the algorithm generates an infinite sequence, not meeting a stopping criterion. Therefore, it is reasonable to think that \( \epsilon_{\text{prec}} \) is null.

**Theorem 8.2** Suppose that the General Assumptions, Assumptions A1–A8 hold, and BIRA does not stop by Restoration Failure. Then,

\[
\lim_{k \to \infty} g(y^k) = 0, \quad \lim_{k \to \infty} g(y^R) = 0, \quad \lim_{k \to \infty} \parallel h(x^k_R, y^k) \parallel = 0, \quad \text{and} \quad \lim_{k \to \infty} \parallel P_{D^k+1}(x^k_R - \nabla_x f(x^k_R, y^{k+1})) - x^k_R \parallel = 0.
\]

**Proof:** Assume, by contradiction that BIRA computes infinitely many iterations and at least one of the sequences \( \{\parallel h(x^k_R, y^k_R) \parallel\} \), \( \{g(y^k)\} \) or \( \{\parallel P_{D^k}(x^k_R - \nabla_x f(x^R_k, y^{k+1})) - x^k_R \parallel\} \) does not converge to zero. To fix ideas, suppose that \( \{g(y^k)\} \) does not converge to zero. Then, there exists \( \varepsilon > 0 \) and infinitely many indices \( K \) such that \( g(y^j) > \varepsilon \) for all \( j \in K \). Therefore, \( g(y^k) > \varepsilon_{\text{feas}} \) occurs infinitely many times if we define \( \epsilon_{\text{prec}} = \varepsilon \). By Theorem \[6.1\], this is impossible and so \( \lim_{k \to \infty} g(y^k) = 0 \). Since \( 0 \leq g(y^k_R) \leq g(y^k) \), we also have that \( \lim_{k \to \infty} g(y^R) = 0 \).

The convergence to zero of the sequences \( \{\parallel h(x^k_R, y^k_R) \parallel\} \) and \( \{\parallel P_{D^k}(x^k_R - \nabla_x f(x^k_R, y^{k+1})) - x^k_R \parallel\} \) is proved in an entirely analogous way using \( \epsilon_{\text{feas}} = \varepsilon \) or \( \epsilon_{\text{opt}} = \varepsilon \), respectively.

\[\square\]
9 Conclusions

Many practical problems require the minimization of functions that are very difficult to evaluate with constraints with the same characteristics. In these cases, common sense indicates that one should try to minimize suitable progressive approximations with the hope that successive partial minimizers would converge to the solution of the original problem. In many cases error bounds are not available, so that we know how to get closer to the true problem but we cannot estimate distances between partial and final solutions.

The natural questions that arise are: With which precision we need to solve each partial problem? and How to choose the approximate problem that should be addressed after finishing each stage of the process? For solving these questions one needs to consider two different objectives: decreasing the objective function and increasing the precision. It is natural to combine these objectives in a single merit function.

The papers [11, 12, 13, 40] suggested that a good framework to address this problem is given by the Inexact Restoration approach of classical constrained optimization. The idea is that “maximal evaluation precision” can be considered as a constraint of the problem depending of a precision variable \( y \) that lies in an abstract set \( Y \). The tools of Inexact Restoration indicate an algorithmic path for modifying \( y \) and decreasing the objective function in such a way that, hopefully, most iterations are performed with moderate precision and the overall computational cost is affordable.

The present paper is the first contribution in which the Inexact Restoration framework is applied to the case in which, not only the objective function but also the constraints are subject to uncertainty. An interesting feature of our approach is that our method applied to the particular case in which exact evaluations are possible (\( Y \) is a singleton, \( g_f(y) = 0 \) and \( g_h(y) = 0 \)) coincides with (a version of) the classical Inexact Restoration method for smooth constrained optimization. Paradoxically, this nice feature motivates a challenging open problem: Is it really necessary to use the IR approach both for the algebraic and the precision constraints? From the aesthetic point of view our “double IR” strategy seems to be attractive but it cannot be discarded that using different underlying strategies for the algebraic constraints could result in more efficient algorithms.

References


