An SDP Relaxation for the Sparse Integer Least Square Problem

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Abstract

In this paper, we study the sparse integer least square problem (SILS), which is the NP-hard variant of the least square problem, where we only consider sparse \( \{0, \pm 1\} \)-vectors. We propose an \( \ell_1 \)-based SDP relaxation to SILS, and provide sufficient conditions for our SDP relaxation to solve SILS. The class of data input which guarantee that SDP solves SILS is broad enough to cover many cases in real-world applications, such as privacy preserving identification, and multiuser detection. To show this, we specialize our sufficient conditions to two special cases of SILS with relevant applications: the feature extraction problem and the integer sparse recovery problem. We show that our SDP relaxation can solve the feature extraction problem with sub-Gaussian data, under some weak conditions on the second moment of the covariance matrix. Next, we show that our SDP relaxation can solve the integer sparse recovery problem under some conditions that can be satisfied both in high and low coherence settings. We also show that in the high coherence setting, our SDP relaxation performs better than other \( \ell_1 \)-based methods, such as Lasso and Dantzig Selector.

Key words: Semidefinite relaxation, Sparsity, Integer least square problem, \( \ell_1 \) relaxation

1 Introduction

The Integer Least Square problem is a fundamental NP-hard optimization problem which arises from many real-world applications, including communication theory, lattice design, Monte Carlo second-moment estimation, and cryptography. We refer the reader to the comprehensive survey [1] and references therein. In the integer least square problem, we are given an \( n \times d \) matrix \( M \), a \( d \)-vector \( b \), and we seek the closest point to \( b \), in the lattice spanned by the columns of \( M \). The ILS problem can be formulated as the following optimization problem:

\[
\min \frac{1}{n} \| Mx - b \|_2^2 \\
\text{s.t. } x \in \mathbb{Z}^d.
\]  

(ILS)

In many scenarios, one is only interested in sparse solutions to (ILS), i.e., vectors \( x \) with a large fraction of entries equal to zero. This is primarily motivated by the need to recover a sparse signal [28, 48], or the need to improve the efficiency of data structure representation [35]. Applications include cyber security [28], array signal processing [48], sparse code multiple access [11]. In this sparse setting, the feasible region is often further restricted to the set \( \{0, \pm 1\}^d \).

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Applications can be found in multiuser detection, where user terminals transmit binary symbols in a code-division multiple access (CDMA) system [52], in sensor networks, where sensors with low duty cycles are either silent (transmit 0) or active (transmit ±1) [40], and in privacy preserving identification, where a sparse vector in \( \{0, \pm 1\} \) is employed to approximate the ‘content’ of feature data [34]. In this paper, we study this version of (ILS), where we only consider sparse solutions with entries in \( \{0, \pm 1\} \). Formally, in the \textit{sparse integer least square problem}, an instance consists of an \( n \times d \) matrix \( M \), a vector \( b \in \mathbb{R}^n \), and a positive integer \( \sigma \leq d \). Our task is to find a vector \( x \) which solves the optimization problem

\[
\min \frac{1}{n} \|Mx - b\|^2_2 \quad \text{s.t.} \quad x \in \{0, \pm 1\}^d, \quad \|x\|_0 = \sigma.
\]

(SILS)

As we will see in Theorem 1 in Section 2, Problem (SILS) is NP-hard in its full generality. In this paper, we are interested in algorithms that obtain an optimal solution to Problem (SILS) in polynomial time, provided some assumptions on the data input are satisfied. To be best of our knowledge, the only known closely related result is in [5], where the authors study the problem where we replace the constraint \( \|x\|_0 = \sigma \) with \( \|x\|_0 \leq \sigma \). The authors propose a sparse sphere decoding algorithm which returns an optimal solution to the problem. They also show that this algorithm has an expected running time which is polynomial in \( d \), in the case where \( M \) has i.i.d. standard Gaussian entries and there exists a sparse integer vector \( z^* \in \{0, \pm 1\}^d \) such that the residual vector \( b - Mz^* \) is comprised of i.i.d. Gaussian entries. However, this algorithm results in an exponential running time at the presence of a nonsparse \( z^* \).

Algorithms for Problem (SILS) with a non-polynomial running time include sparsity-exploiting sphere decoding-based MUD [52] and integer quadratic optimization algorithms (see, e.g., [6] and references therein). Efficient algorithms for Problem (SILS) with no theoretical guarantee on the quality of the solution can be found in [52, 40], where the authors proposed sparsity-exploiting decision-directed MUD, Lasso-based convex relaxation methods, and CoSaMP.

Our contribution. In this paper, we further the understanding of the limits of computations for Problem (SILS) by obtaining a broad class of data input which guarantee that (SILS) can be solved efficiently. To be concrete, in Section 3 we give an \( \ell_1 \)-based semidefinite relaxation to (SILS), denoted by (SILS-SDP). It is known that semidefinite programming (SDP) problems can be solved in polynomial time up to an arbitrary accuracy, by means of the ellipsoid algorithm and interior point methods [42, 26]. Recent studies have witnessed great success of SDP relaxations in (i) solving structured integer quadratic optimization problems in polynomial time, and (ii) finding the hidden sparse structure of a given mathematical object. Examples include clustering [23], sparse principal component analysis [2], sparse support vector machine [10], sub-Gaussian mixture model [16], community detection problem [22], and so on. Note that problem (SILS) is also by nature an integer quadratic optimization problem with a sparsity constraint, so it is natural and well-motivated to seek for an effective SDP relaxation to (SILS). We show that (SILS-SDP) is able to solve (SILS) under several diverse sets of input \((M, b, \sigma)\), suggesting that it is a very flexible relaxation. In this paper, we give both theoretical and computational evidence, aiming to explain the flexibility of (SILS-SDP). In Theorems 2 and 3 in Section 4, we provide sufficient conditions for (SILS-SDP) to find a unique optimal solution to (SILS). To the best of our knowledge, our results are the first ones that study the polynomial solvability of (SILS) in its full generality. Furthermore, we illustrate that our proposed sufficient conditions can be easily verified in some practical situation, based on a quantity known as matrix coherence. To be formal, we define the \textit{coherence} of a positive semidefinite matrix \( \Psi \) to be

\[
\mu(\Psi) := \max_{i \neq j} \frac{|\Psi_{ij}|}{\sqrt{|\Psi_{ii}| |\Psi_{jj}|}}.
\]
where we assume $0/0 = 0$ if necessary. Recently, matrix coherence has aroused much attention in compressed sensing \cite{Donoho2006} and in sparsity-aware learning \cite{Friedlander2012}, thanks to its ease of computation and connection to the ability to recover a sparse optimal solution \cite{Shen2013}. In this paper, we say that a model has high coherence if we have $\mu(M^T M) = \omega(1/\sigma)$, while it has low coherence if we have $\mu(M^T M) = O(1/\sigma)$. In particular, in Theorem 4 we give sufficient conditions for \text{(SILS-SDP)} to solve \text{(SILS)}, which are tailored to low coherence models.

Next, in Sections 5 and 6, we showcase the power and flexibility of \text{(SILS-SDP)}, by showing that it is able to nicely solve two problems of interest related to \text{(SILS)}: the feature extraction problem and the integer sparse recovery problem. All these results will be consequences of Theorems 2 and 3. The input to both the feature extraction problem and the integer sparse recovery problem is the same as the input to \text{(SILS)}: we are given an $n \times d$ matrix $M$, a vector $b \in \mathbb{R}^n$, and a positive integer $\sigma \leq d$. One key difference with \text{(SILS)} is that the data input in these two problems satisfies

$$b = Mz^* + \epsilon, \quad \text{(LM)}$$

for some ground truth vector $z^* \in \mathbb{R}^d$ and for some small noise vector $\epsilon \in \mathbb{R}^n$. Note that, in this setting, $z^*$ and $\epsilon$ are unknown, i.e., they are not part of the input of the problem. The linear model assumption is often present in real-world problems and has been considered in several works in the literature, including \cite{Candes2006, Donoho2006, Liu2010}. Next, we discuss in detail the feature extraction problem and the integer sparse recovery problem.

**Feature extraction problem.** The feature extraction problem is defined as Problem \text{(SILS)}, where \text{(LM)} holds (for a general vector $z^*$). A version of this problem, where the sparsity constraint is replaced with a penalty term in the objective, was studied in \cite{Candes2006}, where an optimal solution to the problem serves as a public storage of the feature vector $z^*$. This justifies the name of the problem, since it can be viewed as a way to extract features from $z^*$. A closely related problem was considered in \cite{Heeger1995} to design an illumination-robust descriptor in face recognition. More generally, the idea of obtaining a sparse estimator from a general vector $z^*$ arises in several areas of research, including subset selection, statistical learning, and face recognition. Some advantages of finding a sparse estimator even if the ground truth $z^*$ is not necessarily sparse are reducing the cost of collecting data, improving prediction accuracy when variables are highly correlated, reducing model complexity, avoiding overfitting, and enhancing robustness \cite{Candes2006, Donoho2006, Needell2009}. As previously discussed, methods proposed in \cite{Candes2006, Donoho2006, Liu2010} can also solve the feature extraction problem. However, they neither have a polynomial running time in the case where $z^*$ is a nonsparse vector, nor they give a quality guarantee on the quality of the obtained solution.

Since the feature extraction problem is a special case of \text{(SILS)}, Theorems 2 and 3 already show that our semidefinite relaxation \text{(SILS-SDP)} can efficiently solve this problem. Next, in Theorem 5 we specialize Theorem 3 to the feature extraction problem, in the case where $M$ and $\epsilon$ have sub-Gaussian entries. The reason we are interested in this setting is that, in many fields of modern research such as compressed sensing \cite{Donoho2006}, computer vision \cite{Sivic2003}, and high dimensional statistics \cite{Vershynin2012}, it is more and more common to assume (sub-)Gaussianity in real-world data distributions. In particular, in Theorem 5 we derive a user-friendly version of our sufficient conditions when the second moment information of $M$ is known.

Next, in Model 1 we give a concrete data model where the rows of $M$ are i.i.d. standard Gaussian vectors. We prove in Theorem 6 that, for this model, \text{(SILS-SDP)} can solve the feature extraction problem with high probability. We also provide numerical results showcasing the empirical probability that \text{(SILS-SDP)} solves the feature extraction problem in this model.

**Integer sparse recovery problem.** In the integer sparse recovery problem, our input satisfies \text{(LM)}, for some $z^* \in \{0, \pm 1\}^d$ with cardinality $\sigma$, and our goal is to recover $z^*$ correctly. This is a well-known problem that arises in many fields, including sensor network \cite{Liu2010}, digital fingerprints \cite{He2009}, array signal processing \cite{D Ruiz2006}, compressed sensing \cite{Candes2006}, and multiuser detection \cite{Kolmogorov2010, Liu2010}. In this paper, we show that we can often efficiently recover $z^*$ by solving
the coherence of $z$, is necessary and sufficient for the recovery of the signed support of criterion $\ell$ -based convex relaxation, in this paper we compare our method with well-known methods in the literature [13, 17]. Since our SDP relaxation (SILS-SDP) is by nature an $\ell_1$-based convex relaxation techniques do not attain a satisfactory performance under high coherence models [2, 38, 19]. In this paper, we show computationally that, for this model, (SILS-SDP) recovers $z^*$ correctly with high probability. The fact that, for this model, (SILS-SDP) recovers $z^*$ correctly with high probability in this model, thanks to Theorem 8, thus the sufficient conditions in Theorem 7 cannot imply any (known or unknown) sufficient condition for the sparse recovery problem.

We note that the integer sparse recovery problem is, in fact, a special case of the sparse recovery problem, which is a fundamental problem that has aroused much attention from different fields of research in the past decades, including compressed sensing [9, 14], high dimensional statistical analysis [8, 15], and wavelet denoising [12]. In the sparse recovery problem, our input satisfies (LM), for some $z^* \in \mathbb{R}^d$ with cardinality $\sigma$, and our goal is to recover the signed support of $z^*$. For details on the sparse recovery problem, we refer interesting readers to the excellent review [13]. Observe that, under the assumptions of the integer sparse recovery problem, i.e., $z^* \in \{0, \pm 1\}^d$, determining the signed support of $z^*$ is equivalent to determining $z^*$ itself. A large number of algorithms for sparse recovery problem have been introduced and studied in the literature [13, 17]. Since our SDP relaxation (SILS-SDP) is by nature an $\ell_1$-based convex relaxation, in this paper we compare our method with well-known $\ell_1$-based convex relaxation algorithms. In particular we consider Lasso [2] and Dantzig Selector [8] (definitions are given in Section 2), and we see how they compare with (SILS-SDP) in solving the integer sparse recovery problem. Theoretical guarantees for Lasso and Dantzig Selector have been extensively studied in the literature. For Lasso, a condition known as mutual incoherence [35] or irrepresentable criterion [50], is necessary and sufficient for the recovery of the signed support of $z^*$. In [27], the authors show that when the coherence of matrix $M^T M$ is less than $1/(\sigma \epsilon)$, Lasso converges to $z^*$, provided some additional assumptions are met. Similarly, it was studied in [30] that when the coherence of $M^T M$ is of order $O(1/\sigma)$, Dantzig Selector is guaranteed to converge to $z^*$ as well. For high coherence models, there have been several studies conducted for Lasso and Dantzig Selector. For example, the restricted isometry property (RIP) or the null space property (NSP) guarantee that Lasso and Dantzig Selector obtain a relatively good convergence to $z^*$. We refer interested readers to [51] and references therein for details and more sufficient conditions. As discussed in [35], however, all these assumptions are often violated in many real-world applications, and oftentimes these convex relaxation techniques do not attain a satisfactory performance under high coherence models [2, 38, 19]. In this paper, we show computationally that, under Model 2, Lasso and Dantzig Selector perform poorly, yet (SILS-SDP) recovers $z^*$ with high probability. The fact that, for this model, (SILS-SDP) recovers $z^*$ with high probability is implied by Theorem 8, thus the sufficient conditions in Theorem 7 cannot imply any (known or unknown) sufficient condition for the sparse recovery problem.

**Organization of this paper.** In Section 2, we show that (SILS) is NP-hard. In Section 3 we present our SDP relaxation (SILS-SDP). In Section 4 we provide our general sufficient conditions for (SILS-SDP) to solve (SILS). In Section 5 we apply these sufficient conditions...
to the scenarios where \([LM]\) holds, and discuss the implications for the feature extraction problem and the integer sparse recovery problem. In Section 6, we present the numerical results. To streamline the presentation, we defer some proofs to Sections 7 and 8. We conclude the introduction with the notation that will be used in this paper.

**Notation: constants.** In this paper, we say that a number in \(\mathbb{R}\) is a constant if it only depends on the input of the problem, including its dimension. We say that a number in \(\mathbb{R}\) is an absolute constant if it is a fixed number that does not depend on anything at all.

**Notation: vectors.** \(0_d\) denotes the \(d\)-vector of zeros, \(1_d\) denotes the \(d\)-vector of ones. For any positive integer \(d\), we define \([d] := \{1, 2, \ldots, d\}\). Let \(x\) be a \(d\)-vector. The support of \(x\) is the set \(\text{Supp}(x) := \{i \in [d] : x_i \neq 0\}\). We denote by \(\text{diag}(x)\) the diagonal \(d \times d\) matrix with diagonal entries equal to the components of \(x\). For an index set \(I \subseteq [d]\), we denote by \(x_I\) the subvector of \(x\) whose entries are indexed by \(I\). We say that \(x\) is a unit vector if \(\|x\|_2 = 1\), and we define the unit sphere in \(\mathbb{R}^d\) as \(S^{d-1} := \{x \in \mathbb{R}^d : \|x\|_2 = 1\}\). For \(1 \leq p \leq \infty\), we denote the \(p\)-norm of \(x\) by \(\|x\|_p\). The 0-(pseudo)norm of \(x\) is \(\|x\|_0 := |\text{Supp}(x)|\).

**Notation: matrices.** \(I_n\) denotes the \(n \times n\) identity matrix. \(O_n\) denotes the \(n \times n\) zero matrix, and \(O_{m \times n}\) denotes the \(m \times n\) zero matrix. We denote by \(S^n\) the set of all \(n \times n\) symmetric matrices. Let \(M\) be a \(m \times n\) matrix. Given two index sets \(I \subseteq [m]\), \(J \subseteq [n]\), we denote by \(M_{I,J}\) the submatrix of \(M\) consisting of the entries in rows \(I\) and columns \(J\). We denote by \(|M|\) the matrix obtained from \(M\) by taking the absolute values of the entries. We denote the rows of \(M\) by \(m_1, m_2, \ldots, m_n\), and its columns by \(M_1, M_2, \ldots, M_d\). For two \(m \times n\) matrices \(M\) and \(N\), we write \(M \preceq N\) when each entry of \(M\) is at most the corresponding entry of \(N\). If \(M, N \in S^n\), we use \(M \succeq N\) to denote that \(M - N\) is a positive semidefinite matrix. Let \(R\) be a \(m \times m\) positive semidefinite matrix. We denote by \(\lambda_j(R)\) the \(i\)-th smallest eigenvalue of \(R\), and by \(v_j(R)\) the (right) eigenvector corresponding to \(\lambda_j(R)\). The minimum eigenvalue is also denoted by \(\lambda_{\min}(R) := \lambda_1(R)\). We denote by \(\text{diag}(R)\) the \(m\)-vector \((R_{11}, R_{22}, \ldots, R_{mm})^\top\). If \(X\) is a matrix, we denote by \(X^\top\) the Moore-Penrose generalized inverse of \(X\). Let \(f(X) : \mathbb{R}^{n \times n} \to \mathbb{R}\) be a convex function and let \(X_0 \in \mathbb{R}^{n \times n}\). We denote by \(\partial f(X_0)\) the subdifferential (which is the set of subgradients) of \(f\) at \(X_0\), i.e., \(\partial f(X_0) := \{G \in \mathbb{R}^{n \times n} : f(Y) \geq f(X_0) + \text{tr}(G(Y - X_0)), \forall Y \in \mathbb{R}^{n \times n}\}\). The \(p\)-to-\(q\) norm of a matrix \(P\), where \(1 \leq p \leq q \leq \infty\), is defined as \(\|P\|_{p \to q} := \min_{\|x\|_p = 1} \|Px\|_q\). The 2-norm of a matrix \(P\) is defined by \(\|P\|_2 = \|P\|_{2 \to 2}\). The infinity norm, also known as Chebyshev norm, of \(P\) is defined by \(\|P\|_{\infty} := \max_{i,j} |P_{ij}|\). For a rank-one matrix \(P = uv^\top\), clearly \(\|P\|_{\infty} = \|u\|_{\infty}\|v\|_{\infty}\).

**Notation: probability.** We denote the expected value by \(\mathbb{E}(\cdot)\). A random vector \(X \in \mathbb{R}^d\) is centered if \(\mathbb{E}(X) = 0_d\). We denote the (multivariate) Gaussian distribution by \(N(\theta, \Sigma)\), where \(\theta\) is the mean and \(\Sigma\) is the covariance matrix. We abbreviate ‘independent and identically distributed’ with ‘i.i.d.’, and ‘with high probability’ with ‘w.h.p.’, meaning with probability at least \(1 - O(1/d) - O(\exp(-\sigma^2))\) in this paper. We say that a random variable \(X \in \mathbb{R}\) is sub-Gaussian with parameter \(L\) if \(\mathbb{E}\exp\{t(X - \mathbb{E}X)\} \leq \exp\left(t^2 L^2/2\right)\), for every \(t \in \mathbb{R}\), and we write \(X \sim SG(L^2)\). We say a centered random vector \(X \in \mathbb{R}^d\) is sub-Gaussian with parameter \(L\) if \(\mathbb{E}\exp (tX^\top x) \leq \exp(2t^2/2)\), for every \(t \in \mathbb{R}\) and for every \(x\) such that \(\|x\|_2 = 1\). With a little abuse of notation, we also write \(X \sim SG(L^2)\). We say that a random variable \(X \in \mathbb{R}\) is sub-exponential if \(\mathbb{E}\exp\{t|X|\} \leq \exp(2Kt)\), for every \(|t| \leq 1/K\), for some constant \(K\). For a sub-exponential random variable \(X\), the Orlicz norm of \(X\) is defined as \(\|X\|_{\phi_1} := \inf\{t > 0 : \mathbb{E}\exp(|X|/t) \leq 2\}\). For more details, and for properties of sub-Gaussian and sub-exponential random variables (or vectors), we refer readers to the book [14].

## 2 NP-hardness

In this section, we show that \([\text{SILS}]\) is NP-hard. To prove NP-hardness, we give a polynomial reduction from **Exact Cover by 3-sets** \((\text{X3C})\). An instance of this decision problem consists of a set \(S\) and a collection \(C\) of 3-element subsets of \(S\). The task is to decide whether \(C\) contains
an exact cover for \( S \), i.e., a sub-collection \( \hat{C} \) of \( C \) such that every element of \( S \) occurs exactly once in \( \hat{C} \). See [18] for details.

**Theorem 1.** Problem \((\text{SILS})\) is NP-hard.

**Proof.** First, we define the decision problem \((\text{SILS}_0)\). An instance consists of the same data as in \((\text{SILS})\), and our task is that of deciding whether there exists \( x \in \mathbb{R}^d \) such that

\[
Mx = b \quad x \in \{0, \pm 1\}^d \quad ||x||_0 = \sigma.
\]

\((\text{SILS}_0)\) can be trivially solved by \((\text{SILS})\) since \((\text{SILS}_0)\) is feasible if and only if the optimal value of \((\text{SILS})\) is zero. Hence, to prove the theorem it is sufficient to show that \((\text{SILS}_0)\) is NP-hard. In the remainder of the proof, we show that \((\text{SILS}_0)\) is NP-hard by giving a polynomial reduction from X3C.

We start by showing how to transform an instance of X3C to an instance of \((\text{SILS}_0)\). Consider an instance of X3C given by a set \( S = \{s_1, s_2, \ldots, s_n\} \) and a collection \( C = \{c_1, c_2, \ldots, c_d\} \) of 3-element subsets of \( S \). Without loss of generality we can assume that \( n \) is a multiple of 3, since otherwise there is trivially no exact cover. Let \( b \) be the vector \( 1_n \) of \( n \) ones. The matrix \( M \) has \( d \) column vectors, one for each set in \( C \). Specifically, the \( j \)-th column of \( M \) has entries \((z_1, z_2, \ldots, z_n)\) where \( z_i = 1 \) if \( s_i \in c_j \) and \( z_i = 0 \) otherwise. Finally, we set \( \sigma := n/3 \). To conclude the proof we show that the constructed instance of \((\text{SILS}_0)\) is feasible if and only if \( C \) contains an exact cover for \( S \).

If \( C \) contains an exact cover for \( S \), say \( \hat{C} \), then consider the vector \( \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_d) \), where \( \bar{x}_j = 1 \) if \( c_j \in \hat{C} \), and \( \bar{x}_j = 0 \) otherwise. Then we have \( M\bar{x} = b \) and \( ||\bar{x}||_0 = n/3 = \sigma \), thus \((\text{SILS}_0)\) is feasible.

Conversely, assume the \((\text{SILS}_0)\) is feasible and let \( \bar{x} \) be a feasible solution. Now consider the subcollection \( \hat{C} \) of \( C \), consisting of those sets \( c_j \) such that \( \bar{x}_j \) is nonzero. We wish to prove that \( \hat{C} \) is an exact cover for \( S \). \( M\bar{x} = b \) implies that each element of \( S \) is contained in at least one set in \( \hat{C} \). Since \( \bar{x} \) has exactly \( n/3 \) nonzero entries, we have that \( \hat{C} \) contains exactly \( n/3 \) subsets of \( S \). Therefore each element of \( S \) is contained in exactly one set in \( \hat{C} \) and so \( \hat{C} \) is an exact cover for \( S \). \( \square \)

### 3 A semidefinite programming relaxation

In this section, we introduce our SDP relaxation of problem \((\text{SILS})\). We define the \( n \times (1 + d) \) matrix \( A := (-b \ M) \). We are now ready to define our SDP relaxation:

\[
\begin{align*}
\min & \quad \frac{1}{n} \text{tr}(A^T AW) \\
\text{s.t.} & \quad W \succeq 0, \\
& \quad W_{11} = 1, \\
& \quad \text{tr}(W_x) = \sigma, \\
& \quad 1_d^T |W_x|_1 \leq \sigma^2, \\
& \quad \text{diag}(W_x) \leq 1_d.
\end{align*}
\]

\((\text{SILS-SDP})\)

In this model, the decision variable is the \( (1 + d) \times (1 + d) \) matrix of variables \( W \). The matrix \( W_x \) is the submatrix of \( W \) obtained by dropping its first row and column.

**Proposition 1.** Problem \((\text{SILS-SDP})\) is an SDP relaxation of Problem \((\text{SILS})\). Precisely:
(i) Let \( x \) be a feasible solution to Problem (SILS), let \( w \) be obtained from \( x \) by adding a new first component equal to one, and let \( W := ww^\top \). Then, \( W \) is feasible to Problem (SILS-SDP) and has the same cost as \( x \).

(ii) Let \( W \) be a feasible solution to Problem (SILS-SDP), and let \( x \) be obtained from the first column of \( W \) by dropping the first entry. If \( \text{rank}(W) = 1 \) and \( x \in \{0, \pm 1\}^d \), then \( x \) is feasible to Problem (SILS) and has the same cost as \( W \).

**Proof.** (i). Let \( x, w, W \) be as in the statement. To show that \( W \) is feasible to (SILS-SDP), we first see that \( W = ww^\top \geq 0 \) and \( W_{11} = 1 \cdot 1 = 1 \). Then, by direct calculation, \( \text{tr}(W_x) = \text{tr}(xx^\top) = \|x\|_2^2 = \sigma \) and \( \text{diag}(W_x) \leq 1_d \) hold true. Lastly, for a \( \sigma \)-dimensional vector \( z \), we have \( \|z\|_1 \leq \sqrt{\sigma}\|z\|_2 \) by Cauchy-Schwartz inequality. Thus \( \|x\|_0 = \sigma \) implies \( \|x\|_1 \leq \sqrt{\sigma}\|x\|_2 \), and we obtain
\[
1_d^\top W_x 1_d = \|x\|_1^2 \leq \sigma \|x\|_2^2 = \sigma^2.
\]
Regarding the costs of the solutions, we have
\[
\frac{1}{n}\|Mx - b\|_2^2 = \frac{1}{n}\|Aw\|_2^2 = \frac{1}{n}\text{tr}(w^\top A^\top Aw) = \frac{1}{n}\text{tr}(A^\top Aw) = \frac{1}{n}\text{tr}(A^\top AW). \quad (2)
\]

(ii). Let \( W \) and \( x \) be as in the statement and assume \( \text{rank}(W) = 1 \) and \( x \in \{0, \pm 1\}^d \). We write \( W = ww^\top \) for some \((d + 1)\)-vector \( w \). Given \( W_{11} = 1 \), we either have \( w_1 = 1 \) or \( w_1 = -1 \). In the case \( w_1 = 1 \) we have \( W = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x^\top \end{pmatrix} \). In the case \( w_1 = -1 \) we have \( W = \begin{pmatrix} -1 & -1 \\ -x & -x \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x^\top \end{pmatrix} \). From \( x \in \{0, \pm 1\}^d \) and \( \text{tr}(W_x) = \sigma \), we obtain in both cases \( \|x\|_0 = \sigma \), and so \( x \) is feasible to (SILS). Regarding the costs of the solutions, in both cases we have that (2) holds.

In the remainder of the paper, we say that (SILS-SDP) recovers \( x^* \), if \( x^* \in \{0, \pm 1\}^d \), and (SILS-SDP) admits a unique rank-one optimal solution \( W^* := \begin{pmatrix} 1 \\ x^* \end{pmatrix} \begin{pmatrix} 1 \\ x^\top \end{pmatrix} \). Due to Proposition 1, the vector \( x^* \) is then optimal to (SILS), and hence we also say that (SILS-SDP) solves (SILS) if there exists a vector \( x^* \in \{0, \pm 1\}^d \) such that (SILS-SDP) recovers \( x^* \). We remark that, if (SILS-SDP) solves (SILS), then (SILS) can be indeed solved in polynomial time by solving (SILS-SDP), because we can obtain \( x^* \) by checking the first column of \( W^* \).

### 4 Sufficient conditions for recovery

In this section, we present Theorems 2 and 3, which are two of the main results of this paper. In both theorems, we provide sufficient conditions for (SILS-SDP) to solve (SILS), which are primarily focused on the input \( A = (M, -b) \) and \( \sigma \). The statements require the existence of two parameters \( \mu_2^* \) and \( \delta \), and in Theorem 2 we also require the existence of a decomposition of a specific matrix \( \Theta \). Therefore, both theorems below can help us identify specific classes of problem (SILS) that can be solved by (SILS-SDP). As a corollary to Theorem 3, we then obtain Theorem 4, where we show that in a low coherence model, (SILS) can be solved by (SILS-SDP) under certain conditions.

It is worth to note that, although the linear model assumption (LM) is often present in the literature in integer least square problems (see, e.g., [22]), in this section we consider the general setting where we do not make this assumption. Note that we only provide here the statements of our sufficient conditions. The proofs are mainly based on Karush–Kuhn–Tucker (KKT) conditions [25] and can be found in Section 7.
Theorem 2. Let \( x^* \in \{0, \pm 1\}^d \), define \( S := \text{Supp}(x^*) \), and assume \( |S| = \sigma \). Define \( y^* := -M^\top b/n, Y_{11}^* := -(y_S^*)^\top x_S^* \), and assume \( Y_{11}^* > 0 \). Then, `SILS-SDP` recovers \( x^* \), if there exists a constant \( \delta > 0 \) such that the following conditions are satisfied:

\[
A1. \frac{1}{n}(M^\top M)_{SS,SS} + \frac{1}{\sigma}y_{SS}^\top \leq \mu_3^*, \text{ where } \mu_3^* \text{ is defined by }
\[
\mu_3^* := \frac{1}{\sigma} \left\{ \lambda_{\text{min}}\left( \frac{1}{n}(M^\top M)_{SS,SS} \right) - \delta + \min_{i \in S}\left| -y_i^* - \frac{1}{n}(M^\top M)_{SS}x^*_S \right|_1/x_i^* \right\}; \tag{3}
\]

\[
A2. \text{There exists } \mu_2^* \in (-\infty, -\lambda_{\text{min}}\left( \frac{1}{n}(M^\top M)_{SS,SS} \right) + \delta] \text{ such that the matrix }
\]

\[
\Theta := \frac{1}{n}(M^\top M)_{SS,SS} + \mu_2^*I_d - \frac{1}{y_{11}^*}y_{SS}^\top(y_{SS}^*)^\top
- \frac{1}{n}(M^\top M)_{SS,SS} - \frac{1}{y_{11}^*}y_{SS}^\top(y_{SS}^*)^\top \left[ (I_{SS} - \frac{1}{\sigma}x^*_S(x^*_S)^\top) \right] \frac{1}{n}(M^\top M)_{SS,SS} - \frac{1}{y_{11}^*}y_{SS}^\top(y_{SS}^*)^\top
\]

\[
\text{can be written as the sum of two matrices } \Theta_1 + \Theta_2, \text{ with } \Theta_1 > 0, \|\Theta_2\|_\infty \leq \mu_3^* \text{ or } \Theta_1 \geq 0, \|\Theta_2\|_\infty < \mu_3^*.
\]

Remark. Condition \( A2 \) in Theorem 2 is not as strong as it might seem. This condition asks for a decomposition of \( \Theta \) into the sum of a positive definite \( \Theta_1 \) and another matrix \( \Theta_2 \) with infinity norm upper bounded by \( \mu_3^* \). To construct \( \Theta_1 \), the following informal idea may be helpful. By Lemma 3 (which can be found in Section 7), \( M^\top M \geq 0 \) implies

\[
(M^\top M)_{SS,SS} \succeq (M^\top M)_{SS,SS} + \|M\|_2 M^\top M M_{SS,SS}.
\]

Therefore, if \( (M^\top M)_{SS,SS} \) is large enough and \( \delta \) is chosen wisely, the matrix

\[
\frac{1}{n}(M^\top M)_{SS,SS} - \frac{1}{n}(M^\top M)_{SS,SS} \left[ I_{SS} - \frac{1}{\sigma}x^*_S(x^*_S)^\top \right] \frac{1}{n}(M^\top M)_{SS,SS} - \frac{1}{y_{11}^*}y_{SS}^\top(y_{SS}^*)^\top
\]

is positive semidefinite and can be used to construct the positive semidefinite matrix \( \Theta_1 \).

Numerically, we found that such decomposition \( \Theta = \Theta_1 + \Theta_2 \) often exists for several different instances; however, it can be challenging to write it down explicitly. A specific instance is given in Model 2 in Section 5. In particular, it is an interesting open problem to obtain a simple sufficient condition which guarantees the existence of such decomposition.

In the next theorem, the sufficient conditions are easier to check than those in Theorem 2. This is because the statement of Theorem 3 only requires the existence of two parameters \( \mu_2^* \) and \( \delta \).

Theorem 3. Let \( x^* \in \{0, \pm 1\}^d \), define \( S := \text{Supp}(x^*) \), and assume \( |S| = \sigma \). Define \( y^* := -M^\top b/n, Y_{11}^* := -(y_S^*)^\top x_S^* \), and assume \( Y_{11}^* > 0 \). Let \( \theta := \arccos \left( \frac{y_S^*}{\|y_S^*\|_2} \right) \). Then, `SILS-SDP` recovers \( x^* \), if there exists a constant \( \delta > 0 \) such that the following conditions are satisfied:

\[
B1. \frac{1}{n}(M^\top M)_{SS,SS} + \frac{1}{\sigma}y_{SS}^\top \leq \mu_3^*, \text{ where } \mu_3^* \text{ is defined by (3)};
\]

\[
B2. \text{There exists } \mu_2^* \in (-\infty, -\lambda_{\text{min}}\left( \frac{1}{n}(M^\top M)_{SS,SS} \right) + \delta] \text{ such that such that the following conditions are satisfied}
\]

\[
\| \frac{1}{n}(M^\top M)_{SS,SS} + \mu_2^*I_d - \| + \| \frac{1}{y_{11}^*}y_{SS}^\top(y_{SS}^*)^\top \|| + \frac{1}{\sigma\delta\cos^2(\theta)} \|y_{SS}^\top\|_\infty^2 < \mu_3^*.
\]

Next, we give a corollary to Theorem 3, which shows that the assumptions of Theorem 3 can be fulfilled in models with low coherence.
Corollary 4. Let \( x^* \in \{0, \pm 1 \}^d \), define \( S := \text{Supp}(x^*) \), and assume \( |S| = \sigma \). Define \( y^* := -M^Tb/n \), \( Y_{i1}^* := -(y_{i1}^*)^T x_{i1}^* \), and assume \( Y_{i1}^* > 0 \). Let \( \theta := \arccos \left( \frac{(y_{i1}^*)^T x_{i1}^*}{\sigma \|y_{i1}^*\|_2} \right) \). Denote \( \Delta_1 := \min_{i\in S} (-y_i^*/x_i^*) - \|y_{S^c}^*\|_\infty \) and \( \Delta_2 := \min_{i\in S} (-y_i^*/x_i^*) - \sigma \|y_{S^c}^*\|_\infty /Y_{i1}^* + 1 - \cos^2(\theta) \|y_{S^c}^*\|_2^2. \) Suppose that the columns of \( M \) are normalized such that \( \max_{i\in [d]} \|M_i\|_2 \leq 1 \). Then, \( \text{(SILS-SDP)} \) recovers \( x^* \), if there exists a constant \( \delta > 0 \) such that the following conditions are satisfied:

**C1.** \( \lambda_{\min}\left( \frac{1}{n}(M^T M)_{S,S} \right) - \delta - \| \frac{1}{n}(M^T M)_{S,S} x_{S}^* \|_\infty \geq \min_{j=1,2} \Delta_j \geq \Delta > 0 \) for some constant \( \Delta \);

**C2.** There exists \( \mu_2^* \in (-\infty, -\lambda_{\min}\left( \frac{1}{n}(M^T M)_{S,S} \right) + \delta] \) such that \( \| \text{diag} \left( \frac{1}{n}(M^T M)_{S^c,S^c} + \mu_2^* I_{d-\sigma} \right) \|_\infty < \frac{\Delta}{\sigma} \);

**C3.** \( \mu(M^T M) < \frac{\Delta}{\sigma} \), where \( \mu(\cdot) \) is defined in \( (1) \).

**Proof.** We define \( \mu_3^* \) as in \( (3) \). From **C3** we obtain that \( \max_{i\neq j} \| (M^T M/n)_{ij} \| \leq \mu(M^T M/n) = \mu(M^T M) \leq \frac{\Delta}{\sigma} \). Then, we observe that

\[
\mu_3^* \geq \frac{1}{\sigma} \left\{ \lambda_{\min}\left( \frac{1}{n}(M^T M)_{S,S} \right) - \delta - \min_{i\in S} (-y_i^*/x_i^*) - \| \frac{1}{n}(M^T M)_{S,S} x_{S}^* \|_\infty \right\}
\]

and

\[
\| \frac{1}{n}(M^T M)_{S,S^c} + \frac{1}{\sigma} y_{S^c}^* (x_{S}^*)^T \|_\infty \leq \| \frac{1}{n}(M^T M)_{S,S^c} \|_\infty + \frac{1}{\sigma} \| y_{S^c}^* \|_\infty.
\]

Combining these facts with **C1**, we see that **B1** holds. If, in addition, **C2** holds, we obtain **B2**. \( \square \)

**Remark.** Theorem 4.4 shows that, if our model has low coherence, \( \text{(SILS-SDP)} \) can solve \( \text{(SILS)} \) well under the two conditions **C1** and **C2**. In this remark, we informally illustrate how these two conditions can be easily fulfilled in certain scenarios. Observe that **C1** and **C2** hold if \( \min_{j=1,2} \Delta_j \) is sufficiently large, and it is indeed possible to obtain a large \( \min_{j=1,2} \Delta_j \) intuitively. A large \( \Delta_1 \) can be obtained if, for example, there is a set \( S \) with cardinality \( \sigma \) such that \( \min_{i\in S} \| y_i^* \|_\infty - \| y_{S^c}^* \|_\infty \) is large, and \( x_{S}^* = \text{sign}(y_{S}^*). \) Also the requirement that \( \Delta_2 \) is large is not as restrictive as it might seem. In particular, if \( \cos(\theta) \) is close to one, we easily obtain a large \( \Delta_2 \) if we secure a large \( \Delta_1 \). Indeed, since \( \sigma \| y_{S^c}^* \|_\infty /Y_{i1}^* = \sigma \| y_{S^c}^* \|_\infty /(-\sum_{i\in S} y_i^* x_i^*) \) each term in the summation on the denominator is always greater than \( \| y_{S^c}^* \|_\infty \) if \( \Delta_1 \) is large. Thus, this term is in fact upper bounded by \( \| y_{S^c}^* \|_\infty. \) As another term \( [1 - \cos^2(\theta)] / \cos^2(\theta) \cdot \| y_{S^c}^* \|_\infty \) vanishes given that \( \cos(\theta) \) is close to one, we thus obtain that \( \Delta_2 \approx \Delta_1 \), and so \( \Delta_2 \) is also large.

While the above ideas on how **C1** and **C2** can be satisfied are not very precise, they can be further formalized and used in proofs for some concrete data models, including those given in the next section.

5 Consequences for linear data models

In this section, we showcase the power of Theorems 2 and 3 by presenting some of their implications for the feature extraction problem and the integer sparse recovery problem, as defined in Section 1. First, note that we can directly employ these two theorems and Theorem 4 in the specific settings of the two problems, in order to obtain corresponding sufficient conditions for \( \text{(SILS-SDP)} \) to solve these problems. To avoid repetition, we do not present these specialized sufficient conditions, and we leave their derivation to the interested reader. Instead, we focus on the consequences of Theorems 2 and 3 for these two problems, that we believe are the most significant. In Section 5.1, we consider the feature extraction problem, where \( M \) and \( \epsilon \) have sub-Gaussian entries. We specialize Theorem 3 to this setting, and thereby obtain Theorem 5 where we give user-friendly sufficient conditions based on second moment information. In Section 5.1.1
we then give a concrete data model for the feature extraction problem. In particular, the feature extraction problem under this data model can be solved by \((\text{SILS-SDP})\) due to Theorem \(5\). Next, in Section \(5.2\) we consider the integer sparse recovery problem. We present Theorem \(7\) which is obtained by specializing Theorem \(2\) to this problem. We then consider two concrete data models for the integer sparse recovery problem, which can be solved by \((\text{SILS-SDP})\). The first one, presented in Section \(5.2.1\) has high coherence, while the second one, in Section \(5.2.2\) has low coherence.

5.1 Feature extraction problem with sub-Gaussian data

In this section, we consider the feature extraction problem, and we assume that \(M\) and \(\epsilon\) have sub-Gaussian entries. Recall that the feature extraction problem is Problem \((\text{SILS})\), where \((\text{LM})\) holds (for a general vector \(z^*\)). We first give a technical lemma, which gives high-probability upper bounds for distances between some random variables and their means. This lemma is due to known results in probability and statistics.

**Lemma 1.** Suppose that \(M\) consists of centered row vectors \(m_i \overset{i.i.d.}{\sim} \mathcal{S}_n(L^2)\) for some \(L > 0\) and \(i \in [n]\), and denote the covariance matrix of \(m_i\) by \(\Sigma\). Assume the noise vector \(\epsilon\) is a centered sub-Gaussian random vector independent of \(M\), with \(\epsilon_i \overset{i.i.d.}{\sim} \mathcal{S}_n(\varrho^2)\) for \(i \in [n]\). Then, the following statements hold:

1A. Suppose \(\sigma/n \to 0\). Then, there exists an absolute constant \(c_1 > 0\) such that \(\|\frac{1}{n}(M^\top M)_{S,S} - \Sigma_{S,S}\|_2 \leq c_1 L \sqrt{\sigma/n}\) holds w.h.p. as \((n, \sigma) \to \infty\);

1B. Suppose \(\log(d)/n \to 0\) and let \(F := \frac{1}{n} M^\top M - \Sigma\). Then, there exists an absolute constant \(B\) such that \(\|F\|_{\infty} \leq BL^2 \sqrt{\log(d)/n}\) holds w.h.p. as \((n, d) \to \infty\);

1C. Suppose \(\log(d)/n \to 0\) and let \(F := \frac{1}{n} M^\top M - \Sigma\). Let \(x^* \in \{0, \pm 1\}^d\), define \(S := \text{Supp}(x^*)\), and assume \(|S| = \sigma\). Then, there exists an absolute constant \(B_1\) such that \(\|Fx^*\|_{\infty} = \|F_{S,S} x^*_S\|_{\infty} \leq B_1 L^2 \sqrt{\sigma \log(d)/n}\) holds w.h.p. as \((n, d) \to \infty\);

1D. Suppose \(\log(d)/n \to 0\) and let \(F := \frac{1}{n} M^\top M - \Sigma\). Let \(z^* \in \mathbb{R}^d\). Then, there exists an absolute constant \(B_2\) such that \(\|Fz^* + \frac{1}{n} M^\top \epsilon\|_{\infty} < B_2 L \sqrt{(\varrho^2 + L^2 \|z^*\|_2^2) \log(d)/n}\) holds w.h.p. as \((n, d) \to \infty\).

**Proof.** 1A follows from Proposition 2.1 in \([43]\).

To show 1B, we first observe that each entry in \(F\) is a sub-exponential random variable with Orlicz norm upper bounded by an absolute constant multiple of \(L^2\) (by Lemma 2.7.7 and Exercise 2.7.10 in \([43]\)). Since \(\log(d)/n \to 0\), by Bernstein inequality (see, e.g., Theorem 2.8.1 in \([43]\), we obtain that there exists an absolute constant \(c > 0\) such that for \(i, j \in [d]\),

\[
\mathbb{P}(|F_{ij}| > t) \leq 2 \exp\left(-\frac{c nt^2}{L^4}\right).
\]

Then, using the union bound, we see that

\[
\mathbb{P}(\|F\|_{\infty} > t) \leq 2d^2 \exp\left(-\frac{c nt^2}{L^4}\right) = 2 \exp\left(2 \log(d) - \frac{c nt^2}{L^4}\right).
\]

Taking \(t = BL^2 \sqrt{\log(d)/n}\) for some large absolute constant \(B > 0\), we see that \(\mathbb{P}(\|F\|_{\infty} > t) \leq O(1/d)\). Note that, in the previous argument, although we cannot apply Theorem 2.8.1 directly because we do not know the exact Orlicz norms, the statement is true if we replace \(\sum_{i=1}^n \|X_i\|_{\psi_1}^2\) and \(\max_i \|X_i\|_{\psi_1}\) with their upper bounds, which are \(nK^4 L^4\) and \(K^2 L^2\), for an absolute constant \(K > 0\). The reason is that the proof of Theorem 2.8.1 still works with such
replacement, although we get a slightly worst bound. We will use Bernstein inequality similarly later in the proof.

To show 1C we observe that for any nonzero vector $x \in \mathbb{R}^d$, we have $\sum_{j=1}^{d} x_j m_{kj} \sim \text{SG}(\|x\|_2^2 L^2)$. Then $(F x)_i = \frac{1}{n} \sum_{j=1}^{d} \sum_{k=1}^{n} m_{ki} m_{kj} x_j - (\Sigma x)_i = \frac{1}{n} \sum_{k=1}^{n} m_{ki} \sum_{j=1}^{d} x_j m_{kj} - (\Sigma x)_i$, and we can view the first term in $(F x)_i$ as a sum of independent sub-Gaussian products. Therefore, again by Lemma 2.7.7 and Exercise 2.7.10 in [44], we see that $(F x)_i$ is the average of sub-exponential random variables that have Orlicz norms upper bounded by an absolute constant multiple of $\|x\|_2^2 L^2$. Hence, by Bernstein inequality,

$$\Pr\left( \left| \sum_{j=1}^{d} F_{ij} x_j \right| > t \right) \leq 2 \exp\left( -c \frac{n t^2}{L^4 \|x\|_2^2} \right).$$

Then, using the union bound,

$$\Pr(\|F x\|_\infty > t) \leq 2d \exp\left( -c \frac{n t^2}{L^4 \|x\|_2^2} \right). \quad (5)$$

Taking $x = x^*$ and $t = B_1 L^2 \sqrt{\sigma \log(d)/n}$, for some large absolute constant $B_1 > 0$, we obtain 1C.

For 1D we first observe that each entry of $M^\top \epsilon$ is sub-exponential with Orlicz norm upper bounded by an absolute constant multiple of $L \varrho$. Then, we can show that there exist absolute constants $C_1, C_2 > 0$ such that $\left\| \frac{1}{n} M^\top \epsilon \right\|_\infty \leq C_1 \sqrt{\frac{n \rho^2 L^2 \log(d)}{n}}$ and $\|F x^*\|_\infty \leq C_2 \sqrt{L^2 \|x^*\|_2^2 \log(d)/n}$ hold w.h.p., similarly to the proof of 1B and 1C. By taking a large enough absolute constant $B_2 > 0$, e.g., $B_2 = 2 \max\{C_1, C_2\}$, we obtain 1D.

We are now ready to present our sufficient conditions for solving the feature extraction problem with sub-Gaussian data. The proof of Theorem 5 is based on Theorem 3.

**Theorem 5.** Let $x^* \in \{0, \pm 1\}^d$, define $S := \text{Supp}(x^*)$, and assume $|S| = \sigma$. Assume [LM] holds. In addition, suppose that $M$ consists of centered row vectors $m_i \iid \text{SG}(L^2)$ for some $L > 0$ and $i \in [n]$, and we denote the covariance matrix of $m_i$ by $\Sigma$. Assume the noise vector $\epsilon$ is a centered sub-Gaussian random vector independent of $M$, with each $\epsilon_i \iid \text{SG}(\varrho^2)$ for $i \in [n]$. Let the constants $c_1, B, B_1, B_2$ be the same as in Lemma 2. Define $\hat{y}^* := -\Sigma z^*$, $\hat{y}_{11}^* := -(\hat{y}^*)^\top x^*_S$, $\hat{\theta} := \text{arccos}\left(\frac{(\hat{y}^*)^\top x^*_S}{\|x^*_S\|_2}\right)$, and assume $\hat{y}_{11}^* > 0$ and $\frac{1}{\sigma} \hat{y}_{11}^* = \Omega(1)$. Suppose there exist $\delta > 0$ such that the following conditions are satisfied:

**D1.** The function $f_n(x) := \sqrt{\frac{\|x\|_2^2}{(x^\top x_n^* x^*_S)^2}} - \frac{1}{\sigma}$ is $\frac{1}{\sqrt{\sigma}}$-Lipschitz continuous at the point $\hat{y}_S^*$ for some constant $\ell_n$;

**D2.** $\left\| \Sigma_{S,S'} + \frac{1}{\sigma} \hat{y}_{S'}^* (x^*_S)^\top \right\|_\infty + BL^2 \sqrt{\log(d)/n} + \frac{1}{\sigma} \lambda_n \leq \hat{\mu}_3^*$ holds, where $\lambda_n := B_2 L \sqrt{\left(\frac{\sigma^2 L^2 \|x^*_S\|_2^2}{n}\right) \log(d)}$ and $\hat{\mu}_3^* := \frac{1}{\sigma} \left\{ \min_{S} \left( \min_{i \in S} (-\hat{y}^* - \Sigma_{S,S'} x^*_S)_i / x^*_S - \lambda_n - B_1 L^2 \frac{\sigma \log(d)}{n} - c_1 L \sqrt{\frac{n}{\sigma}} \right) \right\}$;

**D3.** There exists $\hat{\mu}_2^* \in (-\infty, -\min_{S} \min_{i \in S} (-\hat{y}^* - \Sigma_{S,S'} x^*_S)_i / x^*_S - \lambda_n - B_1 L^2 \frac{\sigma \log(d)}{n} - c_1 L \sqrt{\frac{n}{\sigma}} + \delta]$ such that $\|\Sigma_{S',S'} + \hat{y}_{S'}^* I_{d-\sigma}\|_\infty + BL^2 \sqrt{\frac{\log(d)}{n}} + \left(\frac{\|\hat{y}_{S'}^*\|_\infty + \lambda_n}{\hat{y}_{11}^* - \sigma \lambda_n}\right)^2 + \gamma_n \leq \hat{\mu}_3^*$, where $\gamma_n := (\|f_n(\hat{y}_S^*) + \ell_n \lambda_n\|^2 + \|\hat{y}_S^*\|_\infty + \lambda_n)^2$.

Then, there exists a constant $C = C(\Sigma, z^*, x^*, \sigma)$ such that when

$$n \geq C L^2 (\varrho^2 + L^2 \|z^*\|_2^2 + \sigma) \log(d),$$

[SILS-SDP] recovers $x^*$ w.h.p. as $(n, \sigma, d) \to \infty$. 

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Proof. It is sufficient to check that all assumptions in this theorem imply all assumptions in Theorem \[3\]. We define $Y_{11}$, $\hat{y}_S$, $\mu_3$, $\cos(\theta)$ as in Theorem \[3\]. We define $\mu_3^*$ with the same $\delta$ here, and we take $\mu_2^* = \hat{\mu}_2$. Throughout the proof, we take $n \geq C L^2 (\rho^2 + L^2 \|z^*\|^2_2 + \sigma) \log(d)$ for some constant $C$, so $\sigma/n \to 0$, $\log(d)/n \to 0$, and we can apply Lemma \[1\]. Consequently, in the rest of the proof, we assume that (1A), (1D), and (3A) in Lemma \[1\] hold.

We now check $Y_{11} > 0$. We have

$$
\frac{1}{\sigma} |Y_{11}^* - \hat{Y}_{11}^*| = \frac{1}{\sigma} \left( - (y_S^* - \hat{y}_S)^\top x_i^* \right) \leq \|y_S^* - \hat{y}_S^*\| \leq 2 B_2 L \sqrt{\sigma \log(d) / n}.
$$

Thus, if we take $C = C(\Sigma, z^*, x^*, \sigma)$ large enough such that $B_2 L \sqrt{(\rho^2 + L^2 \|z^*\|^2_2) \log(d) / n} \leq \hat{Y}_{11}^* / \sigma = \Omega(1)$, we see $Y_{11}^* / \sigma$ and $\hat{Y}_{11}^* / \sigma$ have the same sign, while $Y_{11}^* / \sigma > 0$ is guaranteed by $\hat{Y}_{11}^* > 0$.

Next, we show (1D) implies (1A). We see that

$$
\frac{1}{n} (M^\top M)_{S,S} + \frac{1}{\sigma} y_S^*(x_S^*)^\top - \Sigma_{S,S} - \frac{1}{\sigma} \hat{y}_S^*(x_S^*)^\top \leq \frac{1}{n} (M^\top M)_{S,S} - \Sigma_{S,S} + \frac{1}{\sigma} \|y_S^* - \hat{y}_S^*\|,
$$

where we use (1B) and (1D) at the last inequality. To show that (1B) is true, we first obtain

$$
\frac{1}{\sigma} \left( \lambda_{\min}(\Sigma_{S,S}) - \lambda_{\min}(\frac{1}{n} (M^\top M)_{S,S}) \right) + | \min_{i \in S} [y_S^* - \Sigma_{S,S} x_S^* i / x_i^*] - \min_{i \in S} [- y_i^* - \frac{1}{n} (M^\top M)_{S,S} x_S^* i / x_i^*] \right) 

\leq \frac{c_1 L}{\sigma} \sqrt{\sigma / n} + \frac{1}{\sigma} \left( \max_{i \in S} [\hat{y}_i^* - y_i^*] + \max_{i \in S} [\|\Sigma_{S,S} x_S^* i / x_i^*\|] \right)

\leq \frac{c_1 L}{\sigma} \sqrt{\sigma / n} + \frac{1}{\sigma} B_2 L \sqrt{(\rho^2 + L^2 \|z^*\|^2_2) \log(d) / n} + \frac{1}{\sigma} B_1 L^2 \sqrt{\sigma \log(d) / n},
$$

where we use (1A) and the fact that $| \min_{i \in S} (a_i + b_i) - \min_{i \in S} (c_i + d_i) | \leq \max_{i \in S} |a_i - c_i| + \max_{i \in S} |b_i - d_i|$ in the first inequality, and (1C), (1D) in the last inequality.

Thus,

$$
\hat{\mu}_3^* = \frac{1}{\sigma} \left\{ \lambda_{\min}(\Sigma_{S,S}) - \delta + \min_{i \in S} [- y_i^* - (\Sigma_{S,S} x_S^* i / x_i^*) - BL \sqrt{\|z^*\|^2_2 \log(d) / n}] - B_1 L \sqrt{\sigma \log(d) / n} - c_1 L \sqrt{\sigma / n} \right\}

\leq \frac{1}{\sigma} \left\{ \lambda_{\min}(\frac{1}{n} (M^\top M)_{S,S}) - \delta + \min_{i \in S} [- y_i^* - \frac{1}{n} (M^\top M)_{S,S} x_S^* i / x_i^*] \right\} = \mu_3^*.
$$

From the triangle inequality and (7), we obtain

$$
\frac{1}{n} (M^\top M)_{S,S} + \frac{1}{\sigma} y_S^*(x_S^*)^\top \|\| \leq \|\Sigma_{S,S} + \frac{1}{\sigma} \hat{y}_S^*(x_S^*)^\top \| \leq \|\Sigma_{S,S} + \frac{1}{\sigma} \hat{y}_S^*(x_S^*)^\top \| + B_2 L \sqrt{\sigma \log(d) / n} + \frac{1}{\sigma} B_2 L \sqrt{(\rho^2 + L^2 \|z^*\|^2_2) \log(d) / n}

\leq \hat{\mu}_3 \leq \mu_3^*.
$$

Next, we show that (1B) and (1D) lead to (1B). From (1D) we obtain that $\|\hat{y}_S^* - y_S^*\| < B_2 L \sqrt{(\rho^2 + L^2 \|z^*\|^2_2) \log(d) / n}$, which implies

$$
\|y_S^* - (\hat{y}_S^*)^\top \|_\infty \leq \|y_S^*\|_\infty \leq (\|y_S^*\|_\infty + \|y_S^* - \hat{y}_S^*\|_\infty) \times (\|y_S^*\|_\infty + B_2 L \sqrt{(\rho^2 + L^2 \|z^*\|^2_2) \log(d) / n})^2.
$$

Combining with (7), we see

$$
\frac{1}{\sigma} \|y_S^* - (\hat{y}_S^*)^\top \|_\infty \leq \frac{1}{\sigma} \|y_S^* - (\hat{y}_S^*)^\top \|_\infty \leq \hat{Y}_{11}^* - \hat{y}_S^* \leq \frac{\|y_S^*\|_\infty + \sigma B_2 L \sqrt{(\rho^2 + L^2 \|z^*\|^2_2) \log(d) / n}}{\hat{Y}_{11}^* - \sigma B_2 L \sqrt{(\rho^2 + L^2 \|z^*\|^2_2) \log(d) / n}}.
$$

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Next, \( \sqrt{1 - \cos^2(\theta)} = \sigma f_n(\hat{y}_S^*) \), and Lipschitzness of \( f_n \) in \([1-D]\) together with \([1-D]\) give
\[
\sqrt{\sigma} |f_n(\hat{y}_S^*) - f(\hat{y}_S^*)| \leq \ell_n \|y_S^* - \hat{y}_S^*\|_2 < \sqrt{\sigma} \ell_n B_2 L \sqrt{\left( \frac{\sigma^2 + L^2 \|z_*\|_2^2}{n} \right) \log(d) / n},
\]
and \([1-B]\) gives
\[
\|\Sigma_{S^c,S^c} - \frac{1}{n} (M^T M)_{S^c,S^c}\|_\infty \leq BL^2 \sqrt{\frac{\log(d)}{n}}.
\]
Combining the above three inequalities, we obtain
\[
\|\frac{1}{n} (M^T M)_{S^c,S^c} + \mu_2^* I_{d-\sigma}\|_\infty + \|\frac{1}{Y_{11}^*} y_{S^c}^* (y_{S^c}^*)^\top\|_\infty + \frac{1}{\sigma} \cos^2(\theta) \|y_{S^c}^*\|_2^2
\]
\[
< \|\Sigma_{S^c,S^c} + \hat{\mu}_2^* I_{d-\sigma}\|_\infty + B L \sqrt{\frac{\log(d)}{n}} + \left( \|y_{S^c}^*\|_\infty + B_2 L \sqrt{\left( \frac{\sigma^2 + L^2 \|z_*\|_2^2}{n} \right) \log(d) / n} \right) \left( \|y_{S^c}^*\|_\infty + B_2 L \sqrt{\left( \frac{\sigma^2 + L^2 \|z_*\|_2^2}{n} \right) \log(d) / n} \right)^2
\]
\[
\leq \hat{\mu}_3^* \leq \mu_3^*.
\]
We have shown that the second part of \([B2]\) is true. \( \mu_2^* \in (-\infty, -\lambda_{\min}(\frac{1}{n} (M^T M)_{S,S}) + \delta) \) follows from \([1-A]\) and hence the first part of \([B2]\) is also true, which concludes the proof.

**Remark.** Condition \([D1]\) is not very restrictive. In fact, in some cases, it can be easily fulfilled. For example, in the case where \( x_S^* = \text{sign}(\hat{y}_S) \), the assumption \( \frac{1}{\sigma} Y_{11}^* = \frac{1}{\sigma} (-\hat{y}_S^*)^\top x_S^* = \Omega(1) \) in Theorem 5 guarantees condition \([D1]\). Indeed, we see
\[
\nabla_i f(x) = \frac{1}{2} \frac{2 x_i [x^\top x_S^*]_2 - 2 x_i [x^\top x_S^*]_2 \|x\|_2^2}{[x^\top x_S^*]_1^3} \frac{\sigma}{\|x\|_2^2 - \|x^\top x_S^*\|_1^2 - \|x^\top x_S^*\|_1},
\]
and hence \( \|\nabla f_n(x)\|_2 = \sqrt{\sigma} \|x\|_2 \). Using Taylor’s expansion, there exists some \( \eta \in [0, 1] \) such that \( |f_n(\hat{y}_S^*) - f_n(\hat{y}_S)\|_2 \leq \|\nabla f_n(\hat{y}_S^* + \eta(\hat{y}_S^* - \hat{y}_S))\|_2 \|\hat{y}_S^* - \hat{y}_S\|_2 \). As long as \( \|\hat{y}_S^* - \hat{y}_S\|_\infty \) is sufficiently small such that \( \text{sign}(\hat{y}_S) = \text{sign}(\hat{y}_S^*) \) and \( \frac{1}{\sigma} [\hat{y}_S^* + \eta(\hat{y}_S^* - \hat{y}_S)]^\top x_S^* = \Omega(1) \), we have
\[
\|\nabla f_n(\hat{y}_S^* + \eta(\hat{y}_S^* - \hat{y}_S))\|_2 = \frac{\sqrt{\sigma}}{\|\hat{y}_S^* + \eta(\hat{y}_S^* - \hat{y}_S)\|_1} \frac{\|\hat{y}_S^* + \eta(\hat{y}_S^* - \hat{y}_S)\|_1}{\|\hat{y}_S^* + \eta(\hat{y}_S^* - \hat{y}_S)\|_1} = O(\frac{1}{\sqrt{\sigma}}),
\]
and hence we obtain \([D1]\).

In the opposite case, where \( x_S^* \neq \text{sign}(\hat{y}_S^*) \), some additional but realistic conditions can be assumed to guarantee \([D1]\). A possible case is that the function \( g(x) := \frac{\|x\|_2^2}{\|x\|_2^2} \) is upper bounded by some absolute constant \( c > 0 \) at \( x = \hat{y}_S^* \), and \( \min_{i \in S} |\hat{y}_i^*| = \Omega(1) \). Intuitively, the first assumption is equivalent to saying that the unit direction vector of \( \hat{y}_S^* \) is not nearly orthogonal to \( x_S^* \), and the second assumption is equivalent to saying that the vector \( \Sigma_{S^c,|d|} z_* \) is bounded away from zero. Since \( \min_{i \in S} |\hat{y}_i^*| = \Omega(1) \), when \( \|\hat{y}_S^* - \hat{y}_S\|_\infty \) is sufficiently small,
then \([\hat{y}_S^* + \eta(\hat{y}_S^* - y_S^*)] \tau x_S^* \geq \frac{1}{2} |(\hat{y}_S^*) \tau x_S^*|\) and \(\|\hat{y}_S^* + \eta(\hat{y}_S^* - y_S^*)\|_2 \leq 2\|\hat{y}_S\|_2\) hold. Combining the assumption \(\frac{1}{\delta} Y_{11} = \Omega(1)\), we obtain \(D1\) from the fact

\[
\|\nabla f_{\eta}(\hat{y}_S^* + \eta(\hat{y}_S^* - y_S^*))\|_2 = \frac{\sqrt{\sigma}}{|(\hat{y}_S^*) \tau x_S^*|} \cdot g(\hat{y}_S^* + \eta(\hat{y}_S^* - y_S^*)) \\
\leq \frac{2\sqrt{\sigma}}{|(\hat{y}_S^*) \tau x_S^*|} \cdot \frac{4\|\hat{y}_S^*\|_2}{|(\hat{y}_S^*) \tau x_S^*|} \leq \frac{8c}{\Omega(\sqrt{\sigma})}.
\]

5.1.1 A data model for the feature extraction problem

In this section, we study a concrete data model for the feature extraction problem and we show that it can be solved by \(\text{SILS-SDP}\) with high probability, due to Theorem 5. We now define our first data model, in which the \(m_i\)'s are standard Gaussian vectors.

Model 1. Assume that \(\text{LM} \) holds, where the input matrix \(M\) consist of i.i.d. centered random entries drawn from \(SG(1)\), the ground truth vector is \(z_i^* = \{\pm 2, i \leq \sigma, \quad \text{with arbitrary signs, and where the noise vector } \epsilon \text{ is centered and is sub-Gaussian independent of } M, \quad \text{with } \epsilon_i \text{i.i.d. } SG(\sigma^2)\).

Model 1 can be viewed as follows: \(M\) is a normalized real-world sub-Gaussian data matrix (for each entry of the real-world data matrix, we subtract the column mean and then divide by the column standard deviation) with independent columns, and \(z^*\) is a simplified feature vector, with the \(\sigma\) most significant features being \(\pm 2\), and the remaining features being \(\pm 1\). In computer vision, we can view a Gaussian \(M\) as an image, which is a simplified yet natural assumption \(34\), and we view the vector \(z^*\) as the (simplified) relationship among the center pixel and the pixels around \(47\). It is worthy pointing out that, although Model 1 has a simplified \(z^*\), it is hard to solve \(\text{SILS}\), and existing algorithms generally take an exponential running time \(5, 52\) due to the fact that \(z^*\) is not sparse.

Note that, in Model 1, it is not realistic to assume that the \(\pm 2\) components of \(z^*\) are all in the first \(\sigma\) components. Rather, we should consider the more general model where the components of \(z^*\) are arbitrarily permuted. However, this assumption on \(z^*\) in the model can be made without loss of generality. In fact, \(\text{SILS-SDP}\) can solve Model 1 if and only if it can solve the more general model. This is because both \(\text{SILS-SDP}\) and the model are invariant under permutation of variables. A similar note applies to Models 2 and 3 that will be considered later.

In our next theorem, we show that \(\text{SILS-SDP}\) solves \(\text{SILS}\) with high probability provided that \(n\) is sufficiently large. The numerical performance of \(\text{SILS-SDP}\) under Model 1 will be demonstrated and discussed in Section 6.1.

**Theorem 6.** Consider the feature extraction problem under Model 1. Then, there exists an absolute constant \(C\) such that when

\[
n \geq C(\sigma^2 + d + \varrho^2) \log(d),
\]

\(\text{SILS-SDP}\) solves \(\text{SILS}\) w.h.p. as \((n, d) \rightarrow \infty\).

**Proof.** Let \(x_i^* = \{\text{sign}(z_i^*), i \leq \sigma, \quad \text{and } S := [\sigma]\}.\) In this proof, we employ Theorem 5 to prove that \(\text{SILS}\) recovers \(x^*\) when \(n\) is large enough, by checking all the assumptions therein. We observe that \(L = 1\) when \(\Sigma = I_d\). We also have \(\hat{y}_S^* = -z_S^*\) and \(Y_{11} = (x_S^*)^\tau I_d z_S^*/\sigma = 2 = \Omega(1)\). Throughout the proof, we take \(n \geq C(\|z^*\|_2^2 + \sigma^2 + \varrho^2) \log(d)\) for some absolute constant \(C > 0\). For brevity, we say that \(n\) is sufficiently large if we take a sufficiently large \(C\). For \(D1\)
we first show that \( l_n = \mathcal{O}(1/\sqrt{\sigma}) \) if \( n \) is large enough. By the remark after Theorem 5, we see that for some \( \eta \in [0, 1] \),
\[
\frac{\sigma}{\|\tilde{y}^*_S + \eta(\tilde{y}^*_S - y^*_S)\|_1} \cdot \|\tilde{y}^*_S + \eta(\tilde{y}^*_S - y^*_S)\|_2.
\]
From 1D we obtain
\[
\|\tilde{y}^*_S + \eta(\tilde{y}^*_S - y^*_S)\|_2 \cdot \|\tilde{y}^*_S + \eta(\tilde{y}^*_S - y^*_S)\|_1 \leq \|\tilde{y}^*_S\|_2 + \|\tilde{y}^*_S - y^*_S\|_2 \leq \sqrt{2\sigma + \sqrt{\|\tilde{y}^*_S - y^*_S\|_\infty}} \leq \frac{1}{\sqrt{\sigma}} \frac{\sqrt{2} + B_2\sqrt{(\sigma^2 + \|z^*_S\|_2^2)\log(d)}}{2 - B_2\sqrt{(\sigma^2 + \|z^*_S\|_2^2)\log(d)}}.
\]
Combining the above two inequalities, we see \( l_n = \mathcal{O}(1/\sqrt{\sigma}) \) when \( n \) is sufficiently large.

For 2D we set \( \delta = 1/2 \), and obtain
\[
\hat{\mu}_3^* = \frac{1}{\sigma} \left( \frac{3}{2} - B_2\sqrt{\left(\sigma^2 + \|z^*_S\|_2^2\right)\log(d)}}{n} - B_1\sqrt{\frac{\sigma\log(d)}{n} - c_1\sqrt{\frac{\sigma}{n}}} \right) > \frac{5}{4\sigma}
\]
if \( n \) is sufficiently large. Since we have \( |\tilde{y}^*_S| = 1_{d-\sigma} \) and \( \Sigma_{S,S^c} = O_{\sigma \times (d-\sigma)} \), we see that 2D is true for a sufficiently large \( n \).

To show 3D, we set \( \hat{\mu}_2^* = -1 \) and we see that \( \hat{\mu}_2^* = -1 \leq -1/2 - c_1\sqrt{\sigma/n} \) holds for large \( n \).
Therefore, \( \Sigma_{S^c,S^c} + \hat{\mu}_2^*I_{d-\sigma} = O_{(d-\sigma) \times (d-\sigma)} \). Moreover, \( x^*_S = \text{sign}(z^*_S) \) leads to \( \cos(\theta) = 1 \), and it remains to check \( B\sqrt{\frac{\log(d)}{n} + \frac{(1 + \lambda_n)^2}{2\log(d)}} + 2\epsilon_2^{\lambda_n^2} (1 + \lambda_n)^2 \leq \hat{\mu}_3^* \). We see that
\[
B\sqrt{\frac{\log(d)}{n} + \frac{(1 + B_2\sqrt{\left(\sigma^2 + \|z^*_S\|_2^2\right)\log(d)}}{n}^2 \left(2 - B_2\sqrt{\left(\sigma^2 + \|z^*_S\|_2^2\right)\log(d)}} \right)^2
\]
\[
\leq \frac{5}{4\sigma} < \mu_3^*
\]
is true for a sufficiently large \( n \). Finally, we observe that \( \|z^*_S\|_2^2 = d + 3\sigma \), and this concludes the proof.

In the proof of Theorem 6 we showed that, if \( n \geq C(\sigma^2 + d + \sigma^2) \log(d) \), then \( \text{SILS-SDP} \) solves \( \text{SILS} \) by recovering a special \( x^* \), which is supported on \( [\sigma] \). As we will see in Section 6.1 we observe from numerical tests that \( \text{SILS-SDP} \) solves \( \text{SILS} \) even for smaller values of \( n \), and the recovered sparse integer vector is not necessarily supported on \( [\sigma] \). A possible explanation of this phenomenon is that the upper bounds given in Lemma 1, and used in the proof, can be large even when \( n \) is not necessarily supported. The terms related to \( n \) in conditions 2D - 3D in Theorem 5 will no longer vanish and may become the dominating terms, causing the support set \( S \) of the optimal solution to possibly change.

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5.2 Integer sparse recovery problem

In the realm of communications and signal processing, reconstruction of sparse signals has become a prominent and essential subject of study. In this section, we aim to solve the integer sparse recovery problem. Recall that, in this problem, our input is a constant $\delta > 0$ and our goal is to recover $z^\ast$ correctly. As mentioned in Section 1, solving the well-known sparse recovery problem is equivalent to solving the well-known integer sparse recovery problem.

We first give sufficient conditions for (SILS-SDP) to recover $z^\ast$. For brevity, we denote by $H^0 := I_\sigma - z_\delta^*(z_\delta^*)^T/\sigma$ and define

\[
\Theta := \frac{1}{n}(M^T M)_{S^c,S^c} + \mu_2 I_{d - \sigma} - \frac{1}{Y_{11}} \left( \frac{1}{n} M^T \epsilon \right)_{S^c} \left( \frac{1}{n} M^T \epsilon \right)_{S^c}^T - \frac{1}{\delta(nY_{11})^2} \left( M^T \epsilon \right)_{S^c} \left( y_{S^c}^* (M^T \epsilon)_{S^c}^T H^0 y_{S^c}^* \right) \left( M^T \epsilon \right)_{S^c}^T
\]

\[
- \frac{1}{Y_{11}} \left( \frac{1}{n} M^T \epsilon \right)_{S^c} \left( \frac{1}{n} (M^T M)_{S^c,S^c} z_{S^c}^* \right) - \frac{1}{Y_{11}} \left( \frac{1}{n} (M^T M)_{S^c,S^c} z_{S^c}^* \right) \left( \frac{1}{n} M^T \epsilon \right)_{S^c}^T
\]

\[
- \frac{1}{n} (M^T M)_{S^c,S^c} \left( I_\sigma + \frac{1}{Y_{11}} z_{S^c}^* (y_{S^c}^*)^T \right) \frac{1}{\delta} H^0 \left( I_\sigma + \frac{1}{Y_{11}} y_{S^c}^* (z_{S^c}^*)^T \right) + \frac{1}{Y_{11}} \left( \frac{1}{n} (M^T M)_{S^c,S^c} \right) \left( \frac{1}{n} (M^T M)_{S^c,S^c} \right)^T.
\]

In light of Theorem 2 and the model assumption (LM), we are able to derive the following sufficient conditions for recovering $z^\ast$.

**Theorem 7.** Consider the integer sparse recovery problem. Define $S := \text{Supp}(z^\ast)$, $y^\ast := -M^T b/n$, $Y_{11} := -(y_{S^c}^*)^T z_{S^c}^*$, and assume $Y_{11} > 0$. Then (SILS-SDP) recovers $z^\ast$, if there exists a constant $\sigma > 0$ such that the following conditions are satisfied:

**E1.** $\frac{1}{\sigma} \|M^T \epsilon\|_{S^c} \leq \mu_3^\ast$, where $\mu_3^\ast := \frac{1}{\sigma} \{ \lambda_{\min}(\frac{1}{n} (M^T M)_{S,S}) - \delta + \min_{i \in S}(\frac{1}{\delta} M^T \epsilon_i)/z_i^\ast \};$

**E2.** There exists $\mu_2^\ast \in (-\infty, -\min_{(M^T M)_{S,S}} + \delta]$ such that the matrix $\Theta$ defined in (8) can be written as the sum of two matrices $\Theta_1 + \Theta_2$, with $\Theta_1 > 0$, $\|\Theta_2\|_{\infty} \leq \mu_2^\ast$ or $\Theta_1 \geq 0$, $\|\Theta_2\|_{\infty} < \mu_3^\ast$.

**Proof.** We intend to use Theorem 2 with $x^* = z^\ast$, hence we need to prove that conditions E1-E2 imply A1-A2. Recall that we have $b = Mz^\ast + \epsilon$, and $|S| = |\text{Supp}(z^\ast)| = \sigma$. To show A1 we only need to observe that

\[
y^\ast - \frac{1}{n} (M^T M)_{S,S} x^* = \frac{1}{n} M^T (Mx^* + \epsilon) - \frac{1}{n} (M^T M)_{S,S} x_{S^c}^* = \frac{1}{n} M^T \epsilon,
\]

so A1 coincides with E1 in this setting. Then, a direct calculation shows that $\Theta$ in this theorem coincides with the one in Theorem 2 by expanding $y_{S^c}^*$. \qed

We observe that the assumptions in Theorem 7 do not imply that $M^T M$ has low coherence, the RIP, the NSP, or any other property which guarantees that Lasso or Dantzig Selector solve the sparse recovery problem. This will be evident from our computational results in Section 6.2.

**Remark.** The assumptions of Theorem 7 can be easily fulfilled in some scenarios. We start by claiming that $Y_{11} > 0$ and E1 are essentially weak and natural: the former asks $Y_{11} > 1/n \cdot (M^T b)^T x_{S^c} = -1/n \cdot (M^T Mx^* + M^T \epsilon)^T x_{S^c} > 0$, and the latter asks that $1/n \cdot \|M^T \epsilon\|_{\infty}$ is less than or equal to $\sigma \mu_3^\ast$. Such conditions are met in the case where $Mx^* \neq 0$, $\epsilon$ is a random noise vector independent of $M$ when $n$ is large, and $\lambda_{\min}(\frac{1}{n} (M^T M/n)_{S,S})$ is lower bounded by some positive constant. In addition, E1 is quite similar to the constraint in the definition of Dantzig Selector (DS), but here we only require this type of constraint for the $S^c$ block of $M^T \epsilon/n$. Next, Condition E2 asks to construct $\Theta_1$ in a way such that $\|\Theta_2\|_{\infty}$ is small. Note that,
although $\Sigma^2$ is complicated and sometimes it can be challenging to give such decomposition of $\Theta$, this assumption holds in an ideal scenario, where $(M^\top M)_{S^c,S^c}$ is large enough such that $\lambda_{\min}(\Theta) \geq 0$.

### 5.2.1 A data model with high coherence for the integer sparse recovery problem

In this section, we introduce a data model for the integer sparse recovery problem that admits high coherence. We will prove that our SDP relaxation (SILS-SDP) can solve the integer sparse recovery problem under this model with high probability, as a consequence of Theorem 7. To be concrete, we study the following data model.

**Model 2.** Assume that $[LM]$ holds, where the rows $m_1, m_2, \ldots, m_n$ of the input matrix $M$ are random vectors drawn from i.i.d. $\mathcal{N}(0_d, \Sigma)$, with

$$\Sigma := \begin{pmatrix} cI_\sigma & 1_\sigma 1_{d-\sigma}^\top \\ 1_{d-\sigma} 1_\sigma^\top & c'\sigma 1_{d-\sigma} 1_{d-\sigma}^\top + c''I_{d-\sigma} \end{pmatrix} = \begin{pmatrix} cI_\sigma & 1_\sigma 1_{d-\sigma}^\top \\ 1_{d-\sigma} 1_\sigma^\top & c'\sigma 1_{d-\sigma} 1_{d-\sigma}^\top \end{pmatrix} + \begin{pmatrix} O_\sigma & 0 \\ 0 & c''I_{d-\sigma} \end{pmatrix} = \Sigma_1 + \Sigma_2$$

for $c > 1$, $c' > 1$ and $c'' > 0$. The ground truth vector is $z^* = \begin{pmatrix} a \\ 0_{d-\sigma} \end{pmatrix}$, with $a \in \{\pm 1\}^\sigma$, and the noise vector $\epsilon$ is centered and is sub-Gaussian independent of $M$, with $\epsilon_i \overset{i.i.d.}{\sim} \mathcal{S}\mathcal{G}(\sigma^2)$.

Note that, in Model 2, $\Sigma$ is indeed positive semidefinite due to Lemma 3 in Section 7 and the fact that $\Sigma_{S^c,S^c} \succeq (\sigma/c)^1_{1_{d-\sigma}}$. Moreover, Model 2 is a model with highly correlated variables, and $\mu(M^\top M) = \Omega(1)$ when $n$ is sufficiently large, so Model 2 does not have low coherence. Furthermore, Model 2 does not satisfy the mutual incoherence property, since $\|(M^\top M)_{S^c,S}(M^\top M)_{S^c,S}^{-1}\|_{\infty \to \infty} = \Omega(\sigma) > 1$ when $n$ is sufficiently large. The above two facts follow from [LA] and [LB] in Lemma 1. The aforementioned properties are known to be crucial for $\ell_1$-based convex relaxation algorithms like Dantzig Selector and Lasso to recover $z^*$. We find that numerically, these two algorithms indeed give a high prediction error in Model 2, as we will discuss in Section 6.2. However, the following theorem shows that our semidefinite relaxation (SILS-SDP) can recover $z^*$ with high probability.

**Theorem 8.** Consider the integer sparse recovery problem under Model 2. Then, there exists a constant $C = C(c, c', c'')$ such that when

$$n \geq C\sigma^2 \sigma^2 \log(d),$$

(SILS-SDP) recovers $z^*$ w.h.p. as $(n, \sigma, d) \to \infty$.

The proof of Theorem 8 is given in Section 8 and the numerical performance of (SILS-SDP) under Model 2 is presented in Section 6.2.

### 5.2.2 A data model with low coherence for the integer sparse recovery problem

In this section, we show that (SILS-SDP) can solve the integer sparse recovery problem also under some data models with low coherence. Here, we focus on the following data model, which is a generalized version of the model studied in [36].

**Model 3.** Assume that $[LM]$ holds, where the input matrix $M$ consist of i.i.d. random entries drawn from $\mathcal{S}\mathcal{G}(1)$, the ground truth vector is $z^* = \begin{pmatrix} a \\ 0_{d-\sigma} \end{pmatrix}$, with $a \in \{\pm 1\}^\sigma$, and the noise vector $\epsilon$ is centered and is sub-Gaussian independent of $M$, with $\epsilon_i \overset{i.i.d.}{\sim} \mathcal{S}\mathcal{G}(\sigma^2)$.

From [LA] and [LB] in Lemma 1, we can see that when $n = \Omega(\sigma^2 \log(d))$, the mutual incoherence property holds in Model 3, i.e., $\|(M^\top M)_{S^c,S}(M^\top M)_{S^c,S}^{-1}\|_{\infty \to \infty} < 1$. At the same time, Model 3 admits low coherence, so it is known that algorithms like Lasso and Dantzig Selector can
recover \( z^* \) efficiently \([32, 27]\). As a similar result, we show in the next theorem that (SILS-SDP) can recover \( z^* \) when \( n = \Omega((\sigma^2 + \varrho^2) \log(d)) \). While this result can be proven using Theorem 4 or Theorem 7 in our proof we use Theorem 5 instead. This is because, although Theorem 5 is tailored to the feature extraction problem, it leads to a cleaner proof. In Section 6.3, we will demonstrate the numerical performance of (SILS-SDP) under Model 3 and we will compare it with (Lasso) and (DS).

**Theorem 9.** Consider the integer sparse recovery problem under Model 3. There exists an absolute constant \( C \) such that when

\[
n \geq C(\sigma^2 + \varrho^2) \log(d),
\]

(SILS-SDP) recovers \( z^* \) w.h.p. as \((n, d) \to \infty\).

**Proof.** We prove this proposition using Theorem 5. Note that \( L = \mathcal{O}(1) \) when \( \Sigma = I_d \). We first see \( \hat{y}_S^* = -z_S^* \) and then \( \hat{Y}^*_I/\sigma = (z_S^*)^T I_d z_S^*/\sigma = 1 \). Throughout the proof, we take \( n \geq C(\sigma^2 + \varrho^2) \log(d) \) for some absolute constant \( C \). For brevity, we say that \( n \) is sufficiently large if we take a sufficiently large \( C \).

For condition \( D_1 \), we can show that \( l_n = \mathcal{O}(1/\sqrt{\sigma}) \) if \( n \) is large enough, in a similar way to the proof of Theorem 6.

For \( D_2 \), we set \( \delta = 1/2 \), and obtain

\[
\hat{\mu}_3 = \frac{1}{\sigma} \left( \frac{1}{2} B_2 \sqrt{\sigma^2 + \|z^*\|^2} \log(d) \right) - B_1 \sqrt{\frac{\sigma \log(d)}{n}} - c_1 \sqrt{\frac{\sigma}{n}} > \frac{1}{4\sigma}
\]

when \( n \) is sufficiently large. Since \( \hat{y}_S^{**} = 0_{d-\sigma} \) and \( \Sigma_{S,S} = O_{\sigma \times (d-\sigma)} \), \( D_2 \) indeed holds for \( n \) sufficiently large.

To show \( D_3 \), we first set \( \hat{\mu}_2 = -1 \). This is indeed a valid choice because we require \( \hat{\mu}_2 \leq -1/2 - c_1 \sqrt{\sigma/n} \), and when \( n \) is sufficiently large we can enforce \( c_1 \sqrt{\sigma/n} \leq 1/4 \). Therefore, we see \( \Sigma_{S,S} + \hat{\mu}_2^* I_{d-\sigma} = O_{d-\sigma} \). Furthermore, since \( x_S^* = z_S^* \), we have \( \cos(\hat{\theta}) = 1 \), and it remains to check whether \( B \sqrt{\frac{\log(d)}{n}} + \lambda_3^2 \bar{\Lambda}_{y_1} - \sigma \lambda_n + 2 \ell_2^2 \bar{\lambda}_n \leq \hat{\mu}_3^* \). It is clear that

\[
B \sqrt{\frac{\log(d)}{n}} + \frac{B_2^2 (\sigma + \varrho)^2 \log(d)}{\sigma(1 - B_2^2 (\sigma^2 + \varrho^2) \log(d))} + 2 \ell_2^2 B_4^2 (\sigma^2 + \varrho)^2 \log^2(d) \leq \frac{1}{4\sigma} < \mu_3^*
\]

is indeed true for a sufficiently large \( n \). \( \square \)

An algorithm proposed in \([32]\) shows that it is possible to recover \( z^* \) efficiently when the entries of \( M \) are i.i.d. standard Gaussian random variables at sample complexity \( n = \Omega(\sigma \log(ed/\sigma) + \varrho^2 \log(d)) \). In Theorem 9, we show that we need \( n = \Omega((\sigma^2 + \varrho^2) \log(d)) \) many samples. The differences between these results are that: (1) we recover the integer vector \( z^* \) exactly, while \([32]\) recovers an estimator of \( z^* \); (2) our method is more general, since theirs may not extend to the sub-Gaussian setting. We view such difference of sample complexity as a trade off to obtain integrality and a more general setting.

6 Numerical tests

In this section, we discuss the numerical performance of our SDP relaxation (SILS-SDP). We report the numerical performance of (SILS-SDP) under the data models which are studied in Section 5. We also compare the performance of (SILS-SDP) with other known convex relaxation algorithms. Recall that, in Section 5, we considered two problems: the feature extraction
problem and the integer sparse recovery problem. For the feature extraction problem, we are not aware of other convex relaxation algorithms, and hence we solely report the performance of (SILS-SDP) for Model 1. For the integer sparse recovery problem, we report the performance of (SILS-SDP) for Model 2 and Model 3, and we compare its performance with Lasso and Dantzig Selector, which are the most studied convex relaxation algorithms for the sparse recovery problem. Lasso and Dantzig Selector are defined by

\[
z_{\text{Lasso}} := \arg \min \frac{1}{2n} \| Mz - b \|_2^2 + \lambda \| z \|_1, \quad (\text{Lasso})
\]

\[
z_{\text{DS}} := \arg \min \| z \|_1, \quad \text{s.t.} \quad \| M^T (Mz - b) \|_\infty \leq \eta \quad (\text{DS})
\]

where \( \lambda \) and \( \eta \) are parameters to be chosen. All the calculations are made via CVX v2.2, a package for solving convex optimization problems [21] implemented in Matlab, with solver Mosek 9.2 [3].

### 6.1 Performance under Model 1

In this part, we show the numerical performance of (SILS-SDP) in the feature extraction problem under Model 1 as studied in Section 5.1.1. We assume that the entries of \( M \) in Model 1 are i.i.d. standard Gaussian, and \( \epsilon \sim \mathcal{N}(0_d, \varrho^2 I_d) \). In Figure 1, we first validate Theorem 6 numerically, by plotting the empirical probability of recovery, i.e., the percentage of times (SILS-SDP)
solves (SILS) over 100 instances, for each \( n = \lceil cd \log(d) \rceil \), with control parameter \( c \) ranging from 0.25 to 4. Note that, here, \( d \log(d) \) is the dominating term in the lower bound on \( n \) in Theorem 6. As discussed after Theorem 6, for small values of \( n \), the recovered sparse integer vector is not necessarily the vector \( x^* \) in the proof of Theorem 6. In Figure 2, we then plot the empirical probability of recovery of \( x^* \), i.e., the percentage of times (SILS-SDP) recovers \( x^* \) over 100 instances. The instances considered in Figure 2 are identical to those considered in Figure 1. As shown in Figures 1 and 2, both the empirical probability of recovery and the empirical probability of recovery of \( x^* \) go to 1 as \( c \) grows larger. However, the empirical probability of recovery is much closer to one also for small values of \( c \).

### 6.2 Performance under Model 2

In this part, we show how (SILS-SDP) performs numerically in the integer sparse recovery problem under Model 2 as studied in Section 5.2.1. We take \( c = 1.2, c' = 1.05 \), and \( c'' = 1 \) in the covariance matrix \( \Sigma \), and we take \( \epsilon \sim \mathcal{N}(0, \varrho^2 I_d) \).

In Figure 3, we study the setting where \( z^* = (a_{0 \leq \sigma}^\top a) \) with \( a \) uniformly drawn in \( \{\pm 1\}^7 \). We plot the empirical probability of recovery of \( z^* \) for each \( n = \lceil c\varrho^2\sigma^2 \log(d) \rceil \), with control parameter \( c \) ranging from 1 to 15. As predicted in Theorem 8, when \( c \) is large enough, the empirical probability of recovery of \( z^* \) goes to 1 as the control parameter \( c \) increases. Empirically, we also observe there is a transition to failure of recovery when the control parameter \( c \) is sufficiently small.

In Figure 4, we restrict ourselves to the setting where \( z^* = (1_\sigma 0_{d-\sigma}) \), and we compare the performance of (SILS-SDP), (Lasso), and (DS). We are particularly interested in this setting as it is explicitly shown in [45] that Lasso is not guaranteed to perform well. This is still a high coherence model and no guarantee on the performance of Dantzig Selector is known for this model. The parameters \( \lambda \) in (Lasso) and \( \eta \) in (DS) are determined via a 10-fold cross-validation on a held out validation set, as suggested in [6]. We report three significant quantities for sparse recovery problems, which evaluate the quality of the solution vector \( z \) returned by the algorithm. For (SILS-SDP), the vector \( z \) that we evaluate is the vector \( w^* \) obtained from the first column of the optimal solution \( W^* \) to (SILS-SDP), by deleting its first entry equal to one. The first quantity that we report is the number of nonzeros, which is \( |\text{Supp}(z)| \) and measures how sparse a solution is. The second quantity that we report is the true positive rate, defined as

\[
\text{true positive rate}(z) := \frac{|\text{Supp}(z^*) \cap \text{Supp}(z)|}{|\text{Supp}(z^*)|}.
\]
This quantity measures how well $z$ recovers the ground truth sparse vector $z^*$ by evaluating how much their support sets overlap. The last quantity that we report, which is suggested in [6], is known as prediction error, which is defined as

$$\text{prediction error}(z) := \frac{\|M(z - z^*)\|_2^2}{\|Mz^*\|_2^2}.$$  

As discussed in [6], the prediction error takes into account the correlation of features and is a meaningful measure of error for algorithms that do not have performance guarantee. We report these three quantities under different signal-to-noise ratios, i.e.,

$$\text{signal-to-noise ratio} := \frac{\text{Var}(m^T_i z^*)}{\varphi^2} = \frac{\|\Sigma^{-1/2}_i z^*_S\|_2^2}{\varphi^2}.$$  

In Figure 4, we study two sets of $(d, \sigma)$, namely, $(d, \sigma) \in \{(100, 5), (40, 2)\}$, with $\varphi \in \{0.5, 1, 1.5\}$, and we fix our choice of $n$ to be $\lceil 2\sigma^2 \log(d) \rceil$. We remark that, Model 2 is just one example of a high coherence model for the sparse recovery problem under which (SILS-SDP) works better than (Lasso) and (DS). For instance, we observe the same behavior in a model introduced in [6] (see Example 1 therein for details). On the other hand, for (SILS-SDP), as $n$ grows, the empirical probability of recovery of $z^*$ tends to one, and the conditions in Theorem 7 can be satisfied.

In Figure 4, we study two sets of $(d, \sigma)$, namely, $(d, \sigma) \in \{(100, 5), (40, 2)\}$, with $\varphi \in \{0.5, 1, 1.5\}$, and we fix our choice of $n$ to be $\lceil 2\sigma^2 \log(d) \rceil$. In an underdetermined system ($d > n$), plotted in the first row of Figure 4, we conclude that the probability that Lasso and Dantzig Selector recover the true support $|\sigma|$ of $z^*$ is low, while (SILS-SDP) nearly always recovers the true support, even when signal-to-noise ratio is low. In an overdetermined system ($d < n$), plotted in the second row of Figure 4, the true positive rates of Lasso and Dantzig Selector dramatically improve, however they are still inferior to (SILS-SDP) in terms of number of nonzeros and prediction error.
6.3 Performance under Model 3

In this part, we study the numerical performance of (SILS-SDP) in the integer sparse recovery problem under Model 3 as studied in Section 5.2.2. Note that Model 3 has low coherence when \( n \geq \sigma^2 \log(d) \). We restrict ourselves to the scenario where each entry of \( M \) is i.i.d. standard Gaussian, \( z^* = (a_{d-\sigma}) \) with \( a \) uniformly drawn in \( \{\pm 1\}^\sigma \), and \( \epsilon \sim \mathcal{N}(0, \varrho^2 I_d) \).

In Figure 5, we plot the empirical probability of recovery of \( z^* \), for each \( n = \lceil c(\sigma^2 + \varrho^2 \log(d)) \rceil \) with control parameter \( c \) ranging from 1/8 to 2. As predicted in Proposition 9, when \( c \) grows, the probability that (SILS-SDP) recovers \( z^* \) goes to 1. Empirically, we also observe that there is a transition to failure of recovery when the control parameter \( c \) is sufficiently small.

In Figure 6, we compare the numerical performance of (SILS-SDP), (Lasso), and (DS) under Model 3. From [45] and [27], we know that also (Lasso) and (DS) converge to \( z^* \), provided that we set
\[ \lambda = 2\sqrt{\log(d)/n} \] in (\text{Lasso}) and \[ \eta = 2\theta(\frac{\lambda}{4} + \sqrt{\log(d)}) \] in (\text{DS}). Hence, we set the parameters \( \lambda \) and \( \eta \) to these values without performing cross-validation. In Figure 6 we report three significant quantities: the first two are the number of nonzeros and the true positive rate, as defined in Section 6.2. The third one is the successful recovery rate, defined as

\[
\text{successful recovery rate}(z) := \frac{|\text{Supp}(z^*) \cap S_{\max}^n(z)|}{|\text{Supp}(z^*)|},
\]

where \( S_{\max}^n(z) \) is the set indices corresponding to the top \( \sigma \) entries of \( z \) having largest absolute values. The reason we consider here the successful recovery rate instead of the prediction error, considered for Model 2, is that in all three algorithms \( z \) converges to \( z^* \) in Model 3. Hence, for \( n \) large enough, \( |z_i| \) is close to 0 when \( z_i^* = 0 \), and \( |z_j| \) is close to one if \( z_j^* \neq \pm 1 \). Hence, we can recover \( z^* \) by simply looking at the \( \sigma \) largest entries of \( |z| \). We conclude from Figure 6 that all three algorithms obtain great results in Model 3 and this is mainly due to the low coherence of the model. Since all three algorithms perform well, (\text{Lasso}) and (\text{DS}) should be preferred since they run significantly faster than (\text{SILS-SDP}). In particular, (\text{SILS-SDP}) can be solved in about one second with \( d = 40 \) and in about one minute with \( d = 100 \), while the other two can be solved in less than 0.1 second in both cases.

7 Proofs of Theorems 2 and 3

In this section, we prove Theorem 2 and Theorem 3 stated in Section 4. We start by providing sufficient conditions which guarantee that (\text{SILS-SDP}) recovers a sparse vector \( x^* \in \{0, \pm 1\}^d \) in Proposition 2 and then we use it to prove Theorem 2 in Section 7.1 and Theorem 3 in Section 7.2.

In this section we will be using the convex function \( f : \mathbb{R}^{(1+d)\times(1+d)} \to \mathbb{R} \) defined by \( f(Z) := (0,1_d^\top)|Z|\begin{pmatrix} 0 \\ 1_d \end{pmatrix} \). Note that

\[
\partial f(Z) = \begin{cases}
U \in \mathbb{R}^{(1+d)\times(1+d)} : U_{ij} = \begin{cases}
0, & \text{if at least one of } i, j \leq 1,
\text{sign}(Z_{ij}), & \text{if both of } i, j \geq 2 \text{ and } Z_{ij} \neq 0,
[-1, 1], \quad \text{otherwise}.
\end{cases}
\end{cases}
\]

Proposition 2. Let \( x^* \in \{0, \pm 1\}^d \), define \( S := \text{Supp}(x^*) \), and assume \( |S| = \sigma \). Define \( y^* := -M^\top b/n, Y_{11}^* := -(y^*)^\top x^*_S, \) and assume \( Y_{11}^* > 0 \). Let \( \delta > 0, \mu_3 \in (-\infty, -\lambda_\min(\frac{1}{n}(M^\top M)_{S,S}) + \delta], \mu_4 \) be defined by \([3]\), \( p^* \in \mathbb{R}^d \) be a vector with \( p_{S^c}^* := 0_{d-\sigma} \) and \( p_S^* \) satisfying

\[
\text{diag}(p_S^*)x^*_S = -\frac{1}{n}(M^\top M)_{S,S}x^*_S - \sigma \mu_3 x^*_S - y^*_S - \mu_2 x^*_S.
\]

Let \( Y_x^* \in \mathbb{R}^{d\times d} \) be a matrix that satisfies

\[
-\mu_3^* x^*_S (y^*)^\top = \frac{1}{n} M^\top M - Y_x^* + \text{diag}(p^* + \mu_3^* 1_d)|_{S,S}.
\]

and let \( H := Y_x^* - \frac{1}{n} y^*(y^*)^\top \). Then we have \( p^* \geq 0_d, \lambda_2(H_{S,S}) \geq \delta, \) and \( H_{S,S} \geq 0 \).

Assume, in addition, that the following conditions are satisfied:

1. \( H_{S^c,S^c} \geq H_{S^c,S}H_{S,S}^\top H_{S^c,S^c} \);  
2. \( H_{S^c,S} x^*_S = 0_{d-\sigma} \);  
3. \( \|\frac{1}{n} (M^\top M - Y_x^*)|_{S^c,S}\|_\infty \leq \mu_3^* \);  
4. \( \|\frac{1}{n} (M^\top M - Y_x^*)|_{S^c,S^c} + \mu_3^* I_{d-\sigma}\|_\infty \leq \mu_3^* \).
Lemma 3 (10). By (10), the unique optimal solution to (SILS-SDP). Furthermore, if we also assume that λ_2(H) > 0, then W^* is the unique optimal solution to (SILS-SDP).

To show Proposition 2, we need two lemmas.

Lemma 2 (20, Section 5). Let D = diag(d_i) be a diagonal matrix of order n, and let C = D + auuᵀ with a < 0 and u an n-vector. Denote the eigenvalues of C by λ_1, λ_2, ..., λ_n and assume λ_i ≤ λ_{i+1}, d_i ≤ d_{i+1}. We have d_1 + a∥u∥² ≤ λ_1 ≤ d_1, and d_{i-1} ≤ λ_i ≤ d_i for i ≥ 2.

Lemma 3 (2, Appendix A.5.5). Let P be a symmetric matrix written as a 2 × 2 block matrix

\[ P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}. \]

The following are equivalent:

1. \( P \succeq 0 \).
2. \( P_{11} \succeq 0, (I - P_{11}^TP_{11})P_{12} = O, \) and \( P_{22} \succeq P_{12}^TP_{11}P_{12} \).

We are now ready to prove Proposition 2.

Proof of Proposition 2. We divide the proof into three steps. In Step A, we show \( p^* \succeq 0_d \), \( \lambda_2(H_{S,S}) \geq \delta \), and \( H_{S,S} \succeq 0 \). In Step B, we show that if in addition, \( F1, F4 \) hold, then \( W^* \) is optimal to \( \text{SILS-SDP} \). In Step C, we show that if furthermore \( \lambda_2(H) > 0 \) holds, then \( W^* \) is the unique optimal solution to \( \text{SILS-SDP} \).

Step A. We first show \( p^* \succeq 0_d \). Since \( p^*_S = 0_{d-\sigma} \), it suffices to prove \( p^*_S \succeq 0_\sigma \). We have

\[
\min_{i \in S} p_i^* = -\sigma \mu^*_S - \mu^*_S + \min_{i \in S} \left[ \left( -\frac{1}{n} (M^TM)_{S,S} x^*_i - y_i^* \right) / x^*_i \right] - \lambda_{\min} \left( \frac{1}{n} (M^TM)_{S,S} \right) + \delta - \mu^*_2 \succeq 0,
\]

where the last inequality is due to \( \mu^*_S \in (\infty, -\lambda_{\min}(\frac{1}{n} (M^TM)_{S,S}) + \delta] \).

Next, we show \( \lambda_2(H_{S,S}) \geq \delta \). To see this, (11) gives

\[
H_{S,S} = (Y_x^*)^T S_u - \frac{1}{Y_{11}} y^*_S (y^*_S)^T = \frac{1}{n} (M^TM)_{S,S} + \mu^*_S x^*_S (x^*_S)^T + \text{diag}(p^*_S + \mu^*_2 I_\sigma) - \frac{1}{Y_{11}} y^*_S (y^*_S)^T.
\]

By (10), \( x^*_S \) is an eigenvector of \( H_{S,S} \) corresponding to the zero eigenvalue. Therefore, to show \( \lambda_2(H_{S,S}) \geq \delta \), it is sufficient to show that for any unit vector \( a \in \text{Span} \{ x^*_S \} \), we have \( a^T H_{S,S} a \geq \delta \). We obtain

\[
a^T H_{S,S} a = a^T \left( \frac{1}{n} (M^TM)_{S,S} + \mu^*_S x^*_S (x^*_S)^T + \text{diag}(p^*_S + \mu^*_2 I_\sigma) - \frac{1}{Y_{11}} y^*_S (y^*_S)^T \right) a
\]

\[
= a^T \left( \frac{1}{n} (M^TM)_{S,S} + \text{diag}(p^*_S + \mu^*_2 I_\sigma) - \frac{1}{Y_{11}} y^*_S (y^*_S)^T \right) a.
\]

We then define the following two auxiliary matrices:

\[
R := \frac{1}{n} (M^TM)_{S,S} + \mu^*_2 I_\sigma + \text{diag}(p^*_S) - \frac{1}{Y_{11}} y^*_S (y^*_S)^T, \quad P := \frac{1}{n} (M^TM)_{S,S} + \mu^*_2 I_\sigma + \text{diag}(p^*_S).
\]

To prove \( a^T H_{S,S} a \geq \delta \), it is sufficient to show \( \lambda_{\min}(P) \geq \delta \). Indeed, by Lemma 2, we see \( \lambda_2(R) \geq \lambda_{\min}(P) \geq \delta \). From (10), \( x^*_S \) is an eigenvector of \( R \) corresponding to eigenvalue \( -\sigma \mu^*_S \leq 0 \), so it is an eigenvector corresponding to the smallest eigenvalue of \( R \), which then implies \( a^T H_{S,S} a = a^T R a \geq \delta \). We now check \( \lambda_{\min}(P) \geq \delta \). Recall again \( \min_{i \in S} p_i^* = -\lambda_{\min}(\frac{1}{n} (M^TM)_{S,S}) + \delta - \mu^*_2 \).

We obtain

\[
P = \frac{1}{n} (M^TM)_{S,S} + \mu^*_2 I_\sigma + \text{diag}(p^*_S) \succeq \left( \lambda_{\min}(\frac{1}{n} (M^TM)_{S,S}) + \mu^*_2 + \min_{i \in S} p_i^* \right) I_\sigma = \delta I_\sigma.
\]
This concludes the proof that \( \lambda_{\min}(P) \geq \delta \), and therefore \( \lambda_2(H_{S,S}) \geq \delta \).

Finally, \( H_{S,S} \geq 0 \) follows easily if one observes that \( \lambda_{\min}(H_{S,S}) = 0 \). Indeed, direct calculation and (10) gives \( H_{S,S}x_S^* = 0_\sigma \), which gives our desired property.

**Step B.** In this part, we first show that \( H \geq 0 \). From Lemma 3 it suffices to show the following three facts: (i) \( H_{S,S} \geq 0 \), (ii) \( (I_\sigma - H_{S,S}H_{S,S}^H)H_{S,S^c} = O_{\sigma \times (d-\sigma)} \), and (iii) \( H_{S,S^c} \geq H_{S,S}H_{S,S}^H H_{S,S^c} \). Note that (i) holds by part (a) and (iii) holds by (F1) so it remains to show (ii). We see

\[
(I_\sigma - H_{S,S}H_{S,S}^H)H_{S,S^c} = \frac{1}{\sigma}x_S^\top(x_S^*)^\top H_{S,S^c} = \frac{1}{\sigma}x_S^\top 0_{d-\sigma}^\top = O_{\sigma \times (d-\sigma)},
\]

where we have used the facts \( \lambda_2(H_{S,S}) \geq \delta > 0 \) from part (a) and \( H_{S,S}x_S^* = 0_\sigma \) in the first equality.

We define \( Y^* := \begin{pmatrix} Y_{11}^* \\ y^* \\ y^* \end{pmatrix} \) and \( \mu_1^* := Y_{11}^* - 1/n \cdot b^\top b \). We observe that \( Y^* \geq 0 \) again by Lemma 3 due to the facts \( H \geq 0 \) and \( Y_{11}^* > 0 \). We define the function \( \mathcal{L} : \mathbb{R}^{(1+d) \times (1+d)} \rightarrow \mathbb{R} \) as follows:

\[
\mathcal{L}(W) := \frac{1}{n} \operatorname{tr}(A^\top AW) - \operatorname{tr}(Y^*W) + \mu_1^*(W_{11} - 1) + \mu_2^*(\operatorname{tr}(W_x) - \sigma) + \mu_3^*(1_d^\top |W_x|1_d - \sigma^2) + \operatorname{tr}(\operatorname{diag}(p^*)(W_x - I_d)).
\]

Since \( \mathcal{L} \) is a summation of convex functions, we can apply the subdifferential sum rule for convex functions (see, e.g., [371]), and obtain the subdifferential of \( \mathcal{L} \) at \( W \):

\[
\partial \mathcal{L}(W) = \left\{ \frac{1}{n} A^\top A + \operatorname{diag}(\begin{pmatrix} \mu_1^* \\ p^* \end{pmatrix}) + \mu_2^* \begin{pmatrix} 0 \\ I_d \end{pmatrix} + \mu_3^* U - Y^* : U \in \partial f(W) \right\}.
\]

We now show \( O_{d+1} \in \partial \mathcal{L}(W^*) \). From (9) and the definition of \( \mu_1^* \) and \( W^* \), it is equivalent to showing

\[
-\mu_3^* x_S^*(x_S^*)^\top = \frac{1}{n} M^\top M - Y_x^* + \operatorname{diag}(p^* + \mu_4^* 1_d),
\]

\[
\mu_3^* 1_d^\top 1_d \geq \frac{1}{n} M^\top M - Y_x^* + \operatorname{diag}(p^* + \mu_2^* 1_d). \tag{14}
\]

We see that (13) coincides with (11), and (14) is implied by (11), the fact that \( p_{S^c}^* = 0_{d-\sigma} \), (F3) and (F4). Therefore we obtain that for any feasible solution \( W_0 \) to \( \text{SILS-SDP} \)

\[
\mathcal{L}(W_0) - \mathcal{L}(W^*) \geq \operatorname{tr}(O_{d+1}(W_0 - W^*)) = 0.
\]

We now expand \( \mathcal{L}(W_0) \) and \( \mathcal{L}(W^*) \). Direct calculation gives \( \mu_1^*((W_0)_{11} - 1) = 0, \mu_2^*(\operatorname{tr}((W_0)_x) - \sigma) = 0, \mu_3^*(1_d^\top |(W_0)_x|1_d - \sigma^2) \leq 0, \operatorname{tr}(\operatorname{diag}(p^*)((W_0)_x - I_d)) \leq 0, \) and \( \operatorname{tr}(Y^*W_0) \geq 0 \), given the facts that \( W_0 \) is feasible to \( \text{SILS-SDP} \), \( p^* \geq 0, \mu_4^* \geq 0, \) and \( Y^* \geq 0 \). Similarly, \( \mu_1^*((W^*)_x)_{11} - 1) = 0, \mu_2^*(\operatorname{tr}((W^*)_x) - \sigma) = 0, \mu_3^*(1_d^\top |(W^*)_x|1_d - \sigma^2) = 0, \) and \( \operatorname{tr}(\operatorname{diag}(p^*)((W^*)_x - I_d)) = 0, \) \( \operatorname{tr}(Y^*W^*) = 0 \). The last equality is due to \( Y_{w^*} = 0_{d+1} \), which is implied by \( Hx^* = 0_d \). Indeed, \( H_{S,S}x_S^* = 0_\sigma \) is given by (10) and (11), and \( H_{S,S^c}x_S^* = 0_{d-\sigma} \) is given by (F2). To sum up, we obtain the following fact

\[
\frac{1}{n} \operatorname{tr}(A^\top AW_0) \geq \frac{1}{n} \operatorname{tr}(A^\top AW_0) + \mu_2^*(1_d^\top |(W_0)_x|1_d - \sigma^2) + \operatorname{tr}(\operatorname{diag}(p^*)((W_0)_x - I_d)) - \operatorname{tr}(Y^*W_0)
\]

\[
= \mathcal{L}(W_0) \geq \mathcal{L}(W^*) = \frac{1}{n} \operatorname{tr}(A^\top AW^*), \tag{15}
\]
and hence $W^*$ is optimal to (SILS-SDP).

**Step C.** Finally, we show that $W^*$ is the unique optimal solution if we additionally assume $\lambda_2(H) > 0$. First, note that $\lambda_2(H) > 0$ implies $\lambda_2(Y^*) > 0$ due to the fact that

$$Y^* = \begin{pmatrix} Y^*_{11} & (y^*)^\top \\ y^* & Y^*_x \end{pmatrix} = \begin{pmatrix} 1 & I_d \\ \frac{1}{\sigma^2} y^* & 0 \end{pmatrix} \begin{pmatrix} Y^*_{11} & (y^*)^\top \\ y^* & Y^*_x - \frac{1}{\sigma^2} y^*(y^*)^\top \end{pmatrix} \begin{pmatrix} 1 & I_d \end{pmatrix}^\top.$$

To proceed with the proof, we further assume $W_0$ in part (b) is an arbitrary optimal solution to (SILS-SDP), and we show $W_0 = W^*$. From the optimality of $W_0$, all inequalities in (15) must be equalities, which implies that the following conditions must hold:

$$\text{tr}(Y^*W_0) = 0, \quad \mu_{3,1}(\sigma_1) \geq 0, \quad \text{tr}(\text{diag}(p^*)(W_0^* - I_d)) = 0.$$

We will use the facts $\text{tr}(Y^*W_0) = 0$, $\lambda_2(Y^*) > 0$, and $Y^*w^* = 0$ to show $W_0 = w^*(w^*)^\top$. Let $W_0 = \sum_{i=1}^{1+d} \lambda_i(W_0)v_i(W_0)v_i(W_0)^\top$ be the singular value decomposition of $W_0$. We then have

$$0 = \text{tr}(Y^*W_0) = \sum_{i=1}^{1+d} \lambda_i(W_0)\text{tr}(Y^*v_i(W_0)v_i(W_0)^\top) = \sum_{i=1}^{1+d} \lambda_i(W_0)v_i(W_0)^\top Y^*v_i(W_0).$$

Since $Y^*, W_0 \geq 0$, we see $\lambda_i(W_0) \geq 0$ and $v_i(W_0)^\top Y^*v_i(W_0) \geq 0$, and thus $\lambda_i(W_0)v_i(W_0)^\top Y^*v_i(W_0) = 0$ for any $1 \leq i \leq 1 + d$.

We now show that $W_0$ is a rank-one matrix. First, $\text{rank}(W_0) \geq 1$, since $O_{1+d}$ is not feasible to (SILS-SDP). Now, for a contradiction, suppose $\text{rank}(W_0) \geq 2$. This means that the two largest singular values of $W_0$ are positive, i.e., $\lambda_1(W_0) > 0$ and $\lambda_d(W_0) > 0$. Therefore, we obtain $v_{1+d}(W_0)^\top Y^*v_{1+d}(W_0) = v_{1+d}(W_0)^\top Y^*v_{1+d}(W_0) = 0$, which is equivalent to $Y^*v_{1+d}(W_0) = Y^*v_{1+d}(W_0) = 0_{1+d}$ since $Y^* \geq 0$. This is a contradiction to the fact $\lambda_2(Y) > 0$, since $\{v_i(W_0)\}_{i=1}^{1+d}$ is an orthogonal basis of $\mathbb{R}^{1+d}$. We have thereby shown that $W_0$ is a rank-one matrix.

Hence we have $W_0 = \lambda_{1+d}(W_0)v_{1+d}(W_0)v_{1+d}(W_0)^\top$ and $Y^*v_{1+d}(W_0) = 0_{1+d}$. Again by $\lambda_2(Y^*) > 0$ and $Y^*w^* = 0$, we obtain $v_{1+d}(W_0) = \frac{1}{\sqrt{\sigma + 1}}w^*$. Therefore, $W_0 = \frac{\lambda_{1+d}(W_0)}{\sigma + 1}w^*(w^*)^\top$. Combining with $\|v_i(W_0)\|_2 = 1$ and $\text{tr}(W_0) = \sigma + 1$, leads us to $W_0 = w^*(w^*)^\top$, which concludes our proof.

In the remainder of the section we prove Theorems 2 and 3. We start with a useful lemma, which introduces the Schur complement of a positive semidefinite matrix. This result follows from Lemma 3.

**Lemma 4.** For a positive semidefinite matrix $H \in \mathbb{R}^{d \times d}$, and a set of indices $S \subseteq [d]$, we have

$$
\begin{pmatrix}
H_{S,S} & H_{S, S^c} \\
H_{S^c, S} & H_{S^c, S^c}
\end{pmatrix}
= 
\begin{pmatrix}
I_\sigma \\
H_{S^c, S^c}^\top H_{S^c, S}^\dagger I_{d-\sigma}
\end{pmatrix}
\begin{pmatrix}
H_{S,S} & H_{S, S^c, S}^\top \\
H_{S^c, S} & H_{S^c, S}^\top H_{S^c, S^c} H_{S, S^c}
\end{pmatrix}
\begin{pmatrix}
I_\sigma \\
H_{S^c, S}^\dagger H_{S^c, S^c}
\end{pmatrix}.
$$

**7.1 Proof of Theorem 2**

In this proof we intend to use Proposition 2 thus we check that all assumptions in Proposition 2 are satisfied. In particular, we take $(Y^*_x)_{S,S}$ as per (11), $p^*_S$ as per (10), and $p^*_S = 0_{d-\sigma}$, as in the statement of Proposition 2. Note that since $Y^*_x$ is not completely determined, we also need to define its missing parts, i.e., its $(S^c, S)$ and $(S^c, S^c)$ blocks. For brevity, we denote $H^0 := I_\sigma - x^*_S(x^*_S)^\top/\sigma$ and $P := (M^\top M)_{S, S^c}/n - y^*_S(y^*_S)^\top/\sigma$. We take

$$(Y^*_x)_{S^c, S} := \frac{1}{n}(M^\top M)_{S^c, S} - \frac{1}{n\sigma}(M^\top M)_{S^c, S} x^*_S - \frac{1}{\sigma \gamma_1} y^*_S(y^*_S)^\top x^*_S(x^*_S)^\top.$$

(16)
where the last inequality is due to the facts that 

In this proof we use Proposition 2, thus we check that all assumptions in Proposition 2 are 

\[ (Y_x^*)_{S^c,S^c} := \Theta_1 + \nu I_{d-\sigma} + \frac{1}{Y_{11}}y_{S^c}^*(y_{S^c}^*)^\top + \frac{1}{\delta} P^\top H^0 P, \]  

(17)

where we set \( \nu := \mu_3^* - \|\Theta_2\|_\infty \geq 0 \). As in Proposition 2, we define \( H := Y_x^* - y^*(y^*)^\top / Y_{11}^* \).

Next, we show that \( F_1, F_2 \) are implied due to our choice of \( p^* \) and \( Y_x^* \), and conditions \( A_1 - A_2 \). This will show that \( W^* := \begin{pmatrix} 1 \\ x^* \end{pmatrix} \begin{pmatrix} 1 \\ x^* \end{pmatrix}^\top \) is optimal to \((\text{SILS-SDP})\). After that, we show that \( A_2 \) automatically implies \( \lambda_2(H) > 0 \), which additionally guarantees the uniqueness of \( W^* \), and we conclude that \((\text{SILS-SDP})\) recovers \( x^* \).

We now check that \( F_1 \) holds. By direct calculation,

\[ H_{S^c,S^c} \geq \frac{1}{\delta} P^\top H^0 P \geq H_{S^c,S} H_{S^c,S}^\top, \]

where the last inequality is due to the facts that \( P = H^0 H_{S^c,S}^\top, (H^0)^2 = H^0 \), and \( H_{S,S} \geq \delta (I_\sigma - x_S^* (x_S^*)^\top / \sigma) \). The last fact is due to \( \lambda_2(H_{S,S}) \geq \delta \) and \( H_{S,S} x_S^* = 0_\sigma \).

Next, we prove that \( F_2 \) is satisfied. From (16), we obtain

\[
H_{S^c,S^c} x_S^* = \frac{1}{n} (M^T M)_{S^c,S^c} - \frac{1}{Y_{11}} y_{S^c}^*(y_{S^c}^*)^\top (I_\sigma - \frac{1}{\sigma} x_S^* (x_S^*)^\top) x_S^*
\]

\[
= \frac{1}{n} (M^T M)_{S^c,S^c} - \frac{1}{Y_{11}} y_{S^c}^*(y_{S^c}^*)^\top 0_\sigma = 0_{d-\sigma}.
\]

\( F_3 \) is true because we have

\[
\| (\frac{1}{n} M^T M - Y_x^*)_{S^c,S^c} \|_\infty = \| (\frac{1}{n\sigma} (M^T M)_{S^c,S^c} x_S^* - \frac{1}{Y_{11}\sigma} y_{S^c}^*(y_{S^c}^*)^\top x_S^*) (x_S^*)^\top \|_\infty \leq \mu_3^*.
\]

Consider now \( F_4 \). Due to (17),

\[
\| (\frac{1}{n} M^T M - Y_x^*)_{S^c,S^c} + \mu_3^* I_{d-\sigma} \|_\infty = \| -\nu I_{d-\sigma} + \Theta_2 \|_\infty \leq \nu + \| \Theta_2 \|_\infty \leq \mu_3^*.
\]

Finally, we will show \( A_2 \) implies \( \lambda_2(H) > 0 \). Lemma 4 shows that \( \lambda_2(H) > 0 \) is equivalent to \( H_{S^c,S^c} - H_{S^c,S} H_{S^c,S}^\top H_{S^c,S} H_{S^c,S^c} \geq 0 \), due to the facts \( \lambda_{\min}(H_{S,S}) = 0 \) and \( \lambda_2(H_{S,S}) \geq \delta > 0 \). Finally, we observe that \( H_{S^c,S^c} - H_{S^c,S} H_{S^c,S}^\top H_{S^c,S} H_{S^c,S^c} \geq H_{S^c,S^c} - H_{S^c,S^c} (1/\delta) \cdot (I_\sigma - x_S^* (x_S^*)^\top / \sigma) H_{S^c,S^c} = \Theta_1 + \nu I_{d-\sigma} \geq 0 \) as desired.

7.2 Proof of Theorem 3

In this proof we use Proposition 2, thus we check that all assumptions in Proposition 2 are satisfied. We fix \( (Y_x^*)_{S,S} \) as per (11), \( p_x^* \) as per (10), and \( p_S^* = 0_{d-\sigma} \). Note that we still need to define the missing parts of \( Y_x^* \), namely, its \((S^c,S)\) and \((S^c,S^c)\) blocks. We take

\[
(Y_x^*)_{S^c,S} := -\frac{1}{\sigma} y_{S^c}^*(x_S^*)^\top,
\]

(18)

\[
(Y_x^*)_{S^c,S^c} := \nu I_{d-\sigma} + \frac{1}{Y_{11}} y_{S^c}^*(y_{S^c}^*)^\top + H_{S^c,S} H_{S^c,S}^\top.
\]

(19)

With a little abuse of notation, we denote by \( \nu > 0 \) the slack in the inequality introduced in \( B_2 \).

As in Proposition 2, we define \( H := Y_x^* - y^*(y^*)^\top / Y_{11}^* \).

Next, we check \( F_1, F_4 \) and \( \lambda_2(H) > 0 \). Similarly to the proof of Theorem 2, we show that \( F_1, F_4 \) are implied by our choice of \( p^* \) and \( Y_x^* \), and conditions \( B_1, B_2 \). This will show that

\[
W := \begin{pmatrix} 1 \\ x^* \end{pmatrix} \begin{pmatrix} 1 \\ x^* \end{pmatrix}^\top
\]

is optimal to \((\text{SILS-SDP})\). After that, we show that \( B_2 \) implies \( \lambda_2(H) > 0 \),
which additionally guarantees the uniqueness of $W^*$, and we conclude that \( \text{SILS-SDP} \) recovers $x^*$.

We now check that $\mathbf{F1}$ holds. From (19),
\[
H_{S^c, S^c} = \nu I_{d-\sigma} + H_{S^c, S} H^\dagger_{S,S} H_{S^c, S} \succeq H_{S^c, S} H^\dagger_{S,S} H_{S^c, S}.
\]

Next, we prove that $\mathbf{F2}$ is satisfied. From (18), we obtain
\[
H_{S^c, S^c} x^*_S = [-\frac{1}{\sigma} y_{S^c}(x^*_S) - \frac{1}{Y_{11}^*} y_{S^c}(y^*_S)] x^*_S = -y^*_S + y^*_c = 0_{d-\sigma}.
\]

$\mathbf{F3}$ is true because
\[
\|\frac{1}{n} M^\top M - Y^*_x\|_{S^c, S^c} = \|\frac{1}{n} (M^\top M)_{S^c, S^c} + \frac{1}{\sigma} y_{S^c}(x^*_S)^\top \|_\infty \leq \mu_3^*.
\]

Consider now $\mathbf{F4}$. From (19),
\[
\|\frac{1}{n} (M^\top M - Y^*_x)_{S^c, S^c} + \mu_3^* I_{d-\sigma}\|_\infty
\]
\[
\leq \|\frac{1}{n} (M^\top M)_{S^c, S^c} + \mu_3^* I_{d-\sigma}\|_\infty + \|H_{S^c, S} H^\dagger_{S,S} H_{S^c, S}\|_\infty + \|\frac{1}{Y_{11}^*} y_{S^c}(y^*_S)^\top \|_\infty + \nu
\]
\[
\leq \|\frac{1}{n} (M^\top M)_{S^c, S^c} + \mu_3^* I_{d-\sigma}\|_\infty + \|\frac{1}{Y_{11}^*} y_{S^c}(y^*_S)^\top \|_\infty + \nu + \frac{1}{\delta} \| - \frac{1}{\sigma} x^*_S - \frac{1}{Y_{11}^*} y^*_S\|_2^2 \|y^*_S (y^*_S)^\top \|_\infty
\]
\[
= \|\frac{1}{n} (M^\top M)_{S^c, S^c} + \mu_3^* I_{d-\sigma}\|_\infty + \|\frac{1}{Y_{11}^*} y_{S^c}(y^*_S)^\top \|_\infty + \nu + \frac{1 - \cos^2(\theta)}{\delta \sigma \cos^2(\theta)} \| y^*_S \|_2^2 B_2 \mu_3^*.
\]

where we used the triangle inequality in the first inequality, the fact that $\|H_{S,S}^\dagger\|_2 \leq \frac{1}{\delta}$ in the second inequality, and the fact that
\[
\| - \frac{1}{\sigma} x^*_S - \frac{1}{Y_{11}^*} y^*_S\|_2^2 = \frac{1}{\sigma} + \frac{2 (x^*_S)^\top y^*_S}{Y_{11}^* \sigma} + \|\frac{1}{Y_{11}^*} y^*_S\|_2^2 = \frac{1}{\sigma} + \|\frac{1}{Y_{11}^*} y^*_S\|_2^2 = \frac{1 - \cos^2(\theta)}{\sigma \cos^2(\theta)}
\]
in the penultimate equality.

Finally, we show that $B_2$ implies $\lambda_2(H) > 0$. From Lemma 4, it suffices to show $\lambda_{\min}(H_{S^c, S^c} - H_{S,S}^\dagger H_{S,S}^\top H_{S,S}^\dagger H_{S,S})$ is positive. By definition of $H_{S^c, S^c}$, we obtain that $H_{S^c, S^c} - H_{S,S}^\dagger H_{S,S}^\top H_{S,S} = \nu I_{d-\sigma} > 0$.

\section{Proof of Theorem 8}

Before proving Theorem 8, we need some detailed analysis of our covariance matrix $\Sigma$ and some useful probabilistic inequalities. We will use them to evaluate norms of some matrices, which are useful for the construction of the decomposition $\Theta = \Theta_1 + \Theta_2$ in Theorem 7.

Throughout the section, we use the same definitions as in the statement of Theorem 7, i.e., $S := \text{Supp}(z^*)$, $y^* := -M^\top b/n$, $Y_{11}^* := -(y^*_S)^\top z^*_S$, and $\mu_3 = 1/\sigma \cdot \{\lambda_{\min}((M^\top M)/n)_{S,S} - \delta + \min_{i \in S}[M^\top c]/(nx_i^*)\}$. Furthermore, we use the notation introduced in Model 2 and we introduce some additional notation that is specific for it. Let $y_i', y_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0_d, I_d)$. We observe that $m_i$ has the same distribution as another random vector $\Sigma_2^2 y_i' + \Sigma_2 y_i''$. For the ease of notation, we write $M_1^\top := \Sigma_2^2 (y_1', \ldots, y_m)$ and $M_2^\top := \Sigma_2^2 (y_1'', \ldots, y_m'')$. Hence we assume $M = M_1 + M_2$. Observe that $\Sigma_2^2 = \left(\begin{array}{cc}
\Sigma_1^2 & \sqrt{\sigma} \\
\sqrt{\sigma} & \sigma I_{d-\sigma} \end{array}\right)$, so $M_2$ is an $n \times d$ matrix with the first $\sigma$ columns being zero.

In Lemma 5 below, we show that $\Sigma_2^2$ has a simple structure.
Lemma 5. The matrix $\Sigma_1$ in Model 3 satisfies $\Sigma_1^2 = \begin{pmatrix} A_{11} & a_1b_1^\top \\ a_1d_\sigma b_1^\top & b_1d_\sigma \end{pmatrix}$ for some matrix $A_{11} \in \mathbb{R}^{d \times \sigma}$ and $a, b \in \mathbb{R}$.

Proof. Since the characteristic polynomial of $\Sigma_1$ is $(x-c)^{d-1}x^{d-1}1[x^2 - (c+c'(d-\sigma))x + \sigma(c'c' - 1)(d-\sigma)]$, we conclude that $\Sigma_1$ has eigenvalues $c$ with multiplicity $\sigma - 1$, and has eigenvalues $\lambda_1, \lambda_2$ with multiplicity 1, where $\lambda_1$ and $\lambda_2$ are the two distinct roots of $x^2 - (c+c'(d-\sigma))x + \sigma(c'c' - 1)(d-\sigma)$.

It is clear that every eigenvector corresponding to $c$ is a $\sigma$-sparse vector supported on $\{c\}$, since the equation $(\Sigma_1-cI)d\sigma = 0$ forces $1_{d_\sigma}^\top w_{\{c\}} = 0$ and $(1_{d_\sigma}^\top w_{\{c\}})_i = 0_{d_\sigma}$, which implies $w_{\{c\}} = 0_{d_\sigma}$. Furthermore, direct calculation shows that the structure of eigenvectors corresponding to $\lambda_i$ (for $i = 1, 2$) must be $u_i = (a_i, b_i, b_i)$, for some constants $a_i, b_i$ such that $\rho(a_i^2 + (d-\sigma)b_i^2) = 1$. Therefore, the $(\{c\}, \{c\})$ block of $\Sigma$ is solely contributed by the corresponding block of $\sqrt{\lambda_1 \lambda_2} u_1 u_1^\top + \sqrt{\lambda_2} u_2 u_2^\top$, and the corresponding entries are all equal to $a = \sqrt{\lambda_1} a_1 b_1 + \sqrt{\lambda_2} a_2 b_2$. Similarly, we can show $b = \sqrt{\lambda_1} b_1^2 + \sqrt{\lambda_2} b_2^2$. □

By Lemma 5, we observe that $(M_1^T M_1)_{\sigma \sigma}, (M_1^T M_2)_{\sigma \sigma}$, and $(M_1^T M_1)_{\sigma \sigma}$ are rank-one matrices. In fact, there exist vectors $u \in \mathbb{R}^\sigma$, $v \in \mathbb{R}^{d_\sigma}$, and a scalar $c$ such that $(M_1^T M_1)_{\sigma \sigma}/n = u^\top u$, $(M_1^T M_2)_{\sigma \sigma}/n = u^\top v$, and $(M_1^T M_1)_{\sigma \sigma}/n = c_1 u^\top u$. In the next lemma, we provide some probabilistic upper bounds.

Lemma 6. Consider Model 3 and suppose $\log(d)/n \to 0$ and $(n, d, \sigma) \to \infty$. Let $u, v, c \in \mathbb{R}$ as defined above. Then, the following properties hold with probability at least $1 - O(1/d)$:

6A. $\exists$ constant $C_1 = C_1(c, c')$ such that $\| (M_1^T M_1/n)_{\sigma \sigma} \|_\infty \leq C_1 \sqrt{\log(d)}/n$;
6B. $\exists$ constant $C_2 = C_2(c, c')$ such that $\| (M_1^T M_1)_{d_{\sigma} d_{\sigma}} \|_\infty \leq C_2 \sigma \log(d)/n$;
6C. $\exists$ constant $C_3 = C_3(c, c', c'')$ such that $\| (M_1^T \epsilon/n)_{\sigma \sigma} \|_\infty \leq C_3 \sqrt{c'' \sigma \log(d)/n}$;
6D. $\exists$ constant $C_4 = C_4(c)$ such that $\| (M_1^T \epsilon/n)_{\sigma \sigma} \|_\infty \leq C_4 \sqrt{c'' \sigma \log(d)/n}$;
6E. $\exists$ constant $C_5 = C_5(c, c')$ such that $\| v \|_\infty \leq C_5 \sqrt{\log(d)/n}$;
6F. $\exists$ constant $C_6 = C_6(c, c')$ such that $\| (M_1^T M_1/n)_{d_{\sigma} d_{\sigma}} \|_2 \to_\infty \leq C_6(\sqrt{\log(d)} + \sqrt{\sigma})/\sqrt{n}$;
6G. $\exists$ constant $C_7 = C_7(c, c')$ such that $\| u - 1_{\sigma} \|_\infty \leq C_7 \sqrt{\log(d)/n}$;
6H. $\exists$ constant $C_8 = C_8(c')$ such that $| c_1 - c' \sigma | \leq C_8 \sigma \sqrt{\log(d)/n}$.

Proof. We note that most of these properties are due to Bernstein inequality or similar derivations in Lemma 1 and in 2. To avoid repetition, we only show why Bernstein inequality can be applied, provide upper bounds for the Orlicz norm (see 44 for definition) of some sub-exponential random variables that are of interest, and then briefly discuss how 6A - 6H can be shown.

We first show that the entries of $M_1^T M_1$, $M_1^T M_1$, and $M_1^T \epsilon$ are sums of sub-exponential random variables. Indeed, this is due to the fact that the product of two sub-Gaussian random variables is sub-exponential (see, e.g., Lemma 2.7.7 in 44). Then, we can safely apply Bernstein inequality. Upper bound the Orlicz norms of their entries, and discuss the derivations of 6A - 6H. We start by showing 6A and then we illustrate that the proofs of 6B - 6H can be obtained in a similar way.

The proof of 6A is very similar to the proof of 1B in Lemma 1. We first notice that $(M_1^T M_1)_{d_{\sigma} d_{\sigma}} = \Sigma_{\gamma}^{\frac{1}{2}} (y''_1, \ldots, y''_a) (y'_1, \ldots, y'_a)^\top (\Sigma_{\gamma}^{\frac{1}{2}})_{d_{\sigma} d_{\sigma}}$, thus every entry in $(M_1^T M_1)_{d_{\sigma} d_{\sigma}}$ is the sum of products of two independent centered Gaussian variables with variance upper bounded by $c''$ and $c$, respectively (since $\Sigma_{\gamma}^{\frac{1}{2}} y''_i \sim \mathcal{N}(0_d, c'' I_d)$ and $(\Sigma_{\gamma}^{\frac{1}{2}})_{d_{\sigma} d_{\sigma}} y'_i \sim \mathcal{N}(0_{d_{\sigma}} I_{d_{\sigma}})$). Then, from Lemma 2.7.7 in 44, we see that $(M_2^T M_1)_{d_{\sigma} d_{\sigma}}$ has entries that are the sums of sub-exponential
random variables with Orlicz norm upper bounded by a constant multiple of $\sqrt{cc'}$. Next, applying Bernstein inequality, for each entry of $(M^T_2 M_1)_{s'}S$, applying a union bound argument to upper bound the probability of the random event $R_1(t) := \{\|\frac{1}{n}(M^T_2 M_1)_{s'}S\|_\infty > t\}$, and then by setting $t = C'_1 \sqrt{cc'} \sqrt{\log(d)/n}$, for some large absolute constant $C'_1 > 0$, we obtain $\mathbb{P}(R_1(t)) \leq O(1/d)$.

For 6B, we have shown that each entry in $(M^T_2 M_1)_{d'}S$ is the sum of sub-exponential variables with Orlicz norm upper bounded by an absolute constant multiple of $\sqrt{cc'}$. Mirroring the proof of 1C in Lemma 1, we can show that there exists an absolute constant $c_0 > 0$ such that for every nonzero vector $x \in \mathbb{R}^\sigma$,

$$\mathbb{P}\left( \frac{1}{n}(M^T_2 M_1)_{s'}Sx \|_\infty > t \right) \leq 2(d - \sigma) \exp\left( - c_0 \frac{n t^2}{\alpha c \|x\|_2^2} \right),$$

(20)

and hence we obtain 6B by taking a sufficiently large absolute constant $C_1$. For 6C, 6D, 6E, and 6H we can similarly obtain that entries of $(M^T)_{s'}S$, $(M^T)_{s'}S$, $(M^T)_{s'}S$, are sums of sub-exponential variables with Orlicz norm upper bounded by an absolute constant multiple of $\sqrt{cc'}$, $\sqrt{cc'}$, $\sqrt{cc'}$, and $\sqrt{cc'}$, respectively. Then, we can apply Bernstein inequality and the union bound to derive these five properties, similarly to the proof of 6A. For 6F, the derivation is the same as the proof of Lemma 15 in [2], if we replace eq. (82) therein by (20), and proceed with the arguments after Lemma 16 therein.

In the following, we define some matrices that will be used in the proof of Theorem 8 for the construction of $\Theta_1$ and $\Theta_2$ in Theorem 7. Recall that $H^0 = I_d - z^*_S(z^*_S)^T/\sigma$. For simplicity, we denote $B := [I_d + z^*_S(y^*_S)^T/Y_{11}]^{(1/\delta)} H^0 [I_d + y^*_S(z^*_S)^T/Y_{11}] + z^*_S(z^*_S)^T/Y_{11}$, and we define

$$\Theta^A_2 := -\frac{1}{Y_{11}} \left( \frac{1}{n} (M^T)_{s'}S \left( \frac{1}{n} (M^T)_{s'} \right)^T \right),$$

$$\Theta^B_1 := \left( \sqrt{c} 1_{d - \sigma} + \frac{u^T x_s^*}{Y_{11} \sqrt{c}} \right) \left( \frac{1}{n} (M^T)_{s'} \right)^T,$$

$$\Theta^C_2 := -\frac{1}{Y_{11}} \left( \frac{1}{n} (M^T)_{s'} \left( \frac{1}{n} (M^T)_{1} \right)^T \right) - \frac{1}{Y_{11}} \left( \frac{1}{n} (M^T)_{s'} \right)^T,$$

$$\Theta^D_2 := \frac{1}{\delta n^2 Y_{11}^2} \left[ (M^T)_{s'}S \left( I_d + \frac{z^*_S(y^*_S)^T}{Y_{11}} \right) H^0 y^*_S (M^T)_{s'}S - (M^T)_{s'}S (y^*_S)^T H^0 \left( I_d + \frac{z^*_S(z^*_S)^T}{Y_{11}} \right) (M^T)_{s'}S \right],$$

$$\Theta^E_1 := \frac{1}{\delta n Y_{11}^2} \left( (M^T)_{s'}S \right)^T H^0 y^*_S (M^T)_{s'}S,$$

$$\Theta^F_1 := \frac{1}{\delta n Y_{11}^2} \left( (M^T)_{s'}S \right)^T H^0 y^*_S (M^T)_{s'}S,$$

$$\Theta^F_2 := \left( \frac{1}{\delta} - \frac{1}{\sqrt{c}} \right) \left( \frac{1}{n} (M^T_{s'}M_1)_{s'}S B \right) \left( \frac{1}{\sqrt{c}} \right) \left( \frac{1}{n} (M^T_{s'}M_1)_{s'}S B \right) \left( \frac{1}{\sqrt{c}} \right),$$

for some proper positive constants $\tilde{c}, \tilde{c}$, and $\tilde{c}$ such that $\Theta^B_1, \Theta^E_1$ and $\Theta^F_1$ are positive semidefinite matrices. The high-level idea in the proof of Theorem 8 is to take $\Theta_1 = \Theta^B_1 + \Theta^E_1 + \Theta^F_1$ and
\(\Theta_2 = \Theta_2^A + \Theta_2^B + \Theta_2^C + \Theta_2^D + \Theta_2^E + \Theta_2^F\), and to directly check that such \(\Theta_1\) and \(\Theta_2\) add up to \(\Theta\) in Theorem 7. Before proving Theorem 8, we need two lemmas: Lemma 7 gives some useful results that will be used repeatedly in the proofs of Lemma 8 and Theorem 8, and Lemma 8 gives upper bounds on the infinity norms of the matrices defined above that contribute to \(\Theta_2\).

**Lemma 7.** There exists a constant \(C = C(c,c',c'') > 0\) such that when \(n \geq C\sigma^2\sigma^2\log(d)\), the following properties hold w.h.p. as \((n,\sigma, d) \to \infty:\)

7A. \(Y_{11}^* \geq \sigma/2;\)

7B. \(||y_S^* - z_S^*||_2 \leq 1/2;\)

7C. \(||u^T(I_\sigma + y_S^*(z_S^* )^\top / Y_{11}^*)||_2 \leq 6\sqrt{\sigma};\)

7D. \(||H^0 y_S^*||_2 \leq 1/2.\)

**Proof.** For brevity, in this proof, we say that \(n\) is sufficiently large if we take a sufficiently large \(C\).

For 7A, we observe that \(Y_{11}^* = -(z_S^*)^\top y_S^* = (z_S^*)^\top (M^\top M/n)_{SS} y_S^* - (z_S^*)^\top (M^\top \epsilon/n)_S\), and hence from 1A and 6D, \(Y_{11}^* \geq \sigma - c_1 \sigma \sqrt{n}/(n/2)\). The statement simply follows from \(\Theta_4\).

For 7B, we observe that \(||((M^\top M)_{SS}/n - I_\sigma)z_S^* + (M^\top \epsilon/n)_S||_2 \leq ||(M^\top M)_{SS}/n - I_\sigma||_2 z_S^*||_2 + \sqrt{\gamma}(1 + C_1 \sigma \epsilon \log(d)/n)\) for sufficiently large \(n\).

Finally, for 7D, we observe that \(H^0 z_S^* = (I_\sigma - z_S^* (z_S^*)^\top) z_S^* = 0\), thus \(||H^0 y_S^*||_2 = \|H^0 (y_S^* - z_S^*)||_2 \leq \|H^0||_2 \|y_S^* - z_S^*||_2 \leq 1/2\) by the fact that \(||H^0||_2 = 1\) and 7B.

**Lemma 8.** There exists a constant \(C = C(c,c',c'') > 0\) such that when \(n \geq C\sigma^2\sigma^2\log(d)\), the following properties hold w.h.p. as \((n,\sigma, d) \to \infty:\)

8A. \(||\Theta_2^A||_\infty = \mathcal{O}\left(\sigma^2\log(d)/n\right);\)

8B. \(||\Theta_2^B||_\infty = \mathcal{O}\left(\sigma\log(d)/n + \sigma^2\log(d)/(\tilde{c}n)\right);\)

8C. \(||\Theta_2^C||_\infty = \mathcal{O}\left(\sqrt{\sigma^2\log(d)/n}/\delta + \sqrt{\sigma\log(d)/n}/(\delta n)\right);\)

8D. \(||\Theta_2^D||_\infty = \mathcal{O}\left(\sigma^2\log(d)/(n\delta)\right);\)

8E. \(||\Theta_2^E||_\infty = \mathcal{O}\left((\sqrt{\sigma^2\log(d)/n} + \sigma^2 n)/\tilde{c}\right).\)

**Proof.** For brevity, in this proof, we say that \(n\) is sufficiently large if we take a sufficiently large \(C\). In the proof, we will repeatedly use the fact that for a rank-one matrix \(P = ab^\top\), \(||P||_\infty = ||a||_\infty ||b||_\infty\).

8A. The statement simply follows from 7A and 6C.

8B. Observe that, among the three terms in \(\Theta_2^B\), the first term is the transpose of the second term, so it is sufficient to upper bound the infinity norm of the first term, since the same bound holds for the second. From 6G and 7A, we have that \(||1/Y_{11}^* (M^\top \epsilon/n)_S - (M^\top M/n)_{SS} z_S^* ||_\infty\) is upper bounded by \(2C_2^3 C_3 \sigma\log(d)/n\). Then, from 6G and 7A, we obtain that the infinity norm of \((u^\top z_S^2)/(||Y_{11}^*||_2^2)\), \((M^\top \epsilon/n)_S - (M^\top \epsilon/n)_S z_S^*||_\infty\) is upper bounded by \(4C^3_2 \sigma^2 \log(d)/(\tilde{c}n)\).

8C. Note that the first term in the definition of \(\Theta_2^C\) is the transpose of the second term, thus it is sufficient to upper bound the infinity norm of the first term. We write \((M^\top M/n)_{SS} (I_\sigma + \tilde{c})/(\tilde{c}n)\).
$z_5(y_S^\top/Y_{11}^*)H^0y_S^* (M^\top \epsilon/n)_S^* = 1_{d-\sigma} u^\top (I_\sigma + z_5^\top(y_S^* / Y_{11}^*))H^0y_S^* (M^\top \epsilon/n)_S^* + (M^\top M_1)_{S,S'} (I_\sigma + z_5^\top(y_S^* / Y_{11}^*))H^0y_S^* (M^\top \epsilon/n)_S^* := P_1 + P_2$, since $(M^\top M_2)_{S',S} = (M^\top M_2)_{S,S'} = 0_{(d-\sigma) \times \sigma}$. 

$||P_1||_{\infty} = ||u^\top (I_\sigma + y_S^* (z_5^\top / Y_{11}^*))H^0y_S^* ||(M^\top \epsilon/n)_S^* ||_{\infty} $, by $7C$ $7D$ and $6C$, we see $||P_1||_{\infty} \leq 3C \sqrt{\sigma^2 + \delta^2} \log(d)/n$. 

Next, $||P_2||_{\infty} = ||(M^\top M_1)_{S,S'} (I_\sigma + z_5^\top(y_S^* / Y_{11}^*))H^0y_S^* ||(M^\top \epsilon/n)_S^* ||_{\infty} $, thus from $6F$ and $6C$, we obtain that $||P_2||_{\infty} \leq C_6 \sqrt{\sigma^2 \log(d)/n} \sqrt{\log(d) + \sigma}/\sqrt{n} (||I_\sigma + z_5^\top(y_S^* / Y_{11}^*))H^0y_S^* ||_{\infty}$. By $7D$ $7B$ and $7A$, $||I_\sigma + z_5^\top(y_S^* / Y_{11}^*))H^0y_S^* ||_{\infty} \leq ||H^0y_S^* ||_2 + ||x_S^\top||_2 ||y_S^* ||_2 ||H^0y_S^* ||_2/11 \leq 2$. Hence, we see $||P_2||_{\infty} \leq 2C_6 \sqrt{\sigma^2 \log(d)/n} \sqrt{\log(d) + \sigma}/\sqrt{n}$.

Finally, from $7A$, we obtain $||\Theta_S^F||_{\infty} = O\left(\sqrt{\sigma^2 \log(d)/n} + \sqrt{\sigma} \sigma \log(d)/(\delta n)\right)$.

$8D$, From the fact that $H^0 z_5^\top = 0$, we have $y_S^* H^0y_S^* = (y_S^* - z_5^\top) H^0(y_S^* - z_5^\top)$. By $7B$ and $H^0||_2 = 1$, we see $(y_S^*)^\top H^0y_S^* \leq 1/4$. We are done by combining the above conclusion, $6C$ and $7A$.

$8E$, We start by estimating the infinity norm of the first term in the definition of $\Theta_S^F$. To do so, we first provide an upper bound on $||B||_2$. Write $B = [H^0/\delta + z_5^\top(y_S^* / Y_{11}^*) + (y_S^*)^\top H^0y_S^* - z_5^\top(y_S^* / Y_{11}^*))^\top / (Y_{11}^*)^2 + (z_5^\top(y_S^* / Y_{11}^*))^\top / (\delta Y_{11}^*)] := B_1 + B_2$, and we will upper bound $||B_1||_2$ and $||B_2||_2$. For $B_1$, recall that $H^0 = I_\sigma - z_5^\top(y_S^* / \sigma)$, thus $B_1 = (1/\delta) I_\sigma + (1/\delta) Y_{11} + (y_S^*)^\top H^0y_S^* / (Y_{11}^*)^2 - 1/(\sigma \delta) x_S^\top (x_S^*)^\top$. From $H^0 = (H^0)^2$, $7A$ and $7D$, we see $(y_S^*)^\top H^0y_S^* / (Y_{11}^*)^2 \leq 1/\sigma^2$, and thus $||B_1||_2 \leq 1 + 2 + 1/\sigma + 1/\delta \leq 3 + 2/\delta$. For $B_2$, we only need to upper bound $z_5^\top(y_S^* / Y_{11}^*)^\top H^0y_S^* / (\delta Y_{11}^*)$, since the other term is symmetric. From $7A$ and $7D$, $||z_5^\top(y_S^* / Y_{11}^*)||_2 \leq 2 ||x_S^\top||_2 ||H^0y_S^* ||_2/\delta \leq 1/\delta$. Thus, $||B||_2 \leq 3 + 2/\delta$. Combining this and $6F$ $6G$, we obtain $||M^\top M_1/n||_{S,S} B_{\infty} ||_{\infty} = O\left(\frac{\sigma \log(d) + \sigma}{\delta \sqrt{n}}\right)$.

For the second term in the definition of $\Theta_S^F$, we write $B := H^0/\delta + B_3$, and we give upper bounds on the infinity norms of $(M^\top M_1/n)_{S,S'} B_0(\delta M^\top M_1/n)_{S,S'}$ and $(M^\top M_1/n)_{S,S'} B_3(M^\top M_1/n)_{S,S'}$. We know that the diagonal entries of $H^0$ are $1 - 1/\sigma$, and the off-diagonal entries have an absolute value of $1/\sigma$, thus, along with $6A$, we see $||(M^\top M_1/n)_{S,S'} H^0(\delta M^\top M_1/n)_{S,S'} ||_{\infty} \leq \sigma^2 \sigma (1/\sigma + 1/(\sigma \delta)) = O(\sigma \log(d)/n)$. Next, by $7A$ and $7D$, each entry in $B_3$ is upper bounded by $O(1/\sigma + 1/(\sigma \delta))$. Together with $6A$, we obtain $||(M^\top M_1/n)_{S,S'} B_3(\delta M^\top M_1/n)_{S,S'} ||_{\infty} \sigma^2 \sigma (1/(\sigma \delta) = O(\sigma \log(d)/(\delta n))$. Using the triangle inequality, the second term in $\Theta_S^F$ has infinity norm upper bounded by $O(\sigma \log(d)/(\delta n))$.

We are now ready to prove Theorem $8$ using Theorem $7$.

**Proof of Theorem $8$.** We use Theorem $7$ to prove this proposition. In the proof, we take $n \geq C_8 \sigma^2 \log(d)$ for some constant $C = C(c, c', c'') > 0$. For brevity, we say $n$ is sufficiently large if we take a sufficiently large $C$. Recall that we take $\mu^* = \sigma (1 - 1/\sigma) + \sigma (1/\delta - 1/\sigma) = O(\sigma \log(d)/n)$. Next, we check the remaining conditions required in Theorem $7$.

Note that the assumption $Y_{11}^* > 0$ is automatically true by $7A$. Next, we take $\delta := 1 + \max\{\lambda_{\min}(M^\top M/n)_{S,S} - 1 - c''\} / 2 \geq 1$. $\mu^*$ is indeed nonnegative due to $6D$ and $1A$ with $L^2 = c$, because $\mu^* \geq c(1 - 1)/(2\sigma) > 0$ (if $\delta = 1$) or $\mu^* \geq c''/(2\sigma) > 0$ (if $\delta > 1$) for sufficiently large $n$. From $6C$ $6E$ is true for sufficiently large $n$. Next, we focus on $E$. We take $\mu^* := -c''$, and now show that it is a valid choice by checking $\mu^* \in (-\infty, -\lambda_{\min}(M^\top M/n)_{S,S}) + \delta$. Note that if $\delta = 1$, we have $\lambda_{\min}(M^\top M/n)_{S,S} - 1 - c'' \leq 0$, and therefore $-\lambda_{\min}(M^\top M/n)_{S,S} + \delta \geq -c''$; on the contrary, if $\delta > 1$, we have $\lambda_{\min}(M^\top M/n)_{S,S} + \delta \leq -c''$. This implies that we can take $\mu^* := -c''$ in both cases.

Next, we construct $\Theta_1$ and $\Theta_2$ as required in $E$. We take $\Theta_1 = \Theta_F^B + \Theta_F^C + \Theta_F^D$ and $\Theta_2 = \Theta_F^2 + \Theta_F^D + \Theta_F^E + \Theta_F^F$. It still remains to give valid choices for the constants $c, c, c, \text{ and } \hat{c}$ in $\Theta_F^B, \Theta_F^C, \Theta_F^D$ and $\Theta_F^F$ such that these three matrices are positive semidefinite; (b) show that $\Theta = \Theta_1 + \Theta_2$; and (c) prove that $||\Theta_2||_2 < \mu^*_3$.

For (a), it suffices to show that we can take $\hat{c}, \hat{c}, c, \text{ and } \hat{c}$ in a way such that the first two terms in the definition of $\Theta_F^F$ sum up to a positive semidefinite matrix, and the first two
terms in the definition of $\Theta^F_1$ sum up to a positive semidefinite matrix. From \textbf{6H}, we obtain $(M_1^T M_1/n)_{S^c, S^c} \geq ((\hat{c}' \sigma - C_8 \sigma \sqrt{\log(d)/n})_{1_{d-1}} \top_{1_{d-1}}$, so it suffices to give some choices of these constants such that $\hat{c}' - C_8 \sigma \sqrt{\log(d)/n} - \hat{c} + \hat{c} + \hat{c} + \hat{c} \geq 0$. We first take $\hat{c} = u^T B u$, where the definition of $B$ can be found after the proof of Lemma \textbf{6}. We then validate the choice by showing $u^T B u = \sigma + \mathcal{O}(\sqrt{\sigma})$. Indeed, since $\hat{c}' > 1$, this shows $u^T B u < c' \sigma$ for some moderately large $\sigma$, making it possible to attain a nonnegative $c' \sigma - B \sigma \sqrt{\log(d)/n} - (\hat{c} + \hat{c} + \hat{c} + \hat{c})$, for sufficiently large $n$. Observe that

$$u^T B u = u^T \left( H^0 + \frac{1}{Y_{11}^*} x_S^T (x_S^*)^\top \right) u + 2 \frac{u^T x_S^T (y_S^*)^\top}{(Y_{11}^*)^2} \left( y_S^* \right)^\top \left( y_S^* \right)^\top H^0 y_S^*.$$

By \textbf{1A} and \textbf{6D}, we have $Y_{11}^* \geq \sigma \left( 1 - c_1 \sqrt{\sigma/n} - C_4 \sqrt{g^2 \log(d)/n} \right)$. Thus, when $n$ is large enough, we obtain $1/Y_{11}^* \leq 1/(1 + 2c_1 \sqrt{\sigma/n} + 2C_4 \sqrt{g^2 \log(d)/n})$. Recall that $H^0 = I_{\sigma} - x_S^* (x_S^*)^\top / \sigma$, we then have

$$u^T \left( H^0 + \frac{1}{Y_{11}^*} x_S^T (x_S^*)^\top \right) u \leq \sigma \left( 1 + C_7 \sqrt{\sigma \log(d)/n} \right)^2 + \sigma \cdot \left( 2c_1 \sqrt{\sigma/n} + 2C_4 \sqrt{g^2 \log(d)/n} \right) = \sigma + \mathcal{O}(\sqrt{\sigma}),$$

when $n$ is sufficiently large. For the remaining terms, we see

$$\frac{(u^T x_S^T (y_S^*)^\top H^0 y_S^*)^2}{(Y_{11}^*)^2} \leq \left( 1 + C_7 \sqrt{\sigma \log(d)/n} \right)^2 \left( 1 + 2c_1 \sqrt{\sigma/n} + 2C_4 \sqrt{g^2 \log(d)/n} \right)^2 \cdot \frac{1}{4} \leq O(1).$$

Finally, we take $0 < \hat{c}, \hat{c} \ll 1$ small enough, and $\hat{c}' = c' \sigma - \hat{c} - C_8 \sigma \sqrt{\log(d)/n} - \hat{c} - \hat{c}$, to enforce $c' \sigma - C_8 \sigma \sqrt{\log(d)/n} - (\hat{c} + \hat{c} + \hat{c} + \hat{c}) \geq 0$. We can verify that $\hat{c} > 0$ if $n$ and $\sigma$ are sufficiently large and $\hat{c}, \hat{c}$ are chosen to be sufficiently small.

Checking the validity of (b) is straightforward by direct calculation. For (c), we first show $\|\Theta_2^F\|_\infty = \mathcal{O}(\sigma \log(d)/n + \sqrt{\log(d)/n})$, which is indeed true because $\|\Theta_2^F\|_\infty \leq \|v v^\top / \hat{c} \|_\infty + \|(M_1^T M_1/n)_{S^c, S^c} - c'' I_{d-1}\|_\infty = \mathcal{O}(\sigma \log(d)/n + \sqrt{\log(d)/n})$, where the last equality is due to \textbf{6E} and \textbf{1B} with $L^2 = c'$. Combing this fact and Lemma \textbf{6}, we obtain that

$$\|\Theta_2\|_\infty \leq \|\Theta_1\|_\infty + \|\Theta_2^B\|_\infty + \|\Theta_2^C\|_\infty + \|\Theta_2^D\|_\infty + \|\Theta_2^F\|_\infty + \|\Theta_2^E\|_\infty \leq \mathcal{O} \left( \frac{g^2 \log(d)/n}{\sigma} + \frac{\sigma \log(d)}{n} + \frac{g^2 \sigma \log(d)}{n} + \sqrt{\sigma \log(d)/n} \right) + \mathcal{O} \left( \frac{g^2 \log(d)/n}{n \sigma} + \frac{(\sqrt{\log(d)} + \sigma)^2}{n} + \frac{\sigma \log(d)}{n} \right) + \mathcal{O} \left( \frac{\sigma \log(d)}{n} + \sqrt{\log(d)/n} \right) \leq \frac{1}{4 \sigma} \min\{c-1, c''\} < \frac{1}{2 \sigma} \min\{c-1, c''\} \leq \mu_3^*.$$  

w.h.p. when $n \geq C g^2 \sigma^2 \log(d)$, for some large constant $C = C(c', c'', c''') > 0$. \hfill $\Box$
References


